11 Chapter 11 Solution Set

Problems

11.1 Derive Eqs.(11.15) using Green function techniques.

We use the two equations satisfied by the E field and the Green function dyadic

$$\nabla_{r'} \times \nabla_{r'} \times \overleftrightarrow{\mathbf{G}}(\mathbf{r}, \mathbf{r}', \omega) - k_0^2 \overleftrightarrow{\mathbf{G}}(\mathbf{r}, \mathbf{r}', \omega) = \overleftrightarrow{\mathbf{I}}\,\delta(\mathbf{r} - \mathbf{r}'),$$
$$\nabla_{r'} \times \nabla_{r'} \times \mathbf{E}(\mathbf{r}') - k_0^2 \mathbf{E}(\mathbf{r}') = i\omega\mu_0 \mathbf{J}(\mathbf{r}').$$

If we now take the dot product from the right of the top equation with the E field and the dot product of the bottom equation from the left with the Green function dyadic and subtract the two resulting equations we obtain

$$[\nabla_{r'} \times \nabla_{r'} \times \overleftrightarrow{\mathbf{G}}(\mathbf{r}, \mathbf{r}', \omega)] \cdot \mathbf{E}(\mathbf{r}') - \overleftrightarrow{\mathbf{G}}(\mathbf{r}, \mathbf{r}', \omega) \cdot [\nabla_{r'} \times \nabla_{r'} \times \mathbf{E}(\mathbf{r}')]$$
$$= \overleftrightarrow{\mathbf{I}}\,\delta(\mathbf{r} - \mathbf{r}') \cdot \mathbf{E}(\mathbf{r}') - i\omega\mu_0 \overleftrightarrow{\mathbf{G}}(\mathbf{r}, \mathbf{r}', \omega) \cdot \mathbf{J}(\mathbf{r}')$$

We now integrate over a volume τ to obtain

$$\overbrace{\int_\tau d^3r' \left\{[\nabla_{r'} \times \nabla_{r'} \times \overleftrightarrow{\mathbf{G}}(\mathbf{r}, \mathbf{r}', \omega)] \cdot \mathbf{E}(\mathbf{r}') - \overleftrightarrow{\mathbf{G}}(\mathbf{r}, \mathbf{r}', \omega) \cdot [\nabla_{r'} \times \nabla_{r'} \times \mathbf{E}(\mathbf{r}')]\right\}}^{I_\tau(\mathbf{r})}$$
$$= \mathbf{E}(\mathbf{r}) - i\omega\mu_0 \int_\tau d^3r' \, \overleftrightarrow{\mathbf{G}}(\mathbf{r}, \mathbf{r}', \omega) \cdot \mathbf{J}(\mathbf{r}')$$

where we assume that the field point $\mathbf{r} \in \tau$. We now show that the quantity $I_\tau(\mathbf{r})$ vanishes in the limit where τ becomes infinite and the Green function dyadic and E field are outgoing at infinity (satisfy the Sommerfeld radiation condition). This then yields the desired result showing that the E field is given in terms of the Green function dyadic by Eqs.(11.15a). A similar calculation yields the equation for the H field.

There are a number of approaches that can be employed to evaluate $I_\tau(\mathbf{r})$ two of the best being in Chew's book (Chew, 1990). Perhaps the easiest is the use of the vector identity (Chew, 1990)

$$[\nabla_{r'} \times \nabla_{r'} \times \overleftrightarrow{\mathbf{G}}(\mathbf{r}, \mathbf{r}', \omega)] \cdot \mathbf{E}(\mathbf{r}') - \overleftrightarrow{\mathbf{G}}(\mathbf{r}, \mathbf{r}', \omega) \cdot [\nabla_{r'} \times \nabla_{r'} \times \mathbf{E}(\mathbf{r}')]$$
$$= -\nabla \cdot \{\mathbf{E}(\mathbf{r}') \times \nabla_{r'} \times \overleftrightarrow{\mathbf{G}}(\mathbf{r}, \mathbf{r}', \omega) + \nabla_{r'} \times \mathbf{E}(\mathbf{r}') \times \overleftrightarrow{\mathbf{G}}(\mathbf{r}, \mathbf{r}', \omega)\}$$

to convert the volume integral to a surface integral using Gauss' divergence theorem. Using this identity we find that

$$I_\tau(\mathbf{r}) = -\int_{\partial\tau} dS'\,\hat{\mathbf{n}}'\cdot\{\mathbf{E}(\mathbf{r}')\times\nabla_{r'}\times\overleftrightarrow{\mathbf{G}}(\mathbf{r},\mathbf{r}',\omega)+\nabla_{r'}\times\mathbf{E}(\mathbf{r}')\times\overleftrightarrow{\mathbf{G}}(\mathbf{r},\mathbf{r}',\omega)\},$$

where $\hat{\mathbf{n}}'$ is the unit vector along $\mathbf{r}'$.

