Chapter 7: Sequences and Series

Part A: Sequences

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Sequences of Real Numbers

A **sequence** is an unending list of real numbers, such as:

1. $1, 2, 3, 4, \ldots$
2. $1, 1, 1, 1, \ldots$
3. $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$
4. $\sqrt{1}, -\sqrt{2}, \sqrt{3}, -\sqrt{4}, \ldots$
5. $3, 1, 4, 1, 5, 9, \ldots$
6. $0.1, -0.23, \pi, \sqrt{2}, e, \ldots$

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4. Example (6) is acceptable only if it is part of some complete assignment of real numbers to positions in the sequence.

Describing a Sequence

The general notation for a sequence is to label its members by their position, such as: $a_1, a_2, a_3, \ldots$. A more compact representation is $(a_n)_{n=1}^{\infty}$ or even just (a_n).

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Example 1

Here are some examples of describing a sequence by giving the form of its n^{th} term:

$$\begin{array}{ll}
1, 2, 3, 4, \ldots & a_n = n \\
1, 1, 1, 1, \ldots & a_n = 1 \\
1, -1, 1, -1, \ldots & a_n = (-1)^{n+1} \\
1, 1/2, 1/3, 1/4, \ldots & a_n = 1/n \\
\sqrt{1}, \sqrt{2}, \sqrt{3}, \sqrt{4}, \ldots & a_n = \sqrt{n}
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This is the most satisfactory way of describing a sequence, although it is not always possible.

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Formally, a sequence is a function $f : \mathbb{N} \to \mathbb{R}$. Such a function generates numbers $a_1 = f(1)$, $a_2 = f(2)$, $a_3 = f(3)$, ...

Limit of A Sequence

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Let (a_n) be a sequence of real numbers, and L a real number. We say that (a_n) **converges** to L if for every real number $\epsilon > 0$ there is $N \in \mathbb{N}$ such that $n \geq N$ implies $|a_n - L| < \epsilon$. The number L is called the **limit** of the sequence.

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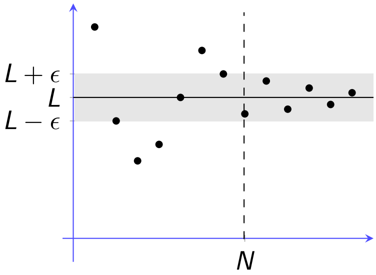

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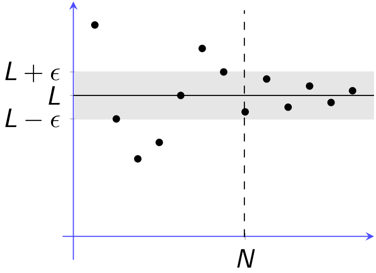


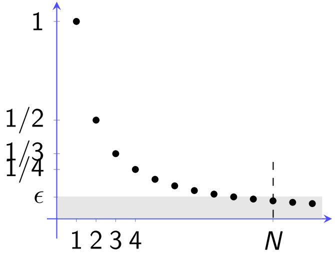
If (a_n) converges to L we say $a_n \to L$ as $n \to \infty$, or $\lim_{n\to\infty} a_n = L$. More briefly, we may just say $a_n \to L$ or $\lim a_n = L$. If a sequence does not converge, we say it **diverges**.

Example

Example 2

Let us show that $\lim_{n\to\infty} \frac{1}{n} = 0$. Consider any $\epsilon > 0$. Then $1/\epsilon > 0$. By the Archimedean Property, there is a natural number N such that $N > 1/\epsilon$. Hence $\frac{1}{N} < \epsilon$. This N works for us: If $n > N$ then

$$|a_n - L| = \left|\frac{1}{n} - 0\right| = \frac{1}{n} < \frac{1}{N} < \epsilon.$$



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Let us show that $\lim_{n\to\infty} r^n = 0$ if $|r| < 1$. Note that $|r| < 1$ implies $\frac{1}{|r|} > 1$. So we can write $\frac{1}{|r|} = 1 + h$ with $h > 0$. Hence $\frac{1}{|r|^n} = (1+h)^n > nh$ and so $|r|^n < \frac{1}{nh}$. Consider any $\epsilon > 0$. By the Archimedean Property, there is a natural number N such that $\frac{1}{N} < h\epsilon$. This N works: If $n > N$ then

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Task 1

Show that the limit of a sequence is unique, if it exists.

