Credit Risk

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Solutions to Exercises

Chapter 1

1.1. If $V(0)(1+U)(1+D) < F < V(0)(1+U)^2$, then

$$F = V(0)(1+U)^2 - \frac{1}{q^2}E(0)(1+R)^2.$$

If $V(0)(1+D)^2 < F < V(0)(1+U)(1+D)$, then

$$F = \frac{q^2 V(0)(1+U)^2 + 2q(1-q)V(0)(1+U)(1+D) - E(0)(1+R)^2}{q^2 + 2q(1-q)}.$$

The expected return on equity is 100.45%, and the standard deviation is 88.29%. For debt the corresponding values are 48.03% and 7.64%.

1.2. In the Black-Scholes model the value of a call option increases as a the volatility σ of the underlying asset increases, all other parameters being constant. This is because the partial derivative of the option price with respect to σ, i.e. the vega of the option is positive,

$$\text{vega} = V(0)\sqrt{\frac{T}{2\pi}}e^{-d_+^2/2} > 0;$$

see [BSM]. Moreover, the value of a call option decreases if the strike price increases, all remaining parameters being constant.

As a result, since initial equity $E(0)$ is the value of a call option with strike F, to keep $E(0)$ fixed F must increase when σ increases. It follows that

$$k_D = \frac{1}{T}\ln\frac{F}{D(0)}$$

is an increasing function of σ.

For instance, for the data in Example 1.8 the value of k_D increases from 5.15% to 5.45% as σ increases from 30% to 35%.

1.3. For a company with $w_E = 40\%$ equity financing we obtain $F =$

63.5453 and compute the expected returns on equity and debt $\mu_E = 18.22\%$, $\mu_D = 5.38\%$.

For one with $w_E = 60\%$ equity financing, the corresponding values are $F = 42.0571$ and $\mu_E = 14.11\%$, $\mu_D = 5.13\%$.

1.4. For a company with 40% financing by equity we get $\sigma_E = 83.54\%$ and $\sigma_D = 3.18\%$. For one with 60% equity, $\sigma_E = 56.51\%$ and $\sigma_D = 0.35\%$.

1.5. From portfolio theory we have the formula

$$\sigma_V^2 = w_E^2\sigma_E^2 + w_D^2\sigma_D^2 + 2w_E w_D \sigma_E \sigma_E \rho_{ED}$$

for the variance of the return on the value of the company assets. It follows that the correlation between the returns on equity and debt can be written as

$$\rho_{ED} = \frac{\sigma_V^2 - w_E^2\sigma_E^2 - w_D^2\sigma_D^2}{2w_E w_D \sigma_E \sigma_E}.$$

Formulae for σ_E and σ_D are derived in Chapter 1. To use the formula

$$\begin{aligned}\sigma_V^2 &= \mathbb{E}_P\left(\left(\frac{V(T)-V(0)}{D(0)}\right)^2\right) - \mu_V^2 \\ &= \frac{\mathbb{E}_P(V(T)^2)}{V(0)^2} - \frac{2\mathbb{E}_P(V(T))}{V(0)} + 1 - \mu_V^2\end{aligned}$$

for σ_V we find

$$\begin{aligned}\mathbb{E}_P(V(T)) &= V(0)e^{(\mu-\frac{1}{2}\sigma^2)T}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}e^{\sigma\sqrt{T}x}dx \\ &= V(0)e^{(\mu-\frac{1}{2}\sigma^2)T}e^{\frac{1}{2}\sigma^2 T}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(x-\sigma\sqrt{T})^2}dx \\ &= V(0)e^{\mu T}\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}_P(V(T)^2) &= V(0)^2e^{(2\mu-\sigma^2)T}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}e^{2\sigma\sqrt{T}x}dx \\ &= V(0)^2e^{(2\mu-\sigma^2)T}e^{2\sigma^2 T}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(x-2\sigma\sqrt{T})^2}dx \\ &= V(0)^2e^{(2\mu+\sigma^2)T}.\end{aligned}$$

This makes it possible to compute the correlation for various levels of the debt ratio w_D using the data from Example 1.9:

w_D	0.8	0.7	0.6	0.5	0.4	0.3
ρ_{ED}	0.4162	0.3364	0.2359	0.1316	0.0496	0.0091

1.6. We start with the case when $l = 1$. In this case

$$d_+(T) = \frac{1}{2}\sqrt{T}\sigma, \quad d_-(T) = -\frac{1}{2}\sqrt{T}\sigma$$

and

$$\lim_{T\to 0} d_+(T) = \lim_{T\to 0}\frac{1}{2}\sqrt{T}\sigma = 0, \quad \lim_{T\to 0} d_-(T) = -\lim_{T\to 0}\frac{1}{2}\sqrt{T}\sigma = 0.$$

As a result,

$$\lim_{T\to 0}(N(-d_+(T)) + N(d_-(T))) = N(0) + N(0) = \frac{1}{2} + \frac{1}{2} = 1$$

and

$$\lim_{T\to 0}\ln(N(-d_+(T)) + N(d_-(T))) = 0.$$

It means that we can apply l'Hôpital's rule to compute the limit

$$\begin{aligned}\lim_{T\to 0} s(T) &= -\lim_{T\to 0}\frac{\frac{d}{dT}(\ln(N(-d_+(T)) + N(d_-(T))))}{\frac{d}{dT}(T)} \\ &= -\lim_{T\to 0}\frac{\frac{d}{dT}(N(-d_+(T)) + N(d_-(T)))}{N(-d_+(T)) + N(d_-(T))} \\ &= -\lim_{T\to 0}\frac{d}{dT}(N(-d_+(T))) - \lim_{T\to 0}\frac{d}{dT}(N(d_-(T))) \\ &= \lim_{T\to 0}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}d_+(T)^2}\frac{1}{4\sqrt{T}}\sigma + \lim_{T\to 0}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}d_-(T)^2}\frac{1}{4\sqrt{T}}\sigma = \infty.\end{aligned}$$

Finally, we take $l > 1$. In this case

$$\lim_{T\to 0} d_+(T) = \lim_{T\to 0}\frac{-\ln l + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} = -\infty,$$

$$\lim_{T\to 0} d_-(T) = \lim_{T\to 0}\frac{-\ln l - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} = -\infty.$$

As a result,

$$\lim_{T\to 0}\ln\left(\frac{1}{l}N(-d_+(T)) + N(d_-(T))\right) = \ln\frac{1}{l} = -\ln l$$

and

$$\lim_{T\to 0} s(T) = \lim_{T\to 0} \frac{-\ln\left(\frac{1}{l}N(-d_+(T)) + N(d_-(T))\right)}{T} = \infty$$

since the limit in the numerator is $\ln l$, a finite positive number, and the limit in the denominator is ∞.

1.7. Default occurs when $L + \Pi(T) < F$. The probability of this event is

$$P(L + \Pi(T) < F) = P(X < a_2) = N(a_2).$$

1.8. Since,

$$D(T) = F\mathbf{1}_{\{L+\Pi(T)\geq F\}} + (L + \Pi(T))\,\mathbf{1}_{\{0\leq L+\Pi(T)<F\}}.$$

we have

$$\mathbb{E}_P\Big(D(T)^2\Big) = \mathbb{E}_P\Big(F^2\mathbf{1}_{\{L+\Pi(T)\geq F\}} + (L + \Pi(T))^2\,\mathbf{1}_{\{0\leq L+\Pi(T)<F\}}\Big)\,.$$

Here $\Pi(T) = \frac{1}{q}(G(T) - G(0))$ and $G(T) = G(0)e^{(\mu-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}X}$, where X is a random variable with the standard normal distribution under the real-life probability P. When $-qL + G(0) > 0$, we get

$$\begin{aligned}
\mathbb{E}_P\Big(D(T)^2\Big) &= \mathbb{E}_P\left(F^2\mathbf{1}_{\{X\geq a_2\}} + \left(L - \frac{1}{q}G(0) + \frac{1}{q}G(T)\right)^2 \mathbf{1}_{\{a_1\leq X<a_2\}}\right) \\
&= F^2 P(X \geq a_2) + \left(L - \frac{1}{q}G(0)\right)^2 P(a_1 \leq X < a_2) \\
&\quad + \frac{2}{q}\left(L - \frac{1}{q}G(0)\right) G(0)\mathbb{E}_P\Big(e^{(\mu-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}X}\mathbf{1}_{\{a_1\leq X<a_2\}}\Big) \\
&\quad + \frac{1}{q^2}G(0)^2\mathbb{E}_P\Big(e^{(2\mu-\sigma^2)T+2\sigma\sqrt{T}X}\mathbf{1}_{\{a_1\leq X<a_2\}}\Big) \\
&= F^2 N(-a_2) + \left(L - \frac{1}{q}G(0)\right)^2 (N(a_2) - N(a_1)) \\
&\quad + \frac{2}{q}\left(L - \frac{1}{q}G(0)\right) G(0)e^{\mu T}\left(N\left(a_2 - \sigma\sqrt{T}\right) - N\left(a_1 - \sigma\sqrt{T}\right)\right) \\
&\quad + \frac{1}{q^2}G(0)^2 e^{(2\mu+\sigma^2)T}\left(N\left(a_2 - 2\sigma\sqrt{T}\right) - N\left(a_1 - 2\sigma\sqrt{T}\right)\right),
\end{aligned}$$

where a_1, a_2 are given by (1.5) and (1.6). When $-qL + G(0) \leq 0$, the inequality $0 \le L + \Pi(T)$ is always satisfied, and we get

$$\begin{aligned}
\mathbb{E}_P\Big(D(T)^2\Big) &= \mathbb{E}_P\left(F^2 \mathbf{1}_{\{X \ge a_2\}} + \left(L - \frac{1}{q}G(0) + \frac{1}{q}G(T)\right)^2 \mathbf{1}_{\{X < a_2\}}\right) \\
&= F^2 P(X \ge a_2) + \left(L - \frac{1}{q}G(0)\right)^2 P(X < a_2) \\
&\quad + \frac{2}{q}\left(L - \frac{1}{q}G(0)\right)G(0)\mathbb{E}_P\Big(e^{(\mu - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}X}\mathbf{1}_{\{X < a_2\}}\Big) \\
&\quad + \frac{1}{q^2}G(0)^2\mathbb{E}_P\Big(e^{(2\mu - \sigma^2)T + 2\sigma\sqrt{T}X}\mathbf{1}_{\{X < a_2\}}\Big) \\
&= F^2 N(-a_2) + \left(L - \frac{1}{q}G(0)\right)^2 N(a_2) \\
&\quad + \frac{2}{q}\left(L - \frac{1}{q}G(0)\right)G(0)e^{\mu T} N\left(a_2 - \sigma\sqrt{T}\right) \\
&\quad + \frac{1}{q^2}G(0)^2 e^{(2\mu + \sigma^2)T} N\left(a_2 - 2\sigma\sqrt{T}\right).
\end{aligned}$$

Combining these formulae with (1.4) and (1.7) gives

$$\mathrm{Var}_P(D(T)) = \mathbb{E}_P\left(D(T)^2\right) - \mathbb{E}_P(D(T))^2.$$

1.9. For any positive integer n,

$$\begin{aligned}
\{t \in [0,T] : V(t) \le Fe^{-\gamma(T-t)} + \tfrac{1}{n}\} &\supseteq \{t \in [0,T] : V(t) \le Fe^{-\gamma(T-t)} + \tfrac{1}{n+1}\} \\
&\supseteq \{t \in [0,T] : V(t) \le Fe^{-\gamma(T-t)}\},
\end{aligned}$$

hence

$$\tau_n \le \tau_{n+1} \le \tau.$$

It follows that the non-decreasing sequence τ_n has a limit $\lim_{n\to\infty} \tau_n = \sigma$ and $\sigma \le \tau$. Since V has continuous paths,

$$V(\sigma) = \lim_{n\to\infty} V(\tau_n) = \lim_{n\to\infty}\left(Fe^{-\gamma(T-\tau_n)} + \tfrac{1}{n}\right) = Fe^{-\gamma(T-\sigma)},$$

which means that $\tau \le \sigma$. As a result, $\tau = \sigma$. It remains to verify that $\tau_n < \tau$ for each n when $\tau < \infty$. Because V has continuous paths, for any positive integer n there is an $\varepsilon > 0$ (which may depend on $\omega \in \Omega$) such that

$$V(\tau - \varepsilon) \le V(\tau) + \tfrac{1}{n} = Fe^{-\gamma(T-\tau)} + \tfrac{1}{n},$$

which means that

$$\tau_n \le \tau - \varepsilon < \tau.$$

This shows that τ is a predictable stopping time.

1.10. The time T debt payoff is

$$F_{\mathrm{M}}\mathbf{1}_{\{V(T)\geq F_{\mathrm{M}}\}} + V(T)\mathbf{1}_{\{V(T)<F_{\mathrm{M}}\}}$$

in the Merton model, and

$$F_{\mathrm{B}}e^{(r-\gamma)(T-\tau)}\mathbf{1}_{\{\tau\leq T\}} + F_{\mathrm{B}}\mathbf{1}_{\{\tau>T\}}$$

in the barrier model. Hence

$$D(0) = e^{-rT}\mathbb{E}_Q(F_{\mathrm{M}}\mathbf{1}_{\{V(T)\geq F_{\mathrm{M}}\}} + V(T)\mathbf{1}_{\{V(T)<F_{\mathrm{M}}\}})$$

and

$$D(0) = e^{-rT}\mathbb{E}_Q(F_{\mathrm{B}}e^{(r-\gamma)(T-\tau)}\mathbf{1}_{\{\tau\leq T\}} + F_{\mathrm{B}}\mathbf{1}_{\{\tau>T\}}).$$

It follows that

$$\begin{aligned}
F_{\mathrm{M}} &\geq \mathbb{E}_Q(F_{\mathrm{M}}\mathbf{1}_{\{V(T)\geq F_{\mathrm{M}}\}} + V(T)\mathbf{1}_{\{V(T)<F_{\mathrm{M}}\}}) \\
&= \mathbb{E}_Q(F_{\mathrm{B}}e^{(r-\gamma)(T-\tau)}\mathbf{1}_{\{\tau\leq T\}} + F_{\mathrm{B}}\mathbf{1}_{\{\tau>T\}}) \\
&\geq F_{\mathrm{B}}\mathbb{E}_Q(\mathbf{1}_{\{\tau\leq T\}} + \mathbf{1}_{\{\tau>T\}}) \\
&= F_{\mathrm{B}}.
\end{aligned}$$

1.11. When $\gamma = r + \frac{1}{2}\sigma^2$, we have

$$\begin{aligned}
\alpha &= \frac{1}{\sigma^2}\left(r - \gamma - \frac{1}{2}\sigma^2\right) = -1, \\
\beta &= -\sigma(\alpha + 1) = 0,
\end{aligned}$$

and

$$\begin{aligned}
d_3 &= \frac{\ln L + \sigma\beta T}{\sigma\sqrt{T}} = \frac{\ln L}{\sigma\sqrt{T}} = -\frac{a}{\sqrt{T}}, \\
d_4 &= \frac{\ln L - \sigma\beta T}{\sigma\sqrt{T}} = \frac{\ln L}{\sigma\sqrt{T}} = -\frac{a}{\sqrt{T}},
\end{aligned}$$

where

$$a = -\frac{\ln L}{\sigma}.$$

The distribution function of τ is

$$Q(\tau \leq t) = N\left(\frac{-a + \sigma t}{\sqrt{t}}\right) + e^{2\sigma a}N\left(\frac{-a - \sigma t}{\sqrt{t}}\right),$$

and the left-hand side of (1.9) becomes

$$\begin{aligned}
&\mathbb{E}_Q(e^{(\gamma-r)\tau}\mathbf{1}_{\{\tau\le T\}})\\
&=\mathbb{E}_Q(e^{\frac{1}{2}\sigma^2\tau}\mathbf{1}_{\{\tau\le T\}})\\
&=\int_0^T e^{\frac{1}{2}\sigma^2 t}dQ(\tau\le t)\\
&=\int_0^T e^{\frac{1}{2}\sigma^2 t}dN\left(\frac{-a+\sigma t}{\sqrt{t}}\right)+e^{2\sigma a}\int_0^T e^{\frac{1}{2}\sigma^2 t}dN\left(\frac{-a-\sigma t}{\sqrt{t}}\right).
\end{aligned}$$

On the other hand, the right-hand side of (1.9) can be written as

$$L^{-1}N(d_3)+L^{2\alpha+1}N(d_4)=2e^{\sigma a}N\left(-\frac{a}{\sqrt{T}}\right).$$

We compute and compare the derivatives of these two expressions with respect to T. We have

$$\begin{aligned}
&\frac{d}{dT}\mathbb{E}_Q(e^{(\gamma-r)\tau}\mathbf{1}_{\{\tau\le T\}})\\
&=e^{\frac{1}{2}\sigma^2 T}N'\left(\frac{-a+\sigma T}{\sqrt{T}}\right)\frac{a+\sigma T}{2\sqrt{T^3}}+e^{2\sigma a}e^{\frac{1}{2}\sigma^2 T}N'\left(\frac{-a-\sigma T}{\sqrt{T}}\right)\frac{a-\sigma T}{2\sqrt{T^3}}\\
&=\frac{1}{\sqrt{2\pi}}e^{\frac{1}{2}\sigma^2 T}e^{-\frac{(-a+\sigma T)^2}{2T}}\frac{a+\sigma T}{2\sqrt{T^3}}+\frac{1}{\sqrt{2\pi}}e^{2\sigma a}e^{\frac{1}{2}\sigma^2 T}e^{-\frac{(-a-\sigma T)^2}{2T}}\frac{a-\sigma T}{2\sqrt{T^3}}\\
&=\frac{1}{\sqrt{2\pi}}e^{a\sigma}e^{-\frac{a^2}{2T}}\frac{a}{\sqrt{T^3}}.
\end{aligned}$$

On the other hand,

$$\frac{d}{dT}\left(2e^{\sigma a}N\left(-\frac{a}{\sqrt{T}}\right)\right)=2e^{\sigma a}N'\left(-\frac{a}{\sqrt{T}}\right)\frac{a}{2\sqrt{T^3}}=\frac{1}{\sqrt{2\pi}}e^{\sigma a}e^{-\frac{a^2}{2T}}\frac{a}{\sqrt{T^3}}.$$

We can see that these derivatives are the same, hence the left- and right-hand sides of (1.9) can differ only by a constant. To see that this constant is 0 we take the left-limit as $T\searrow 0$ on both sides of (1.9). The left-hand side of (1.9) clearly tends to 0. The right-hand side also tends to 0 because $-\frac{a}{\sqrt{T}}$ tends to $-\infty$ (since $L<1$ so $a=-\frac{1}{\sigma}\ln L>0$), and so $N\left(-\frac{a}{\sqrt{T}}\right)$ tends to 0.