We now have to make use of the radiation conditions satisfied by the field and Green function dyadic. The E and H fields satisfy Eqs.(11.16) while the Green function dyadic satisfies the far field condition

$$\overleftrightarrow{\mathbf{G}}(\mathbf{r},\mathbf{r}',\omega) = [\overleftrightarrow{\mathbf{I}}+\frac{1}{k_0^2}\nabla_{r'}\nabla_{r'}]G_+(\mathbf{r}-\mathbf{r}') \sim -\frac{1}{4\pi}[\overleftrightarrow{\mathbf{I}}-\hat{\mathbf{n}}'\hat{\mathbf{n}}']e^{-ik_0\hat{\mathbf{n}}'\cdot\mathbf{r}}\frac{e^{ik_0r'}}{r'}$$

as $r'\to\infty$. We then find that

$$\nabla_{r'}\times\overleftrightarrow{\mathbf{G}}(\mathbf{r},\mathbf{r}',\omega) \sim -\frac{ik_0}{4\pi}\hat{\mathbf{n}}'\times[\overleftrightarrow{\mathbf{I}}-\hat{\mathbf{n}}'\hat{\mathbf{n}}']e^{-ik_0\hat{\mathbf{n}}'\cdot\mathbf{r}}\frac{e^{ik_0r'}}{r'} = -\frac{ik_0}{4\pi}\hat{\mathbf{n}}'\times\overleftrightarrow{\mathbf{I}}\,e^{-ik_0\hat{\mathbf{n}}'\cdot\mathbf{r}}\frac{e^{ik_0r'}}{r'}.$$

so that

$$\begin{aligned}\mathbf{E}(\mathbf{r}')\times\nabla_{r'}\times\overleftrightarrow{\mathbf{G}}(\mathbf{r},\mathbf{r}',\omega) &\sim -\frac{ik_0}{4\pi}\mathbf{f}_e(\mathbf{n}')\times[\hat{\mathbf{n}}'\times\overleftrightarrow{\mathbf{I}}]e^{-2ik_0\mathbf{n}'\cdot\mathbf{r}}\frac{e^{2ik_0r'}}{r'^2}\\ &= -\frac{ik_0}{4\pi}\mathbf{f}_e(\mathbf{n}')\hat{\mathbf{n}}'e^{-2ik_0\mathbf{n}'\cdot\mathbf{r}}\frac{e^{2ik_0r'}}{r'^2}\end{aligned}$$

since $\hat{\mathbf{n}}'\cdot\mathbf{f}_e(\mathbf{n}')=0$ and where we have used the vector identity $\mathbf{a}\times[\mathbf{b}\times\mathbf{c}] = (\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}$. We also find that

$$\begin{aligned}\nabla_{r'}\times\mathbf{E}(\mathbf{r}')\times\overleftrightarrow{\mathbf{G}}(\mathbf{r},\mathbf{r}',\omega) &\sim -\frac{ik_0}{4\pi}[\hat{\mathbf{n}}'\times\mathbf{f}_e(\mathbf{n}')]\times[\overleftrightarrow{\mathbf{I}}-\hat{\mathbf{n}}'\hat{\mathbf{n}}']e^{-2ik_0\mathbf{n}'\cdot\mathbf{r}}\frac{e^{2ik_0r'}}{r'^2}\\ &= \frac{ik_0}{4\pi}\mathbf{f}_e(\mathbf{n}')\hat{\mathbf{n}}'e^{-2ik_0\mathbf{n}'\cdot\mathbf{r}}\frac{e^{2ik_0r'}}{r'^2}\end{aligned}$$

where we have again used the above vector identity and the fact that $\hat{\mathbf{n}}'\cdot\mathbf{f}_e(\mathbf{n}')=0$. Using the above two results gives

$$\begin{aligned}&\mathbf{E}(\mathbf{r}')\times\nabla_{r'}\times\overleftrightarrow{\mathbf{G}}(\mathbf{r},\mathbf{r}',\omega)+\nabla_{r'}\times\mathbf{E}(\mathbf{r}')\times\overleftrightarrow{\mathbf{G}}(\mathbf{r},\mathbf{r}',\omega)\\ &\sim -\frac{ik_0}{4\pi}[\mathbf{f}_e(\mathbf{n}')\hat{\mathbf{n}}'-\mathbf{f}_e(\mathbf{n}')\hat{\mathbf{n}}']e^{-2ik_0\mathbf{n}'\cdot\mathbf{r}}\frac{e^{2ik_0r'}}{r'^2}=0.\end{aligned}$$

The integrand of the surface integral is thus zero to order r'^2 with the result that the surface integral is zero in the limit $r'\to\infty$ which establishes the desired result.

11.2 Compute the radiation pattern of the Green function dyadic.

See solution to preceding problem.

11.3 Derive Eqs.(11.23).

The easiest approach is to use the Maxwell equation relating the E and H fields and the angular spectrum expansion for the E field given in Eq.(11.22a). Using the Maxwell equation we have that

$$\mathbf{H}_+(\mathbf{r}) = \frac{-i}{\omega\mu_0}\nabla\times\mathbf{E}_+(\mathbf{r})$$

which then yields

$$\begin{aligned}\mathbf{H}_+(\mathbf{r}) &= \frac{-i}{\omega\mu_0}\nabla\times\{\frac{ik_0}{2\pi}\int_{-\pi}^{\pi}d\beta\int_{C_\pm}\sin\alpha d\alpha\,\mathbf{A}_e(k_0\mathbf{s})e^{ik_0\mathbf{s}\cdot\mathbf{r}}\}\\ &= \frac{k_0}{2\pi\omega\mu_0}\int_{-\pi}^{\pi}d\beta\int_{C_\pm}\sin\alpha d\alpha\,\nabla\times\mathbf{A}_e(k_0\mathbf{s})e^{ik_0\mathbf{s}\cdot\mathbf{r}}\\ &= \frac{ik_0^2}{2\pi\omega\mu_0}\int_{-\pi}^{\pi}d\beta\int_{C_\pm}\sin\alpha d\alpha\,\mathbf{s}\times\mathbf{A}_e(k_0\mathbf{s})e^{ik_0\mathbf{s}\cdot\mathbf{r}}\\ &\frac{ik_0}{2\pi}\int_{-\pi}^{\pi}d\beta\int_{C_\pm}\sin\alpha d\alpha\,\overbrace{\frac{k_0}{\omega\mu_0}\mathbf{s}\times\mathbf{A}_e(k_0\mathbf{s})}^{\mathbf{A}_h(k_0\mathbf{s})}\,e^{ik_0\mathbf{s}\cdot\mathbf{r}}\end{aligned}$$

where the last line follows from Eq.(11.24).