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Task 1

Show that the limit of a sequence is unique, if it exists.

Task 2

Let $a_n = c$ be a constant sequence. Show that $a_n \to c$.

Example of Divergence

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Example 4

Consider the sequence given by $a_n = (-1)^n$. The entries $-1, 1, -1, 1, \ldots$ keep switching between ± 1 so the sequence does not settle down and does not have a limit. How do we establish this formally?

Example of Divergence

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Example 4

Consider the sequence given by $a_n = (-1)^n$. The entries $-1, 1, -1, 1, \ldots$ keep switching between ± 1 so the sequence does not settle down and does not have a limit. How do we establish this formally?

We use the idea that if the sequence entries approach a certain number L, then they also approach each other. For example, if some numbers are each within 1 unit of L, then they are also all within 2 units of each other.

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We use the idea that if the sequence entries approach a certain number L, then they also approach each other. For example, if some numbers are each within 1 unit of L, then they are also all within 2 units of each other.

Suppose $a_n \to L$. Take $\epsilon = 1$. There will be an N such that $n \geq N$ implies $|a_n - L| < 1$. In particular, $|a_N - L| < 1$ and $|a_{N+1} - L| < 1$. Therefore, $|a_N - a_{N+1}| \leq |a_N - L| + |a_{N+1} - L| < 2$, which is false as consecutive entries actually have a gap of 2. This contradiction informs us that the sequence diverges.

Exercises

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Let (a_n) be a given sequence and k a fixed natural number. Define a sequence (b_n) by $b_n = a_{n+k}$. (That is, we drop the first k terms of the given sequence to create a new sequence) Show that $\lim b_n = L$ if and only if $\lim a_n = L$.

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Suppose (a_n) *is a converging sequence and* $m \leq a_n \leq M$ *for every* n*. Then* $m \leq \lim_{n\to\infty} a_n \leq M$*.*

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Task 6

Let $|a_n| \to 0$. Show that $a_n \to 0$.

Sandwich or Squeeze Theorem

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Theorem 5 (Sandwich or Squeeze Theorem)

Let (a_n), (b_n), (c_n) be sequences such that for every n, $a_n \leq b_n \leq c_n$. If $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L$ then $\lim\limits_{n\to\infty} b_n = L$.

Sandwich or Squeeze Theorem

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Theorem 5 (Sandwich or Squeeze Theorem)

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Proof. Consider any $\epsilon > 0$. Then

$$a_n \to L \implies \text{there is } N_a \text{ such that if } n > N_a \text{ then } L - \epsilon < a_n < L + \epsilon,$$
$$c_n \to L \implies \text{there is } N_c \text{ such that if } n > N_c \text{ then } L - \epsilon < c_n < L + \epsilon.$$

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Example 6

Consider $a_n = r^n/n!$ where $r > 0$ is fixed. Fix $M \in \mathbb{N}$ such that $M > r$.

$$\text{For } n > M,\ 0 < \frac{r^n}{n!} = \frac{r}{n} \cdots \frac{r}{M+1} \cdot \frac{r^M}{M!} < \frac{1}{n} \cdot \frac{r^{M+1}}{M!} \to 0.$$

Hence $\dfrac{r^n}{n!} \to 0$.

Algebra of Limits

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Theorem 7

Let $a_n \to L$ and $b_n \to M$. Also, let $c \in \mathbb{R}$. Then:

1. $|a_n| \to |L|$.
2. $c\, a_n \to c\, L$.
3. $a_n + b_n \to L + M$.
4. $a_n - b_n \to L - M$.
5. $a_n b_n \to LM$.
6. $a_n / b_n \to L/M$ *if* $M \neq 0$.

Proof. The proofs are similar to the algebra of limits for functions. □

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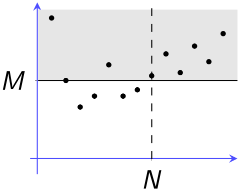
Task 7

Find the following limits.

1. $\displaystyle \lim_{n\to\infty} \frac{5n^2 - 1}{n^2 + 3n - 1000}$.
2. $\displaystyle \lim_{n\to\infty} \frac{\sin n}{n}$.

Infinite Limits

We say that $\lim_{n\to\infty} a_n = \infty$ if for every real number M there is an $N \in \mathbb{N}$ such that $n \geq N$ implies $a_n > M$. Similarly, we say $\lim_{n\to\infty} a_n = -\infty$ if for every real number M there is an $N \in \mathbb{N}$ such that $n \geq N$ implies $a_n < M$.