1.12. According to Theorem 1.29,

$$D(0)=Fe^{-rT}\left(N(-d_1)-L^{2\alpha}N(d_2)\right)+V(0)\left(N(d_3)+L^{2\alpha+2}N(d_4)\right),$$

where

$$L = \frac{Fe^{-\gamma T}}{V(0)},$$
$$\alpha = \frac{r - \gamma - \frac{1}{2}\sigma^2}{\sigma^2},$$

and

$$d_1 = \frac{\ln L - (r - \gamma - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} = \frac{\ln\frac{F}{V(0)} - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}},$$
$$d_2 = \frac{\ln L + (r - \gamma - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} = \frac{\ln\frac{F}{V(0)} + (r - 2\gamma - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}},$$
$$d_3 = \frac{\ln L - (r - \gamma + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} = \frac{\ln\frac{F}{V(0)} - (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}},$$
$$d_4 = \frac{\ln L + (r - \gamma + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} = \frac{\ln\frac{F}{V(0)} + (r - 2\gamma + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}.$$

To show that $D(0)$ is a decreasing function of γ when the remaining parameters $F, V(0), r, \sigma, T$ are fixed, we shall show that the partial derivative of $D(0)$ with respect to γ is negative. Since d_1 and d_2 do not depend on γ,

$$\begin{aligned}\frac{\partial}{\partial\gamma}D(0) &= \frac{\partial}{\partial\gamma}\left(-Fe^{-rT}L^{2\alpha}N(d_2) + V(0)L^{2\alpha+2}N(d_4)\right)\\ &= -Fe^{-rT}\frac{\partial}{\partial\gamma}\left(L^{2\alpha}\right)N(d_2) + V(0)\frac{\partial}{\partial\gamma}\left(L^{2\alpha+2}\right)N(d_4)\\ &\quad - Fe^{-rT}L^{2\alpha}\frac{\partial}{\partial\gamma}N(d_2) + V(0)L^{2\alpha+2}\frac{\partial}{\partial\gamma}N(d_4).\end{aligned}$$

Observe that the expression in the last line is 0,

$$
\begin{aligned}
&-Fe^{-rT}L^{2\alpha}\frac{\partial}{\partial\gamma}N(d_2)+V(0)L^{2\alpha+2}\frac{\partial}{\partial\gamma}N(d_4)\\
&=-Fe^{-rT}L^{2\alpha}N'(d_2)\frac{\partial d_2}{\partial\gamma}+V(0)L^{2\alpha+2}N'(d_4)\frac{\partial d_4}{\partial\gamma}\\
&=-Fe^{-rT}L^{2\alpha}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}d_2^2}\left(-\frac{2\sqrt{T}}{\sigma}\right)+V(0)L^{2\alpha+2}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}d_4^2}\left(-\frac{2\sqrt{T}}{\sigma}\right)\\
&=\frac{2\sqrt{T}}{\sigma\sqrt{2\pi}}FL^{2\alpha}\left(e^{-rT}e^{-\frac{\left(\ln L+(r-\gamma-\frac{1}{2}\sigma^2)T\right)^2}{2\sigma^2T}}-e^{-\gamma T}Le^{-\frac{\left(\ln L+(r-\gamma+\frac{1}{2}\sigma^2)T\right)^2}{2\sigma^2T}}\right)\\
&=\frac{2\sqrt{T}}{\sigma\sqrt{2\pi}}FL^{2\alpha}e^{-\frac{(\ln L+rT-\gamma T)^2+\left(\frac{1}{2}\sigma^2T\right)^2}{2\sigma^2T}}\left(e^{-rT}e^{\frac{\ln L+rT-\gamma T}{2}}-e^{-\gamma T}Le^{-\frac{\ln L+rT-\gamma T}{2}}\right)\\
&=\frac{2\sqrt{T}}{\sigma\sqrt{2\pi}}FL^{2\alpha}e^{-\frac{(\ln L+rT-\gamma T)^2+\left(\frac{1}{2}\sigma^2T\right)^2}{2\sigma^2T}}\left(e^{\frac{\ln L-rT-\gamma T}{2}}-e^{\frac{\ln L-rT-\gamma T}{2}}\right)\\
&=0.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\frac{\partial}{\partial\gamma}D(0)&=-Fe^{-rT}\frac{\partial}{\partial\gamma}\left(L^{2\alpha}\right)N(d_2)+V(0)\frac{\partial}{\partial\gamma}\left(L^{2\alpha+2}\right)N(d_4)\\
&=-Fe^{-rT}\left(\frac{\partial(2\alpha)}{\partial\gamma}L^{2a}\ln L+2\alpha\frac{\partial L}{\partial\gamma}L^{2\alpha-1}\right)N(d_2)\\
&\quad+V(0)\left(\frac{\partial(2\alpha+2)}{\partial\gamma}L^{2\alpha+2}\ln L+(2\alpha+2)\frac{\partial L}{\partial\gamma}L^{2\alpha+1}\right)N(d_4)\\
&=\frac{2}{\sigma^2}Fe^{-rT}L^{2a}\left(\ln L+\left(r-\gamma-\frac{1}{2}\sigma^2\right)T\right)N(d_2)\\
&\quad-\frac{2}{\sigma^2}V(0)L^{2a+2}\left(\ln L+\left(r-\gamma+\frac{1}{2}\sigma^2\right)T\right)N(d_4)\\
&=\frac{2}{\sigma^2}(\ln L+(r-\gamma)T)L^{2a}e^{-rT}F\left(N(d_2)-e^{\ln L+(r-\gamma)T}N(d_4)\right)\\
&\quad-Fe^{-rT}L^{2a}TN(d_2)-V(0)L^{2a+2}TN(d_4)\\
&<\frac{2}{\sigma^2}(\ln L+(r-\gamma)T)L^{2a}e^{-rT}F\left(N(d_2)-e^{\ln L+(r-\gamma)T}N(d_4)\right).
\end{aligned}
$$

It remains to show that

$$N(d_2)-e^{\ln L+(r-\gamma)T}N(d_4)<0.$$

With $x = \frac{\ln L+(r-\gamma)T}{\sigma\sqrt{T}}$ and $a = \frac{1}{2}\sigma\sqrt{T}$ this can be written as

$$N(x-a) - e^{2ax}N(x+a) < 0.$$

In fact, the last inequality holds for every $x \in \mathbb{R}$ and $a > 0$. This is so because the limit of the left-hand side as $x \to -\infty$ is 0, and the derivative with respect to x is negative,

$$\begin{aligned}
&\frac{d}{dx}\left(N(x-a) - e^{2ax}N(x+a)\right) \\
&= N'(x-a) - e^{2ax}N'(x+a) - 2ae^{2ax}N(x+a) \\
&= \frac{1}{\sqrt{2\pi}}e^{-\frac{(x-a)^2}{2}} - e^{2ax}\frac{1}{\sqrt{2\pi}}e^{-\frac{(x+a)^2}{2}} - 2ae^{2ax}N(x+a) \\
&= \frac{1}{\sqrt{2\pi}}\left(e^{-\frac{x^2+a^2-2ax}{2}} - e^{2ax}e^{-\frac{x^2+a^2+2ax}{2}}\right) - 2ae^{2ax}N(x+a) \\
&= -2ae^{2ax}N(x+a) < 0.
\end{aligned}$$

1.13. To show that $D(0)$ is an increasing function of F when the remaining parameters $\gamma, V(0), r, \sigma, T$ are fixed, we shall show that the partial derivative of $D(0)$ with respect to F is negative. Differentiating the expression for $D(0)$ in Theorem 1.29, we get

$$\begin{aligned}
\frac{\partial}{\partial F}D(0) &= \frac{\partial}{\partial F}\left(Fe^{-rT}\left(N(-d_1) - L^{2\alpha}N(d_2)\right) + V(0)\left(N(d_3) + L^{2\alpha+2}N(d_4)\right)\right) \\
&= e^{-rT}\left(N(-d_1) - L^{2\alpha}N(d_2)\right) \\
&\quad + Fe^{-rT}\left(\frac{\partial}{\partial F}N(-d_1) - \frac{\partial L^{2\alpha}}{\partial F}N(d_2) - L^{2\alpha}\frac{\partial}{\partial F}N(d_2)\right) \\
&\quad + V(0)\left(\frac{\partial}{\partial F}N(d_3) + \frac{\partial L^{2\alpha+2}}{\partial F}N(d_4) + L^{2\alpha+2}\frac{\partial}{\partial F}N(d_4)\right) \\
&= e^{-rT}\left(N(-d_1) - L^{2\alpha}N(d_2)\right) - Fe^{-rT}\frac{\partial L^{2\alpha}}{\partial F}N(d_2) + V(0)\frac{\partial L^{2\alpha+2}}{\partial F}N(d_4) \\
&= e^{-rT}N(-d_1) - (2\alpha+1)e^{-rT}L^{2\alpha}N(d_2) + (2\alpha+2)\frac{V(0)}{F}L^{2\alpha+2}N(d_4) \\
&= e^{-rT}N(-d_1) + \frac{V(0)}{F}L^{2\alpha+2}N(d_4) \\
&\quad + (2\alpha+1)\left(-e^{-rT}L^{2\alpha}N(d_2) + \frac{V(0)}{F}L^{2\alpha+2}N(d_4)\right) \\
&> (2\alpha+1)\left(-e^{-rT}L^{2\alpha}N(d_2) + \frac{V(0)}{F}L^{2\alpha+2}N(d_4)\right).
\end{aligned}$$

This is because

$$\begin{aligned}
&Fe^{-rT}\frac{\partial}{\partial F}N(-d_1)+V(0)\frac{\partial}{\partial F}N(d_3)\\
&=-\frac{1}{\sigma\sqrt{2\pi T}}\left(e^{-rT}e^{-\frac{1}{2}d_1^2}-\frac{V(0)}{F}e^{-\frac{1}{2}d_3^2}\right)\\
&=-\frac{1}{\sigma\sqrt{2\pi T}}\left(e^{-rT}e^{-\frac{\left(\ln\frac{F}{V(0)}-rT+\frac{1}{2}\sigma^2T\right)^2}{2\sigma^2T}}-\frac{V(0)}{F}e^{-\frac{\left(\ln\frac{F}{V(0)}-rT-\frac{1}{2}\sigma^2T\right)^2}{2\sigma^2T}}\right)\\
&=-\frac{1}{\sigma\sqrt{2\pi T}}e^{-\frac{\left(\ln\frac{F}{V(0)}-rT\right)^2+\left(\frac{1}{2}\sigma^2T\right)^2}{2\sigma^2T}}\left(e^{-rT}e^{\frac{-\ln\frac{F}{V(0)}+rT}{2}}-e^{-\ln\frac{F}{V(0)}}e^{\frac{\ln\frac{F}{V(0)}-rT}{2}}\right)\\
&=0
\end{aligned}$$

and

$$\begin{aligned}
&-Fe^{-rT}L^{2\alpha}\frac{\partial}{\partial F}N(d_2)+V(0)L^{2\alpha+2}\frac{\partial}{\partial F}N(d_4)\\
&=\frac{1}{\sigma\sqrt{2\pi T}}L^{2\alpha}\left(-e^{-rT}e^{-\frac{1}{2}d_2^2}+\frac{V(0)}{F}L^2e^{-\frac{1}{2}d_4^2}\right)\\
&=\frac{1}{\sigma\sqrt{2\pi T}}L^{2\alpha}\left(-e^{-rT}e^{-\frac{\left(\ln\frac{F}{V(0)}-2\gamma T+rT-\frac{1}{2}\sigma^2T\right)^2}{2\sigma^2T}}+\frac{V(0)}{F}L^2e^{-\frac{\left(\ln\frac{F}{V(0)}-2\gamma T+rT+\frac{1}{2}\sigma^2T\right)^2}{2\sigma^2T}}\right)\\
&=\frac{1}{\sigma\sqrt{2\pi T}}L^{2\alpha}e^{-\frac{\left(\ln\frac{F}{V(0)}-2\gamma T+rT\right)^2+\left(\frac{1}{2}\sigma^2T\right)^2}{2\sigma^2T}}\left(-e^{-rT}e^{\frac{\ln\frac{F}{V(0)}-2\gamma T+rT}{2}}+e^{\ln\frac{F}{V(0)}-2\gamma T}e^{\frac{-\ln\frac{F}{V(0)}+2\gamma T-rT}{2}}\right)\\
&=0.
\end{aligned}$$

Next, putting $x=\frac{\ln\frac{F}{V(0)}-2\gamma T+rT}{\sigma\sqrt{T}}$ and $a=\frac{1}{2}\sigma\sqrt{T}$, we can write

$$\begin{aligned}
\frac{\partial}{\partial F}D(0)&>(2\alpha+1)\left(-e^{-rT}L^{2\alpha}N(d_2)+\frac{V(0)}{F}L^{2\alpha+2}N(d_4)\right)\\
&=(2\alpha+1)L^{2\alpha}e^{-rT}\left(-N(x-a)+e^{2ax}N(x+a)\right)\\
&>0.
\end{aligned}$$

where the inequality

$$-N(x-a)+e^{2ax}N(x+a)>0$$

was proved in Solution 1.12. Hence, we have demonstrated that $\frac{\partial D(0)}{\partial F}>0$, which shows that $D(0)$ increases as F increases (with the other parameters fixed).

1.14. According to Theorems 1.29 and 1.30,

$$D(0) = Fe^{-rT}\left(N(-d_1) - L^{2\alpha}N(d_2)\right) + V(0)\left(N(d_3) + L^{2\alpha+2}N(d_4)\right),$$
$$E(0) = V(0)\left((N(-d_3) - L^{2\alpha+2}N(d_4)\right) - e^{-rT}F\left(N(-d_1) - L^{2\alpha}N(d_2)\right),$$

where

$$d_1 = \frac{\ln\frac{F}{V(0)} - \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}},$$
$$d_2 = \frac{\ln\frac{F}{V(0)} + \left(r - 2\gamma - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}},$$
$$d_3 = \frac{\ln\frac{F}{V(0)} - \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$
$$d_4 = \frac{\ln\frac{F}{V(0)} + \left(r - 2\gamma + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

Hence, in the limit as $\gamma \to \infty$ we have $d_2 \to -\infty$ and $d_4 \to -\infty$, and so $N(d_2) \to 0$ and $N(d_4) \to 0$. It follows that

$$D(0) \to Fe^{-rT}N(-d_1) + V(0)N(d_3),$$
$$E(0) \to V(0)N(-d_3) - e^{-rT}FN(-d_1)$$

as $\gamma \to \infty$, which is consistent with the formulae for the Merton model in Section 1.2.

1.15. Consider the function of two variables

$$\Phi(F,\gamma) = Fe^{-rT}\left(N(-d_1) - L^{2\alpha}N(d_2)\right) + V(0)\left(N(d_3) + L^{2\alpha+2}N(d_4)\right),$$

where $L, \alpha, d_1, d_2, d_3, d_4$ (which also depend on F, γ) are given in Solution 1.12, with the parameters $V(0), r, \sigma, T$ fixed. According to Theorem 1.29,

$$\Phi(F,\gamma) = D(0).$$

In Exercises 1.12 and 1.13 we saw that

$$\frac{\partial\Phi}{\partial\gamma} < 0 \quad \text{and} \quad \frac{\partial\Phi}{\partial F} > 0.$$

Now consider F as a function of γ such that

$$\Phi(F(\gamma),\gamma) = D(0)$$

for a fixed value of $D(0)$. It follows that

$$\frac{dF}{d\gamma}\frac{\partial \Phi}{\partial F} + \frac{\partial \Phi}{\partial \gamma} = 0,$$

hence

$$\frac{dF}{d\gamma} = -\frac{\frac{\partial \Phi}{\partial \gamma}}{\frac{\partial \Phi}{\partial F}} > 0.$$

This means that F, and therefore $s = \frac{1}{T}\ln\frac{F}{D(0)} - r$, is an increasing function of γ when the parameters $D(0), V(0), r, \sigma, T$ are fixed.

Chapter 2

2.1. If $\bar{\tau}$ is defined as $\bar{\tau}(\omega) = \sup\{t \geq 0 : F(t) < \omega\}$, then

$$\{\bar{\tau} \leq t\} = \{\omega \in [0,1] : \tau(\omega) \leq t\} = [0, F(t)].$$

It follows that

$$P(\bar{\tau} \leq t) = m([0, F(t)]) = F(t),$$

hence τ and $\bar{\tau}$ have the same probability distribution.

2.2. For any $a \in [0,1)$ take $b = F(a) = 1 - e^{-\lambda a}$. Then

$$[0, a] = \{\bar{\tau} \leq b\} \in \sigma(\bar{\tau}).$$

Because the intervals $[0, a]$ for $a \in [0,1)$ generate the σ-field of Borel sets in $[0,1]$, it follows that $\sigma(\bar{\tau})$ contains all such Borel sets.

Next, for any $b \geq 0$ take $a = \bar{\tau}(b) = -\frac{1}{\lambda}\ln(1-b)$. Then

$$\{\bar{\tau} \leq b\} = [0, a]$$

is a Borel set on $[0,1]$. Because sets of the form $\{\bar{\tau} \leq b\}$ for $b \geq 0$ generate the σ-field $\sigma(\bar{\tau})$, it follows that $\sigma(\bar{\tau})$ is contained in the family of Borel sets in $[0,1]$.

We conclude that $\sigma(\bar{\tau})$ is equal to the family of Borel sets in $[0,1]$.

2.3. The expected default time within the 20-year period is

$$\mathbb{E}\left(\tau \mathbf{1}_{\{\tau \leq 20\}}\right) = \int_0^{20} t\lambda e^{-\lambda t} dt = -20e^{-20\lambda} - \frac{1}{\lambda}e^{-20\lambda} + \frac{1}{\lambda}.$$

If 10 companies have been observed over 20 years, among which 2 companies defaulted in year 5 and 3 companies defaulted in year 12, then an estimate of the expected default time $\tau \mathbf{1}_{\{\tau \leq 2\}}$ within the 20 year period is

$$\frac{2}{10} \times 5 + \frac{3}{10} \times 12 = 4.6.$$

This gives

$$-20e^{-20\lambda} - \frac{1}{\lambda} e^{-20\lambda} + \frac{1}{\lambda} = 4.6.$$

Solving this equation numerically, we get $\lambda = 0.19626$.

For this value of λ we can compute the probability that a given company will survive beyond 5 years as

$$P(5 < \tau) = e^{-5\lambda} = e^{-5 \times 0.19626} = 0.37482.$$

The expected time of default is

$$\mathbb{E}(\tau) = \frac{1}{\lambda} = \frac{1}{0.19626} = 5.0953.$$

2.4. Since

$$\{\tau \leq t\} = \{I(t) = 1\} \in \mathcal{I}_t$$

for each $t \geq 0$, it follows that τ is a stopping time with respect to the filtration $(\mathcal{I}_t)_{t \geq 0}$.

Now suppose that τ is a stopping time with respect to some filtration $(\mathcal{F}_t)_{t \geq 0}$. It means that $\{\tau \leq s\} \in \mathcal{F}_s \subset \mathcal{F}_t$ and therefore also $\{s < \tau\} \in \mathcal{F}_t$ for each $s \in [0, t]$. This in turn means that $I(s)$ is $\mathcal{F}_t$-measurable for each $s \in [0, t]$. Because $\mathcal{I}_t$ is the smallest σ-field such that $I(s)$ is $\mathcal{I}_t$-measurable for each $s \in [0, t]$, it follows that $\mathcal{I}_t \subset \mathcal{F}_t$ for each $t \geq 0$.

This proves that $(\mathcal{I}_t)_{t \geq 0}$ is the smallest filtration with respect to which τ is a stopping time.

2.5. The random variable $I(t)$ can take two values only, namely 1 or 0. We have

$$\{I(t) = 1\} = \{\tau \leq t\},$$
$$\{I(t) = 0\} = \{t < \tau\}.$$

It follows that the σ-field $\sigma(I(t))$ consist of four events $\emptyset, \Omega, \{\tau \leq t\}, \{t < \tau\}$ when $t > 0$. For $t = 0$ we have $\{\tau \leq 0\} = \emptyset$ and $\{0 < \tau\} = \Omega$, so $\sigma(I(0))$ consists of just two events $\emptyset, \Omega$.

2.6. Take a sequence of sets $B_1, B_2, \ldots \in \mathcal{D}_1$. By the definition of the family $\mathcal{D}_1$, each of these sets is of the form $B_n = A_n \cap \{\tau \leq t\}$ for some $A_n \in \sigma(\tau)$, where $n = 1, 2, \ldots$. Then $A_1 \cup A_2 \cup \cdots \in \sigma(\tau)$, hence

$$B_1 \cup B_2 \cup \cdots = (A_1 \cup A_2 \cup \cdots) \cap \{\tau \le t\} \in \mathcal{D}_1.$$

Similarly, for a sequence of sets $B_1, B_2, \ldots \in \mathcal{D}_2$, which by the definition of $\mathcal{D}_2$ are of the form $B_n = A_n \cup \{t < \tau\}$, where $A_n \in \sigma(\tau)$ for each $n = 1, 2, \ldots$, we have $A_1 \cup A_2 \cup \cdots \in \sigma(\tau)$, and so

$$B_1 \cup B_2 \cup \cdots = (A_1 \cup A_2 \cup \cdots) \cup \{t < \tau\} \in \mathcal{D}_2.$$

2.7. Suppose that $C_1, C_2, \ldots \in \mathcal{D}$. By the definition of $\mathcal{D}$, we have $C_n \cap \{\tau \le t\} \in \mathcal{I}_t$ for each $n = 1, 2, \ldots$. Then

$$(C_1 \cap C_2 \cap \cdots) \cap \{\tau \le t\} = (C_1 \cap \{\tau \le t\}) \cap (C_2 \cap \{\tau \le t\}) \cap \cdots \in \mathcal{I}_t,$$

which means that $C_1 \cap C_2 \cap \cdots \in \mathcal{D}$.

2.8. Take any $s \in \mathbb{R}$. Then $\{\tau \wedge t \le s\} = \Omega \in \mathcal{I}_t$ if $t \le s$, and $\{\tau \wedge t \le s\} = \{\tau \le s\}$ if $t > s$. In either case $\{\tau \wedge t \le s\} \in \mathcal{I}_t$. Because the sets $\{\tau \wedge t \le s\}$ for $s \in \mathbb{R}$ generate the σ-field $\sigma(\tau \wedge t)$, it follows that $\sigma(\tau \wedge t) \subset \mathcal{I}_t$. Moreover, since $\{t < \tau\} \in \mathcal{I}_t$, it follows that

$$\sigma(\{t < \tau\}, \sigma(\tau \wedge t)) \subset \mathcal{I}_t.$$

Now suppose that $\mathcal{G}$ is a σ-field such that $\{t < \tau\} \in \mathcal{G}$ and $\sigma(\tau \wedge t) \subset \mathcal{G}$. We will show that $\{\tau \le s\} \in \mathcal{G}$ for every $s \in [0, t]$. Because $\mathcal{I}_t$ is the smallest σ-field containing $\{\tau \le s\}$ for every $s \in [0, t]$, this implies that $\mathcal{I}_t \subset \mathcal{G}$. Hence $\mathcal{I}_t$ is the smallest σ-field such that $\{t < \tau\} \in \mathcal{I}_t$ and $\sigma(\tau \wedge t) \subset \mathcal{I}_t$. As a result,

$$\sigma(\{t < \tau\}, \sigma(\tau \wedge t)) = \mathcal{I}_t.$$

It remains to show that $\{\tau \le s\} \in \mathcal{G}$ for every $s \in [0, t]$. If $s \in [0, t)$, then $\{\tau \le s\} = \{\tau \wedge t \le s\} \in \mathcal{G}$ since $\sigma(\tau \wedge t) \subset \mathcal{G}$. For $s = t$ we have $\{\tau \le t\} \in \mathcal{G}$ since $\{t < \tau\} \in \mathcal{G}$, which completes the argument.