11.4 Derive Eqs.(11.26) from Eqs.(11.22b) and (11.23b). These derivations are easily accomplished.

11.5 Prove the identities:

$$\mathbf{L}\cdot[\mathbf{s}\times\mathbf{L}] = [\mathbf{s}\times\mathbf{L}]\cdot\mathbf{L} = 0, \quad [\mathbf{s}\times\mathbf{L}]\cdot[\mathbf{s}\times\mathbf{L}] = L^2.$$

To prove these identities we make use of the definition of the angular momentum operator in terms of the derivatives w.r.t. α and β given in the book

$$\mathbf{L} = -ik_0\mathbf{s}\times\nabla_{k_0\mathbf{s}} = i[\hat{\boldsymbol{\alpha}}\frac{1}{\sin\alpha}\frac{\partial}{\partial\beta} - \hat{\boldsymbol{\beta}}\frac{\partial}{\partial\alpha}],$$

and the fact that the spherical coordinate system with unit vectors $\mathbf{s}, \hat{\boldsymbol{\alpha}}, \hat{\boldsymbol{\beta}}$ is a right-handed orthogonal system and that the unit vectors satisfy the equations

$$\frac{\partial}{\partial\alpha}\hat{\boldsymbol{\alpha}} = -\mathbf{s}, \quad \frac{\partial}{\partial\alpha}\hat{\boldsymbol{\beta}} = 0, \quad \frac{\partial}{\partial\beta}\hat{\boldsymbol{\alpha}} = \cos\alpha\hat{\boldsymbol{\beta}}, \quad \frac{\partial}{\partial\beta}\hat{\boldsymbol{\beta}} = -\cos\alpha\hat{\boldsymbol{\alpha}} - \sin\alpha\mathbf{s}. \tag{11.1}$$

We will only prove the first the identity $\mathbf{L}\cdot[\mathbf{s}\times\mathbf{L}] = 0$ since the proof of the others follows entirely parallel lines. We have that

$$\mathbf{s}\times\mathbf{L} = i\mathbf{s}\times[\hat{\boldsymbol{\alpha}}\frac{1}{\sin\alpha}\frac{\partial}{\partial\beta} - \hat{\boldsymbol{\beta}}\frac{\partial}{\partial\alpha}] = i[\hat{\boldsymbol{\beta}}\frac{1}{\sin\alpha}\frac{\partial}{\partial\beta} + \hat{\boldsymbol{\alpha}}\frac{\partial}{\partial\alpha}]$$

We then find that

$$\begin{aligned}\mathbf{L}\cdot[\mathbf{s}\times\mathbf{L}] &= -[\hat{\boldsymbol{\alpha}}\frac{1}{\sin\alpha}\frac{\partial}{\partial\beta} - \hat{\boldsymbol{\beta}}\frac{\partial}{\partial\alpha}]\cdot[\hat{\boldsymbol{\beta}}\frac{1}{\sin\alpha}\frac{\partial}{\partial\beta} + \hat{\boldsymbol{\alpha}}\frac{\partial}{\partial\alpha}]\\ &= -\{\hat{\boldsymbol{\alpha}}\frac{1}{\sin\alpha}\frac{\partial}{\partial\beta}\cdot\hat{\boldsymbol{\beta}}\frac{1}{\sin\alpha}\frac{\partial}{\partial\beta} + \hat{\boldsymbol{\alpha}}\frac{1}{\sin\alpha}\frac{\partial}{\partial\beta}\cdot\hat{\boldsymbol{\alpha}}\frac{\partial}{\partial\alpha}\}\\ &+\{\hat{\boldsymbol{\beta}}\frac{\partial}{\partial\alpha}\cdot\hat{\boldsymbol{\beta}}\frac{1}{\sin\alpha}\frac{\partial}{\partial\beta} + \hat{\boldsymbol{\beta}}\frac{\partial}{\partial\alpha}\cdot\hat{\boldsymbol{\alpha}}\frac{\partial}{\partial\alpha}\}\end{aligned}$$

We now carry out the various differentiations and make use of the identities given in Eq.(11.1) to find that

$$\hat{\boldsymbol{\alpha}}\frac{1}{\sin\alpha}\frac{\partial}{\partial\beta}\cdot\hat{\boldsymbol{\beta}}\frac{1}{\sin\alpha}\frac{\partial}{\partial\beta} = -\frac{\cos\alpha}{\sin^2\alpha}\frac{\partial}{\partial\beta}$$
$$\hat{\boldsymbol{\alpha}}\frac{1}{\sin\alpha}\frac{\partial}{\partial\beta}\cdot\hat{\boldsymbol{\alpha}}\frac{\partial}{\partial\alpha} = \frac{1}{\sin\alpha}\frac{\partial^2}{\partial\alpha\partial\beta}$$
$$\hat{\boldsymbol{\beta}}\frac{\partial}{\partial\alpha}\cdot\hat{\boldsymbol{\beta}}\frac{1}{\sin\alpha}\frac{\partial}{\partial\beta} = -\frac{\cos\alpha}{\sin^2\alpha}\frac{\partial}{\partial\beta}+\frac{1}{\sin\alpha}\frac{\partial^2}{\partial\alpha\partial\beta}$$
$$\hat{\boldsymbol{\beta}}\frac{\partial}{\partial\alpha}\cdot\hat{\boldsymbol{\alpha}}\frac{\partial}{\partial\alpha} = 0.$$

We then obtain

$$\mathbf{L}\cdot[\mathbf{s}\times\mathbf{L}] = \frac{\cos\alpha}{\sin^2\alpha}\frac{\partial}{\partial\beta} - \frac{1}{\sin\alpha}\frac{\partial^2}{\partial\alpha\partial\beta} - \frac{\cos\alpha}{\sin^2\alpha}\frac{\partial}{\partial\beta} + \frac{1}{\sin\alpha}\frac{\partial^2}{\partial\alpha\partial\beta} = 0$$

11.6 Derive Eqs.(11.30) from Eqs.(11.29).

This follows directly by making use of the identities given in the derivation.

