February 2, 2015

Problems for Chapter 7 of *Advanced Mathematics for Applications*

Spherical Systems

by Andrea Prosperetti

1 Algebraic radial dependence

1. The velocity field $\mathbf{v}(\mathbf{x},t)$ inside a rigid sphere translating with velocity $\mathbf{w}$ and rotating with angular velocity Ω is $\mathbf{v} = \mathbf{w} + \Omega \times \mathbf{x}$, where $\mathbf{x}$ is measured from the center of the sphere. Express this velocity field in terms of suitably normalized spherical harmonics.

2. Solve Laplace's equation $\nabla^2 u = 0$ subject, on the sphere $r = a$, to the condition
$$u = u_0 \quad \text{for} \quad 0 \le \theta < \pi/2, \qquad u = -u_0 \quad \text{for} \quad \pi/2 < \theta \le \pi,$$
where u_0 is a constant, for any ϕ. Find separately the solution of the interior problem such that u is bounded at $r = 0$ and of the exterior problem satisfying $u \to 0$ at infinity.

3. If $u(\mathbf{x})$ is a harmonic function (i.e., it satisfies Laplace's equation $\nabla^2 u = 0$), then so is the function
$$v(\mathbf{x}) = \frac{1}{|\mathbf{x}|}\, u\left(\frac{\mathbf{x}}{|\mathbf{x}|^2}\right).$$
(a) Verify the previous statement if $u = r^n\, P_n(\cos\theta)$.
(b) Prove the previous statement in general by expanding u in spherical harmonics. This is an alternate proofs of *Kelvin's inversion theorem* given on p. 186.

4. It is shown on p. 402 that, if u is harmonic in a region bounded by the surface S, then
$$u(\mathbf{x}) = \frac{1}{4\pi}\int_S \left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\frac{\partial u}{\partial n'} - u(\mathbf{x}')\,\frac{\partial}{\partial n'}\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) dS'.$$
Let S be a sphere of radius a and consider its interior. Verify the previous relation for the function
$$u = r^n\, P_n(\cos\theta)$$
in which $r = |\mathbf{x}|$. (Note that the coordinate system is given, so that $\mathbf{x}$ is prescribed and cannot be chosen so as to simplify the calculation.) Then calculate the integral for the same u and the same surface taking $\mathbf{x}$ *outside* the sphere.

5. Solve the axi-symmetric Laplace equation $\nabla^2 u = 0$ inside the spherical shell $a < r < b$. The boundary conditions are $u(a,\theta) = f_a(\theta)$, $u(b,\theta) = f_b(\theta)$, with f_a and f_b given.

6. Find the solution of Laplace's equation $\nabla^2 u = 0$ which is regular inside the spherical region $0 < r < a$ and equals A exp kz on $-a \le z \le a$ with A and k constants.

7. Consider a scalar field satisfying Laplace's equation $\nabla^2 u = 0$ and given, in spherical coordinates, by

$$u_0 = r^n\, Y_n^m(\theta, \phi),$$

where Y_n^m is a spherical harmonic. Suppose now that a sphere of radius a centered at the origin $r = 0$ is inserted into this field so that the potential is modified to the form $u = u_0 + u_1$. If the situation is such that the gradient of the total field u normal to the sphere must vanish, determine the correction u_1 outside the sphere.

8. Prove that the solution $u(r, \theta, \phi)$ to the interior Neumann problem for the unit sphere

$$\nabla^2 u = 0 \;\text{ for }\; 0 < r < 1, \qquad \frac{\partial u}{\partial r} = f(\theta, \phi) \;\text{ on } r = 1,$$

regular within the sphere can be expressed as

$$u = \int_0^1 v(rt, \theta, \phi) \frac{dt}{t} + \text{const.}$$

where v is the (regular) solution of the interior Dirichlet problem

$$\nabla^2 v = 0 \quad \text{for} \quad 0 < r < 1, \qquad v(1, \theta, \phi) = f(\theta, \phi).$$

9. Solve the axi-symmetric Laplace equation $\nabla^2 u = 0$ in the half space $0 < \theta < \frac{\pi}{2}, a < r < \infty$. The boundary conditions are $u \to 0$ at infinity, $u = f(\theta)$, given, on the hemisphere $r = a$, $u = 0$ on the plane $\theta = \frac{\pi}{2}$ for $a < r$. For consistency, $f(\pi/2) = 0$. After finding the general solution, consider the special case $f(\theta) = \cos^2\theta$. How many coefficients of the series will be non-zero in this case?