2.9. By Corollary 2.20, $\mathcal{I}_t \subset \sigma(\tau)$ for each $t \ge 0$, so $\mathcal{I}_\infty \subset \sigma(\tau)$.

We are going to show that $\{\tau \le t\} \in \mathcal{I}_\infty$ for every $t \in \mathbb{R}$. Indeed, $\{\tau \le t\} = \varnothing \in \mathcal{I}_\infty$ for any $t < 0$, and $\{\tau \le t\} \in \mathcal{I}_t \subset \mathcal{I}_\infty$ for any $t \ge 0$. Because $\sigma(\tau)$ is the smallest σ-field containing the sets $\{\tau \le t\}$ for all $t \in \mathbb{R}$, it follows that $\sigma(\tau) \subset \mathcal{I}_\infty$.

We have proved that $\sigma(\tau) = \mathcal{I}_\infty$.

2.10. Let $0 < t < u$. Then, clearly, $\{t < \tau \le u\} \in \sigma(\tau)$. However, since $\{t < \tau\}$ is an atom in $\mathcal{I}_t$ containing $\{t < \tau \le u\}$ and (under the assumptions adopted in Chapter 2) $\{t < \tau \le u\}$ is non-empty and differs from $\{t < \tau\}$, it follows that $\{t < \tau \le u\} \notin \mathcal{I}_t$.

2.11. Since σ is a stopping time with respect to the filtration $(\mathcal{I}_t)_{t\geq 0}$ and $0 \leq \sigma \leq T$, it follows that $\sigma = \sigma \wedge T$ is $\mathcal{I}_T$-measurable, hence by Proposition 2.22 it can be written as

$$\sigma = \eta(\tau)\mathbf{1}_{\{\tau\leq T\}} + c\mathbf{1}_{\{T<\tau\}}$$

for some Borel function $\eta : \mathbb{R} \to \mathbb{R}$ and some constant $c \in \mathbb{R}$. Since $0 \leq \sigma \leq T$, we have $c \in [0, T]$.

Now suppose that $\tau\wedge c \leq \sigma$ does not hold, that is, the set $\{\sigma < \tau \wedge c\}$ is non-empty. Then there is a $t \in \mathbb{R}$ such that $\{\sigma \leq t\} \cap \{t < \tau\} \supset \{\sigma \leq t < \tau \wedge c\}$ is non-empty. Because $\{\sigma \leq t\} \in \mathcal{I}_t$ and $\{t < \tau\}$ is an atom in $\mathcal{I}_t$, it follows that $\{t < \tau\} \subset \{\sigma \leq t\}$. Since $t < c \leq T$, we therefore have $\{T < \tau\} \subset \{t < \tau\} \subset \{\sigma \leq t\} \subset \{\sigma < c\}$. But $\sigma = c$ on the non-empty set $\{T < \tau\}$, a contradiction, which proves that $\tau \wedge c \leq \sigma$.

Finally, suppose that $c < \sigma$ on $\{c < \tau\}$, that is, $\{c < \sigma\} \cap \{c < \tau\}$ is non-empty. Because $\{c < \sigma\} \in \mathcal{I}_c$ and $\{c < \tau\}$ is an atom in $\mathcal{I}_c$, it follows that $\{c < \tau\} \subset \{c < \sigma\}$. But this is impossible because $c \leq T$, so $\{T < \tau\} \subset \{c < \tau\} \subset \{c < \sigma\}$ and $\sigma = c$ on the non-empty set $\{T < \tau\}$. It proves that $\sigma \leq c$ on $\{c < \tau\}$. Because $\tau \wedge c \leq \sigma$, it follows that $\sigma = c$ on $\{c < \tau\}$.

2.12. By definition, the σ-field $\mathcal{I}_\sigma$ consists of all events $A \subset \Omega$ such that $A \cap \{\sigma \leq t\} \in \mathcal{I}_t$ for each $t \geq 0$. From Exercise 2.11 we know that there is a deterministic constant $c \in [0, T]$ such that $\tau \wedge c \leq \sigma$ and $\sigma = c$ on $\{c < \tau\}$. It follows that $\{\sigma < \tau\} = \{c < \tau\}$. It also follows that $\{c < \tau\} \cap \{\sigma \leq t\} = \varnothing \in \mathcal{I}_t$ if $t < c$, and $\{c < \tau\} \cap \{\sigma \leq t\} = \{c < \tau\} \in \mathcal{I}_c \subset \mathcal{I}_t$ if $c \leq t$. This means that $\{c < \tau\} \in \mathcal{I}_\sigma$. Now suppose that $A \in \mathcal{I}_\sigma$ and $A \subset \{c < \tau\}$. Since $\{c < \tau\} \subset \{\sigma \leq c\}$, we have $A = A \cap \{\sigma \leq c\} \in \mathcal{I}_c$. Because $\{c < \tau\}$ is an atom in $\mathcal{I}_c$, we therefore have $A = \varnothing$ or $A = \{c < \tau\}$, proving that $\{c < \tau\}$ is an atom in $\mathcal{I}_\sigma$.

2.13. Let $\varphi = (\varphi_B, \varphi_D)$ be a strategy such that there is a sequence $0 = \sigma_0 < \sigma_1 < \cdots < \sigma_n = T$ of $(\mathcal{I}_t)_{t\geq 0}$-stopping times with the following properties:

(i) the processes $\varphi_B(t), \varphi_D(t)$ are constant on $(\sigma_{k-1}, \sigma_k]$ for each $k = 1, \ldots, n$;

(ii) the random variables $\varphi_B(\sigma_k), \varphi_D(\sigma_k)$ are $\mathcal{I}_{\sigma_{k-1}}$-measurable for each $k = 1, \ldots, n$, and $\varphi_B(0), \varphi_D(0)$ are $\mathcal{I}_0$-measurable.

Moreover, suppose that the strategy satisfies the self-financing condition

$$V_\varphi(\sigma_k) = \varphi_B(\sigma_{k+1})B(\sigma_k, T) + \varphi_D(\sigma_{k+1})D(\sigma_k, T)$$

for each $k = 0, \ldots, n-1$, where

$$V_\varphi(t) = \varphi_B(t)B(t, T) + \varphi_D(t)D(t, T)$$

is the value of the strategy at any time $t \in [0, T]$.

We are going to show that if $V_\varphi(0) = 0$ and $V_\varphi(T) \neq 0$ with positive probability, then $V_\varphi(T) > 0$ with positive probability and $V_\varphi(T) < 0$ with positive probability, proving that arbitrage is impossible to achieve using this kind of strategy.

Take the smallest k such that $V_\varphi(\sigma_k) \neq 0$ on a set of positive probability. Clearly, such k exists because $V_\varphi(T) \neq 0$ with positive probability. We have $k > 1$ because $V_\varphi(0) = 0$. The self-financing condition at time σ_{k-1} gives

$$0 = V_\varphi(\sigma_{k-1}) = \varphi_B(\sigma_k)B(\sigma_{k-1}, T) + \varphi_D(\sigma_k)D(\sigma_{k-1}, T).$$

On $\{\tau \leq \sigma_{k-1}\}$ we therefore have $\varphi_B(\sigma_k) = 0$ because $D(\sigma_{k-1}, T) = 0$. Since $\varphi_B(\sigma_k)$ is $\mathcal{I}_{\sigma_{k-1}}$-measurable, it must be constant on $\{\sigma_{k-1} < \tau\}$, which is an atom in $\mathcal{I}_{\sigma_{k-1}}$ according to Exercise 2.12. This constant must be non-zero or else $\varphi_B(\sigma_k) = 0$, hence $\varphi_D(\sigma_k) = 0$ and so $V_\varphi(\sigma_k) = 0$ everywhere, contradicting the choice of k. There are two possibilities:

Case 1: $\varphi_B(\sigma_k) > 0$ on $\{\sigma_{k-1} < \tau\}$.

On $\{\sigma_{k-1} < \tau \leq \sigma_k\}$ we have $D(\sigma_k, T) = 0$, and it follows that $V_\varphi(\sigma_k) = \varphi_B(\sigma_k)B(\sigma_k, T) > 0$. For all later times the defaultable bonds remain worthless, so the value of the strategy must remain positive on $\{\sigma_{k-1} < \tau \leq \sigma_k\}$. In particular, $V_\varphi(T) > 0$ on the event $\{\sigma_{k-1} < \tau \leq \sigma_k\}$, which we can show to have positive probability. Indeed, according to Exercise 2.11, there are deterministic constants $c_{k-1}, c_k \in [0, T]$ such that $\sigma_i \geq \tau$ on $\{\tau \leq c_i\}$ and $\sigma_i = c_i$ on $\{c_i < \tau\}$ for $i = k-1, k$. From Exercise 2.12 it follows that $\{\sigma_{k-1} < \tau\} = \{c_{k-1} < \tau\}$ and $\{\sigma_k < \tau\} = \{c_k < \tau\}$. By taking complements, we can write the last equality as $\{\tau \leq \sigma_k\} = \{\tau \leq c_k\}$. As a result, we have $\{\sigma_{k-1} < \tau \leq \sigma_k\} = \{c_{k-1} < \tau \leq c_k\}$. Because $\sigma_{k-1} < \sigma_k$, we have $c_{k-1} < c_k$, and so the event $\{\sigma_{k-1} < \tau \leq \sigma_k\} = \{c_{k-1} < \tau \leq c_k\}$ has positive probability in view of the assumptions about τ adopted in Chapter 2.

On $\{\sigma_k < \tau\}$ the long position $\varphi_B(\sigma_k) > 0$ in non-defaultable bonds is balanced by the short position $\varphi_D(\sigma_k) = -\frac{B(\sigma_{k-1},T)}{D(\sigma_{k-1},T)}\varphi_B(\sigma_k) < 0$

in defaultable bonds so that $V_\varphi(\sigma_{k-1}) = 0$ at time σ_{k-1}. Prior to a default the defaultable bond grows faster than the non-defaultable one, so at time σ_k the long position will become dominated by the short one, hence $V_\varphi(\sigma_k) < 0$ on $\{\sigma_k < \tau\}$. If $k = n$, then $\sigma_k = T$, so $V_\varphi(T) < 0$ on the set $\{T < \tau\}$ of positive probability. If $k < n$, then by the self-financing condition $\varphi_B(\sigma_{k+1})B(\sigma_k, T) = V_\varphi(\sigma_k) < 0$ on the event $\{\sigma_k < \tau \leq \sigma_{k+1}\}$, so $\varphi_B(\sigma_{k+1}) < 0$ and $V_\varphi(\sigma_{k+1}) = \varphi_B(\sigma_{k+1})B(\sigma_{k+1}, T) < 0$ on $\{\sigma_k < \tau \leq \sigma_{k+1}\}$. For all times later than σ_{k+1} the defaultable bonds remain worthless, hence the value of the strategy must remain negative on the event $\{\sigma_k < \tau \leq \sigma_{k+1}\}$. In particular, $V_\varphi(T) < 0$ on $\{\sigma_k < \tau \leq \sigma_{k+1}\}$, which we have shown to have positive probability. This completes the argument in Case 1.

Case 2: $\varphi_B(\sigma_k) < 0$ on $\{\sigma_{k-1} < \tau\}$.

Taking the self-financing simple strategy $-\varphi(t) = (-\varphi_B(t), -\varphi_D(t))$ for $t \in [0, T]$ reduces this to Case 1.

2.14. For any $t \in (-\infty, 0)$

$$\begin{aligned} F_Q(t) &= Q(\bar\tau \leq t) = Q(\tau \leq t) = 0, \\ G_Q(t) &= 1 - F_Q(t) = 1, \\ \Gamma_Q(t) &= -\ln G_Q(t) = 0. \end{aligned}$$

For any $t \in [0, T]$

$$\begin{aligned} F_Q(t) &= Q(\bar\tau \leq t) = Q(\tau \leq t) = 1 - e^{-g(0)+g(t)} = 1 - e^{-\lambda t}, \\ G_Q(t) &= 1 - F_Q(t) = e^{-\lambda t}, \\ \Gamma_Q(t) &= -\ln G_Q(t) = \lambda t. \end{aligned}$$

For any $t \in (T, \infty)$

$$\begin{aligned} F_Q(t) &= Q(\bar\tau \leq t) = 1 - Q(t < \bar\tau) = 1 - Q(T < \tau) = 1 - e^{-\lambda T}, \\ G_Q(t) &= 1 - F_Q(t) = e^{-\lambda T}, \\ \Gamma_Q(t) &= -\ln G_Q(t) = \lambda T. \end{aligned}$$

Chapter 3

3.1. For $r = 0.05$, $T_0 = 0$, $T_1 = \frac{1}{2}$, $T_2 = \frac{3}{4}$, $T_3 = 1$ and $D(0, T_0) = 1$, $D(0, T_1) = 0.9268$, $D(0, T_2) = 0.8487$, $D(0, T_3) = 0.7635$ we obtain the following piecewise linear expression for $D(0,T)$:

$$D(0,T) = \begin{cases} \frac{T_1-T}{T_1-T_0}D(0,T_0) + \frac{T-T_0}{T_1-T_0}D(0,T_1) & \text{for } T_0 \leq T \leq T_1, \\ \frac{T_2-T}{T_2-T_1}D(0,T_1) + \frac{T-T_1}{T_2-T_1}D(0,T_2) & \text{for } T_1 < T \leq T_2, \\ \frac{T_3-T}{T_3-T_2}D(0,T_2) + \frac{T-T_2}{T_3-T_2}D(0,T_3) & \text{for } T_2 < T \leq T_3. \end{cases}$$

$$= \begin{cases} 1 - 0.1464T & \text{for } 0 \leq T \leq \frac{1}{2}, \\ 1.083 - 0.3124T & \text{for } \frac{1}{2} < T \leq \frac{3}{4}, \\ 1.1043 - 0.3408T & \text{for } \frac{3}{4} < T \leq 1. \end{cases}$$

Since

$$D(0,T) = e^{-rT}e^{-\Gamma(T)},$$

we have

$$\Gamma(T) = -rT - \ln D(0,T)$$

and

$$\gamma(T) = \frac{d\Gamma(T)}{dT} = -r - \frac{1}{D(0,T)}\frac{dD(0,T)}{dT}$$

$$= \begin{cases} -0.05 + \frac{0.1462}{1-0.1464T} & \text{for } 0 \leq T \leq \frac{1}{2}, \\ -0.05 + \frac{0.3124}{1.083-0.3124T} & \text{for } \frac{1}{2} < T \leq \frac{3}{4}, \\ -0.05 + \frac{0.3408}{1.1043-0.3408T} & \text{for } \frac{3}{4} < T \leq 1. \end{cases}$$

3.2. If τ is exponentially distributed under Q with parameter λ, then

$$F(T) = 1 - e^{-\lambda T}.$$

On the other hand,

$$F(T) = 1 - e^{rT}D(0,T).$$

Hence

$$\lambda = -\frac{1}{T}\ln D(0,T) - r = -\frac{1}{\frac{1}{2}}\ln 0.9133 - 0.05 = 0.1314.$$

We can now compute

$$D(0,\tfrac{1}{4}) = e^{-\frac{1}{4}r}Q(\tfrac{1}{4} < \tau) = e^{-\frac{1}{4}(r+\lambda)} = e^{-\frac{1}{4}(0.05+0.1314)} = 0.9557.$$

3.3. With $T_1 = \frac{1}{2}$, $T_2 = 1$ and $D(0,T_1) = 0.8679$, $D(0,T_2) = 0.7055$, the system of equations

$$aT_1^2 + bT_1 = -\ln D(0,T_1) - rT_1,$$
$$aT_2^2 + bT_2 = -\ln D(0,T_2) - rT_2,$$

becomes

$$\tfrac{1}{4}a + \tfrac{1}{2}b = 0.11668,$$
$$a + b = 0.29885,$$

which gives

$$a = 0.13098, \quad b = 0.16787.$$

We can now compute $D(0, T_3)$ for $T_3 = \frac{3}{4}$:

$$D(0, T_3) = e^{-rT_3} e^{-\Gamma(T_3)} = e^{-rT_3} e^{-aT_3^2 - bT_3}$$
$$= e^{-0.05 \times \frac{3}{4}} e^{-0.13098 \times \left(\frac{3}{4}\right)^2 - 0.16787 \times \frac{3}{4}} = 0.7889.$$

3.4. Let $T_1 = \frac{1}{2}$ and $T_2 = 1$. Suppose the hazard rate is constant on $(0, T_1], (T_1, T_2]$ with values γ_1, γ_2, respectively. Then

$$\gamma_1 = \frac{1}{T_1} \ln \frac{1}{D(0, T_1)} - r = \frac{1}{\frac{1}{2}} \ln \frac{1}{0.85} - 0.05 = 0.2750,$$
$$\gamma_2 = \frac{1}{T_2 - T_1} \ln \frac{D(0, T_1)}{D(0, T_2)} - r = \frac{1}{1 - \frac{1}{2}} \ln \frac{0.85}{0.80} - 0.05 = 0.0712.$$

Since $T = \frac{3}{4}$ belongs to the interval $(T_1, T_2]$,

$$\Gamma(T) = \int_0^T \gamma(t) dt = \gamma_1 T_1 + \gamma_2 (T - T_1)$$
$$= 0.2750 \times \tfrac{1}{2} + 0.0712(\tfrac{3}{4} - \tfrac{1}{2}) = 0.1553,$$

and so

$$D(0, T) = e^{-rT} e^{-\Gamma(T)} = e^{-0.05 \times \frac{3}{4}} e^{-0.1553} = 0.8247.$$

3.5. Let $T_1 = \frac{1}{2}$, $T_2 = \frac{3}{4}$ and $T_3 = 1$. Suppose the hazard rate is constant on $(0, T_1], (T_1, T_2], (T_2, T_3]$ with values $\gamma_1, \gamma_2, \gamma_3$, respectively. Then

$$\gamma_1 = \frac{1}{T_1} \ln \frac{1}{D(0, T_1)} - r = \frac{1}{\frac{1}{2}} \ln \frac{1}{0.9037} - 0.05 = 0.1525,$$
$$\gamma_2 = \frac{1}{T_2 - T_1} \ln \frac{D(0, T_1)}{D(0, T_2)} - r = \frac{1}{\frac{3}{4} - \frac{1}{2}} \ln \frac{0.9037}{0.8609} - 0.05 = 0.1441,$$
$$\gamma_3 = \frac{1}{T_3 - T_2} \ln \frac{D(0, T_2)}{D(0, T_3)} - r = \frac{1}{1 - \frac{3}{4}} \ln \frac{0.8609}{0.7724} - 0.05 = 0.3839.$$

3.6. We found in Exercise 3.4 that

$$\gamma(t) = \begin{cases} 0.2750 & \text{on } [0, \frac{1}{2}], \\ 0.0712 & \text{on } (\frac{1}{2}, 1]. \end{cases}$$

It follows that

$$\hat{D}(t,1) = e^{-\int_t^1 (r+\gamma(s))ds} = \begin{cases} e^{0.3250t-0.2231} & \text{for } t \in [0, \frac{1}{2}], \\ e^{0.1212t-0.1212} & \text{for } t \in (\frac{1}{2}, 1]. \end{cases}$$

3.7. Let $0 \le s < t$. Since the Poisson process has independent increments, $N(t) - N(s)$ is independent of the σ-field $\mathcal{N}_s$. It follows that

$$\mathbb{E}(N(t) - N(s)|\mathcal{N}_s) = \mathbb{E}(N(t) - N(s)) = \lambda (t-s).$$

The last equality holds because $N(t) - N(s)$ has the Poisson distribution with parameter $\lambda(t-s)$. As a result,

$$\mathbb{E}(N(t) - \lambda t|\mathcal{N}_s) = N(s) - \lambda s,$$

that is, $N(t) - \lambda s$ is a martingale with respect to the filtration $(\mathcal{N}_t)_{t\ge 0}$.

3.8. Take any $t \ge 0$ and let τ be the time of the first jump of the Poisson process $N(t)$. Observe that $t < \tau$ means that the first jump hasn't yet occurred at time t, which is equivalent to $N(t) = 0$ given that the Poisson process starts with $N(0) = 0$. As a result,

$$Q(t < \tau) = Q(N(t) = 0) = e^{-\lambda t}.$$

Hence τ is exponentially distributed with parameter λ.