11.7 Prove that an EM non-radiating source must have zero total charge.

From the first Maxwell equation we have

$$\epsilon_0\nabla\cdot\mathbf{E}(\mathbf{r}) = \rho(\mathbf{r})$$

which yields

$$\epsilon_0\int_{\tau_0} d^3r\,\nabla\cdot\mathbf{E}(\mathbf{r}) = \epsilon_0\int_{\partial\tau_0} dS'\,\mathbf{n}\cdot\mathbf{E}(\mathbf{r}) = \int_{\tau_0} d^3r\,\rho(\mathbf{r}) = Q$$

where Q is the total charge inclosed in the source volume τ_0 and $\partial\tau_0$ is the boundary of τ_0 and $\mathbf{n}$ the outward directed normal to $\partial\tau_0$. An NR source generates zero field outside the source region from which it follows that the surface integral vanishes[1] and the total charge has to be zero.

11.8 Derive a general expression for a non-radiating EM source supported within a spherical region.

From Example 11.5 we can express the transverse current in the Debye representation

$$\mathbf{J}_T(\mathbf{r}) = \nabla\times\mathbf{r}Q_h^d(\mathbf{r}) + \frac{i}{\omega\epsilon_0}\nabla\times\nabla\times\mathbf{r}Q_e^d(\mathbf{r})$$

where the two scalar source components $Q_h^d(\mathbf{r})$ and $Q_e^d(\mathbf{r})$ have spatial Fourier transforms that satisfy the equations

$$\tilde{Q}_h^d(k_0\mathbf{s}) = \frac{-4\pi i}{\omega\mu_0}f_h^d(\mathbf{s}),\quad \tilde{Q}_e^d(k_0\mathbf{s}) = \frac{-4\pi i}{\omega\mu_0}f_e^d(\mathbf{s}),$$

where $f_h^d(\mathbf{s})$ and $f_e^d(\mathbf{s})$ are the magnetic and electric scalar radiation patterns of the Debye representation of the magnetic and electric field radiation patterns defined in Eqs.(11.42). An NR source has to have zero electric and

[1] We assume at least a piecewise continuous source so that the field is continuous.

magnetic field radiation patterns from which it follows that the two boundary values of the scalar source components $\tilde{Q}_h^d(k_0\mathbf{s})$ and $\tilde{Q}_e^d(k_0\mathbf{s})$ must also vanish. This, in turn, requires these two scalar sources to be classical NR scalar sources that are reviewed extensively in Chapter 1 and 2.

11.9 Derive the most general form of an EM surface source supported over a plane surface.

Here we use the Whittaker representation of a 3D transverse current presented in Example 11.3:

$$\mathbf{J}_T(\mathbf{r}) = \nabla \times \hat{\mathbf{z}} Q_h^w(\mathbf{r}) + \frac{i}{\omega\epsilon_0}\nabla \times \nabla \times \hat{\mathbf{z}} Q_e^w(\mathbf{r}).$$

We now assume that both scalar source components are supported on the plane $z = 0$ and can then be expressed in terms of singlet and doublet components in the form (cf., Sections 1.8 and 2.12 and Problem 5.8)

$$Q_h^w(\mathbf{r}) = Q_{hs}^w(\boldsymbol{\rho})\delta(z) + Q_{hd}^w(\boldsymbol{\rho})\delta'(z), \quad Q_e^w(\mathbf{r}) = Q_{es}^w(\boldsymbol{\rho})\delta(z) + Q_{ed}^w(\boldsymbol{\rho})\delta'(z)$$

where $\boldsymbol{\rho}$ is the position vector on the $z = 0$ plane and we have labeled the "singlet" and "doublet" components with the subscripts s and d, respectively. We then find that

$$\mathbf{J}_T(\mathbf{r}) = \nabla \times \hat{\mathbf{z}} \overbrace{[Q_{hs}^w(\boldsymbol{\rho})\delta(z) + Q_{hd}^w(\boldsymbol{\rho})\delta'(z)]}^{Q_h^w(\mathbf{r})}$$
$$+\frac{i}{\omega\epsilon_0}\nabla \times \nabla \times \hat{\mathbf{z}} \overbrace{[Q_{es}^w(\boldsymbol{\rho})\delta(z) + Q_{ed}^w(\boldsymbol{\rho})\delta'(z)]}^{Q_e^w(\mathbf{r})}$$

as the most general form of a transverse current supported on the plane $z = 0$.

11.10 Determine the relationship between the components of the surface source found in the previous problem for it to be NR throughout one of the two half-spaces bounded by the source plane.

