Chapter 4: Signal Representation using Fourier Series

Problem 4.1

(a) Using Definition 4.4, the CT function $x_1(t)$ can be represented as $x_1(t) = c_1\phi_1(t) + c_2\phi_2(t) + c_3\phi_3(t)$ with the coefficients c_n, for n = 1,2, and 3, given by

$$c_1 = \tfrac{1}{2T}\int_{-T}^{T} x_1(t)\phi_1(t)dt = \tfrac{1}{2T}\int_{-T}^{0}(-A)dt + \tfrac{1}{2T}\int_{0}^{T} Adt = \tfrac{1}{2T}(-AT + AT) = 0\,,$$

$$c_2 = \tfrac{1}{2T}\int_{-T}^{T} x_1(t)\phi_2(t)dt = \tfrac{1}{2T}\int_{-T}^{-T/2}(-A)(-1)dt + \tfrac{1}{2T}\int_{-T/2}^{0}(-A)(1)\phi_2(t)dt + \tfrac{1}{2T}\int_{0}^{T/2} A(1)dt + \tfrac{1}{2T}\int_{T/2}^{T} A(-1)dt$$

$$= \tfrac{1}{2T}\left(\tfrac{AT}{2} - \tfrac{AT}{2} + \tfrac{AT}{2} - \tfrac{AT}{2}\right) = 0,$$

and $$c_3 = \tfrac{1}{2T}\int_{-T}^{T} x_1(t)\phi_3(t)dt = \tfrac{1}{2T}\int_{-T}^{0}(-A)(1)dt + \tfrac{1}{2T}\int_{0}^{T} A(-1)dt = \tfrac{1}{2T}(-AT - AT) = -A\,.$$

In other words, $x_1(t) = -A\phi_3(t)$, which can also be proved by inspection.

(b) By inspection, $x_2(t) = -A\phi_2(t)$, which can also be proven by evaluating the coefficients $c_1 = 0$, $c_2 = -A$, and $c_3 = 0$.

(c) Using Definition 4.4, the CT function $x_3(t)$ can be represented as $x_3(t) = c_1\phi_1(t) + c_2\phi_2(t) + c_3\phi_3(t)$ with the coefficients c_n, for n = 1,2, and 3, given by

$$c_1 = \tfrac{1}{2T}\int_{-T}^{T} x_3(t)\phi_1(t)dt = \tfrac{1}{2T}\int_{-T}^{-T/2}(A)(1)dt + \tfrac{1}{2T}\int_{T/2}^{T}(A)(1)dt = \tfrac{1}{2T}\left(\tfrac{AT}{2} + \tfrac{AT}{2}\right) = \tfrac{A}{2}\,,$$

$$c_2 = \tfrac{1}{2T}\int_{-T}^{T} x_3(t)\phi_2(t)dt = \tfrac{1}{2T}\int_{-T}^{-T/2}(A)(-1)dt + \tfrac{1}{2T}\int_{T/2}^{T}(A)(-1)dt = \tfrac{1}{2T}\left(-\tfrac{AT}{2} - \tfrac{AT}{2}\right) = -\tfrac{A}{2}\,,$$

and $$c_3 = \tfrac{1}{2T}\int_{-T}^{T} x_3(t)\phi_3(t)dt = \tfrac{1}{2T}\int_{-T}^{-T/2}(A)(1)dt + \tfrac{1}{2T}\int_{0}^{T}(A)(-1)dt = \tfrac{1}{2T}(AT - AT) = 0\,.$$

In other words, $x_3(t) = 0.5A(\phi_1(t) - \phi_2(t))$, which can also be proved by inspection. ▌

Problem 4.2

Computing the integral

$$\int_{-\infty}^{\infty}\phi_1(t)\phi_2(t)dt = \int_{-\infty}^{\infty} e^{-2|t|}\left(1 - Ke^{-4|t|}\right)dt$$

Since the function inside the integral is even with respect to t, therefore,

$$\int_{-\infty}^{\infty} \phi_1(t)\phi_2(t)dt = 2\int_0^{\infty} e^{-2t}\left(1 - Ke^{-4t}\right)dt = 2\int_0^{\infty} e^{-2t}\ dt - 2K\int_0^{\infty} e^{-6t}\ dt = 1 - \tfrac{K}{3}$$