3.9. Let $0 \le s < t$. Since $I(t) = \mathbf{1}_{\{\tau \le t\}}$ and τ has the exponential distribution, by Proposition 2.27 applied to the risk-neutral probability Q, we have

$$\begin{aligned}
\mathbb{E}(I(t)|\mathcal{I}_s) &= \mathbb{E}(\mathbf{1}_{\{\tau\le t\}}|\mathcal{I}_s) \\
&= \mathbf{1}_{\{\tau\le s\}}\mathbb{E}(\mathbf{1}_{\{\tau\le t\}}|\sigma(\tau)) + \mathbf{1}_{\{s<\tau\}}\frac{\mathbb{E}(\mathbf{1}_{\{s<\tau\}}\mathbf{1}_{\{\tau\le t\}})}{Q(s<\tau)} \\
&= \mathbf{1}_{\{\tau\le s\}}\mathbf{1}_{\{\tau\le t\}} + \mathbf{1}_{\{s<\tau\}}\frac{Q(s<\tau\le t)}{Q(s<\tau)} \\
&= I(s) + \mathbf{1}_{\{s<\tau\}}\left(1 - e^{-\lambda(t-s)}\right).
\end{aligned}$$

On the other hand, using Proposition 2.27 once again, we have

$$\begin{aligned}
\mathbb{E}(\lambda(t\wedge\tau)|\mathcal{I}_s) &= \lambda\mathbf{1}_{\{\tau\le s\}}\mathbb{E}(t\wedge\tau|\sigma(\tau)) + \lambda\mathbf{1}_{\{s<\tau\}}\frac{\mathbb{E}(\mathbf{1}_{\{s<\tau\}}(t\wedge\tau))}{Q(s<\tau)} \\
&= \lambda\mathbf{1}_{\{\tau\le s\}}(t\wedge\tau) + \lambda\mathbf{1}_{\{s<\tau\}}\frac{se^{-\lambda s} + \frac{1}{\lambda}\left(e^{-\lambda s} - e^{-\lambda t}\right)}{e^{-\lambda s}} \\
&= \lambda\mathbf{1}_{\{\tau\le s\}}\tau + \lambda\mathbf{1}_{\{s<\tau\}}s + \mathbf{1}_{\{s<\tau\}}\left(1 - e^{-\lambda(t-s)}\right) \\
&= \lambda(s\wedge\tau) + \mathbf{1}_{\{s<\tau\}}\left(1 - e^{-\lambda(t-s)}\right)
\end{aligned}$$

since

$$\mathbb{E}\left(\mathbf{1}_{\{s<\tau\}}(t\wedge\tau)\right) = \int_s^\infty (t\wedge u)\,\lambda e^{-\lambda u}du$$
$$= \lambda\int_s^t ue^{-\lambda u}du + \lambda t\int_t^\infty e^{-\lambda u}du$$
$$= se^{-\lambda s} + \frac{1}{\lambda}\left(e^{-\lambda s} - e^{-\lambda t}\right)$$

and $Q(s<\tau) = e^{-\lambda s}$. It follows that

$$\mathbb{E}\left(I(t) - \lambda(t\wedge\tau)\,|\mathcal{I}_s\right) = I(s) - \lambda(s\wedge\tau)\,.$$

Because $\lambda(t\wedge\tau)$ is an $(\mathcal{I}_t)_{t\geq 0}$-adapted process with continuous trajectories, it follows that it is a compensator of $I(t)$.

3.10. We have

$$\mathbb{E}\left(\Gamma(t\wedge\tau)\right) = \int_0^\infty \Gamma(t\wedge u)f(u)du$$
$$= \int_0^t \Gamma(u)f(u)du + \Gamma(t)\int_t^\infty f(u)du$$
$$= \int_0^t \Gamma(u)f(u)du + \Gamma(t)G(t)$$
$$= F(t)$$

since integration by parts gives

$$\Gamma(t)G(t) = \Gamma(t)G(t) - \Gamma(0)G(0)$$
$$= \int_0^t \gamma(u)G(u)du - \int_0^t \Gamma(u)f(u)du$$
$$= \int_0^t f(u)du - \int_0^t \Gamma(u)f(u)du$$
$$= F(t) - \int_0^t \Gamma(u)f(u)du.$$

3.11. By Proposition 2.13

$$\gamma(t) = \frac{f(t)}{1-F(t)}$$

and from Corollary 3.16 we have

$$\Gamma(t\wedge\tau) = \int_0^{t\wedge\tau}\gamma(u)du = \int_0^t (1-I(u))\,\gamma(u)du,$$

hence it follows immediately that

$$\Gamma(t \wedge \tau) = \int_0^{t\wedge\tau} \frac{1}{1-F(u)} f(u)du = \int_0^t \frac{1-I(u)}{1-F(u)} f(u)du.$$

3.12. Since

$$L(t) = (1-I(t))\, e^{\Gamma(t)} = \mathbf{1}_{\{t<\tau\}} e^{\Gamma(t)}$$

is a martingale with respect to the filtration $(\mathcal{I}_t)_{t\geq 0}$, it follows that

$$\begin{aligned}\mathbb{E}\left(\mathbf{1}_{\{T<\tau\}}|\mathcal{I}_t\right) &= e^{-\Gamma(T)}\mathbb{E}\left(L(T)|\mathcal{I}_t\right)\\ &= e^{-\Gamma(T)}L(t)\\ &= \mathbf{1}_{\{t<\tau\}}e^{\Gamma(t)-\Gamma(T)}\\ &= \mathbf{1}_{\{t<\tau\}}e^{-\int_t^T \gamma(s)ds}.\end{aligned}$$

3.13. Let $X(t) = \mathbf{1}_{[0,c]}(t)$ for some $c \geq 0$. By (3.4), we have

$$\begin{aligned}\int_0^t X(u)dM(u) &= \mathbf{1}_{\{\tau\leq t\}}\mathbf{1}_{[0,c]}(\tau) - \int_0^{t\wedge\tau} \mathbf{1}_{[0,c]}(u)\gamma(u)du\\ &= \mathbf{1}_{\{\tau\leq t\wedge c\}} - \int_0^{t\wedge c\wedge\tau}\gamma(u)du.\end{aligned}$$

Now take any $0 \leq s < t$ and consider two cases:
Case 1: $s < c$. Then

$$\mathbb{E}\left(\int_0^t X(u)dM(u)\middle|\mathcal{I}_s\right) = \mathbb{E}\left(\mathbf{1}_{\{\tau\leq t\wedge c\}} - \int_0^{t\wedge c\wedge\tau}\gamma(u)du\middle|\mathcal{I}_s\right)$$

$$= \mathbf{1}_{\{\tau\leq s\}}\mathbb{E}\left(\mathbf{1}_{\{\tau\leq t\wedge c\}} - \int_0^{t\wedge c\wedge\tau}\gamma(u)du\middle|\sigma(\tau)\right) + \mathbf{1}_{\{s<\tau\}}e^{\Gamma(s)}\mathbb{E}\left(\mathbf{1}_{\{s<\tau\}}\left(\mathbf{1}_{\{\tau\leq t\wedge c\}} - \int_0^{t\wedge c\wedge\tau}\gamma(u)du\right)\right)$$

$$= \mathbf{1}_{\{\tau\leq s\}}\left(\mathbf{1}_{\{\tau\leq t\wedge c\}} - \int_0^{t\wedge c\wedge\tau}\gamma(u)du\right) + \mathbf{1}_{\{s<\tau\}}e^{\Gamma(s)}\mathbb{E}\left(\mathbf{1}_{\{s<\tau\leq t\wedge c\}} - \int_0^{t\wedge c}\mathbf{1}_{\{s\vee u<\tau\}}\gamma(u)du\right)$$

$$= \mathbf{1}_{\{\tau\leq s\wedge c\}} - \mathbf{1}_{\{\tau\leq s\}}\int_0^{c\wedge\tau}\gamma(u)du + \mathbf{1}_{\{s<\tau\}}e^{\Gamma(s)}\left(F(t\wedge c) - F(s) - \int_0^{t\wedge c} G(s\vee u)\gamma(u)du\right)$$

$$= \mathbf{1}_{\{\tau\leq s\wedge c\}} - \mathbf{1}_{\{\tau\leq s\}}\int_0^{c\wedge\tau}\gamma(u)du$$

$$+\mathbf{1}_{\{s<\tau\}}e^{\Gamma(s)}\left(F(t\wedge c) - F(s) - \int_0^s G(s)\gamma(u)du - \int_s^{t\wedge c} G(u)\gamma(u)du\right)$$

$$= \mathbf{1}_{\{\tau\leq s\wedge c\}} - \mathbf{1}_{\{\tau\leq s\}}\int_0^{c\wedge\tau}\gamma(u)du - \mathbf{1}_{\{s<\tau\}}\int_0^s\gamma(u)du$$

$$= \mathbf{1}_{\{\tau\leq s\wedge c\}} - \int_0^{s\wedge c\wedge\tau}\gamma(u)du = \int_0^s X(u)dM(u).$$

Case 2: $c \le s$. Then

$$
\begin{aligned}
&\mathbb{E}\left(\int_0^t X(u)dM(u)\middle|\mathcal{I}_s\right) = \mathbb{E}\left(\mathbf{1}_{\{\tau\le t\wedge c\}} - \int_0^{t\wedge c\wedge\tau}\gamma(u)du\middle|\mathcal{I}_s\right)\\
&= \mathbb{E}\left(\mathbf{1}_{\{\tau\le c\}} - \int_0^{c\wedge\tau}\gamma(u)du\middle|\mathcal{I}_s\right)\\
&= \mathbf{1}_{\{\tau\le s\}}\mathbb{E}\left(\mathbf{1}_{\{\tau\le c\}} - \int_0^{c\wedge\tau}\gamma(u)du\middle|\sigma(\tau)\right) + \mathbf{1}_{\{s<\tau\}}e^{\Gamma(s)}\mathbb{E}\left(\mathbf{1}_{\{s<\tau\}}\left(\mathbf{1}_{\{\tau\le c\}} - \int_0^{c\wedge\tau}\gamma(u)du\right)\right)\\
&= \mathbf{1}_{\{\tau\le c\}} - \mathbf{1}_{\{\tau\le s\}}\int_0^{c\wedge\tau}\gamma(u)du + \mathbf{1}_{\{s<\tau\}}e^{\Gamma(s)}\mathbb{E}\left(\mathbf{1}_{\{s<\tau\}}\mathbf{1}_{\{\tau\le c\}} - \mathbf{1}_{\{s<\tau\}}\int_0^{c}\mathbf{1}_{\{u<\tau\}}\gamma(u)du\right)\\
&= \mathbf{1}_{\{\tau<c\}} - \mathbf{1}_{\{\tau\le s\}}\int_0^{c\wedge\tau}\gamma(u)du - \mathbf{1}_{\{s<\tau\}}e^{\Gamma(s)}\int_0^{c}\mathbb{E}\left(\mathbf{1}_{\{s<\tau\}}\right)\gamma(u)du\\
&= \mathbf{1}_{\{\tau<c\}} - \mathbf{1}_{\{\tau\le s\}}\int_0^{c\wedge\tau}\gamma(u)du - \mathbf{1}_{\{s<\tau\}}e^{\Gamma(s)}\int_0^{c}G(s)\gamma(u)du\\
&= \mathbf{1}_{\{\tau<c\}} - \mathbf{1}_{\{\tau\le s\}}\int_0^{c\wedge\tau}\gamma(u)du - \mathbf{1}_{\{s<\tau\}}\int_0^{c\wedge\tau}\gamma(u)du\\
&= \mathbf{1}_{\{\tau<c\}} - \int_0^{c\wedge\tau}\gamma(u)du = \mathbf{1}_{\{\tau<s\wedge c\}} - \int_0^{s\wedge c\wedge\tau}\gamma(u)du = \int_0^s X(u)dM(u).
\end{aligned}
$$

3.14. Let

$$X(t) = Z\mathbf{1}_{(a,b]}(t)$$

for some $0 \le a < b$ and some $\mathcal{I}_a$-measurable random variable Z. By Proposition 2.22,

$$Z = \eta(\tau)\mathbf{1}_{\{\tau\le a\}} + c\mathbf{1}_{\{a<\tau\}}$$

for some Borel function $\eta : \mathbb{R} \to \mathbb{R}$ and a deterministic number $c \in \mathbb{R}$. Then

$$
\begin{aligned}
X(t) &= \left(\eta(\tau)\mathbf{1}_{\{\tau\le a\}} + c\mathbf{1}_{\{a<\tau\}}\right)\mathbf{1}_{(a,b]}(t)\\
&= \eta(\tau)\mathbf{1}_{(a,b]}(t)\mathbf{1}_{\{\tau\le a\}} + c\mathbf{1}_{(a,b]}(t)\mathbf{1}_{\{a<\tau\}}
\end{aligned}
$$

and

$$\begin{aligned}
\int_0^t X(u)dM(u) &= \mathbf{1}_{\{\tau\le t\}}X(\tau) - \int_0^{t\wedge\tau} X(u)\gamma(u)du \\
&= \mathbf{1}_{\{\tau\le t\}}Z\mathbf{1}_{(a,b]}(\tau) - \int_0^{t\wedge\tau} Z\mathbf{1}_{(a,b]}(u)\gamma(u)du \\
&= \mathbf{1}_{\{\tau\le t\}}\left(\eta(\tau)\mathbf{1}_{\{\tau\le a\}} + c\mathbf{1}_{\{a<\tau\}}\right)\mathbf{1}_{(a,b]}(\tau) \\
&\quad - \int_0^{t\wedge\tau} \left(\eta(\tau)\mathbf{1}_{\{\tau\le a\}} + c\mathbf{1}_{\{a<\tau\}}\right)\mathbf{1}_{(a,b]}(u)\gamma(u)du \\
&= c\mathbf{1}_{\{a<\tau\le t\wedge b\}} - c\int_0^{t\wedge\tau} \mathbf{1}_{(a,b]}(u)\gamma(u)du \\
&= \mathbf{1}_{\{\tau\le t\}}c\mathbf{1}_{(a,b]}(\tau) - \int_0^{t\wedge\tau} c\mathbf{1}_{(a,b]}(u)\gamma(u)du \\
&= \int_0^t Y(u)dM(u),
\end{aligned}$$

where

$$Y(t) = c\mathbf{1}_{(a,b]}(t)$$

is a deterministic function. It follows by Lemma 3.19 that

$$\int_0^t X(u)dM(u) = \int_0^t Y(u)dM(u)$$

is a is a martingale with respect to the filtration $(\mathcal{I}_t)_{t\ge0}$.

Chapter 4

4.1. We have

$$H(t) = e^{-rt}D(t,T) = \mathbb{E}\left(e^{-rT}D(T,T)|\mathcal{I}_t\right) = \mathbb{E}\left(e^{-rT}\mathbf{1}_{\{T<\tau\}}|\mathcal{I}_t\right),$$

so taking

$$h(t) = e^{-rT}\mathbf{1}_{\{T<t\}}$$

gives

$$H(t) = e^{-rt}D(t,T) = \mathbb{E}\left(h(\tau)|\mathcal{I}_t\right).$$

Substituting the expression for h into the formula for J gives

$$\begin{aligned}
J(t) &= \mathbb{E}\left(h(\tau)|t<\tau\right) = \mathbb{E}\left(e^{-rT}\mathbf{1}_{\{T<t\}}|t<\tau\right) \\
&= e^{-rT}\frac{Q(T<\tau)}{Q(t<\tau)} = e^{-rT}e^{-\Gamma(T)}e^{\Gamma(t)}.
\end{aligned}$$

Now can we substitute the expressions for h and J into right-hand side of (4.1) and apply formula (3.4) to get for any $t \leq T$

$$
\begin{aligned}
&\mathbb{E}\left(h(\tau)\right) + \int_0^t \left(h(s) - J(s)\right) dM(s) \\
&= \mathbb{E}\left(e^{-rT}\mathbf{1}_{\{T<\tau\}}\right) + \int_0^t \left(e^{-rT}\mathbf{1}_{\{T<s\}} - e^{-rT}e^{-\Gamma(T)}e^{\Gamma(s)}\right) dM(s) \\
&= e^{-rT}e^{-\Gamma(T)} + \mathbf{1}_{\{\tau\leq t\}}\left(e^{-rT}\mathbf{1}_{\{T<\tau\}} - e^{-rT}e^{-\Gamma(T)}e^{\Gamma(\tau)}\right) \\
&\quad - \int_0^{t\wedge\tau} \left(e^{-rT}\mathbf{1}_{\{T<s\}} - e^{-rT}e^{-\Gamma(T)}e^{\Gamma(s)}\right)\gamma(s)ds \\
&= e^{-rT}e^{-\Gamma(T)} - \mathbf{1}_{\{\tau\leq t\}}e^{-rT}e^{-\Gamma(T)}e^{\Gamma(\tau)} + e^{-rT}e^{-\Gamma(T)}\int_0^{t\wedge\tau} d\left(e^{\Gamma(s)}\right) \\
&= e^{-rT}e^{-\Gamma(T)} - \mathbf{1}_{\{\tau\leq t\}}e^{-rT}e^{-\Gamma(T)}e^{\Gamma(\tau)} + e^{-rT}e^{-\Gamma(T)}\left(e^{\Gamma(t\wedge\tau)} - e^{\Gamma(0)}\right) \\
&= -\mathbf{1}_{\{\tau\leq t\}}e^{-rT}e^{-\Gamma(T)}e^{\Gamma(\tau)} + \mathbf{1}_{\{\tau\leq t\}}e^{-rT}e^{-\Gamma(T)}e^{\Gamma(\tau)} + \mathbf{1}_{\{t<\tau\}}e^{-rT}e^{-\Gamma(T)}e^{\Gamma(t)} \\
&= \mathbf{1}_{\{t<\tau\}}e^{-rT}e^{-\Gamma(T)}e^{\Gamma(t)} \\
&= e^{-rt}D(t,T).
\end{aligned}
$$

4.2. By linearity, it is enough to show that if ϕ is a left-continuous deterministic function such that

$$\int_0^t \phi(s)dM(s) = 0$$

for each $t \geq 0$, then $\phi = 0$ on $(0, \infty)$.

By formula (3.4),

$$\int_0^t \phi(s)dM(s) = \mathbf{1}_{\{\tau\leq t\}}\phi(\tau) - \int_0^{t\wedge\tau} \phi(s)\gamma(s)ds = 0.$$

On $\{t < \tau\}$ we have

$$\int_0^t \phi(s)dM(s) = -\int_0^t \phi(s)\gamma(s)ds = 0.$$

As a result,

$$\int_0^t \phi(s)\gamma(s)ds = 0$$

for every $t \geq 0$ since $\{t < \tau\}$ has non-zero probability and $\int_0^t \phi(s)\gamma(s)ds$ is deterministic. Because $\gamma(t) = e^{\Gamma(t)}f(t) > 0$ for almost every $t \in [0, \infty)$, it follows that $\phi(t) = 0$ for almost every $t \in [0, \infty)$. By left-continuity, we can conclude that $\phi = 0$ on $(0, \infty)$.

4.3. Using the self-financing condition

$$V_\varphi(t) = V_\varphi(0) + \int_0^t \varphi_B(u)dB(u,T) + \int_0^t \varphi_B(u)dD(u,T),$$

by Theorem A.9 (the integration-by-parts formula for Lebesgue–Stieltjes integrals) and Theorems A.10 and A.11, we obtain

$$\begin{aligned}
&\tilde{V}_\varphi(t) - V_\varphi(0)\\
&= e^{-rt}V_\varphi(t) - V_\varphi(0)\\
&= \int_0^t e^{-ru}dV_\varphi(u) + \int_0^t V_\varphi(u_-)d\left(e^{-ru}\right)\\
&= \int_0^t e^{-ru}\varphi_B(u)dB(u,T) + \int_0^t e^{-ru}\varphi_D(u)dD(u,T) + \int_0^t V_\varphi(u_-)d\left(e^{-ru}\right)\\
&= r\int_0^t e^{-ru}\varphi_B(u)B(u,T)du + r\int_0^t e^{-ru}\varphi_D(u)D(u,T)du\\
&\quad - \int_0^t e^{-ru}\varphi_D(u)D(u_-,T)dM(u) - r\int_0^t e^{-ru}V_\varphi(u_-)du\\
&= r\int_0^t e^{-ru}V_\varphi(u)du - \int_0^t e^{-ru}\varphi_D(u)D(u_-,T)dM(u) - r\int_0^t e^{-ru}V_\varphi(u_-)du\\
&= -\int_0^t \varphi_D(u)\tilde{D}(u_-,T)dM(u)\\
&= \int_0^t \varphi_D(u)d\tilde{D}(u,T).
\end{aligned}$$

Here we use the fact that

$$V_\varphi(t) = \varphi_B(t)B(t,T) + \varphi_D(t)D(t,T) = V_\varphi(t_-)$$

for almost all $t \in [0,T]$ and apply Proposition 3.23 in the last equality.

4.4. Using the identities $f(t) = \gamma(t)e^{-\Gamma(t)}$ and $G(T) = e^{-\Gamma(t)}$, we compute

$$\begin{aligned}
\mathbb{E}(\tilde{V}_\varphi(T)) &= e^{-rT}e^{-\Gamma(T)}\mathbb{E}(e^{\Gamma(T\wedge\tau)} - 1)\\
&= e^{-rT}e^{-\Gamma(T)}\left(\int_0^T e^{\Gamma(u)}f(u)du + e^{\Gamma(T)}\int_T^\infty f(u)du - 1\right)\\
&= e^{-rT}e^{-\Gamma(T)}\left(\int_0^T \gamma(u)du + e^{\Gamma(T)}G(T) - 1\right)\\
&= e^{-rT}e^{-\Gamma(T)}\Gamma(T),
\end{aligned}$$

which is greater than 0.

4.5. Suppose that $\varphi(t) = (\varphi_B(t), \varphi_D(t))$ is an admissible self-financing strategy such that $V_\varphi(0) = 0$. Admissibility means that $\tilde{V}_\varphi(t)$ is a martingale under the risk-neutral probability Q. It follows that

$$\mathbb{E}\left(V_\varphi(T)\right) = e^{rT}\mathbb{E}\left(\tilde{V}_\varphi(T)\right) = e^{rT}\tilde{V}_\varphi(0) = e^{rT}V_\varphi(0) = 0.$$

It is therefore impossible for $V_\varphi(T)$ to be non-negative almost surely and positive with positive probability. There is no admissible arbitrage strategy.