If we take the 3D spatial Fourier transform of the surface current found in the preceding problem we obtain

$$\widetilde{\mathbf{J}_T}(\mathbf{K}) = i\mathbf{K} \times \hat{\mathbf{z}}[\tilde{Q}_{hs}^w(\mathbf{K}_\rho) + iK_z\tilde{Q}_{hd}^w(\mathbf{K}_\rho)]$$
$$-\frac{i}{\omega\epsilon_0}\mathbf{K} \times \mathbf{K} \times \hat{\mathbf{z}}[\tilde{Q}_{es}^w(\mathbf{K}_\rho) + iK_z\tilde{Q}_{ed}^w(\mathbf{K}_\rho)]$$

The requirement that a source not radiate into a given half-space $z > 0$ or $z < 0$ is that $\widetilde{\mathbf{J}_T}(\mathbf{k}_0^{\pm}) = 0$ which using the above result and the fact that the two vectors $\mathbf{K} \times \hat{\mathbf{z}}$ and $\mathbf{K} \times \mathbf{K} \times \hat{\mathbf{z}}$ are orthogonal to each other translates into (cf., solution to Problem 5.8)

$$\tilde{Q}_{hs}^w(\mathbf{K}_\rho) \pm i\gamma\tilde{Q}_{hd}^w(\mathbf{K}_\rho) = 0, \quad \tilde{Q}_{es}^w(\mathbf{K}_\rho) \pm i\gamma\tilde{Q}_{ed}^w(\mathbf{K}_\rho) = 0,$$

where $\mathbf{k}_0^{\pm} = \mathbf{K}_\rho \pm \gamma\hat{\mathbf{z}}$ and the plus sign results in a source NR in the r.h.s. and the minus sign in the l.h.s.

11.11 Derive the most general form of an EM surface source supported over a sphere centered at the origin.

Here we employ the Debye representation to represent the transverse part of the current as developed in Example 11.5 in the form

$$\mathbf{J}_T(\mathbf{r}) = \nabla \times \mathbf{r} Q_h^d(\mathbf{r}) + \frac{i}{\omega\epsilon_0} \nabla \times \nabla \times \mathbf{r} Q_e^d(\mathbf{r}).$$

We now employ the same general method used in Problem 11.9 to express the two scalar source components as a sum of singlet and doublet components (cf., Sections 1.8 and 2.12 and the solution to Problem 5.9)

$$Q_h^d(\mathbf{r}) = Q_{hs}^d(\theta,\phi)\delta(r-a) + \frac{a^2}{r^2} Q_{hd}^d(\theta,\phi)\delta'(r-a),$$
$$Q_e^d(\mathbf{r}) = Q_{es}^d(\theta,\phi)\delta(r-a) + \frac{a^2}{r^2} Q_{ed}^d(\theta,\phi)\delta'(r-a),$$

where (r,θ,ϕ) are the polar coordinates of $\mathbf{r}$ and we have taken the sphere to have a radius $r = a$ and we have labeled the "singlet" and "doublet" components with the subscripts s and d, respectively. We then find that

$$\mathbf{J}_T(\mathbf{r}) = \nabla \times \mathbf{r} \overbrace{[Q_{hs}^d(\theta,\phi)\delta(r-a) + \frac{a^2}{r^2} Q_{hd}^d(\theta,\phi)\delta'(r-a)]}^{Q_h^d(\mathbf{r})}$$
$$+ \frac{i}{\omega\epsilon_0} \nabla \times \nabla \times \mathbf{r} \overbrace{[Q_{es}^d(\theta,\phi)\delta(r-a) + \frac{a^2}{r^2} Q_{ed}^d(\theta,\phi)\delta'(r-a)]}^{Q_e^d(\mathbf{r})}$$

as the most general form of a transverse current supported on the surface of a sphere having radius $r = a$.

11.12 Determine the relationship between the components of the surface source found in the previous problem for it to be NR throughout the interior (exterior) of the sphere.

This is the vector equivalent of Problem 5.9 of Chapter 5 where these conditions were determined for a scalar source and field. The vector relationship is easy to obtain for the source to NR throughout the exterior since it is requires that the electric and magnetic radiation patterns both vanish which then yields a solution completely parallel to that employed to solve Problem 11.10. In particular, we showed in Example 11.5 that

$$\tilde{Q}_h^d(k_0\mathbf{s}) = \frac{-4\pi i}{\omega\mu_0} f_h^d(\mathbf{s}), \quad \tilde{Q}_e^d(k_0\mathbf{s}) = \frac{-4\pi i}{\omega\mu_0} f_e^d(\mathbf{s}),$$

so that the field will vanish outside the exterior of the sphere if $\tilde{Q}_h^d(k_0\mathbf{s})$ and $\tilde{Q}_e^d(k_0\mathbf{s})$ both vanish.

We found in the preceding problem that the two scalar components of the

Debye representation of the transverse current of a surface source on a sphere are given by

$$Q_h^d(\mathbf{r}) = Q_{hs}^d(\theta,\phi)\delta(r-a) + \frac{a^2}{r^2}Q_{hd}^d(\theta,\phi)\delta'(r-a),$$
$$Q_e^d(\mathbf{r}) = Q_{es}^d(\theta,\phi)\delta(r-a) + \frac{a^2}{r^2}Q_{ed}^d(\theta,\phi)\delta'(r-a),$$

where (r,θ,ϕ) are the polar coordinates of $\mathbf{r}$ and we have taken the sphere to have a radius $r=a$ and where the subscripts s and d, respectively, denote "singlet" and "doublet" components of the scalar source. Taking the Fourier transform of either of the above two Debye components we obtain

$$\tilde{Q}_p^d(k_0\mathbf{s}) = \int d^3r\, Q_p^d(\mathbf{r})e^{-ik_0\mathbf{s}\cdot\mathbf{r}} = \sum_{l,m} q_p^d(l,m)Y_l^m(\mathbf{s}),$$

where p is either h or e and

$$q_p^d(l,m) = 4\pi i^l \int d^3r\, Q_p^d(\mathbf{r})j_l(k_0r)Y_l^{m*}(\theta,\phi),$$