For the functions to be orthogonal, $1 - \tfrac{K}{3} = 0 \;\Rightarrow\; K = 3.$ ▮

Problem 4.3

The following derivation shows that the individual functions $\{P_n(x), n = 0, 1, 2, 3\}$ have nonzero finite energy. We use the notation $P_{m,n}$ to represent the integral

$$P_{m,n} = \int_{-1}^{1} P_m(x)P_n(x)dx \,.$$

Computing the integrals

$$P_{0,0} = \int_{-1}^{1} 1.1dx = [x]_{-1}^{1} = 2,$$

$$P_{1,1} = \int_{-1}^{1} x^2 dx = \left[\tfrac{1}{3}x^3\right]_{-1}^{1} = \tfrac{2}{3},$$

$$P_{2,2} = \tfrac{1}{4}\int_{-1}^{1} (3x^2-1)^2 dx = \tfrac{1}{4}\int_{-1}^{1} \left(9x^4 - 6x^2 + 1\right)dx = \tfrac{1}{4}\left[\tfrac{9}{5}x^5 - 2x^3 + x\right]_{-1}^{1} = \tfrac{1}{2}(\tfrac{9}{5} - 2 + 1) = \tfrac{2}{5},$$

and
$$P_{3,3} = \tfrac{1}{4}\int_{-1}^{1} \left(25x^6 - 30x^4 + 9x^2\right)dx = \tfrac{1}{4}\left[\tfrac{25}{7}x^7 - 6x^5 + 3x^3\right]_{-1}^{1} = \tfrac{1}{2}(\tfrac{25}{7} - 6 + 3) = \tfrac{2}{7},$$

which shows that the functions $P_n(x)$ have nonzero finite energy.

To show that the functions are orthogonal with respect to each other, we determine the integrals

$$P_{0,1} = \int_{-1}^{1} \underbrace{x}_{=odd}\ dx = 0,$$

$$P_{0,2} = \tfrac{1}{2}\int_{-1}^{1} (3x^2 - 1)dx = \tfrac{1}{2}\left[x^3 - x\right]_{-1}^{1} = 0,$$

$$P_{0,3} = \tfrac{1}{2}\int_{-1}^{1} (\underbrace{5x^3 - 3x}_{=odd})dx = 0,$$

$$P_{1,2} = \tfrac{1}{2}\int_{-1}^{1} (\underbrace{3x^3 - x}_{=odd})dx = 0,$$

$$P_{1,3} = \tfrac{1}{2}\int_{-1}^{1} (5x^4 - 3x^2)dx = \tfrac{1}{2}\left[x^5 - x^3\right]_{-1}^{1} = 0,$$

and
$$P_{2,3} = \tfrac{1}{4}\int_{-1}^{1} \left(\underbrace{15x^5 - 14x^3 + 3x}_{=odd}\right)dx = 0\,.$$ ▮

Problem 4.4

The following derivation shows that the individual functions $\{T_n(x), n = 0, 1, 2, 3\}$ have nonzero finite energy. We use the notation $T_{m,n}$ to represent the integral

$$T_{m,n} = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} T_m(x) T_n(x) dx .$$

Computing the integrals

$$T_{0,0} = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} dx = \left[\sin^{-1}(x)\right]_{-1}^{1} = \pi ,$$

$$T_{1,1} = \int_{-1}^{1} \frac{x^2}{\sqrt{1-x^2}} dx = \left[-\tfrac{1}{2}x\sqrt{1-x^2} + \tfrac{1}{2}\sin^{-1}(x)\right]_{-1}^{1} = \frac{\pi}{2} ,$$

$$T_{2,2} = \int_{-1}^{1} \frac{4x^4 - 4x^2 + 1}{\sqrt{1-x^2}} dx = 4\int_{-1}^{1} \frac{x^4}{\sqrt{1-x^2}} dx - 4T_{1,1} + T_{0,0} = 3\pi - 4(0.5\pi) + \pi = 2\pi ,$$

and similarly, the higher order $T_{m,n}$'s can be proven to be nonzero for $m = n$.