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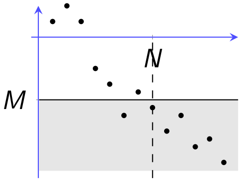

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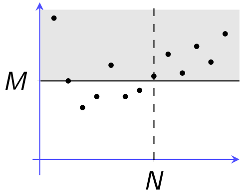

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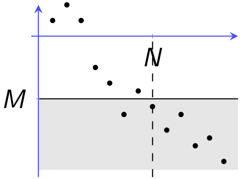

$a_n \to -\infty$

Example 8

We'll show $2^n/n \to \infty$. We have $\frac{2^n}{n} = \frac{(1+1)^n}{n} > \frac{n(n-1)}{2n} = \frac{n-1}{2}$.
So, for any given M, choose $N = 2M + 1$.

Exercises

Task 8

Prove the following.

1. $\lim n = \infty$.
2. *If* $a_n \geq b_n$ *for every* n, *and* $b_n \to \infty$, *then* $a_n \to \infty$.
3. *Suppose* $a_n \neq 0$ *for every* n. *Then* $a_n \to 0$ *if and only if* $|1/a_n| \to \infty$.

Bounded Sequences

Consider a sequence (a_n). It is called

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- **Bounded** if it is both bounded above and bounded below, and
- **Unbounded** if it is not bounded.

Exercise

Task 9

For each given sequence, put a ✓ in each correct category and a ✗ in each incorrect category:

a_n	*Bounded Above*	*Bounded Below*	*Bounded*	*Unbounded*
n				
$-n$				
$(-1)^n$				
$(-1)^n n$				
$1/n$				

Convergent Sequences are Bounded

Theorem 9

Every convergent sequence is bounded.

Proof. Take $\epsilon = 1$.

There will be an N such that $n \geq N$ implies $|a_n - L| < 1$ and so $L - 1 < a_n < L + 1$.

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There will be an N such that $n \geq N$ implies $|a_n - L| < 1$ and so $L - 1 < a_n < L + 1$.

In addition, the entries $a_1, \ldots, a_{N-1}$ are finitely many and have a maximum value M and a minimum value m.

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In addition, the entries $a_1, \ldots, a_{N-1}$ are finitely many and have a maximum value M and a minimum value m.

Then the entire sequence (a_n) lies between $\min\{m, L-1\}$ and $\max\{M, L+1\}$. □

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$(-1)^n$			
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Proof. Suppose (a_n) is increasing and bounded. We'll show it converges to $L = \sup\{a_n : n \in \mathbb{N}\}$.
Consider any $\epsilon > 0$. Then $L - \epsilon$ is not an upper bound for $\{a_n : n \in \mathbb{N}\}$.
Hence there is $N \in \mathbb{N}$ such that $L - \epsilon < a_N \leq L$. This N works.

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Proof. Suppose (a_n) is increasing and bounded. We'll show it converges to $L = \sup\{a_n : n \in \mathbb{N}\}$.
Consider any $\epsilon > 0$. Then $L - \epsilon$ is not an upper bound for $\{a_n : n \in \mathbb{N}\}$.
Hence there is $N \in \mathbb{N}$ such that $L - \epsilon < a_N \leq L$. This N works.
Similarly, if (a_n) is decreasing and bounded, it converges to $\inf\{a_n : n \in \mathbb{N}\}$. $\square$

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It is enough to show that $|r|^n \to 0$. Since $|r| < 1$, the sequence $|r|^n$ is a decreasing sequence, and it is bounded below by 0. So it converges.
Suppose it converges to L.

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Suppose it converges to L.
Now $|r|^{n+1}$ will have the same limit L. But $|r|^{n+1} = |r||r|^n \to |r|L$. This gives $L = |r|L$ and hence $L = 0$.

Example

Example 12

Let (a_n) be a decreasing sequence that converges to 0. We shall show that $2^{a_n} \to 1$.

First, since (a_n) is decreasing, so is 2^{a_n}.

Second, since $a_n \geq 0$, $2^{a_n} \geq 1$. Hence $2^{a_n} \to L \geq 1$.

To complete the proof we need to show that 1 is the greatest lower bound of the set $\{2^{a_n}\}$. We already know that it is a lower bound.