4.6. Using Proposition 2.27, for any $t \in [0, T]$ we compute

$$\begin{aligned}
& e^{-r(T-t)}\mathbb{E}\left(D_\delta(T,T)|\mathcal{I}_t\right) \\
&= e^{-r(T-t)}\mathbb{E}\left(\delta + (1-\delta)\mathbf{1}_{\{T<\tau\}}|\mathcal{I}_t\right) \\
&= \delta e^{-r(T-t)} + (1-\delta)e^{-r(T-t)}\mathbb{E}\left(\mathbf{1}_{\{T<\tau\}}|\mathcal{I}_t\right) \\
&= \delta e^{-r(T-t)} + (1-\delta)e^{-r(T-t)}\left(\mathbf{1}_{\{\tau\le t\}}\mathbb{E}\left(\mathbf{1}_{\{T<\tau\}}|\sigma(\tau)\right) + \mathbf{1}_{\{t<\tau\}}\frac{\mathbb{E}\left(\mathbf{1}_{\{T<\tau\}}\mathbf{1}_{\{t<\tau\}}\right)}{Q(t<\tau)}\right) \\
&= \delta e^{-r(T-t)} + (1-\delta)e^{-r(T-t)}\mathbf{1}_{\{t<\tau\}}\frac{Q(T<\tau)}{Q(t<\tau)} \\
&= \delta e^{-r(T-t)} + (1-\delta)e^{-r(T-t)}\mathbf{1}_{\{t<\tau\}}e^{-(\Gamma(T)-\Gamma(t))}.
\end{aligned}$$

This agrees with the formula for $D_\delta(t, T)$ in Example 4.10.

4.7. In Example 4.10 we saw that

$$D_\delta(t,T) = \delta B(t,T) + (1-\delta)D(t,T).$$

Hence

$$D(t,T) = \frac{1}{1-\delta}D_\delta(t,T) - \frac{\delta}{1-\delta}B(t,T).$$

This means that the zero-recovery bond $D(t, T)$ can be replicated by a portfolio consisting of $\frac{1}{1-\delta}$ positive recovery bonds $D_\delta(t, T)$ and $-\frac{\delta}{1-\delta}$ non-defaultable bonds $B(t, T)$.

4.8. Using (4.4) and Proposition 3.23 with

$$\varphi_D(t) = e^{\Gamma(T)-\Gamma(t)}\left(J(t) - h(t)\right)$$

and

$$\tilde{D}(t,T) = \mathbf{1}_{\{t<\tau\}}e^{-rT}e^{-(\Gamma(T)-\Gamma(t))},$$

we obtain

$$\begin{aligned}
\varphi_B(t) &= \frac{1}{e^{-rt}B(t,T)}\left(V_\varphi(0) + \int_0^t \varphi_D(u)d\tilde{D}(u,T) - \varphi_D(t)\tilde{D}(t,T)\right) \\
&= \frac{1}{e^{-rT}}\left(V_\varphi(0) - \int_0^t \varphi_D(u)\tilde{D}(u_-,T)dM(u) - \varphi_D(t)\tilde{D}(t,T)\right) \\
&= \mathbb{E}\left(h(\tau)\right) - \int_0^t \left(J(u) - h(u)\right)\mathbf{1}_{\{u\leq\tau\}}dM(u) - \left(J(t) - h(t)\right)\mathbf{1}_{\{t<\tau\}} \\
&= \mathbb{E}\left(h(\tau)\right) - \left(J(\tau) - h(\tau)\right)\mathbf{1}_{\{\tau\leq t\}} \\
&\quad + \int_0^{t\wedge\tau} \left(J(u) - h(u)\right)\gamma(u)du - \left(J(t) - h(t)\right)\mathbf{1}_{\{t<\tau\}} \\
&= \mathbb{E}\left(h(\tau)\right) - \left(J(t\wedge\tau) - h(t\wedge\tau)\right) + \int_0^{t\wedge\tau} \left(J(u) - h(u)\right)\gamma(u)du.
\end{aligned}$$

In the proof of Theorem 4.1 (i.e. the martingale representation theorem) it is shown that

$$\begin{aligned}
\int_0^{t\wedge\tau} J(u)\gamma(u)du &= e^{\Gamma(t\wedge\tau)}\int_{t\wedge\tau}^{\infty} h(u)f(u)du - \int_0^{\infty} h(u)f(u)du \\
&\quad + \int_0^{t\wedge\tau} e^{\Gamma(u)}h(u)f(u)du \\
&= J(t\wedge\tau) - \mathbb{E}(h(\tau)) + \int_0^{t\wedge\tau} h(u)\gamma(u)du.
\end{aligned}$$

It follows that

$$\begin{aligned}
\varphi_B(t) &= \mathbb{E}\left(h(\tau)\right) - \left(J(t\wedge\tau) - h(t\wedge\tau)\right) + \int_0^{t\wedge\tau} \left(J(u) - h(u)\right)\gamma(u)du \\
&= h(t\wedge\tau).
\end{aligned}$$

4.9. The payoff of a defaultable bond with constant recovery can be expressed as

$$D_\delta(T,T) = h(\tau),$$

where

$$h(t) = \begin{cases} \delta & \text{if } t \leq T, \\ 1 & \text{if } T < t. \end{cases}$$

According to Theorem 4.11, the replicating strategy is given by

$$\begin{aligned}
\varphi_B(t) &= h(t\wedge\tau), \\
\varphi_D(t) &= e^{\Gamma(T)-\Gamma(t)}\left(J(t) - h(t)\right)
\end{aligned}$$

for any $t \in [0, T]$. It follows that

$$\varphi_B(t) = h(t \wedge \tau) = \delta$$

since $t \wedge \tau \le T$ when $t \le T$. Moreover,

$$\begin{aligned} J(t) = \mathbb{E}\left(h(\tau)|t < \tau\right) &= \frac{\mathbb{E}\left(\mathbf{1}_{\{t<\tau\}}h(\tau)\right)}{Q(t<\tau)} \\ &= \frac{\delta Q(t < \tau \le T) + Q(T < \tau)}{Q(t<\tau)} = \frac{\delta\left(e^{-\Gamma(t)} - e^{-\Gamma(T)}\right) + e^{-\Gamma(T)}}{e^{-\Gamma(t)}}, \end{aligned}$$

so for any $t \le T$

$$\begin{aligned} \varphi_D(t) &= e^{\Gamma(T)-\Gamma(t)}\left(J(t) - h(t)\right) \\ &= e^{\Gamma(T)-\Gamma(t)}\left(\frac{\delta\left(e^{-\Gamma(t)} - e^{-\Gamma(T)}\right) + e^{-\Gamma(T)}}{e^{-\Gamma(t)}} - \delta\right) \\ &= 1 - \delta. \end{aligned}$$

4.10. From Proposition 4.12 we have

$$\begin{aligned} D_\eta(t, T) &= \eta(\tau)\mathbf{1}_{\{\tau \le t\}} \\ &\quad + \mathbf{1}_{\{t<\tau\}}\left(e^{-r(T-t)}e^{-(\Gamma(T)-\Gamma(t))} + e^{-r(T-t)}e^{\Gamma(t)}\int_t^T \eta(s)e^{r(T-s)}f(s)ds\right). \end{aligned}$$

For $t = 0$ this gives

$$\begin{aligned} D_\eta(0, T) &= e^{-rT}e^{-\Gamma(T)} + e^{-rT}\int_0^T \eta(s)e^{r(T-s)}f(s)ds \\ &= e^{-rT}G(T) + \int_0^T \eta(s)e^{-rs}f(s)ds. \end{aligned}$$

4.11. Let

$$h(\tau) = \mathbf{1}_{\{T<\tau\}} + \eta(\tau)e^{r(T-\tau)}\mathbf{1}_{\{\tau \le T\}}.$$

By Proposition 2.27, for any $t \in [0, T]$ we have

$$\begin{aligned}
D_\eta(t,T) &= e^{-r(T-t)}\mathbb{E}\left(h(\tau)|\mathcal{I}_t\right) \\
&= e^{-r(T-t)}\mathbf{1}_{\{\tau\le t\}}\mathbb{E}\left(h(\tau)|\sigma(\tau)\right) + e^{-r(T-t)}\mathbf{1}_{\{t<\tau\}}\frac{\mathbb{E}\left(h(\tau)\mathbf{1}_{\{t<\tau\}}\right)}{Q(t<\tau)} \\
&= e^{-r(T-t)}\mathbf{1}_{\{\tau\le t\}}h(\tau) + e^{-r(T-t)}\mathbf{1}_{\{t<\tau\}}\frac{\mathbb{E}\left(h(\tau)\mathbf{1}_{\{t<\tau\}}\right)}{Q(t<\tau)} \\
&= \mathbf{1}_{\{\tau\le t\}}e^{r(t-\tau)}\eta(\tau) + e^{-r(T-t)}\mathbf{1}_{\{t<\tau\}}\frac{\mathbb{E}\left(\mathbf{1}_{\{T<\tau\}}\right)}{Q(t<\tau)} \\
&\quad + e^{-r(T-t)}\mathbf{1}_{\{t<\tau\}}\frac{\mathbb{E}\left(\eta(\tau)e^{r(T-\tau)}\mathbf{1}_{\{t<\tau\le T\}}\right)}{Q(t<\tau)} \\
&= \mathbf{1}_{\{\tau\le t\}}e^{r(t-\tau)}\eta(\tau) + e^{-r(T-t)}\mathbf{1}_{\{t<\tau\}}e^{-(\Gamma(T)-\Gamma(t))} \\
&\quad + e^{-r(T-t)}\mathbf{1}_{\{t<\tau\}}e^{\Gamma(t)}\int_t^T \eta(s)e^{r(T-s)}f(s)ds.
\end{aligned}$$

4.12. Let

$$H(t) = \mathbb{E}\left(h(\tau)|\mathcal{I}_t\right).$$

By the martingale representation theorem (Theorem 4.1),

$$\begin{aligned}
H(t) &= H(0) + \int_0^t \left(h(s) - J(s)\right)dM(s) \\
&= H(0) + \left(h(\tau) - J(\tau)\right)\mathbf{1}_{\{\tau\le t\}} - \int_0^{t\wedge\tau}\left(h(s) - J(s)\right)\gamma(s)ds.
\end{aligned}$$

It follows that

$$\begin{aligned}
&D_\eta(t,T) - D_\eta(0,T) \\
&= e^{-r(T-t)}H(t) - e^{-rT}H(0) \\
&= \int_0^t e^{-r(T-s)}dH(s) + \int_0^t H(s_-)d\left(e^{-r(T-s)}\right) \\
&= \int_0^t e^{-r(T-s)}h(s)dM(s) - \int_0^t e^{-r(T-s)}J(s)dM(s) \\
&\quad + r\int_0^t H(s_-)e^{-r(T-s)}ds.
\end{aligned}$$

On $\{t < \tau\}$ we have

$$\int_0^t \eta(s)dM(s) = -\int_0^t \eta(s)\gamma(s)\left(1 - I(s)\right)ds$$

and

$$H(t) = H(0) - \int_0^{t\wedge\tau} (h(s) - J(s))\,\gamma(s)ds = H(t_-),$$

so

$$\begin{aligned}
&D_\eta(t,T) - D_\eta(0,T)\\
&= -\int_0^t \eta(s)\gamma(s)\,(1 - I(s))\,ds - \int_0^t e^{-r(T-s)}J(s)dM(s)\\
&\quad +r\int_0^t H(s)e^{-r(T-s)}ds\\
&= -\int_0^t \eta(s)\gamma(s)\,(1 - I(s))\,ds - \int_0^t \hat{D}_\eta(s,T)dM(s)\\
&\quad +r\int_0^t D_\eta(s,T)ds
\end{aligned}$$

since

$$\begin{aligned}
\hat{D}_\eta(t,T) &= e^{-r(T-t)}\left(e^{-(\Gamma(T)-\Gamma(t))} - e^{\Gamma(t)}\int_t^T \eta(s)e^{r(t-s)}f(s)ds\right)\\
&= e^{-r(T-t)}J(t).
\end{aligned}$$

and

$$D_\eta(t,T) = e^{-r(T-t)}\mathbb{E}\,(h(\tau)|\mathcal{I}_t) = e^{-r(T-t)}H(t).$$

This shows that

$$dD_\eta(t,T) = -\eta(t)\gamma(t)\,(1 - I(t))\,dt - \hat{D}_\eta(s,T)dM(t) + rD_\eta(t,T)dt$$

on $\{t < \tau\}$.

4.13. The payoff of a defaultable zero-coupon bond maturing at time S, where $0 < S < T$, is equivalent to a payoff at time T of the form $h(\tau)$, where

$$h(t) = \mathbf{1}_{\{S<t\}}e^{r(T-S)}.$$

By Theorem 4.11, for any $t \in [0,S]$,

$$\varphi_B(t) = h(t\wedge\tau) = \mathbf{1}_{\{S<t\wedge\tau\}}e^{r(T-S)} = 0$$

since $t\wedge\tau \le t \le S$. It means that the bond $D(t,S)$ can be replicated by the $D(t,T)$ bond alone. Since

$$\begin{aligned}
D(t,S) &= \mathbf{1}_{\{t<\tau\}}e^{-r(S-t)}e^{-(\Gamma(S)-\Gamma(t))}\\
&= e^{r(T-S)}e^{(\Gamma(T)-\Gamma(S))}\mathbf{1}_{\{t<\tau\}}e^{-r(T-t)}e^{-(\Gamma(T)-\Gamma(t))}\\
&= e^{r(T-S)}e^{(\Gamma(T)-\Gamma(S))}D(t,T),
\end{aligned}$$

we have

$$\varphi_D(t) = e^{r(T-S)} e^{(\Gamma(T)-\Gamma(S))}.$$

4.14. By Exercise 4.13, for each $k = 1, \ldots, N$ the zero-coupon bond $D(t, T_k)$ with maturity T_k can be replicated by $e^{r(T-T_k)} e^{(\Gamma(T)-\Gamma(T_k))}$ zero-coupon bonds $D(t, T)$ with maturity T. Hence, to replicate the coupon bond we need to buy

$$\sum_{k=1}^{N} C_k e^{r(T-T_k)} e^{(\Gamma(T)-\Gamma(T_k))} + F e^{r(T-T_D)} e^{(\Gamma(T)-\Gamma(T_N))}$$

of the zero-coupon bonds maturing at time T. Then, at time T_k for each $k = 1, \ldots, T_N$ we need to sell $C_k e^{r(T-T_k)} e^{(\Gamma(T)-\Gamma(T_k))}$ of the zero-coupon bonds maturing at T, which produces the defaultable coupon payment

$$C_k e^{r(T-T_k)} e^{(\Gamma(T)-\Gamma(T_k))} D(T_k, T) = C_k \mathbf{1}_{\{T_k<\tau\}}.$$

At time T_N we also need to sell the remaining holdings amounting to $F e^{r(T-T_N)} e^{(\Gamma(T)-\Gamma(T_N))}$ of the zero-coupon bonds maturing at T, which pays

$$F e^{r(T-T_N)} e^{(\Gamma(T)-\Gamma(T_N))} D(T_N, T) = F \mathbf{1}_{\{T_N<\tau\}},$$

i.e. the face value of the defaultable coupon bond.

4.15. Consider a CDS on a defaultable bond $D_\delta(t, T)$ with constant recovery $\delta \in (0, 1)$ paid at maturity T. The recovery leg payment of the CDS is $(1-\delta)\,\mathbf{1}_{\{\tau \le T\}}$ at time T. The premium leg payments are $\alpha \mathbf{1}_{\{t_k<T\}}$ at times t_k for $k = 1, \ldots, N$. Equating the time 0 values of the two legs, we get

$$\mathbb{E}\left(e^{-rT}(1-\delta)\mathbf{1}_{\{\tau\le T\}}\right) = \sum_{k=1}^{N} \mathbb{E}\left(\alpha e^{-rt_k} \mathbf{1}_{\{t_k<T\}}\right).$$

Hence

$$\begin{aligned}\alpha &= \frac{\mathbb{E}\left(e^{-rT}(1-\delta)\mathbf{1}_{\{\tau\le T\}}\right)}{\sum_{k=1}^{N} \mathbb{E}\left(e^{-rt_k}\mathbf{1}_{\{t_k<T\}}\right)} \\ &= \frac{\mathbb{E}\left(e^{-rT}(1-\delta)(1-\mathbf{1}_{\{T<\tau\}})\right)}{\sum_{k=1}^{N} \mathbb{E}\left(e^{-rt_k}\mathbf{1}_{\{t_k<T\}}\right)} = (1-\delta)\,\frac{B(0,T)-D(0,T)}{\sum_{k=1}^{N} D(0,t_k)}.\end{aligned}$$

4.16. The CDS spread is given by

$$\alpha = \frac{B(0,T)-D(0,T)}{\sum_{k=1}^{N} D(0,t_k)} = \frac{e^{-rT} - e^{-rT}e^{-\Gamma(T)}}{\sum_{k=1}^{N} e^{-rt_k}e^{-\Gamma(t_k)}}.$$

When τ is exponentially distributed under Q, we have

$$e^{-\Gamma(t)} = G(t) = Q(t < \tau) = e^{-\lambda t}.$$

This gives

$$\alpha = \frac{e^{-rT} - e^{-rT} e^{-\lambda T}}{\sum_{k=1}^{N} e^{-rt_k} e^{-\lambda t_k}}.$$

For $r = 0.05$, $\lambda = 0.02$, $T = 1$ and $N = 12$ we get

$$\alpha = \frac{e^{-0.05\times 1} - e^{-0.05\times 1} e^{-0.02\times 1}}{\sum_{k=1}^{12} e^{-0.05\times \frac{k}{12}} e^{-0.02\times \frac{k}{12}}} = 0.00163.$$

4.17. If the hazard rate γ is constant, then we have $\Gamma(t) = \gamma t$, so

$$\alpha = \frac{B(0,T) - D(0,T)}{\sum_{k=1}^{N} D(0,t_k)} = \frac{e^{-rT} - e^{-rT} e^{-\Gamma(T)}}{\sum_{k=1}^{N} e^{-rt_k} e^{-\Gamma(t_k)}} = \frac{e^{-rT} - e^{-rT} e^{-\gamma T}}{\sum_{k=1}^{N} e^{-rt_k} e^{-\gamma t_k}}.$$

This nonlinear equation can be solved numerically for γ. When $r = 0.06$, $T = 2$, $N = 24$ and $\alpha = 0.1$, we get $\gamma = 0.127$. This yields the defaultable bond price

$$D(0,T) = e^{-rT} e^{-\Gamma(T)} = e^{-rT} e^{-\gamma T} = 0.6880.$$

4.18. For each $n = 1, \ldots, N$ we have $t_n = \frac{nT}{N}$ and

$$\alpha_n = \frac{e^{-rt_n}(1 - G(t_n))}{\sum_{k=1}^{n} e^{-rt_k} G(t_k)},$$

hence

$$G(t_n) = \frac{1 - \alpha_n \sum_{k=1}^{n-1} e^{r(t_n - t_k)} G(t_k)}{\alpha_n + 1} = \frac{1 - \alpha_n \sum_{k=1}^{n-1} e^{r\frac{(n-k)T}{N}} G(t_k)}{\alpha_n + 1}.$$

Moreover, the piecewise linear hazard function can be written as

$$\Gamma(t_n) = \sum_{k=1}^{n} \gamma_k (t_k - t_{k-1}) = \sum_{k=1}^{n} \gamma_k \frac{T}{N},$$

where $\gamma_1, \ldots, \gamma_N$ are positive constants, so

$$G(t_n) = e^{-\Gamma(t_n)} = e^{-\sum_{k=1}^{n} \gamma_k \frac{T}{N}}.$$

For $T = 1$, $N = 4$, $r = 0.05$ and

$$\alpha_1 = 0.012, \quad \alpha_2 = 0.009, \quad \alpha_3 = 0.011, \quad \alpha_4 = 0.010,$$

we get

$$G(t_1) = \frac{1}{\alpha_1 + 1} = 0.98814,$$

$$G(t_2) = \frac{1 - \alpha_2 e^{r\frac{T}{N}} G(t_1)}{\alpha_2 + 1} = 0.98216,$$

$$G(t_3) = \frac{1 - \alpha_3 \left(e^{r\frac{2T}{4}} G(t_1) + e^{r\frac{T}{4}} G(t_2)\right)}{\alpha_3 + 1} = 0.96728,$$

$$G(t_4) = \frac{1 - \alpha_4 \left(e^{r\frac{3T}{4}} G(t_1) + e^{r\frac{2T}{4}} G(t_2) + e^{r\frac{T}{4}} G(t_1)\right)}{\alpha_4 + 1} = 0.96027.$$

It follows that

$$\gamma_1 = -\frac{N}{T} \ln G(t_1) = 0.047724,$$

$$\gamma_2 = -\frac{N}{T} \ln \frac{G(t_2)}{G(t_1)} = 0.024281,$$

$$\gamma_3 = -\frac{N}{T} \ln \frac{G(t_3)}{G(t_2)} = 0.061065,$$

$$\gamma_4 = -\frac{N}{T} \ln \frac{G(t_4)}{G(t_3)} = 0.029094.$$

Chapter 5

5.1. Since τ is an $(\mathcal{I}_t)_{t\geq 0}$-stopping time and $\mathcal{I}_t \subset \mathcal{G}_t$ for each $t \geq 0$, it follows that τ is a $(\mathcal{G}_t)_{t\geq 0}$-stopping time.