To show that the functions are orthogonal with respect to each other, we determine the integrals

$$T_{0,1} = \int_{-1}^{1} \frac{x}{\sqrt{1-x^2}} dx = -\left[\sqrt{1-x^2}\right]_{-1}^{1} = 0 ,$$

$$T_{0,2} = \int_{-1}^{1} \frac{2x^2 - 1}{\sqrt{1-x^2}} dx = 2T_{1,1} - T_{0,0} = 0 ,$$

$$T_{0,3} = \int_{-1}^{1} \frac{4x^3 - 3x}{\sqrt{1-x^2}} dx = 4\int_{-1}^{1} \frac{x^3}{\sqrt{1-x^2}} dx - 3T_{0,1} = \left[-\sqrt{1-x^2} + \tfrac{1}{3}(1-x^2)^3\right]_{-1}^{1} = 0 ,$$

and similarly, the higher order $T_{m,n}$'s can be proven to be zero for $m \neq n$. ▮

Problem 4.5

Case I ($m = p, n = q$): $$\int_0^1 H_{m,n}(t) H_{p,q}(t) dt = \int_0^1 \left[H_{m,n}(t)\right]^2 dt = \int_0^1 \left[H_{0,0}(2^m t - n)\right]^2 dt$$

Substituting $$x = (2^m t - n) ,$$

we get $$\int_0^1 H_{m,n}(t) H_{p,q}(t) dt = \int_{-n}^{2^m - n} \left[H_{0,0}(x)\right]^2 2^{-m} dx = 2^{-m} \int_0^{2^m - n} \left[H_{0,0}(x)\right]^2 dx .$$

Since $(0 \le n \le 2^m - 1)$, therefore, $(2^m - n) \ge 1$ and

$$\int_0^1 H_{m,n}(t) H_{p,q}(t) dt = 2^{-m} \int_0^1 \left[H_{0,0}(x)\right]^2 dx = 2^{-m} \int_0^{0.5} (1)^2 dx + 2^{-m} \int_{0.5}^1 (-1)^2 dx = 2^{-m} .$$

Case II ($m \neq p, n \neq q$): $$\int_0^1 H_{m,n}(t)H_{p,q}(t)dt = \int_0^1 H_{0,0}(2^m t - n)H_{0,0}(2^p t - q)dt$$

Substituting $$x = (2^m t - n),$$

we get $$\int_0^1 H_{m,n}(t)H_{p,q}(t)dt = 2^{-m}\int_0^{2^m - n} H_{0,0}(x)H_{0,0}(2^{p-m}x + 2^{p-m}n - q)dx.$$

Since ($0 \leq n \leq 2^m - 1$) or ($2^m - n) \geq 1$ and

$$\int_0^1 H_{m,n}(t)H_{p,q}(t)dt = 2^{-m}\int_0^1 H_{0,0}(x)H_{0,0}(2^{p-m}x + 2^{p-m}n - q)dx,$$

or, $$\int_0^1 H_{m,n}(t)H_{p,q}(t)dt = 2^{-m}\int_0^{0.5} H_{0,0}(2^{p-m}x + 2^{p-m}n - q)dx - 2^{-m}\int_{0.5}^1 H_{0,0}(2^{p-m}x + 2^{p-m}n - q)dx = 0.$$

▮

Problem 4.6

(a) By inspection, we note that the time period $T_0 = 2\pi$, which implies that the fundamental frequency $\omega_0 = 1$.

Since the CTFS coefficient a_0 represents the average value of the signal, therefore, $a_0 = 3/2$.

Using Eq. (4.31), the CTFS cosine coefficients a_n's, for ($n \neq 0$), are given by

$$a_n = \tfrac{2}{T_0}\int_0^{T_0} x1(t)\cos(n\omega_0 t)dt = \tfrac{1}{\pi}\int_0^{\pi} 3\cos(n\omega_0 t)dt = \tfrac{1}{\pi}\int_0^{\pi} 3\cos(nt)dt$$
$$= \tfrac{3}{n\pi}\left[\sin(nt)\right]_0^{\pi} = \tfrac{3}{n\pi}\left[\sin(n\pi) - 0\right] = 0$$

Using Eq. (4.32), the CTFS sine coefficients b_n's are given by

$$b_n = \tfrac{2}{T_0}\int_0^{T_0} x1(t)\sin(n\omega_0 t)dt = \tfrac{1}{\pi}\int_0^{\pi} 3\sin(nt)dt = \tfrac{3}{n\pi}\left[-\cos(nt)\right]_0^{\pi} = \tfrac{3}{n\pi}\left[-\cos(n\pi) + \cos(0)\right] = \tfrac{3}{n\pi}\left[1 - (-1)^n\right]$$
$$= \begin{cases} \frac{6}{n\pi} & n = odd \\ 0 & n = even \end{cases}$$

(b) By inspection, we note that the time period $T_0 = 2T$, which implies that the fundamental frequency $\omega_0 = \pi/T$.