and where we have employed the scalar multipole expansion of $\exp(-ik_0\mathbf{s}\cdot\mathbf{r})$ derived in Example 3.4 of Chapter 3 in deriving the result. For the surface source we then find that

$$q_p^d(l,m) = 4\pi i^l \int d^3r\, \overbrace{\left[Q_{ps}^d(\theta,\phi)\delta(r-a) + \frac{a^2}{r^2}Q_{pd}^d(\theta,\phi)\delta'(r-a)\right]}^{Q_p^d(\mathbf{r})} j_l(k_0r)Y_l^{m*}(\theta,\phi)$$
$$= q_{ps}^d(l,m) + q_{pd}^d(l,m)$$

with

$$q_{ps}^d(l,m) = 4\pi i^l \int d^3r\, Q_{ps}^d(\theta,\phi)\delta(r-a)j_l(k_0r)Y_l^{m*}(\theta,\phi)$$
$$= 4\pi i^l a^2 j_l(k_0a)\int d\Omega\, Q_{ps}^d(\theta,\phi)Y_l^{m*}(\theta,\phi)$$
$$q_{pd}^d(l,m) = 4\pi i^l \int d^3r\, \frac{a^2}{r^2}Q_{pd}^d(\theta,\phi)\delta'(r-a)j_l(k_0r)Y_l^{m*}(\theta,\phi)$$
$$= -4\pi i^l a^2 k_0 j_l'(k_0a)\int d\Omega\, Q_{pd}^d(\theta,\phi)Y_l^{m*}(\theta,\phi).$$

The NR conditions then yield

$$\overbrace{\sum_{l,m} q_p^d(l,m)Y_l^m(\mathbf{s})}^{\tilde{Q}_p^d(k_0\mathbf{s})} = 0$$

which require that

$$q_p^d(l,m) = q_{ps}^d(l,m) + q_{pd}^d(l,m) = 0,$$

where, again, p can be h or e. This then leads to the requirements

$$a^2 j_l(k_0a)\int d\Omega\, Q^d_{hs}(\theta,\phi)Y_l^{m*}(\theta,\phi) - a^2k_0 j_l'(k_0a)\int d\Omega\, Q^d_{hd}(\theta,\phi)Y_l^{m*}(\theta,\phi) = 0,$$

$$a^2 j_l(k_0a)\int d\Omega\, Q^d_{es}(\theta,\phi)Y_l^{m*}(\theta,\phi) - a^2k_0 j_l'(k_0a)\int d\Omega\, Q^d_{ed}(\theta,\phi)Y_l^{m*}(\theta,\phi) = 0.$$

These conditions are completely analogous to those found in Problem 11.10 that are satisfied by the Fourier transforms of the two scalar components of a surface source that is NR throughout a half-space. However, the above conditions only guarantee that the spherical surface source is NR throughout the exterior of the sphere.

The conditions required for the spherical shell source to be NR throughout the interior are much more difficult to derive since the angular spectrum expansion cannot be employed. However, they can be derived after a great deal of work from the scalar Green function representations of the vector fields given in Eqs.(11.13). Either of the the fields can be used since the vanishing of either throughout the interior also guarantees the vanishing of the other. We will use the magnetic field here which yields the condition

$$\mathbf{H}_+(\mathbf{r}) = -\int_{\tau_0} d^3r'\,\nabla_{r'}\times\mathbf{J}(\mathbf{r}')G_+(\mathbf{r}-\mathbf{r}') = 0,\quad \mathbf{r}\in\tau_0$$

where τ_0 is the interior of the surface source. If we now use the multipole expansion of the Green function derived in Section 3.4 of Chapter 3 in the above we obtain the NR condition

$$\int_{\tau_0} d^3r'\,\nabla_{r'}\times\mathbf{J}(\mathbf{r}')\overbrace{-ik_0\sum_{l=0}^{\infty}\sum_{m=-l}^{l} j_l(k_0r)h_l^+(k_0a)Y_l^m(\hat{\mathbf{r}})Y_l^{m*}(\hat{\mathbf{r}}')}^{G_+(\mathbf{r}-\mathbf{r}')} = 0,\quad \mathbf{r}\in\tau_0,$$

where we have explicitly assumed that the source is confined to the surface of the sphere of radius $r'=a$ so that the field point $\mathbf{r}$ lies inside the source point $\mathbf{r}'$; i.e., $r<r'=a$. Using the above and integration by parts and making use of the Debye representation of the transverse current given in the preceding problem then yields conditions similar to the exterior NR conditions where, however, $j_l(k_0a)$ and $j_l'(k_0a)$ are replaced by $h_l(k_0a)$ and $h_l'(k_0a)$. This result can also be arrived at intuitively from the results of Problem 5.9.

11.13 Use the results from Example 11.4 to solve the 2D EM ISP for a source compactly supported between two parallel planes in terms of the tangential components of the electric field specified over two bounding parallel planes.

11.14 Fill in the missing steps in the derivation of Eqs.(11.39).

The reader should need no help in solving this problem.