10. Solve the axi-symmetric Laplace equation $\nabla^2 u = 0$ in the half space $0 < \theta < \frac{\pi}{2}, a < r < \infty$. The boundary conditions are $u \to 0$ at infinity, $u(a, \theta) = F(\theta)$ on $r = a$, $\partial u/\partial z = 0$ on the plane $z = 0$. Note that the plane $z = 0$ corresponds to $\theta = \pi/2$, and, on this plane, $\partial/\partial z$ is equal to $(1/r)\partial/\partial\theta$.

11. Solve Laplace's equation $\nabla^2 u = 0$ in the region exterior to a sphere of radius a bounded by the two semi-planes $\phi = 0$, $\phi = \frac{1}{2}\pi$. The boundary conditions are

$$u(a, \theta, \phi) \;=\; F(\theta, \phi)\,, \qquad u(r, \theta, 0) \;=\; u(r, \theta, \pi/2) \;=\; 0$$

and $u \to 0$ at infinity. The solution is easier if the expansion is written in the real form given in (7.2.9) p. 174.

12. In solving the axisymmetric Laplace equation $\nabla^2 u = 0$ in spherical coordinates we used an expansion in Legendre polynomials. (a) Could we have used a Fourier series in θ? (b) If we could have done that, why didn't we do it?

13. The exterior Poisson formula for the Dirichlet problem is given in section 7.8. Find the analogous formula for the interior of a sphere of radius a. Assume that the boundary condition on the surface of the sphere is axisymmetric, $u(r = a, \theta, \phi) = F(\theta)$. Try to calculate the sum of the series solution in closed form.

14. Solve

$$\nabla^2 u = -4\pi q\, \delta^{(3)}(\mathbf{x} - \mathbf{x}_s),$$

subject to $u \to 0$ at infinity, $\mathbf{n} \cdot \nabla u = 0$ on a sphere of radius a placed at a distance $b > a$ from $\mathbf{x}_s$. Try to sum the resulting series in closed form by using the generating function for the Legendre polynomials.

15. Solve, subject to the condition $u \to 0$ at infinity, the equation

$$\nabla^2 u = -4\pi\, q\, [\delta(z-a) - \delta(z+a)]\, \delta(x)\, \delta(y).$$

Expand the result in spherical harmonics for $r < a$ and for $r > a$. Then, keeping the product $2aq = p$ = constant, take the limit $a \to 0$ for $r \neq 0$. The result is by definition the potential produced by a dipole of strength p along the z-axis. Suppose now that the dipole is at the center of a sphere of radius R kept at potential $u = 0$. Find the potential inside the sphere first by using image sources, and then by spherical harmonic expansion. Verify the equality of the two results.

16. Consider the equation

$$\nabla^2 u \;=\; -4\pi \delta^{(3)}(\mathbf{x} - \mathbf{x}_0)$$

inside the sphere of radius a. The point $\mathbf{x}_0$ is internal to the sphere. The solution is regular at the center of the sphere and, on the sphere surface, it satisfies

$$\alpha u + \beta \mathbf{n} \cdot \boldsymbol{\nabla} u \;=\; 0$$

with α and β given constants. Calculate the average value of u over the surface of the sphere. Find the exact solution of the problem and check your answer to the previous question. Does a solution exist if $\alpha = 0$? If not, why?

17. A certain axi-symmetric charge distribution confined within a geometric spherical surface of radius a generates, on the positive z-semi-axis and outside the sphere $r = a$, a potential given by

$$u(x = 0, y = 0, z) \;=\; f(z) \qquad \text{for} \qquad z > a$$

with $f(z)$ a given function tending to 0 for $z \to \infty$. Give an expression for the potential away from the axis. Is any function $f(z)$ decaying at infinity a possible potential? After considering the problem in general consider the specific example

$$u \;=\; \frac{2Q}{b^2}\left(\sqrt{z^2 + b^2} - z\right)$$

with Q and $b \leq a$ given constants. Recall that

$$\sqrt{1+x} \;=\; 1 + \frac{1}{2}x - \sum_{n=2}^{\infty} (-1)^k \frac{(2k-3)!!}{2^k k!} x^k \,.$$

18. A function $u(r, \theta)$ is harmonic inside and outside a spherical surface of radius R. Across this surface the function itself and its radial derivative undergo jumps $u(R+, \theta) - u(R-, \theta) = U(\theta)$ and $\partial_r u|_{R+} - \partial_r u|_{R-} = V(\theta)$ with $U(\theta)$ and $V(\theta)$ known and $u(R\pm, \theta)$ designating the limits as $r \to R$ from the outside and the inside of the sphere. Determine the function u inside and outside the sphere. Calculate the sum of the series on the positive z axis if $U = 0$.