So consider any number $1 + \epsilon$ with $\epsilon > 0$. Then $\log_2(1 + \epsilon) > 0$. Since $a_n \to 0$ we have an N such that $a_N < \log_2(1 + \epsilon)$. Hence $2^{a_N} < 1 + \epsilon$. Therefore $1 + \epsilon$ is not an upper bound for $\{2^{a_n}\}$.

Example

Example 13

Consider the sequence defined recursively by $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2a_n}$. We shall consider two approaches to investigate its limit. In the first approach, we try to obtain a direct formula for a_n. The first few terms are

$$a_1 = 2^{1/2},$$
$$a_2 = \sqrt{2}\sqrt{a_1} = 2^{3/4},$$
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The pattern is $a_n = 2^{1-1/2^n}$. We leave it for you to verify this by mathematical induction. We can now calculate, using the previous example and the fact that $1/2^n \to 0$, that

$$\lim a_n = \lim 2^{1-1/2^n} = \frac{2}{\lim 2^{1/2^n}} = 2.$$

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Example 14

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This can be proved by mathematical induction:

(a) $a_1 = \sqrt{2} < 2$, (b) $a_n < 2 \implies a_{n+1} = \sqrt{2a_n} < \sqrt{2 \times 2} = 2$.

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Hence the sequence is increasing as well as bounded above (by the number 2). Therefore it is convergent. Suppose it converges to L. From the defining relation $a_{n+1} = \sqrt{2a_n}$, we get $a_{n+1}^2 = 2a_n$ and hence $L^2 = 2L$. This implies $L = 0$ or 2. As the sequence has positive and increasing terms it cannot have 0 as a limit. Hence $L = 2$.

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Given a sequence, a **subsequence** is created by dropping some of the terms of the sequence, as long as infinitely many terms still remain.

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Thus, consider a sequence (a_n). Let the first term which is retained be a_{n_1}. Let the second term which is retained be a_{n_2}, with $n_2 > n_1$. In this way we create a new sequence with terms $b_i = a_{n_i}$, and call it a subsequence of the original one.

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If a sequence converges to L, then each of its subsequences also converges to L.

Proof. Let $a_n \to L$. Consider a subsequence $b_k = a_{n_k}$ with $n_1 < n_2 < \cdots$. First, note that $n_k \geq k$. Now, for any $\epsilon > 0$, there is $N \in \mathbb{N}$ such that $n \geq N$ implies $|a_n - L| < \epsilon$. Then $k \geq N$ implies $n_k \geq k \geq N$ implies $|b_k - L| = |a_{n_k} - L| < \epsilon$. □

Subsequences

It may happen that a sequence involves two or more different patterns. For example, the odd terms $a_1, a_3, \dots$ may follow one rule while the even terms $a_2, a_4, \dots$ follow another rule. The concept of subsequences helps in such situations.

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Consider $1, 1, 2, 1/2, 3, 1/3, 4, 1/4, \ldots$. The subsequence $1, 2, 3, 4, \ldots$ diverges and so the original sequence diverges.
Again, consider $1, -1, 1, -1, \ldots$. The subsequence $1, 1, \ldots$ converges to 1. The subsequence $-1, -1, \ldots$ converges to -1. Since the two subsequences have different limits, the original sequence diverges.

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Task 11

1. *Show that* $\lim a_n = L$ *if and only if* $\lim a_{2n+1} = \lim a_{2n} = L$.
2. *Evaluate* $\displaystyle\lim_{n\to\infty} \frac{(-1)^n n}{n+1}$.

Table of Contents

Functions Applied to Sequences

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Take a positive number c and consider the sequence $(c^{1/n})$. Now, the function $f(x) = c^x$ is continuous at every x. Hence,

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Show that $\log a_n \to L \implies a_n \to e^L$.

Sequences and Derivatives

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Let $f(x)$ be differentiable at $x = L$. Then

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Sequences and Derivatives

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Consider the sequence $(1+1/n)^n$. First, we apply the log function to convert it into a product which we can evaluate by the last theorem.

$$\lim \log(1+1/n)^n = \lim n\big(\log(1+1/n) - \log 1\big) = \log' 1 = 1.$$

And now, by the continuity of the exponential function,

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True or False: If $f(x) \to L$ *as* $x \to a$, *and* $a_n \to a$, *then* $f(a_n) \to L$.