Since the CTFS coefficient a_0 represents the average value of the signal, therefore, $a_0 = 0.75$.

Using Eq. (4.31), the CTFS cosine coefficients a_n's, for ($n \neq 0$), are given by

$$a_n = \frac{2}{2T}\int_{-T}^{T} \underbrace{x(t)\cos(n\omega_0 t)}_{even\ function} dt = \frac{2}{T}\int_0^T x(t)\cos(n\omega_0 t)dt = \frac{2}{T}\left[\int_0^{T/2} 0.5\cos(n\omega_0 t)dt + \int_{T/2}^{T}\cos(n\omega_0 t)dt\right]$$

$$= \frac{1}{n\omega_0 T}\left[\sin(n\omega_0 t)\right]_0^{T/2} + \frac{2}{n\omega_0 T}\left[\sin(n\omega_0 t)\right]_{T/2}^{T}$$

$$= \frac{1}{n\pi}\left[\sin(n\omega_0 T/2)\right] + \frac{2}{n\pi}\left[\sin(n\omega_0 T) - \sin(n\omega_0 T/2)\right] \qquad [\because\ \omega_0 T = \pi]$$

$$= \frac{2}{n\pi}\sin(n\pi) - \frac{1}{n\pi}\sin(n\pi/2) = -\frac{1}{n\pi}\sin(n\pi/2)$$

$$= \begin{cases} 0 & n = even \\ -\frac{1}{n\pi} & n = 4k+1 \\ \frac{1}{n\pi} & n = 4k+3 \end{cases}$$

Since $x2(t)$ is even, therefore, the CTFS sine coefficients $b_n = 0$.

(c) By inspection, we note that the time period $T_0 = T$, which implies that the fundamental frequency $\omega_0 = 2\pi/T$.

Since the CTFS coefficient a_0 represents the average value of the signal, therefore, $a_0 = 1/2$.

Since the function $[x3(t) - 0.5]$ is odd, therefore, the CTFS cosine coefficients $a_n = 0$, for $(n \neq 0)$.

Using Eq. (4.32), the CTFS sine coefficients b_n's are given by

$$b_n = \frac{2}{T}\int_0^T\left(1 - \frac{t}{T}\right)\sin(n\omega_0 t)dt$$

$$= \frac{2}{T}\left[\left(1 - \frac{t}{T}\right)\times\frac{-\cos(n\omega_0 t)}{(n\omega_0)} - \left(-\frac{1}{T}\right)\times\frac{-\sin(n\omega_0 t)}{(n\omega_0)^2}\right]_0^T$$

$$= \frac{2}{T}\left[0 - (1)\times\frac{-1}{(n\omega_0)} - \left(\frac{1}{T}\right)\times\frac{\sin(n\omega_0 T)}{(n\omega_0)^2} + -\left(\frac{1}{T}\right)\times\frac{\sin(0)}{(n\omega_0)^2}\right]$$

$$= \frac{2}{n\omega_0 T} = \frac{1}{n\pi}$$

(d) By inspection, we note that the time period $T_0 = 2T$, which implies that the fundamental frequency $\omega_0 = \pi/T$.

Using Eq. (4.30), the CTFS coefficient T_0 is given by

$$a_0 = \frac{1}{2T}\int_{-T}^{T} x(t)dt = \frac{1}{T}\int_0^T x(t)dt = \frac{1}{T}\times\frac{T}{2} = \frac{1}{2}.$$

Using Eq. (4.31), the CTFS cosine coefficients a_n's, for $(n \neq 0)$, are given by

$$a_n = \frac{2}{2T}\int_{-T}^{T} \underbrace{x(t)\cos(n\omega_0 t)}_{even\ \ function}dt = \frac{2}{T}\int_0^T \left(1-\tfrac{t}{T}\right)\cos(n\omega_0 t)dt = \frac{2}{T}\int_0^T \cos(n\omega_0 t)dt - \frac{2}{T^2}\int_0^T t\cos(n\omega_0 t)dt$$