Now let $(\mathcal{H}_t)_{t\geq 0}$ be a filtration such that τ is an $(\mathcal{H}_t)_{t\geq 0}$-stopping time and $\mathcal{F}_t \subset \mathcal{H}_t$ for each $t \geq 0$. Fix any $t \geq 0$. Then, for any $s \leq t$, we have $\{\tau \leq s\} \in \mathcal{H}_s \subset \mathcal{H}_t$ and so $I(s) = \mathbf{1}_{\{\tau \leq s\}}$ is $\mathcal{H}_t$-measurable. As a result, $\mathcal{I}_t \subset \mathcal{H}_t$. It follows that $\mathcal{G}_t \subset \mathcal{H}_t$ since $\mathcal{G}_t = \sigma(\mathcal{F}_t \cup \mathcal{I}_t)$ and $\mathcal{F}_t \cup \mathcal{I}_t \subset \mathcal{H}_t$.

This proves that $(\mathcal{G}_t)_{t\geq 0}$ is the smallest filtration such that τ is an $(\mathcal{G}_t)_{t\geq 0}$-stopping time and $\mathcal{F}_t \subset \mathcal{G}_t$ for each $t \geq 0$.

5.2. Let $0 \leq s < t$. Then $\mathbf{1}_{\{\tau \leq s\}} \leq \mathbf{1}_{\{\tau \leq t\}}$, which gives the result:

$$\begin{aligned}
\mathbb{E}(F(t)|\mathcal{F}_s) &= \mathbb{E}(\mathbb{E}(\mathbf{1}_{\{\tau \leq t\}}|\mathcal{F}_t)|\mathcal{F}_s) \\
&= \mathbb{E}(\mathbf{1}_{\{\tau \leq t\}}|\mathcal{F}_s) \\
&\geq \mathbb{E}(\mathbf{1}_{\{\tau \leq s\}}|\mathcal{F}_s) \\
&= F(s).
\end{aligned}$$

5.3. For any $0 \leq s \leq t$ we have $\{\tau \leq s\} \subset \{\tau \leq t\}$, and so $\mathbf{1}_{\{\tau \leq s\}} \leq \mathbf{1}_{\{\tau \leq t\}}$. By Lemma 5.15,

$$F(s) = \mathbb{E}(\mathbf{1}_{\{\tau \leq s\}} | \mathcal{F}_t) \leq \mathbb{E}(\mathbf{1}_{\{\tau \leq t\}} | \mathcal{F}_t) = F(t),$$

hence $F(t)$ is non-decreasing.

5.4. The process Γ has increasing paths, so for every $t \geq s \geq 0$ we have $\Gamma(t) \geq \Gamma(s)$, which implies that

$$\mathbb{E}(\Gamma(t)|\mathcal{F}_s) \geq \mathbb{E}(\Gamma(s)|\mathcal{F}_s) = \Gamma(s)$$

since $\Gamma(s)$ is $\mathcal{F}_s$-measurable. It follows that Γ is a supermartingale with respect to the filtration $(\mathcal{F}_t)_{t \geq 0}$.

5.5. Let

$$X(t) = vt + \sigma W_Q(t),$$

where

$$v = r - \frac{1}{2}\sigma^2.$$

Then

$$\Gamma(t) = \max_{u \in [0,t]} \left(-\ln \frac{S(t)}{S(0)} \right) = -\min_{u \in [0,t]} X(t).$$

It follows from Lemma A.16 that the cumulative distribution function of $\Gamma(t)$ is

$$\begin{aligned} F_{\Gamma(t)}(x) &= Q(\Gamma(t) \leq x) \\ &= Q(\min_{u \in [0,t]} X(t) \geq -x) \\ &= N\left(\frac{x + vt}{\sigma\sqrt{t}}\right) - e^{-\frac{2vx}{\sigma^2}} N\left(\frac{-x + vt}{\sigma\sqrt{t}}\right) \end{aligned}$$

for any $x \geq 0$, and $F_{\Gamma(t)}(x) = 0$ for $x < 0$. Hence the density of $\Gamma(t)$ is

$$f_{\Gamma(t)}(x) = \frac{1}{\sigma\sqrt{2\pi t}} e^{-\frac{(x+vt)^2}{2\sigma^2 t}} + e^{-\frac{2vx}{\sigma^2}} \frac{1}{\sigma\sqrt{2\pi t}} e^{-\frac{(-x+vt)^2}{2\sigma^2 t}} + \frac{2v}{\sigma^2} e^{-\frac{2vx}{\sigma^2}} N\left(\frac{-x + vt}{\sigma\sqrt{t}}\right)$$

for any $x \geq 0$, and $f_{\Gamma(t)}(x) = 0$ for $x < 0$. It follows that for each $t \geq 0$

$$\begin{aligned}
Q(\tau \leq t) &= \mathbb{E}(\mathbf{1}_{\{\tau\leq t\}}) = \mathbb{E}(\mathbb{E}(\mathbf{1}_{\{\tau\leq t\}}|\mathcal{F}_t)) = \mathbb{E}(F(t)) = \mathbb{E}(e^{-\Gamma(t)}) \\
&= \int_{-\infty}^{\infty} e^{-x} f_{\Gamma(t)}(x)dx \\
&= \frac{1}{\sigma\sqrt{2\pi t}}\int_0^\infty e^{-x}e^{-\frac{(x+\nu t)^2}{2\sigma^2 t}}dx + \frac{1}{\sigma\sqrt{2\pi t}}\int_0^\infty e^{-x}e^{-\frac{2\nu x}{\sigma^2}}e^{-\frac{(-x+\nu t)^2}{2\sigma^2 t}}dx \\
&\quad + \frac{2\nu}{\sigma^2}\int_0^\infty e^{-x}e^{-\frac{2\nu x}{\sigma^2}}N\left(\frac{-x+\nu t}{\sigma\sqrt{t}}\right)dx \\
&= e^{\nu t+\frac{1}{2}\sigma^2 t}N\left(-\frac{\nu t+\sigma^2 t}{\sigma\sqrt{t}}\right) + e^{\nu t+\frac{1}{2}\sigma^2 t}N\left(-\frac{\nu t+\sigma^2 t}{\sigma\sqrt{t}}\right) \\
&\quad + \frac{2\nu}{\sigma^2}\frac{\sigma^2}{2\nu+\sigma^2}N\left(\frac{\nu t}{\sigma\sqrt{t}}\right) - \frac{2\nu}{2\nu+\sigma^2}e^{\nu t+\frac{1}{2}\sigma^2 t}N\left(-\frac{\nu t+\sigma^2 t}{\sigma\sqrt{t}}\right) \\
&= 2e^{\nu t+\frac{1}{2}\sigma^2 t}\frac{\nu+\sigma^2}{2\nu+\sigma^2}N\left(-\frac{\nu t+\sigma^2 t}{\sigma\sqrt{t}}\right) + \frac{2\nu}{2\nu+\sigma^2}N\left(\frac{\nu t}{\sigma\sqrt{t}}\right) \\
&= e^{rt}\frac{2r+\sigma^2}{2r}N\left(-\frac{rt+\frac{1}{2}\sigma^2 t}{\sigma\sqrt{t}}\right) + \frac{2r+\sigma^2}{2r}N\left(\frac{rt-\frac{1}{2}\sigma^2 t}{\sigma\sqrt{t}}\right),
\end{aligned}$$

which gives the probability distribution of τ under Q.

5.6. Suppose that Y is a random variable with the unit exponential distribution. Then for any $x \in [0, 1]$

$$Q(e^{-Y} \leq x) = Q(Y \geq -\ln x) = 1 - F_Y(-\ln x) = e^{\ln x} = x,$$

so e^{-Y} is uniformly distributed in $[0, 1]$.

Conversely, suppose that e^{-Y} is uniformly distributed in $[0, 1]$. Then for any $x \in [0, \infty)$

$$Q(Y \leq x) = Q\left(e^{-Y} \geq e^{-x}\right) = 1 - e^{-x},$$

hence Y has the unit exponential distribution.

5.7. Let

$$\tau_1 = \inf\{t \geq 0 : e^{-\Gamma(t)} \leq X\},$$

where X is a random variable uniformly distributed in $[0, 1]$ and independent of $\mathcal{F}_\infty$, and let

$$\tau_2 = \inf\{t \geq 0 : \Gamma(t) \geq Y\}$$

where Y is a random variable with the unit exponential distribution and independent of $\mathcal{F}_\infty$. By Exercise 5.6, e^{-Y} is uniformly distributed in [0, 1]. It follows that

$$\tau_2 = \inf\{t \geq 0 : \Gamma(t) \geq Y\} = \inf\{t \geq 0 : e^{-\Gamma(t)} \leq e^{-Y}\}$$

has the same distribution as τ_1.

5.8. Since $\emptyset \cap \{\tau \leq t\} = \emptyset$ and $\Omega \cap \{\tau \leq t\} = \{\tau \leq t\}$ belong to $\mathcal{G}_t$, both $\emptyset$ and Ω belong to $\mathcal{A}$. Suppose that $A, B \in \mathcal{A}$. Since $A \cap \{\tau \leq t\}$ and $B \cap \{\tau \leq t\}$ belong to $\mathcal{G}_t$,

$$(A \setminus B) \cap \{\tau \leq t\} = (A \cap \{\tau \leq t\}) \setminus (B \cap \{\tau \leq t\})$$

also belongs to $\mathcal{G}_t$, so $A \setminus B \in \mathcal{A}$. Next, if $A_n \in \mathcal{A}$ and so $A_n \cap \{\tau \leq t\} \in \mathcal{G}_t$ for $n = 1, 2, \ldots$, then

$$\left(\bigcup_{n=1}^{\infty} A_n\right) \cap \{\tau \leq t\} = \bigcup_{n=1}^{\infty} (A_n \cap \{\tau \leq t\})$$

belongs to $\mathcal{G}_t$, so $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$. This shows that $\mathcal{A}$ is a σ-field.

To see that $\sigma(\tau) \subset \mathcal{A}$ we only need to verify that $\{\tau \leq s\} \in \mathcal{A}$ for each $s \geq 0$ since events of the form $\{\tau \leq s\}$ for $s \geq 0$ generate the σ-field $\sigma(\tau)$. Indeed, for any $s \geq 0$ the event $\{\tau \leq s\} \cap \{\tau \leq t\} = \{\tau \leq s \wedge t\}$ belongs to $\mathcal{I}_{s \wedge t} \subset \mathcal{I}_t \subset \mathcal{G}_t$, so $\{\tau \leq s\} \in \mathcal{A}$.

Finally, we take any $A \in \mathcal{F}_t$. Since $\{\tau \leq t\} \in \mathcal{I}_t$, we find that $A \cap \{\tau \leq t\} \in \mathcal{G}_t$, hence $A \in \mathcal{A}$. We have shown that $\mathcal{F}_t \subset \mathcal{A}$.

5.9. Take any $t > 0$. By left-continuity, for any $\varepsilon > 0$ there is a $\delta > 0$ such that $|h(u) - h(t)| < \varepsilon$ when $u \in (t - \delta, t]$. Then for any $n > \max(\frac{s}{\delta}, \frac{t}{s})$ we have $|h_n(t) - h(t)| < \varepsilon$. This proves that $h(t) = \lim_{n\to\infty} h_n(t)$.

5.10. Since $\tau > 0$, we have $\mathbf{1}_{\{\tau \leq 0\}} = 0$ and $\mathbf{1}_{\{0<\tau\}} = 1$, so we just need to show that

$$\mathbb{E}(h(\tau)) = \mathbb{E}\left(\int_0^{\infty} h(u)f(u)du\right).$$

Since h has left-continuous paths, we have, almost surely,

$$h(t) = \lim_{n\to\infty} h_n(t) \quad \text{for each } t > 0,$$

where

$$h_n(t) = \sum_{i=1}^{n^2} h(s_n^{i-1})\mathbf{1}_{(s_n^{i-1}, s_n^i]}(t)$$

with $s_n^i = \frac{i}{n}$ for each $n, i = 1, 2, \ldots$ (see Exercise 5.9 with $s = 1$). It follows that, almost surely,

$$h(\tau) = \lim_{n\to\infty} h_n(\tau).$$

First we verify the desired equality for $h_n(t)$. Indeed,

$$\begin{aligned}
\mathbb{E}(h_n(\tau)) &= \sum_{i=1}^{n^2} \mathbb{E}(h(s_n^{i-1})\mathbf{1}_{(s_n^{i-1},s_n^i]}(\tau)) \\
&= \sum_{i=1}^{n^2} \mathbb{E}(\mathbb{E}(h(s_n^{i-1})\mathbf{1}_{(s_n^{i-1},s_n^i]}(\tau)|\mathcal{F}_{s_n^i})) \\
&= \sum_{i=1}^{n^2} \mathbb{E}(h(s_n^{i-1})\mathbb{E}(\mathbf{1}_{(s_n^{i-1},s_n^i]}(\tau)|\mathcal{F}_{s_n^i})) \\
&= \sum_{i=1}^{n^2} \mathbb{E}(h(s_n^{i-1})(\mathbb{E}(\mathbf{1}_{\{\tau\le s_n^i\}}|\mathcal{F}_{s_n^i}) - \mathbb{E}(\mathbf{1}_{\{\tau\le s_n^{i-1}\}}|\mathcal{F}_{s_n^i}))) \\
&= \sum_{i=1}^{n^2} \mathbb{E}(h(s_n^{i-1})(F(s_n^i) - F(s_n^{i-1}))) \\
&= \mathbb{E}\left(\int_0^\infty h_n(u)f(u)du\right),
\end{aligned}$$

where we use Lemma 5.15 in the fifth equality. Hence, by dominated convergence, for a bounded h we find that

$$\begin{aligned}
\mathbb{E}(h(\tau)) &= \lim_{n\to\infty} \mathbb{E}(h_n(\tau)) \\
&= \lim_{n\to\infty} \mathbb{E}\left(\int_0^\infty h_n(u)f(u)du\right) \\
&= \mathbb{E}\left(\int_0^\infty h(u)f(u)du\right).
\end{aligned}$$

Then, for a non-negative h we obtain the formula by approximating h by a monotone sequence of bounded processes. Finally, for a general h we use the positive and negative parts h^+ and h^- together with the assumption that $h(\tau)$ is integrable, just like in the proof of Theorem 5.34 in the case when $s > 0$.

5.11. Note that

$$L(t) = (1 - \mathbf{1}_{\{\tau\le t\}})e^{\Gamma(t)} = \mathbf{1}_{\{t<\tau\}}e^{\Gamma(t)}.$$

By Theorem 5.35 with $Y = e^{\Gamma(t)}$,

$$\begin{aligned}\mathbb{E}(L(t)|\mathcal{G}_s) &= \mathbb{E}(e^{\Gamma(t)}\mathbf{1}_{\{t<\tau\}}|\mathcal{G}_s)\\ &= \mathbf{1}_{\{s<\tau\}}\mathbb{E}(e^{\Gamma(t)}e^{-(\Gamma(t)-\Gamma(s))}|\mathcal{F}_s)\\ &= \mathbf{1}_{\{s<\tau\}}e^{\Gamma(s)}\\ &= L(s).\end{aligned}$$

5.12. This is again a simple consequence of Theorem 5.35. For $Y = X(t)e^{\Gamma(t)}$, it gives

$$\begin{aligned}\mathbb{E}(X(t)L(t)|\mathcal{G}_s) &= \mathbb{E}(X(t)e^{\Gamma(t)}\mathbf{1}_{\{t<\tau\}}|\mathcal{G}_s)\\ &= \mathbf{1}_{\{s<\tau\}}e^{\Gamma(s)}\mathbb{E}(X(t)|\mathcal{F}_s)\\ &= L(s)X(s).\end{aligned}$$

Chapter 6

6.1. It is shown in Section 6.1 that

$$\begin{aligned}\hat{D}(t,T) = D(0,T) &+ \int_0^t (r+\gamma(u))\hat{D}(u,T)du\\ &+ \int_0^t e^{-r(T-u)}e^{\Gamma(u)}X_{G(T)}(u)dW_Q(u).\end{aligned}$$

By Itô's product rule (see [SCF]), it follows that

$$\begin{aligned}e^{-rt}\hat{D}(t,T) = D(0,T) &+ \int_0^t \gamma(u)e^{-ru}\hat{D}(u,T)du\\ &+ \int_0^t e^{-rT}e^{\Gamma(u)}X_{G(T)}(u)dW_Q(u).\end{aligned}$$

Hence

$$\begin{aligned}e^{-r(t\wedge\tau)}\hat{D}(t\wedge\tau,T) &= D(0,T) + \int_0^{t\wedge\tau} \gamma(u)e^{-ru}\hat{D}(u,T)du\\ &\quad + \int_0^{t\wedge\tau} e^{-rT}e^{\Gamma(u)}X_{G(T)}(u)dW_Q(u)\\ &= D(0,T) + \int_0^t \gamma(u)\tilde{D}(u,T)du\\ &\quad + \int_0^{t\wedge\tau} e^{-rT}e^{\Gamma(u)}X_{G(T)}(u)dW_Q(u).\end{aligned}$$

On the left-hand side we have

$$e^{-r(t\wedge\tau)}\hat{D}(t\wedge\tau,T) = \mathbf{1}_{\{\tau\le t\}}e^{-r\tau}\hat{D}(\tau,T) + \mathbf{1}_{\{t<\tau\}}e^{-rt}\hat{D}(t,T)$$
$$= \mathbf{1}_{\{\tau\le t\}}e^{-r\tau}\hat{D}(\tau,T) + \tilde{D}(t,T).$$

By formula (5.11),

$$\int_0^t \tilde{D}(u_-,T)dM(u) = \mathbf{1}_{\{\tau\le t\}}\tilde{D}(\tau_-,T) - \int_0^{t\wedge\tau}\tilde{D}(u_-,T)\gamma(u)du$$
$$= \mathbf{1}_{\{\tau\le t\}}e^{-r\tau}\hat{D}(\tau,T) - \int_0^t \tilde{D}(u,T)\gamma(u)du.$$

Combining the last three formulae gives

$$\tilde{D}(t,T) = D(0,T) - \int_0^t \tilde{D}(u_-,T)dM(u)$$
$$+ \int_0^{t\wedge\tau} e^{-rT}e^{\Gamma(u)}X_{G(T)}(u)dW_Q(u).$$

6.2. As shown in the proof of Proposition 6.9,

$$V_\varphi(\tau_-) = V_\varphi(\tau) + \varphi_D(\tau)D(\tau_-,T).$$

Taking the left limit as $t \nearrow \tau$ in the expression for $V_\varphi(t)$ in the pre-default region $\{t<\tau\}$ in Definition 6.7, we get

$$\begin{aligned}
V_\varphi(\tau_-) = V_\varphi(0) &+ \int_0^\tau \varphi_B(u)rB(u,T)du \\
&+ \int_0^\tau \varphi_S(u)rS(u)du + \int_0^\tau \varphi_S(u)\sigma S(u)dW_Q(u) \\
&+ \int_0^\tau \varphi_D(u)rD(u,T)du + \int_0^\tau \varphi_D(u)\gamma(u)D(u,T)du \\
&+ \int_0^\tau \varphi_D(u)e^{-r(T-u)}e^{\Gamma(u)}X_{G(T)}(u)dW_Q(u) \\
= V_\varphi(0) &+ \int_0^\tau rV_\varphi(u)du + \int_0^\tau \varphi_S(u)\sigma S(u)dW_Q(u) \\
&+ \int_0^\tau \varphi_D(u)\gamma(u)D(u,T)du \\
&+ \int_0^\tau \varphi_D(u)e^{-r(T-u)}e^{\Gamma(u)}X_{G(T)}(u)dW_Q(u).
\end{aligned}$$

It follows that

$$\begin{aligned}V_\varphi(\tau) &= V_\varphi(\tau_-) - \varphi_D(\tau)D(\tau_-,T)\\ &= V_\varphi(0) + \int_0^\tau rV_\varphi(u)du + \int_0^\tau \varphi_S(u)\sigma S(u)dW_Q(u)\\ &\quad+ \int_0^\tau \varphi_D(u)\gamma(u)D(u,T)du - \varphi_D(\tau)D(\tau_-,T)\\ &\quad+ \int_0^\tau \varphi_D(u)e^{-r(T-u)}e^{\Gamma(u)}X_{G(T)}(u)dW_Q(u).\end{aligned}$$

By inserting this into the expression for $V_\varphi(t)$ in the post-default region $\{\tau \leq \tau\}$ in Definition 6.7, we get

$$\begin{aligned}V_\varphi(t) &= V_\varphi(\tau) + \int_\tau^t rV_\varphi(u)du + \int_\tau^t \varphi_S(u)\sigma S(u)dW_Q(u)\\ &= V_\varphi(0) + \int_0^t rV_\varphi(u)du + \int_0^t \varphi_S(u)\sigma S(u)dW_Q(u)\\ &\quad+ \int_0^\tau \varphi_D(u)\gamma(u)D(u,T)du - \varphi_D(\tau)D(\tau_-,T)\\ &\quad+ \int_0^\tau \varphi_D(u)e^{-r(T-u)}e^{\Gamma(u)}X_{G(T)}(u)dW_Q(u).\end{aligned}$$

Combining this with the expression for $V_\varphi(t)$ in the pre-default region $\{t < \tau\}$ in Definition 6.7, we finally get

$$\begin{aligned}V_\varphi(t) &= V_\varphi(0) + \int_0^t rV_\varphi(u)du + \int_0^t \varphi_S(u)\sigma S(u)dW_Q(u)\\ &\quad- \int_0^t \varphi_D(u)D(u-,T)dM(u)\\ &\quad+ \int_0^{t\wedge\tau} \varphi_D(u)e^{-r(T-u)}e^{\Gamma(u)}X_{G(T)}(u)dW_Q(u).\end{aligned}$$

for all $t \in [0,T]$.