11.15 Derive the expressions for the EM scattering amplitudes given in Eqs.(11.61) from Eqs.(11.60).

We have from Eq.(11.60a) that the scattered electric field vector is given by

$$\mathbf{E}^{(s)}(\mathbf{r},\nu) = -\int_{\tau_0} d^3r'\,[i\omega\mu_0\mathbf{J}_e(\mathbf{r}',\nu) - \frac{1}{\epsilon_0}\nabla\rho_e(\mathbf{r}',\nu) - \nabla\times\mathbf{J}_h(\mathbf{r}',\nu)]G_+(\mathbf{r}-\mathbf{r}')$$
$$\sim \frac{1}{4\pi}\int_{\tau_0} d^3r'\,[i\omega\mu_0\mathbf{J}_e(\mathbf{r}',\nu) - \frac{1}{\epsilon_0}\nabla\rho_e(\mathbf{r}',\nu) - \nabla\times\mathbf{J}_h(\mathbf{r}',\nu)]e^{-ik_0\mathbf{s}\cdot\mathbf{r}'}\frac{e^{ik_0r}}{r}$$
$$= \frac{1}{4\pi}[i\omega\mu_0\widetilde{\mathbf{J}}_e(k_0\mathbf{s},\nu) - \frac{ik_0}{\epsilon_0}\mathbf{s}\widetilde{\rho}_e(k_0\mathbf{s},\nu) - ik_0\mathbf{s}\times\widetilde{\mathbf{J}}_h(k_0\mathbf{s},\nu)]\frac{e^{ik_0r}}{r}$$

which after simplification and using the fact that $\mathbf{s}\times\widetilde{\mathbf{J}}_h(k_0\mathbf{s},\nu) = \mathbf{s}\times\widetilde{\mathbf{J}_{hT}}(k_0\mathbf{s},\nu)$ yields the required result

$$\mathbf{f}_e(\mathbf{s},\nu) = \frac{i\omega\mu_0}{4\pi}\widetilde{\mathbf{J}_{eT}}(k_0\mathbf{s},\nu) - \frac{ik_0}{4\pi}\mathbf{s}\times\widetilde{\mathbf{J}_{hT}}(k_0\mathbf{s},\nu).$$

The scattering amplitude for the magnetic field vector is obtained in an entirely analogous manner.

11.16 Derive Eqs.(11.62).

The Born induced currents and charge distributions are given by

$$\rho_e^B(\mathbf{r},\nu) = -\nabla\cdot[\delta\epsilon(\mathbf{r})\mathbf{E}^{(in)}(\mathbf{r},\nu)], \quad \mathbf{J}_e^B(\mathbf{r},\nu) = -i\omega\delta\epsilon(\mathbf{r})\mathbf{E}^{(in)}(\mathbf{r},\nu),$$
$$\rho_h^B(\mathbf{r},\nu) = -\nabla\cdot[\delta\mu(\mathbf{r})\mathbf{H}^{(in)}(\mathbf{r},\nu)], \quad \mathbf{J}_h^B(\mathbf{r},\nu) = -i\omega\delta\mu(\mathbf{r})\mathbf{H}^{(in)}(\mathbf{r}).$$

We then find that

$$\widetilde{\rho_e^B}(k_0\mathbf{s},\nu) = -ik_0\mathbf{s}\cdot\widetilde{\delta\epsilon\mathbf{E}^{(in)}}(k_0\mathbf{s},\nu), \quad \widetilde{\mathbf{J}_e^B}(k_0\mathbf{s},\nu) = -i\omega\widetilde{\delta\epsilon\mathbf{E}^{(in)}}(k_0\mathbf{s},\nu),$$
$$\widetilde{\rho_h^B}(k_0\mathbf{s},\nu) = -ik_0\mathbf{s}\cdot\widetilde{\delta\mu\mathbf{H}^{(in)}}(k_0\mathbf{s},\nu)], \quad \widetilde{\mathbf{J}_h^B}(k_0\mathbf{s},\nu) = -i\omega\widetilde{\delta\mu\mathbf{H}^{(in)}}(k_0\mathbf{s}).$$

Writing the E field scattering amplitude in the form

$$\mathbf{f}_e(\mathbf{s},\nu) = \frac{1}{4\pi}[i\omega\mu_0\widetilde{\mathbf{J}}_e(k_0\mathbf{s},\nu) - \frac{ik_0}{\epsilon_0}\mathbf{s}\widetilde{\rho}_e(k_0\mathbf{s},\nu) - ik_0\mathbf{s}\times\widetilde{\mathbf{J}}_h(k_0\mathbf{s},\nu)]$$

then yields

$$\mathbf{f}_e^B(\mathbf{s},\nu) = \frac{1}{4\pi}\left[i\omega\mu_0\overbrace{[-i\omega\widetilde{\delta\epsilon\mathbf{E}^{(in)}}(k_0\mathbf{s},\nu)]}^{\widetilde{\mathbf{J}^B}_e(k_0\mathbf{s},\nu)} - \frac{ik_0}{\epsilon_0}\mathbf{s}\overbrace{[-ik_0\mathbf{s}\cdot\widetilde{\delta\epsilon\mathbf{E}^{(in)}}(k_0\mathbf{s},\nu)]}^{\widetilde{\rho}_e^B(k_0\mathbf{s},\nu)}\right]$$
$$-i\frac{k_0}{4\pi}\mathbf{s}\times\overbrace{[-i\omega\widetilde{\delta\mu\mathbf{H}^{(in)}}(k_0\mathbf{s})]}^{\widetilde{\mathbf{J}}_h^B(k_0\mathbf{s},\nu)}$$

which after simplification yields the required result

$$\mathbf{f}_e^B(\mathbf{s},\nu) = -\frac{k_0^2}{4\pi\epsilon_0}\mathbf{s}\times\mathbf{s}\times\widetilde{\delta\epsilon\mathbf{E}^{(in)}}(k_0\mathbf{s}) - \frac{\omega k_0}{4\pi}\mathbf{s}\times\widetilde{\delta\mu\mathbf{H}^{(in)}}(k_0\mathbf{s}).$$

The magnetic field scattering amplitude is similarly obtained.

11.17 Verify that the EM scattering amplitudes satisfy Eqs.(11.18).

The reader should need no help on this problem.