19. A function $u(r, \theta)$ is harmonic inside a sphere of radius R. On the equatorial plane of the sphere, $\theta = \frac{1}{2}\pi$, it is known that $u = U_0 \sin hr/(hr)$ with U_0 and h known constants. What is the most general form of u compatible with this information? [Hint: Recall that $\sin \alpha = \sum_{k=0}^{\infty} \alpha^{2k+1}/(2k+1)!$]

20. By means of a suitable eigenfunction expansion, solve the initial-value problem

$$\frac{\partial^2 u}{\partial t^2} - c_0^2 \frac{\partial}{\partial x}\left[(1 - x^2)\frac{\partial u}{\partial x}\right] \;=\; 0 \qquad 0 < x < 1$$

$$u(x, 0) \;=\; f(x)\,, \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} \;=\; 0\,, \qquad u(0, t) \;=\; 0\,,$$

in which c_0 is a given constant and the function f is also given. The solution u is required to be regular at $x = 1$.

21. By adapting the method explained in section 7.10, find an approximate solution correct to first order in $\epsilon << 1$ of Laplace's equation $\nabla^2 u = 0$ subject, at infinity, to $\nabla u \to \mathbf{U}$, a given constant vector, and $u = u_0$ on the nearly spherical surface given by

$$S(r, \theta, \phi) \equiv r - a[1 + \epsilon P_n(\cos\theta)] = 0.$$

with a and u_0 given constants.

22. Inside the spherical shell bounded by the two spheres of radii a and b, with $a < b$ find an approximate solution to first order in ϵ of the nonlinear equation

$$\nabla^2 u - \epsilon u^2 = 0,$$

where $|\epsilon| \ll 1$ is a given real constant. The boundary conditions are: (i) $\mathbf{n}\cdot\boldsymbol{\nabla} u = \mathbf{V}\cdot\mathbf{n}$ on $r = b$, where $\mathbf{n}$ is the unit normal directed out of the sphere and $\mathbf{V}$ a given constant vector, and (ii) $u = \epsilon P_n(\cos\theta)$ on $r = a$, where P_n, with $n \geq 3$, is a given Legendre polynomial; the angle θ is measured from the direction of $\mathbf{V}$. Let $u = u_0 + \epsilon u_1 + \ldots$.

23. In three-dimensional space, outside the sphere of radius $r =$ a, find an approximate solution, to order ϵ included, of the problem

$$\nabla^2 u + \frac{\epsilon}{a^2} u^2 = 0,$$

where $|\epsilon| \ll 1$, subject to $u \to 0$ for $r \to \infty$, and

$$u(a, \theta) = U \cos\theta.$$

24. Describe a procedure to solve the equation

$$\boldsymbol{\nabla}\cdot[f(r)\boldsymbol{\nabla} u] = 0$$

outside the sphere of radius a. The solution vanishes at infinity and, on the sphere, it satisfies a Dirichlet condition $u = U(\theta)$, where θ is the polar angle. After briefly describing the general procedure, consider the specific case $f(r) = Ar^{-\beta}$ with A and $\beta > 0$ given constants.

2 Spherical Bessel functions

1. A sphere of radius a executes small-amplitude harmonic radial pulsations of frequency ω in a compressible medium in which the speed of sound is c, a given constant (cf. section 1.4). Find the velocity potential $u(\mathbf{x}, t)$ induced in the liquid by letting $u = U(\mathbf{x})\exp(-i\omega t)$ and solving the Helmholtz equation

$$(\nabla^2 + k^2)U = 0,$$

$$U \to 0 \text{ as } r \to \infty, \qquad \frac{\partial U}{\partial r} = -i\epsilon\omega\frac{V_0}{a} \text{ at } r = a,$$

where $k = \omega/c$ is the wavenumber and V_0 is a given constant. Make sure to impose the radiation condition at infinity.

2. A sphere of radius a radiates sound waves of radian frequency ω in a compressible medium in which the speed of sound is c. Find the radiated pressure field $p(\mathbf{x}, t)$ by letting $p = U(\mathbf{x})\exp(-i\omega t)$ and solving

$$\nabla^2 U + k^2 U = 0,$$

$$\frac{\partial U}{\partial r} = -\epsilon i\omega\frac{V_0}{a} P_n(\cos\theta) \quad \text{for} \quad r = a, \qquad U \to 0 \quad \text{for} \quad \mathbf{x} \to \infty.$$

Here $k = \omega/c$ is the wave number. Impose the radiation condition at infinity. For $n = 0$ this reduces to the previous problem. Verify explicitly that your result coincide with the previous ones in this case.