6.3. Suppose that φ is a self-financing strategy such that V_φ is non-negative, hence $\tilde{V}_\varphi$ is non-negative. In the expression (6.4) for the discounted value $\tilde{V}_\varphi(t) = e^{-rt}V_\varphi(t)$ of the strategy in Proposition 6.9 the integrals with respect to W_Q are local martingales. By Remark 5.41, the integral with respect to M in (6.4) is also a local martingale. It follows that $\tilde{V}_\varphi(t)$ is a local martingale. As a non-negative local martingale, $\tilde{V}_\varphi$ must be a supermartingale, hence

$$0 \leq \mathbb{E}(V_\varphi(T)) \leq V_\varphi(0).$$

If φ were an arbitrage strategy, we would have $V_\varphi(0) = 0$ and $\mathbb{E}(V_\varphi(T)) > 0$, so

$$0 < \mathbb{E}(V_\varphi(T)) \leq V_\varphi(0) = 0,$$

a contradiction.

6.4. We need to show that

$$(r + \gamma(t))e^{\Gamma(t)}V_\beta(t) = re^{\Gamma(t)}V_\beta(t) + \varphi_D(t)\gamma(t)D(t,T),$$

that is

$$e^{\Gamma(t)}V_\beta(t) = \varphi_D(t)D(t,T),$$

in the pre-default region $\{t < \tau\}$. The last equality holds because

$$D(t,T) = e^{-r(T-t)}e^{\Gamma(t)}\mathbb{E}(e^{-\Gamma(T)}|\mathcal{F}_t) = e^{\Gamma(t)}V_\alpha(t)$$

in $\{t < \tau\}$, and

$$\varphi_D(u) = \frac{V_\beta(u)}{V_\alpha(u)}.$$

6.5. By Proposition A.15, the joint density of the logarithmic return

$$R(T) = \ln\frac{S(t)}{S(0)} = rt - \frac{1}{2}\sigma^2 t + \sigma W_Q(t)$$

and its minimum $\min_{u\in[0,T]} R(u)$ is

$$f(x,y) = \frac{1}{\sigma\sqrt{2\pi T}}\frac{2(x-2y)}{T\sigma^2}e^{\frac{2y\left(r-\frac{1}{2}\sigma^2\right)}{\sigma^2}}e^{-\frac{\left(2y-x+rT-\frac{1}{2}\sigma^2 T\right)^2}{2\sigma^2 T}}.$$

when $y \leq 0$ and $y \leq x$, and $f(x,y) = 0$ otherwise. According to Example 6.17, the vulnerable call price can be computed as

$$\begin{aligned}
H(0) &= e^{-rT}\mathbb{E}((S(T) - K)^+ e^{-\Gamma(T)}) \\
&= e^{-rT}\mathbb{E}((S(0)e^{R(T)} - K)^+ e^{\min_{u\in[0,T]} R(t)}) \\
&= e^{-rT}\int_{\ln\frac{K}{S(0)}}^{\infty}\left(\int_{-\infty}^{0\wedge x}(S(0)e^x - K)e^y f(x,y)dy\right)dx \\
&= e^{-rT}\int_{\ln\frac{K}{S(0)}}^{\infty}(S(0)e^x - K)\left(\int_{-\infty}^{0\wedge x}e^y f(x,y)dy\right)dx.
\end{aligned}$$

We shall compute $H(0)$ in the case when in the case when $K \geq S(0)$. In this case the upper limit in the inner integral is simply 0, and this integral can be computed as follows:

$$\begin{aligned}
&\int_{-\infty}^{0} e^y f(x,y)dy \\
&= \frac{1}{\sigma\sqrt{2\pi T}} \int_{-\infty}^{0} e^y \frac{2(x-2y)}{\sigma^2 T} e^{\frac{2y\left(r-\frac{1}{2}\sigma^2\right)}{\sigma^2}} e^{-\frac{\left(2y-x+rT-\frac{1}{2}\sigma^2 T\right)^2}{2\sigma^2 T}} dy \\
&= \frac{1}{\sigma\sqrt{2\pi T}} e^{-\frac{r(r-\sigma^2)T}{2\sigma^2}} e^{\frac{rx}{\sigma^2}} \int_{-\infty}^{0} \frac{2(x-2y)}{\sigma^2 T} e^{-\frac{\left(2y-x-\frac{1}{2}\sigma^2 T\right)^2}{2\sigma^2 T}} dy \\
&= \frac{1}{\sigma\sqrt{2\pi T}} e^{-\frac{r(r-\sigma^2)T}{2\sigma^2}} e^{\frac{rx}{\sigma^2}} \int_{-\infty}^{0} \left(\frac{d}{dy}\left(e^{-\frac{\left(2y-x-\frac{1}{2}\sigma^2 T\right)^2}{2\sigma^2 T}} \right) - e^{-\frac{\left(2y-x-\frac{1}{2}\sigma^2 T\right)^2}{2\sigma^2 T}} \right) dy \\
&= \frac{1}{\sigma\sqrt{2\pi T}} e^{-\frac{r(r-\sigma^2)T}{2\sigma^2}} e^{\frac{rx}{\sigma^2}} e^{-\frac{\left(-x-\frac{1}{2}\sigma^2 T\right)^2}{2\sigma^2 T}} - e^{-\frac{r(r-\sigma^2)T}{2\sigma^2}} e^{\frac{rx}{\sigma^2}} N\left(\frac{-x-\frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \right) \\
&= \frac{1}{\sigma\sqrt{2\pi T}} e^{-\frac{\left(x-\left(r-\frac{1}{2}\sigma^2\right)T\right)^2}{2\sigma^2 T}} - e^{-\frac{r(r-\sigma^2)T}{2\sigma^2}} e^{\frac{rx}{\sigma^2}} N\left(\frac{-x-\frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \right).
\end{aligned}$$

Hence, we get

$$\begin{aligned}
&\int_{\ln\frac{K}{S(0)}}^{\infty} \left(\int_{-\infty}^{0} e^y f(x,y)dy \right) dx \\
&= \int_{\ln\frac{K}{S(0)}}^{\infty} \frac{1}{\sigma\sqrt{2\pi T}} e^{-\frac{\left(x-\left(r-\frac{1}{2}\sigma^2\right)T\right)^2}{2\sigma^2 T}} dx - e^{-\frac{r(r-\sigma^2)T}{2\sigma^2}} \int_{\ln\frac{K}{S(0)}}^{\infty} e^{\frac{rx}{\sigma^2}} N\left(\frac{-x-\frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \right) dx \\
&= N\left(\frac{\ln\frac{S(0)}{K} + \left(r-\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} \right) - e^{-\frac{r(r-\sigma^2)T}{2\sigma^2}} \frac{\sigma^2}{r} \int_{\ln\frac{K}{S(0)}}^{\infty} \frac{d}{dx}\left(e^{\frac{rx}{\sigma^2}} \right) N\left(\frac{-x-\frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \right) dx \\
&= N\left(\frac{\ln\frac{S(0)}{K} + \left(r-\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} \right) + e^{-\frac{r(r-\sigma^2)T}{2\sigma^2}} \frac{\sigma^2}{r} e^{\frac{r\ln\frac{K}{S(0)}}{\sigma^2}} N\left(\frac{-\ln\frac{K}{S(0)}-\frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \right) \\
&\quad + e^{-\frac{r(r-\sigma^2)T}{2\sigma^2}} \frac{\sigma^2}{r} \int_{\ln\frac{K}{S(0)}}^{\infty} e^{\frac{rx}{\sigma^2}} \frac{d}{dx} N\left(\frac{-x-\frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \right) dx \\
&= N\left(\frac{\ln\frac{S(0)}{K} + \left(r-\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} \right) + e^{-\frac{r(r-\sigma^2)T}{2\sigma^2}} \frac{\sigma^2}{r} \left(\frac{K}{S(0)} \right)^{\frac{r}{\sigma^2}} N\left(\frac{\ln\frac{S(0)}{K}-\frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \right) \\
&\quad - e^{-\frac{r(r-\sigma^2)T}{2\sigma^2}} \frac{\sigma^2}{r} \int_{\ln\frac{K}{S(0)}}^{\infty} e^{\frac{rx}{\sigma^2}} \frac{1}{\sigma\sqrt{2\pi T}} e^{-\frac{\left(-x-\frac{1}{2}\sigma^2 T\right)^2}{2\sigma^2 T}} dx
\end{aligned}$$

and so

$$
\begin{aligned}
&\int_{\ln \frac{K}{S(0)}}^{\infty} \left(\int_{-\infty}^{0} e^{y} f(x,y) dy \right) dx \\
&= N\left(\frac{\ln \frac{S(0)}{K} + \left(r - \frac{1}{2}\sigma^2\right) T}{\sigma \sqrt{T}} \right) + e^{-\frac{r(r-\sigma^2)T}{2\sigma^2}} \frac{\sigma^2}{r} \left(\frac{K}{S(0)} \right)^{\frac{r}{\sigma^2}} N\left(\frac{\ln \frac{S(0)}{K} - \frac{1}{2}\sigma^2 T}{\sigma \sqrt{T}} \right) \\
&\quad - \frac{\sigma^2}{r} \int_{\ln \frac{K}{S(0)}}^{\infty} \frac{1}{\sigma \sqrt{2\pi T}} e^{-\frac{\left(x - rT + \frac{1}{2}\sigma^2 T\right)^2}{2\sigma^2 T}} dx \\
&= N\left(\frac{\ln \frac{S(0)}{K} + \left(r - \frac{1}{2}\sigma^2\right) T}{\sigma \sqrt{T}} \right) + e^{-\frac{r(r-\sigma^2)T}{2\sigma^2}} \frac{\sigma^2}{r} \left(\frac{K}{S(0)} \right)^{\frac{r}{\sigma^2}} N\left(\frac{\ln \frac{S(0)}{K} - \frac{1}{2}\sigma^2 T}{\sigma \sqrt{T}} \right) \\
&\quad - \frac{\sigma^2}{r} N\left(\frac{\ln \frac{S(0)}{K} + \left(r - \frac{1}{2}\sigma^2\right) T}{\sigma \sqrt{T}} \right) \\
&= \frac{r - \sigma^2}{r} N\left(\frac{\ln \frac{S(0)}{K} + \left(r - \frac{1}{2}\sigma^2\right) T}{\sigma \sqrt{T}} \right) + e^{-\frac{r(r-\sigma^2)T}{2\sigma^2}} \frac{\sigma^2}{r} \left(\frac{K}{S(0)} \right)^{\frac{r}{\sigma^2}} N\left(\frac{\ln \frac{S(0)}{K} - \frac{1}{2}\sigma^2 T}{\sigma \sqrt{T}} \right).
\end{aligned}
$$

We also get

$$
\begin{aligned}
&\int_{\ln \frac{K}{S(0)}}^{\infty} e^{x} \left(\int_{-\infty}^{0} e^{y} f(x,y) dy \right) dx \\
&= \int_{\ln \frac{K}{S(0)}}^{\infty} e^{x} \frac{1}{\sigma \sqrt{2\pi T}} e^{-\frac{\left(x - \left(r - \frac{1}{2}\sigma^2\right)T\right)^2}{2\sigma^2 T}} dx - e^{-\frac{r(r-\sigma^2)T}{2\sigma^2}} \int_{\ln \frac{K}{S(0)}}^{\infty} e^{x} e^{\frac{rx}{\sigma^2}} N\left(\frac{-x - \frac{1}{2}\sigma^2 T}{\sigma \sqrt{T}} \right) dx \\
&= e^{rT} \int_{\ln \frac{K}{S(0)}}^{\infty} \frac{1}{\sigma \sqrt{2\pi T}} e^{-\frac{\left(x - \left(r + \frac{1}{2}\sigma^2\right)T\right)^2}{2\sigma^2 T}} dx - e^{-\frac{r(r-\sigma^2)T}{2\sigma^2}} \frac{\sigma^2}{r + \sigma^2} \int_{\ln \frac{K}{S(0)}}^{\infty} \frac{d}{dx} \left(e^{\frac{\left(r+\sigma^2\right)x}{\sigma^2}} \right) N\left(\frac{-x - \frac{1}{2}\sigma^2 T}{\sigma \sqrt{T}} \right) dx \\
&= e^{rT} N\left(\frac{\ln \frac{S(0)}{K} + \left(r + \frac{1}{2}\sigma^2\right) T}{\sigma \sqrt{T}} \right) + e^{-\frac{r(r-\sigma^2)T}{2\sigma^2}} \frac{\sigma^2}{r + \sigma^2} e^{\frac{\left(r+\sigma^2\right) \ln \frac{K}{S(0)}}{\sigma^2}} N\left(\frac{-\ln \frac{K}{S(0)} - \frac{1}{2}\sigma^2 T}{\sigma \sqrt{T}} \right) \\
&\quad + e^{-\frac{r(r-\sigma^2)T}{2\sigma^2}} \frac{\sigma^2}{r + \sigma^2} \int_{\ln \frac{K}{S(0)}}^{\infty} e^{\frac{\left(r+\sigma^2\right)x}{\sigma^2}} \frac{d}{dx} N\left(\frac{-x - \frac{1}{2}\sigma^2 T}{\sigma \sqrt{T}} \right) dx \\
&= e^{rT} N\left(\frac{\ln \frac{S(0)}{K} + \left(r + \frac{1}{2}\sigma^2\right) T}{\sigma \sqrt{T}} \right) + e^{-\frac{r(r-\sigma^2)T}{2\sigma^2}} \frac{\sigma^2}{r + \sigma^2} \left(\frac{K}{S(0)} \right)^{\frac{r+\sigma^2}{\sigma^2}} N\left(\frac{\ln \frac{S(0)}{K} - \frac{1}{2}\sigma^2 T}{\sigma \sqrt{T}} \right) \\
&\quad - e^{-\frac{r(r-\sigma^2)T}{2\sigma^2}} \frac{\sigma^2}{r + \sigma^2} \int_{\ln \frac{K}{S(0)}}^{\infty} e^{\frac{\left(r+\sigma^2\right)x}{\sigma^2}} \frac{1}{\sigma \sqrt{2\pi T}} e^{-\frac{\left(-x - \frac{1}{2}\sigma^2 T\right)^2}{2\sigma^2 T}} dx
\end{aligned}
$$

and so

$$\int_{\ln\frac{K}{S(0)}}^{\infty} e^x \left(\int_{-\infty}^{0} e^y f(x,y)dy\right)dx$$

$$= e^{rT} N\left(\frac{\ln\frac{S(0)}{K} + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right) + e^{-\frac{r(r-\sigma^2)T}{2\sigma^2}} \frac{\sigma^2}{r+\sigma^2}\left(\frac{K}{S(0)}\right)^{\frac{r+\sigma^2}{\sigma^2}} N\left(\frac{\ln\frac{S(0)}{K} - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right)$$

$$-e^{rT}\frac{\sigma^2}{r+\sigma^2}\int_{\ln\frac{K}{S(0)}}^{\infty}\frac{1}{\sigma\sqrt{2\pi T}}e^{-\frac{\left(x-rT-\frac{1}{2}\sigma^2 T\right)^2}{2\sigma^2 T}}dx$$

$$= e^{rT} N\left(\frac{\ln\frac{S(0)}{K} + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right) + e^{-\frac{r(r-\sigma^2)T}{2\sigma^2}} \frac{\sigma^2}{r+\sigma^2}\left(\frac{K}{S(0)}\right)^{\frac{r+\sigma^2}{\sigma^2}} N\left(\frac{\ln\frac{S(0)}{K} - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right)$$

$$-e^{rT}\frac{\sigma^2}{r+\sigma^2}N\left(\frac{\ln\frac{S(0)}{K} + rT + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right)$$

$$= e^{rT}\frac{r}{r+\sigma^2} N\left(\frac{\ln\frac{S(0)}{K} + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right) + e^{-\frac{r(r-\sigma^2)T}{2\sigma^2}} \frac{\sigma^2}{r+\sigma^2}\left(\frac{K}{S(0)}\right)^{\frac{r+\sigma^2}{\sigma^2}} N\left(\frac{\ln\frac{S(0)}{K} - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right).$$

It follows that in the case when $K \geq S(0)$

$$H(0) = e^{-rT}\int_{\ln\frac{K}{S(0)}}^{\infty}(S(0)e^x - K)\left(\int_{-\infty}^{0} e^y f(x,y)dy\right)dx$$

$$= e^{-rT}S(0)\left[e^{rT}\frac{r}{r+\sigma^2} N\left(\frac{\ln\frac{S(0)}{K} + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right) + e^{-\frac{r(r-\sigma^2)T}{2\sigma^2}} \frac{\sigma^2}{r+\sigma^2}\left(\frac{K}{S(0)}\right)^{\frac{r+\sigma^2}{\sigma^2}} N\left(\frac{\ln\frac{S(0)}{K} - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right)\right]$$

$$-e^{-rT}K\left[\frac{r-\sigma^2}{r}N\left(\frac{\ln\frac{S(0)}{K} + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right) + e^{-\frac{r(r-\sigma^2)T}{2\sigma^2}}\frac{\sigma^2}{r}\left(\frac{K}{S(0)}\right)^{\frac{r}{\sigma^2}} N\left(\frac{\ln\frac{S(0)}{K} - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right)\right]$$

$$= S(0)\frac{r}{r+\sigma^2}N\left(\frac{\ln\frac{S(0)}{K} + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right) - e^{-rT}K\frac{r-\sigma^2}{r}N\left(\frac{\ln\frac{S(0)}{K} + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)$$

$$-K\frac{\sigma^4}{r(r+\sigma^2)}e^{-\frac{r\left(r+\sigma^2\right)T}{2\sigma^2}}\left(\frac{K}{S(0)}\right)^{\frac{r}{\sigma^2}} N\left(\frac{\ln\frac{S(0)}{K} - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right).$$

6.6. Since the discounted defaultable bond price process is a $(\mathcal{G}_t)_{t\geq 0}$-martingale,

$$H(t) = e^{-rt}D(t,T) = \mathbb{E}(e^{-rT}D(T,T)|\mathcal{G}_t) = \mathbb{E}(e^{-rT}\mathbf{1}_{\{T<\tau\}}|\mathcal{G}_t)$$

for each $t \in [0,T]$. Hence, putting

$$h(t) = e^{-rT}\mathbf{1}_{\{T<t\}},$$

we have

$$H(t) = e^{-rt}D(t,T) = \mathbb{E}(h(\tau)|\mathcal{G}_t).$$

Next,

$$\begin{aligned}
J(t) &= e^{\Gamma(t)}\mathbb{E}(\mathbf{1}_{\{t<\tau\}}h(\tau)|\mathcal{F}_t)\\
&= e^{\Gamma(t)}\mathbb{E}(\mathbf{1}_{\{t<\tau\}}e^{-rT}\mathbf{1}_{\{T<\tau\}}|\mathcal{F}_t)\\
&= e^{-rT}e^{\Gamma(t)}\mathbb{E}(\mathbf{1}_{\{T<\tau\}}|\mathcal{F}_t)\\
&= e^{-rT}e^{\Gamma(t)}\mathbb{E}(\mathbb{E}(\mathbf{1}_{\{T<\tau\}}|\mathcal{F}_T)|\mathcal{F}_t)\\
&= e^{-rT}e^{\Gamma(t)}\mathbb{E}(e^{-\Gamma(T)}|\mathcal{F}_t)\\
&= e^{-rT}e^{\Gamma(t)}\mathbb{E}(G(T)|\mathcal{F}_t).
\end{aligned}$$

This can be expressed in terms of the value $V_\alpha(t) = \alpha_B(t)B(t,T) + \alpha_S(t)S(t)$ of the admissible self-financing strategy (α_B, α_S) replicating a derivative security with payoff $G(T)$ and exercise time T in the Black–Scholes model consisting of the assets B, S. Namely,

$$e^{-rt}\mathbb{E}(G(T)|\mathcal{F}_t) = V_\alpha(t),$$

so

$$J(t) = e^{-r(T-t)}e^{\Gamma(t)}V_\alpha(t).$$

6.7. We take

$$h(t) = X\mathbf{1}_{\{T<t\}},$$

so that

$$H(t) = \mathbb{E}(X\mathbf{1}_{\{T<\tau\}}|\mathcal{G}_t) = \mathbb{E}(h(\tau)|\mathcal{G}_t).$$

Moreover,

$$\begin{aligned}
J(t) &= e^{\Gamma(t)}\mathbb{E}(\mathbf{1}_{\{t<\tau\}}h(\tau)|\mathcal{F}_t)\\
&= e^{\Gamma(t)}\mathbb{E}(\mathbf{1}_{\{t<\tau\}}X\mathbf{1}_{\{T<\tau\}}|\mathcal{F}_t)\\
&= e^{\Gamma(t)}\mathbb{E}(X\mathbf{1}_{\{T<\tau\}}|\mathcal{F}_t)\\
&= e^{\Gamma(t)}\mathbb{E}(X\mathbb{E}(\mathbf{1}_{\{T<\tau\}}|\mathcal{F}_T)|\mathcal{F}_t)\\
&= e^{\Gamma(t)}\mathbb{E}(Xe^{-\Gamma(T)}|\mathcal{F}_t)\\
&= e^{\Gamma(t)}\mathbb{E}(XG(T)|\mathcal{F}_t).
\end{aligned}$$

This can be expressed in terms of the value $V_\beta(t) = \beta_B(t)B(t,T) + \beta_S(t)S(t)$ of the admissible self-financing strategy (β_B, β_S) replicating a derivative security with payoff $XG(T)$ and exercise time T in the Black–Scholes model consisting of the assets B, S. Namely,

$$e^{-r(T-t)}\mathbb{E}(XG(T)|\mathcal{F}_t) = V_\beta(t),$$

so

$$J(t) = e^{r(T-t)}e^{\Gamma(t)}V_\beta(t).$$

6.8. Let (α_B, α_S) and (β_B, β_S) be the admissible self-financing strategies that replicate the derivative securities with exercise time T and payoff $G(T)$ and, respectively, $XG(T)$ in the Black–Scholes model consisting of the assets B, S. From Exercise 6.7, for any $t \in [0, T]$ we have

$$\begin{aligned} h(t) &= X\mathbf{1}_{\{T<t\}} = 0, \\ J(t) &= e^{r(T-t)}e^{\Gamma(t)}V_\beta(t). \end{aligned}$$

By Theorem 6.20, tt follows that

$$\begin{aligned} \varphi_D(t) &= \frac{e^{-r(T-t)}\left(J(t) - h(t)\right)}{e^{\Gamma(u)}V_\alpha(t)} \\ &= \frac{V_\beta(t)}{V_\alpha(t)}. \end{aligned}$$

Further,

$$\begin{aligned} \varphi_S(t) &= \mathbf{1}_{\{t\leq\tau\}}\left(e^{\Gamma(t)}\beta_S(t) - \frac{e^{-r(T-t)}\left(J(t) - h(t)\right)}{V_\alpha(t)}\alpha_S(t)\right) \\ &= \mathbf{1}_{\{t\leq\tau\}}e^{\Gamma(t)}\left(\beta_S(t) - \frac{V_\beta(t)}{V_\alpha(t)}\alpha_S(t)\right) \\ &= \mathbf{1}_{\{t\leq\tau\}}e^{\Gamma(t)}\left(\beta_S(t) - \varphi_D(t)\alpha_S(t)\right). \end{aligned}$$

Finally,

$$\begin{aligned} \varphi_B(t) &= h(t\wedge\tau) - e^{r(T-t)}\varphi_S(t)S(t) \\ &= -e^{r(T-t)}\varphi_S(t)S(t). \end{aligned}$$

These are the same formulae as (6.7), (6.8) and (6.11) in Section 6.3.

