Mathematical appendix 7

Tidal asymmetry

Question 9.2(c) requires the evaluation over a tidal cycle of the expression:

$$K\int_0^{2\pi} U^3 dt = K\int_0^{2\pi} (U_0 + U_2\cos 2\omega_1 t + U_4\cos 4\omega_1 t)^3\, dt$$

We rewrite the integral as:

$$U_2^3\int_0^{2\pi}\left(\frac{U_0}{U_2} + \cos 2\omega_1 t + \frac{U_4}{U_2}\cos 4\omega_1 t\right)^3 dt$$

noting that U_0/U_2 and $U_4/U_2 \ll 1$. Multiplying the cubed term in the brackets and ignoring terms $(U_0/U_2)^2$ and $(U_4/U_2)^2$ and higher:

$$U_2^3\int_0^{2\pi}\Big(\cos^3 2\omega_1 t \qquad (1)$$

$$+\,3\cos^2 2\omega_1 t\cdot\frac{U_0}{U_2} \qquad (2)$$

$$+\,3\cos^2 2\omega_1 t\cos 4\omega_1 t\cdot\frac{U_4}{U_2}\Big)\, dt \qquad (3)$$

Evaluating these terms individually:

(1) $\int_0^{2\pi}\cos^3 2\omega_1 t\, dt$ is zero over a full cycle.

(2) $\int_0^{2\pi} 3\cos^2 2\omega_1 t = \frac{3}{2}$

(3) $\int_0^{2\pi} 3\cos^2 2\omega_1 t\cdot\cos 4\omega_1 t\, dt$

$$= 3\int_0^{2\pi}\cos^2 2\omega_1 t(2\cos^2 2\omega_1 t - 1)\, dt$$

$$= 6\int_0^{2\pi}\cos^4 2\omega_1 t\, dt - \frac{3}{2}$$

$$\left[\text{and using the identity} \int\cos^n x\, dx = \frac{\sin x\cos^{n-1}x}{n} + \frac{n-1}{n}\int\cos^{n-2}x\, dx\right.$$

$$\left.\text{over a cycle} \int\cos^4 x\, dx = \frac{3}{4}\int\cos^2 x\, dx\right]$$

$$= 6\cdot\frac{3}{4}\cdot\frac{1}{2} - \frac{3}{2}$$

$$= \frac{3}{4}$$

(1) + (2) + (3) gives

$$U_2^3\left(\frac{3}{2}\frac{U_0}{U_2}+\frac{3}{4}\frac{U_4}{U_2}\right)$$

and hence the value is:

$$KU_2^2\left(\frac{3}{2}U_0+\frac{3}{4}U_4\right)$$