Sequences from Real Functions

Theorem 21

Let $f(x)$ be a real function with domain $[1, \infty)$ and let $\lim_{x\to\infty} f(x) = L$. Suppose $a_n = f(n)$ for $n \in \mathbb{N}$. Then $\lim_{n\to\infty} a_n = L$.

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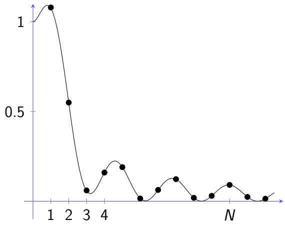

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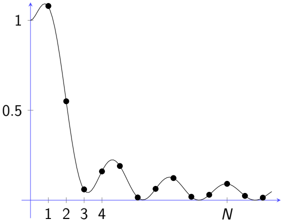

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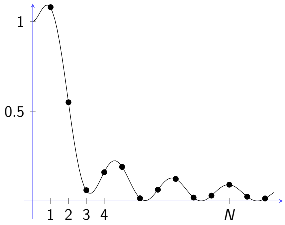

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Examples

Example 22

We will calculate the limit of $a_n = n \sin(1/n)$.

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Example 23

Consider the sequence $1/(\arctan n)^n$. First, we note that $\lim \arctan n = \lim_{x\to\infty} \arctan x = \pi/2$. Hence $1/(\arctan n) \to 2/\pi < 1$.

Examples

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L'Hôpital's Rule

A major gain from the last theorem is that the results for functions, such as L'Hôpital's Rule, can be applied to sequences.

Example 24

Consider the sequence $(n^{1/n})$. We start by applying log to convert to a ratio: $a_n = \log(n^{1/n}) = \dfrac{\log n}{n}$. Since $\lim\limits_{x\to\infty} \dfrac{\log x}{x} = 0$, we have $\lim \dfrac{\log n}{n} = 0$. Hence, $\lim n^{1/n} = e^0 = 1$.

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Task 15

Find the limits of the following sequences.

1. $\dfrac{e^n}{n^{100}}$.
2. $\dfrac{\log n}{\sqrt{n}}$.

Stirling's Approximation

Our later study of 'infinite series' will bring up the sequence $(n!)^{1/n}$.

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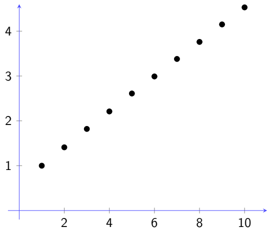

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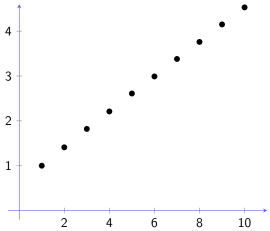

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n	10	100	1000	10000
slope	0.453	0.3799	0.3695	0.3681
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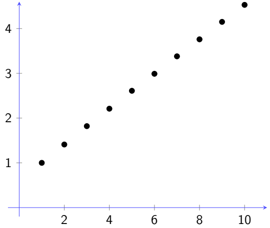

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The reciprocals could be approaching $e \approx 2.718\ldots$.

Stirling's Approximation

Theorem 25

$$\lim_{n\to\infty} \frac{(n!)^{1/n}}{n/e} = 1, \textit{ i.e. } (n!)^{1/n} \approx \frac{n}{e} \textit{ for large } n.$$

Stirling's Approximation

Theorem 25

$$\lim_{n\to\infty} \frac{(n!)^{1/n}}{n/e} = 1, \textit{ i.e. } (n!)^{1/n} \approx \frac{n}{e} \textit{ for large } n.$$

Proof. Consider $\log n! = \sum_{k=1}^{n} \log k$.

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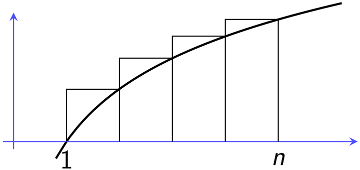
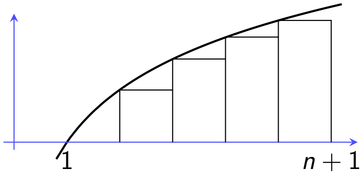
Proof. Consider $\log n! = \sum_{k=1}^{n} \log k$.

It is an upper sum for $\int_1^n \log x \, dx$ and a lower sum for $\int_1^{n+1} \log x \, dx$.



$\int_1^n \log x \, dx < \sum_{k=1}^{n} \log k$



$\sum_{k=1}^{n} \log k < \int_1^{n+1} \log x \, dx$

(continued...)