6.11. Since

$$D_\eta(0, T) = D(0, T) + V_\varphi(0),$$

where $(\varphi_B, \varphi_S, \varphi_D)$ is the strategy in Theorem 6.21, we just need to compute $V_\varphi(0)$. By Corollary 6.22,

$$V_\varphi(0) = V_\beta(0),$$

where (β_B, β_S) is the strategy replicating the payoff $\mathbb{E}(h(\tau)|\mathcal{F}_T)$ with

$$h(t) = \mathbf{1}_{\{t \le T\}} e^{r(T-t)} \eta(t).$$

It follows that

$$\begin{aligned} V_\varphi(0) &= V_\beta(0) = e^{-rT} \mathbb{E}(h(\tau)) \\ &= e^{-rT} \mathbb{E}\left(\int_0^\infty h(u) f(u) du \right) = \mathbb{E}\left(\int_0^T e^{-ru} \eta(u) f(u) du \right). \end{aligned}$$

6.9. The payoff of a zero-recovery defaultable bond with maturity S, where $0 < S < T$, is $\mathbf{1}_{\{S<\tau\}}$, which will grow to become $e^{r(T-S)}\mathbf{1}_{\{S<\tau\}}$ at time T if invested in the non-defaultable bond at time S. Hence we put

$$h(t) = e^{r(T-S)} \mathbf{1}_{\{S<t\}}.$$

The corresponding processes $m(t)$ and $J(t)$ are

$$\begin{aligned} m(t) &= \mathbb{E}(h(\tau)|\mathcal{F}_t) \\ &= e^{r(T-S)} \mathbb{E}(\mathbf{1}_{\{S<\tau\}}|\mathcal{F}_t) \\ &= e^{r(T-S)} e^{-\Gamma(S)} \mathbf{1}_{\{S<t\}} + e^{r(T-S)} \mathbb{E}(e^{-\Gamma(S)}|\mathcal{F}_t) \mathbf{1}_{\{t \le S\}} \end{aligned}$$

and

$$\begin{aligned} J(t) &= e^{\Gamma(t)} \mathbb{E}(\mathbf{1}_{\{t<\tau\}} h(\tau)|\mathcal{F}_t) \\ &= e^{r(T-S)} e^{\Gamma(t)} \mathbb{E}(\mathbf{1}_{\{t<\tau\}} \mathbf{1}_{\{S<\tau\}}|\mathcal{F}_t) \\ &= e^{r(T-S)} e^{\Gamma(t)} \mathbb{E}(\mathbf{1}_{\{t<\tau\}}|\mathcal{F}_t) \mathbf{1}_{\{S<t\}} + e^{r(T-S)} e^{\Gamma(t)} \mathbb{E}(\mathbf{1}_{\{S<\tau\}}|\mathcal{F}_t) \mathbf{1}_{\{t \le S\}} \\ &= e^{r(T-S)} \mathbf{1}_{\{S<t\}} + e^{r(T-S)} e^{\Gamma(t)} \mathbb{E}(e^{-\Gamma(S)}|\mathcal{F}_t) \mathbf{1}_{\{t \le S\}}. \end{aligned}$$

By Theorem 6.20, for any $t \in [0, S]$, this gives

$$\varphi_D(t) = \frac{e^{-r(T-t)} (J(t) - h(t))}{e^{\Gamma(t)} V_\alpha(t)} = \frac{e^{-r(S-t)} \mathbb{E}(e^{-\Gamma(S)}|\mathcal{F}_t)}{V_\alpha(t)}.$$

Taking (β_B, β_S) to be the admissible self-financing strategy that replicates the derivative security with payoff $m(T) = e^{r(T-S)} e^{-\Gamma(S)}$ and exercise time T in the Black–Scholes model consisting of the assets B and S, we then obtain, for any $t \in [0, S]$,

$$\begin{aligned} \varphi_S(t) &= \mathbf{1}_{\{t \le \tau\}} e^{\Gamma(t)} (\beta_S(t) - \varphi_D(t) \alpha_S(t)), \\ \varphi_B(t) &= h(t \wedge \tau) - e^{r(T-t)} \varphi_S(t) S(t) = -e^{r(T-t)} \varphi_S(t) S(t), \end{aligned}$$

which implies that

$$\varphi_B(t) B(t, T) + \varphi_S(t) S(t) = 0.$$

It follows that

$$\begin{aligned}D(t,S) &= V_\varphi(t)\\ &= \varphi_B(t)B(t,T)+\varphi_S(t)S(t)+\varphi_D(t)D(t,T)\\ &= \frac{e^{-r(S-t)}\mathbb{E}(e^{-\Gamma(S)}|\mathcal{F}_t)}{V_\alpha(t)}D(t,T)\\ &= \frac{e^{-r(S-t)}\mathbb{E}(e^{-\Gamma(S)}|\mathcal{F}_t)}{V_\alpha(t)}\mathbf{1}_{\{t<\tau\}}e^{\Gamma(t)}V_\alpha(t)\\ &= e^{-r(S-t)}\mathbf{1}_{\{t<\tau\}}e^{\Gamma(t)}\mathbb{E}(e^{-\Gamma(S)}|\mathcal{F}_t)\end{aligned}$$

for any $t\in[0,S]$. It is the same formula as (6.1) for $D(t,T)$, with T replaced by S.

6.10. The payoff of a zero-recovery defaultable bond with maturity S, where $0<S<T$, is $\mathbf{1}_{\{S<\tau\}}$. Hence, by using Theorem 5.35, we get

$$\begin{aligned}D(t,S) &= e^{-r(S-t)}\mathbb{E}(\mathbf{1}_{\{S<\tau\}}|\mathcal{G}_t)\\ &= e^{-r(S-t)}\mathbf{1}_{\{t<\tau\}}e^{\Gamma(t)}\mathbb{E}(e^{-\Gamma(S)}|\mathcal{F}_t)\end{aligned}$$

for any $t\in[0,S]$, that is, the same result as in Exercise 6.9.

6.11. The payoff of the defaultable bond is

$$\begin{aligned}D_\eta(T,T) &= \mathbf{1}_{\{T<\tau\}}+\mathbf{1}_{\{\tau\le T\}}e^{r(T-\tau)}\eta(\tau)\\ &= D(T,T)+h(\tau),\end{aligned}$$

where

$$h(t)=\mathbf{1}_{\{t\le T\}}e^{r(T-t)}\eta(t).$$

It follows that

$$\begin{aligned}D_\eta(0,T) &= D(0,T)+e^{-rT}\mathbb{E}\left(h(\tau)\right)\\ &= D(0,T)+e^{-rT}\mathbb{E}\left(\int_0^\infty h(u)f(u)du\right)\\ &= D(0,T)+e^{-rT}\mathbb{E}\left(\int_0^\infty \mathbf{1}_{\{u\le T\}}e^{r(T-u)}\eta(u)f(u)du\right)\\ &= D(0,T)+\mathbb{E}\left(\int_0^T e^{-ru}\eta(u)f(u)du\right).\end{aligned}$$

6.12. In Example 6.25 it is shown that

$$
\begin{aligned}
D(0,T) &= Fe^{-(\lambda+r)T}(1-F_\theta(T)) + Re^{-rT}\left(1-e^{-\lambda T}+e^{-\lambda T}F_\theta(T)\right) \\
&\quad - Re^{-rT}\int_0^T e^{-\lambda t}dF_\theta(t) + Fe^{-\gamma T}\int_0^T e^{-(\lambda+r-\gamma)t}dF_\theta(t) \\
&= Re^{-rT} + (F-R)\,e^{-(\lambda+r)T}(1-F_\theta(T)) \\
&\quad - Re^{-rT}\int_0^T e^{-\lambda t}dF_\theta(t) + Fe^{-\gamma T}\int_0^T e^{-(\lambda+r-\gamma)t}dF_\theta(t),
\end{aligned}
$$

where

$$
F_\theta(t) = N(d_1(t)) + L^{2\alpha}N(d_2(t))
$$

with

$$
\begin{aligned}
L &= \frac{Fe^{-\gamma T}}{V(0)} < 1, \\
\alpha &= \frac{r-\gamma-\frac{1}{2}\sigma^2}{\sigma^2}, \\
d_1(t) &= \frac{\ln L - \left(r-\gamma-\frac{1}{2}\sigma^2\right)t}{\sigma\sqrt{t}} = \frac{\sigma^{-1}\ln L - \sigma\alpha t}{\sqrt{t}}, \\
d_2(t) &= \frac{\ln L + \left(r-\gamma-\frac{1}{2}\sigma^2\right)t}{\sigma\sqrt{t}} = \frac{\sigma^{-1}\ln L + \sigma\alpha t}{\sqrt{t}}.
\end{aligned}
$$

We use formula (A.1) in Exercise A.12 to compute the integrals with respect to $F_\theta(t)$. Namely,

$$
\begin{aligned}
\int_0^T e^{-\lambda t}dF_\theta(t) &= \int_0^T e^{-\lambda t}dN(d_1(t)) + L^{2\alpha}\int_0^T e^{-\lambda t}dN(d_2(t)) \\
&= \int_0^T e^{ct}dN\left(\frac{-a-bt}{\sqrt{t}}\right) + e^{-2ab}\int_0^T e^{ct}dN\left(\frac{-a+bt}{\sqrt{t}}\right),
\end{aligned}
$$

where

$$
a = -\sigma^{-1}\ln L, \quad b = \sigma\alpha, \quad c = -\lambda.
$$

We have $a > 0$ and $d = \sqrt{b^2-2c}$, where

$$
b^2 - 2c = \sigma^2\alpha^2 + 2\lambda > 0,
$$

and formula (A.1) gives

$$\begin{aligned}
&\int_0^T e^{-\lambda t} dF_\theta(t) \\
&= \frac{d+b}{2d} e^{-a(b-d)} N\left(\frac{-a-dT}{\sqrt{T}}\right) + \frac{d-b}{2d} e^{-a(b+d)} N\left(\frac{-a+dT}{\sqrt{T}}\right) \\
&\quad + e^{-2ab}\left(\frac{d-b}{2d} e^{-a(-b-d)} N\left(\frac{-a-dT}{\sqrt{T}}\right) + \frac{d+b}{2d} e^{-a(-b+d)} N\left(\frac{-a+dT}{\sqrt{T}}\right)\right) \\
&= e^{-a(b-d)} N\left(\frac{-a-dT}{\sqrt{T}}\right) + e^{-a(b+d)} N\left(\frac{-a+dT}{\sqrt{T}}\right) \\
&= L^{\alpha-\sqrt{\alpha^2+2\lambda/\sigma^2}} N\left(\frac{\ln L - T\sqrt{\alpha^2+2\lambda/\sigma^2}}{\sigma\sqrt{T}}\right) \\
&\quad + L^{\alpha+\sqrt{\alpha^2+2\lambda/\sigma^2}} N\left(\frac{\ln L + T\sqrt{\alpha^2+2\lambda/\sigma^2}}{\sigma\sqrt{T}}\right).
\end{aligned}$$

We also have

$$\begin{aligned}
&\int_0^T e^{-(\lambda+r-\gamma)t} dF_\theta(t) \\
&= \int_0^T e^{-(\lambda+r-\gamma)t} dN(d_1(t)) + L^\alpha \int_0^T e^{-(\lambda+r-\gamma)t} dN(d_2(t)) \\
&= \int_0^T e^{ct} dN\left(\frac{-a-bt}{\sqrt{t}}\right) + e^{-2ab}\int_0^T e^{ct} dN\left(\frac{-a+bt}{\sqrt{t}}\right),
\end{aligned}$$

where

$$a = -\sigma^{-1}\ln L, \quad b = \sigma\alpha, \quad c = -(\lambda + r - \gamma).$$

We have $\alpha > 0$ and $d = \sqrt{b^2 - 2c}$, where

$$\begin{aligned}
b^2 - 2c &= \sigma^2\alpha^2 + 2(\lambda + r - \gamma) \\
&= \sigma^{-2}\left(r - \gamma - \frac{1}{2}\sigma^2\right)^2 + 2(\lambda + r - \gamma) \\
&= \sigma^{-2}\left(r - \gamma + \frac{1}{2}\sigma^2\right)^2 + 2\lambda \\
&= \sigma^{-2}(\alpha + 1)^2 + 2\lambda > 0.
\end{aligned}$$

Formula (A.1) gives

$$\begin{aligned}
&\int_0^T e^{-(\lambda+r-\gamma)t} dF_\theta(t) \\
&= e^{-a(b-d)} N\left(\frac{-a-dT}{\sqrt{T}}\right) + e^{-a(b+d)} N\left(\frac{-a+dT}{\sqrt{T}}\right) \\
&= L^{\alpha-\sqrt{(\alpha+1)^2+2\lambda/\sigma^2}} N\left(\frac{\ln L - T\sqrt{(\alpha+1)^2+2\lambda/\sigma^2}}{\sigma\sqrt{T}}\right) \\
&\quad + L^{\alpha+\sqrt{(\alpha+1)^2+2\lambda/\sigma^2}} N\left(\frac{\ln L + T\sqrt{(\alpha+1)^2+2\lambda/\sigma^2}}{\sigma\sqrt{T}}\right).
\end{aligned}$$

It follows that

$$\begin{aligned}
&D(0,T) \\
&= Re^{-rT} + (F-R)\,e^{-(\lambda+r)T}\left(1 - N\left(\frac{\ln L - \sigma^2\alpha T}{\sigma\sqrt{T}}\right) - L^{2\alpha} N\left(\frac{\ln L + \sigma^2\alpha T}{\sigma\sqrt{T}}\right)\right) \\
&\quad - Re^{-rT} L^{\alpha}\left(L^{-\sqrt{\alpha^2+2\lambda/\sigma^2}} N\left(\frac{\ln L - T\sqrt{\alpha^2+2\lambda/\sigma^2}}{\sigma\sqrt{T}}\right)\right. \\
&\qquad\qquad \left. + L^{\sqrt{\alpha^2+2\lambda/\sigma^2}} N\left(\frac{\ln L + T\sqrt{\alpha^2+2\lambda/\sigma^2}}{\sigma\sqrt{T}}\right)\right) \\
&\quad + Fe^{-\gamma T} L^{\alpha}\left(L^{-\sqrt{(\alpha+1)^2+2\lambda/\sigma^2}} N\left(\frac{\ln L - T\sqrt{(\alpha+1)^2+2\lambda/\sigma^2}}{\sigma\sqrt{T}}\right)\right. \\
&\qquad\qquad \left. + L^{\sqrt{(\alpha+1)^2+2\lambda/\sigma^2}} N\left(\frac{\ln L + T\sqrt{(\alpha+1)^2+2\lambda/\sigma^2}}{\sigma\sqrt{T}}\right)\right).
\end{aligned}$$

Appendix A

A.1. Let $a > 0$, and take a sequence $x_n > 0$ such that $x_n \nearrow a$ and so $F(x_n) \nearrow F(a_-)$ as $n \to \infty$. Since

$$\{a\} = \bigcap_{n=1}^{\infty} (x_n, a]$$

and $(x_{n+1}, a] \subset (x_n, a]$ for each n, it follows that

$$\mu_F(\{a\}) = \lim_{n\to\infty} \mu_F((x_n, a]) = \lim_{n\to\infty} (F(a) - F(x_n)) = F(a) - F(a_-).$$

A.2. Using Exercise A.1, we obtain

$$\begin{aligned}\mu_F([a,b]) &= \mu_F(\{a\} \cup (a,b]) \\ &= \mu_F(\{a\}) + \mu_F((a,b]) \\ &= (F(a) - F(a_-)) + (F(b) - F(a)) \\ &= F(b) - F(a_-)\end{aligned}$$

since $\{a\} \cap (a,b] = \emptyset$. Next,

$$\begin{aligned}\mu_F([a,b)) &= \mu_F([a,b] \setminus \{b\}) \\ &= \mu_F([a,b]) - \mu_F(\{b\}) \\ &= (F(b) - F(a_-)) - (F(b) - F(b_-)) \\ &= F(b_-) - F(a_-)\end{aligned}$$

since $\{b\} \subset [a,b]$. Finally, if $b > a$, then $\{a\} \subset [a,b)$ and so

$$\begin{aligned}\mu_F((a,b)) &= \mu_F([a,b) \setminus \{a\}) \\ &= \mu_F([a,b)) - \mu_F(\{a\}) \\ &= (F(b_-) - F(a_-)) - (F(a) - F(a_-)) \\ &= F(b_-) - F(a).\end{aligned}$$

A.3. To show that for each $a \in \mathbb{R}$, $b > 0$ and $y > 0$

$$\int_0^y x dN\left(\frac{\ln x + a}{b}\right) = e^{\frac{1}{2}b^2 - a} N\left(\frac{\ln y + a - b^2}{b}\right)$$

we start by computing the derivative of the right-hand side with respect to y:

$$\begin{aligned}\frac{d}{dy} e^{\frac{1}{2}b^2 - a} N\left(\frac{\ln y + a - b^2}{b}\right) &= e^{\frac{1}{2}b^2 - a} N'\left(\frac{\ln y + a - b^2}{b}\right)\frac{1}{yb} \\ &= \frac{1}{\sqrt{2\pi}} e^{\frac{1}{2}b^2 - a} e^{-\frac{1}{2}\left(\frac{\ln y + a - b^2}{b}\right)^2} \frac{1}{yb} \\ &= \frac{1}{\sqrt{2\pi}} e^{-\frac{(\ln y + a)^2}{2b^2}} \frac{1}{b} \\ &= N'\left(\frac{\ln y + a}{b}\right)\frac{1}{b},\end{aligned}$$

which is clearly equal to the derivative of the left-hand side with respect to y. Hence, the expressions on either side of the equality differ just by a constant. To see that this constant is 0, hence the equality holds, observe that the right-limit as $y \searrow 0$ of either side of the equality is the same (namely 0).

In the same manner, we can show that

$$\int_0^y x dN\left(\frac{-\ln x + a}{b}\right) = -e^{\frac{1}{2}b^2+a} N\left(\frac{\ln y - a - b^2}{b}\right).$$

Alternatively, this formula can be verified by using the equality already proved above:

$$\begin{aligned}\int_0^y x dN\left(\frac{-\ln x + a}{b}\right) &= \int_0^y x d\left(1 - N\left(\frac{\ln x - a}{b}\right)\right) \\ &= -\int_0^y x dN\left(\frac{\ln x - a}{b}\right) \\ &= -e^{\frac{1}{2}b^2+a} N\left(\frac{\ln y - a - b^2}{b}\right).\end{aligned}$$