3. Find the eigenvalues and eigenfunctions of the Laplacian operator inside a spherical region of radius a, if the eigenfunctions are required to be regular at $r = 0$ and to vanish at $r = a$. No need to calculate the normalization constant, but do pay attention to degeneracies and describe them.

4. Determine all the eigenvalues and eigenfunctions of the Laplace operator ∇^2 inside the spherical shell bounded by concentric spheres of radii a and b with $a < b$, with homogeneous Dirichlet conditions on the bounding surfaces. Establish the degree of degeneracy of each eigenvalue. (Cf. section 3.8 for a similar problem.)

5. Calculate the acoustical normal modes of a spherical shell bounded by rigid surfaces, i.e., find the values of k such that the problem
$$\nabla^2 u + k^2\, u = 0,$$
subject to
$$\frac{\partial u}{\partial r} = 0 \ \text{ at } \quad r = R_1,\ r = R_2,$$
has non-trivial solutions. From the general result, find the limit case for a shell of large radius, i.e. such that both R_1 and $R_2 \to \infty$ while $R_2 - R_1$ remains finite. For a given order ℓ, how many distinct modes $k_n^{(\ell)}$ are there? Is there a largest value of $k_n^{(\ell)}$? For the same order ℓ, what happens to the nodes of the normal modes, (i.e., the zeros of the eigenfunctions), in going from the $n-$th mode to the $(n+1)$-st mode?

6. The surface temperature T_s of a spherical cavity of radius a embedded in an infinite medium having the thermal diffusivity D oscillates sinusoidally according to
$$T_s - T_0 = \Delta T\, P_n(\cos\theta)\, \exp(i\omega t), \qquad\qquad (*)$$
where ΔT, T_0 and ω are constants. Find the resulting temperature distribution $T(\mathbf{x}, t)$ in the medium at steady state by letting
$$T(\mathbf{x}, t) - T_0 = \Delta T\, U(\mathbf{x})\, \exp(i\omega t)$$
where $U(\mathbf{x})$ satisfies the equation (cf. section 1.3)
$$D\nabla^2 U = i\omega U,$$
subject to the condition $U \to 0$ at infinity and, at $r = a$, to the condition stemming from (*). Calculate the amplitude $\frac{\partial T}{\partial r}$ of the temperature gradient at the surface of the cavity.

7. A spherical source of waves placed at $\mathbf{x}_s$, is at a distance d from a "soft" sphere of radius a. The source size is very small compared with d, a and the wavelength, and it is therefore reasonable to model it as point-like. Find the effect of the sphere on the wave emitted by the source by solving
$$\nabla^2 u + k^2 u = -4\pi\, \delta^{(3)}(\mathbf{x} - \mathbf{x}_s),$$
$$u \to \frac{1}{|\mathbf{x} - \mathbf{x}_s|}\ \exp(ik|\mathbf{x} - \mathbf{x}_s|) \quad \text{as} \quad \mathbf{x} \to \mathbf{x}_s,$$
and $u = 0$ on the sphere.

8. Inside the sphere of radius a consider the system
$$\nabla^2 u + b^2 v \;=\; 0\,, \qquad \nabla^2 v + b^2 u \;=\; 0$$
where b is a constant, with u, v regular at the center of the sphere, and $u \;=\; f(\mathbf{x})$, $v = 0$ on the sphere surface. For simplicity, consider only the axisymmetric case and describe how you would go about solving the problem. Carry the calculation as far as you can. There are different ways to attack this problem, and some lead to easier algebra than others.

9. The speed of sound in a non-uniform compressible medium is not constant but varies slightly from place to place. This leads to a perturbed Helmholtz equation of the form

$$\nabla^2 u + k^2[1 - \epsilon\, f(\mathbf{x})]u = 0,$$

where $f(\mathbf{x})$ is given. By letting $u = u_0 + \epsilon u_1 + \dots$ find an approximate general solution of this equation having the form of outgoing waves at infinity and subject to the boundary conditions $u \to 0$ as $|\mathbf{x}| \to \infty$ and $u = V(\theta, \phi)$ on the sphere $|\mathbf{x}| = R$. Carry out the calculations to the point that, given a particular form of f, all that needs to be done is to calculate some integrals and evaluate the series expansion in spherical harmonics.