11.18 Compute $\mathbf{f}_e^{TM}$ and $\mathbf{f}_h^{TE}$ within the Born approximation in terms of $\tilde{\delta\epsilon}$ and $\tilde{\delta\mu}$.

This parallels the computation of $\mathbf{f}_e^{TE}$ and $\mathbf{f}_h^{TM}$ presented in the book.

11.19 Express the TE and TM scattering amplitudes within the Born approximation in terms of scattered field data specified over a spherical surface surrounding the scattering volume.

This problem is solved using the multipole expansions of the electric and magnetic field vectors given in Eqs.(11.49) and the fact that the multipole fields can be shown to satisfy the equations (cf., Jackson (1998))

$$\int d\Omega\, \mathbf{E}^{h^*}_{l,m}(\mathbf{r}) \cdot \mathbf{E}^{h}_{l',m'}(\mathbf{r}) = l(l+1)|h_l^+(k_0 r)|^2 \delta_{l,l'}\delta_{m,m'}$$
$$\int d\Omega\, \mathbf{H}^{e^*}_{l,m}(\mathbf{r}) \cdot \mathbf{H}^{e}_{l',m'}(\mathbf{r}) = l(l+1)|h_l^+(k_0 r)|^2 \delta_{l,l'}\delta_{m,m'}$$
$$\int d\Omega\, \mathbf{E}^{h^*}_{l,m}(\mathbf{r}) \cdot \mathbf{E}^{e}_{l,m}(\mathbf{r}) = \int d\Omega\, \mathbf{H}^{e^*}_{l,m}(\mathbf{r}) \cdot \mathbf{H}^{h} e_{l',m'}(\mathbf{r}) = 0,$$

where the integrals are over the unit sphere. On making use of the above orthogonality conditions we find from the multipole expansions that

$$q^h_{l,m} = \frac{1}{l(l+1)|h_l^+(k_0 r)|^2} \int d\Omega\, \mathbf{E}^{h^*}_{l,m}(\mathbf{r}) \cdot \mathbf{E}_+(\mathbf{r}),$$
$$q^e_{l,m} = \frac{1}{l(l+1)|h_l^+(k_0 r)|^2} \int d\Omega\, \mathbf{H}^{e^*}_{l,m}(\mathbf{r}) \cdot \mathbf{H}_+(\mathbf{r}),$$

where the integral is over any sphere outside the support of the scatterer. The final step is to express the scattering amplitude in terms of the multipole moments via Eqs.(11.53) which holds both within the Born approximation and in general.

11.20 Derive a general expression for non-scattering material parameters $\delta\epsilon$ and $\delta\mu$ within the Born approximation for plane wave incidence.

This can be done immediately using the expressions given in Eqs.(11.64) for the Fourier transforms of the two material parameters in terms of the TE and TM scattering amplitudes. It follows from these equations that the material parameters are non-scattering they must both have transforms that vanish over an Ewald sphere; i.e.,

$$\widetilde{\delta\epsilon}[k_0(\mathbf{s}-\mathbf{s}_0)] = 0, \quad \widetilde{\delta\mu}[k_0(\mathbf{s}-\mathbf{s}_0)] = 0.$$

This then requires that they both be classical scalar non-scattering potentials.

We can also derive the above results directly from the TE scattering amplitude alone. The Born scattering amplitude for the TE case is given in

Eq.(11.63a) which we can write in the form

$$\mathbf{f}_e^{TE}(\mathbf{s},\mathbf{s}_0) = -\frac{k_0^2}{4\pi}\mathbf{s}\times\mathbf{s}\times\mathbf{A}_0(\mathbf{s}_0)\frac{\widetilde{\delta\epsilon}[k_0(\mathbf{s}-\mathbf{s}_0)]}{\epsilon_0}$$
$$-\frac{k_0^2}{4\pi}\mathbf{s}\times\mathbf{s}_0\times\mathbf{A}_0(\mathbf{s}_0)\frac{\widetilde{\delta\mu}[k_0(\mathbf{s}-\mathbf{s}_0)]}{\mu_0}$$
$$= -\frac{k_0^2}{4\pi}\mathbf{s}\times\overbrace{[\mathbf{s}\times\mathbf{A}_0(\mathbf{s}_0)\frac{\widetilde{\delta\epsilon}[k_0(\mathbf{s}-\mathbf{s}_0)]}{\epsilon_0}-\mathbf{s}_0\times\mathbf{A}_0(\mathbf{s}_0)\frac{\widetilde{\delta\mu}[k_0(\mathbf{s}-\mathbf{s}_0)]}{\mu_0}]}^{\mathbf{V}}.$$

For the scattering amplitude to vanish for all scattering directions $\mathbf{s}$ either the vector $\mathbf{V}$ must be in the direction $\mathbf{s}\ \ \forall\mathbf{s}$ or the vector itself must vanish. The first option is clearly not possible while the second yields

$$\mathbf{V} = \mathbf{s}\times\mathbf{A}_0(\mathbf{s}_0)\frac{\widetilde{\delta\epsilon}[k_0(\mathbf{s}-\mathbf{s}_0)]}{\epsilon_0}-\mathbf{s}_0\times\mathbf{A}_0(\mathbf{s}_0)\frac{\widetilde{\delta\mu}[k_0(\mathbf{s}-\mathbf{s}_0)]}{\mu_0} = 0, \quad \forall\mathbf{s}$$

Taking the dot product with $\mathbf{s}_0$ then requires that $\widetilde{\delta\epsilon}[k_0(\mathbf{s}-\mathbf{s}_0)]$ vanish which then requires that $\widetilde{\delta\mu}[k_0(\mathbf{s}-\mathbf{s}_0)]$ also vanish. Thus we conclude that the two material parameters must be classical scalar NR potentials.