Stirling's Approximation

(...continued)

$$\begin{aligned}
\int_1^n \log x\,dx < &\quad \log n! &< \int_1^{n+1} \log x\,dx \\
\implies n\log n - n + 1 < &\quad \log n! &< (n+1)\log(n+1) - n \\
\implies \log\left(\frac{n^n}{e^{n-1}}\right) < &\quad \log n! &< \log\left(\frac{(n+1)^{n+1}}{e^n}\right) \\
\implies \frac{n^n}{e^{n-1}} < &\quad n! &< \frac{(n+1)^{n+1}}{e^n} \\
\implies e^{1/n} < &\quad \frac{(n!)^{1/n}}{n/e} &< (1+1/n)(n+1)^{1/n}.
\end{aligned}$$

Now, $e^{1/n} \to 1$ and $(1+\frac{1}{n})^{1+1/n} \to 1$ and $n^{1/n} \to 1$. Apply Sandwich theorem to finish the proof. $\square$

Example

Example 26

Here is an application of Stirling's Approximation.

$$(n!)^{1/n^2} = \left(\frac{n!^{1/n}}{n/e}\right)^{1/n} \left(\frac{n}{e}\right)^{1/n} = \left(\frac{n!^{1/n}}{n/e}\right)^{1/n} \frac{n^{1/n}}{e^{1/n}}.$$

Now $a_n = \dfrac{n!^{1/n}}{n/e} \to 1$. Hence $a_n^{1/n} \to 1$ (To prove this, apply log).

We already know that $n^{1/n} \to 1$ and $e^{1/n} \to 1$.

Hence, $(n!)^{1/n^2} \to 1$.

Table of Contents

Newton-Raphson Method

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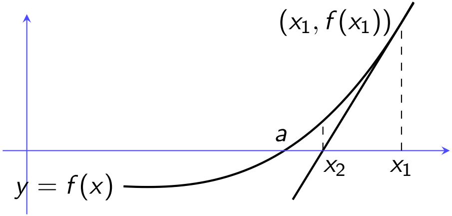
Suppose we have an equation such as $x^3 - 3 = x^2 + x$ which we have to solve for x. We move every term to the left side to put it in the form $f(x) = 0$.



We wish to estimate the point a where $f(a) = 0$. Imagine you are at a point $(x_1, f(x_1))$ on the graph of the function f. In which direction should you move to move towards a? One idea is to generate the tangent line at $(x_1, f(x_1))$ and see where it cuts the x-axis. If it does so at x_2, we repeat the process from the point $(x_2, f(x_2))$.

Newton-Raphson Method

The equation of the tangent line at $(x_1, f(x_1))$ is

$$y = f(x_1) + f'(x_1)(x - x_1).$$

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We have created a sequence $x_1, x_2, x_3, \ldots$. Does it converge? And does it converge to a?
The good news is that if it converges at all, then it converges to a solution. If $x_n \to L$ then

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \implies L = L - \frac{f(L)}{f'(L)} \implies f(L) = 0.$$

Example

Example 27

Suppose the equation we wish to solve is $x^2 = N$, that is, we want to estimate a square root.

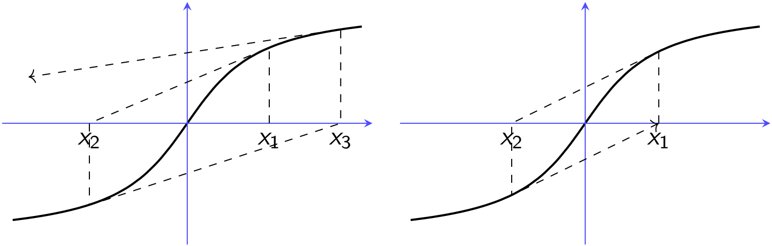
We rearrange this as $f(x) = 0$, where $f(x) = x^2 - N$.

Then $f'(x) = 2x$, and the Newton-Raphson iteration is

$$x_{n+1} = x_n - \frac{x_n^2 - N}{2x_n} = \frac{x_n^2 + N}{2x_n} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right).$$

Let $N = 2$. If we start with $x_1 = 1$, we get $x_2 = 3/2 = 1.5$, $x_3 = 17/12 = 1.417$, and so on.

Failure of Convergence



In the first example, the sequence moves away from the solution of $f(x) = 0$. In the second example, it cycles repeatedly through the same two values.