$$= \frac{2}{n\omega_0 T}\left[\sin(n\omega_0 t)\right]_0^T - \frac{2}{(n\omega_0)^2 T^2}\left[\cos(n\omega_0 t) + n\omega_0 t\sin(n\omega_0 t)\right]_0^T$$

$$= \frac{2}{n\pi}\left[\sin(n\omega_0 T) - 0\right] - \frac{2}{n^2\pi^2}\left[\cos(n\omega_0 T) + n\omega_0 T\sin(n\omega_0 T) - 1\right] \qquad \left[\because\ \omega_0 T = \pi\right]$$

$$= \frac{2}{n\pi}\underbrace{\sin(n\pi)}_{=0} - \frac{2}{n^2\pi^2}\left[\cos(n\pi) + n\pi\sin(n\pi) - 1\right]$$

$$= \frac{2}{n^2\pi^2}\left[1-(-1)^n\right] = \begin{cases} 0 & n = even \\ \frac{4}{n^2\pi^2} & n = odd \end{cases}$$

Since $x4(t)$ is even, therefore, the CTFS sine coefficients $b_n = 0$.

(e) By inspection, we note that the time period $T_0 = 2T$, which implies that the fundamental frequency $\omega_0 = \pi/T$.

Using Eq. (4.30), the CTFS coefficient T_0 is given by

$$a_0 = \frac{1}{2T}\int_0^{2T} x(t)dt = \frac{1}{2T}\int_0^{T}\left[1 - 0.5\sin\left(\tfrac{\pi t}{T}\right)\right]dt = \frac{1}{2T}\int_0^T dt - \frac{1}{4T}\int_0^T \sin\left(\tfrac{\pi t}{T}\right)dt$$

$$= \frac{1}{2} + \frac{1}{4T}\times\frac{1}{\pi/T}\left[\cos\left(\tfrac{\pi t}{T}\right)\right]_0^T = \frac{1}{2} + \frac{1}{4\pi}\left[\cos(\pi) - \cos(0)\right] = \frac{1}{2} - \frac{1}{2\pi} = \frac{\pi-1}{2\pi}$$

Using Eq. (4.31), the CTFS cosine coefficients a_n's, for $(n \neq 0)$, are given by

$$a_n = \frac{2}{2T}\int_0^T \left[1 - 0.5\sin\left(\tfrac{\pi t}{T}\right)\right]\cos(n\omega_0 t)dt = \underbrace{\frac{1}{T}\int_0^T \cos(n\omega_0 t)dt}_{=A} - \underbrace{\frac{1}{2T}\int_0^T \sin\left(\tfrac{\pi t}{T}\right)\cos(n\omega_0 t)dt}_{=B}$$

where Integrals A and B are simplified as

$$A = \frac{1}{n\omega_0 T}\left[\sin(n\omega_0 t)\right]_0^T = \frac{1}{n\pi}\left[\sin(n\omega_0 T) - 0\right] = \frac{1}{n\pi}\left[\sin(n\pi) - 0\right] = 0$$

and

$$B = \frac{1}{2T}\int_0^T \sin\left(\tfrac{\pi t}{T}\right)\cos(n\omega_0 t)dt = \frac{1}{2T}\int_0^T \sin\left(\tfrac{\pi t}{T}\right)\cos\left(\tfrac{n\pi t}{T}\right)dt = \frac{1}{4T}\int_0^T \left[\sin\tfrac{\pi t}{T}(n+1) - \sin\left(\tfrac{\pi t}{T}(n-1)\right)\right]dt$$

$$= \frac{1}{4T}\times\frac{-1}{\pi(n+1)/T}\left[\cos\tfrac{\pi t}{T}(n+1)\right]_0^T + \frac{1}{4T}\times\frac{1}{\pi(n-1)/T}\left[\cos\tfrac{\pi t}{T}(n-1)\right]_0^T \qquad \left[\text{for }\ n \neq 1\right]$$

$$= \frac{1}{4\pi(n+1)}\left[1 - \cos\pi(n+1)\right] - \frac{1}{4\pi(n-1)}\left[1 - \cos\pi(n-1)\right]$$

$$= \begin{cases} 0 & 1 \neq n = odd \\ \frac{2}{4\pi(n+1)} - \frac{2}{4\pi(n-1)} & n = even \end{cases} = \begin{cases} 0 & 1 \neq n = odd \\ -\frac{1}{\pi(n^2-1)} & n = even \end{cases}$$

For $$n = 1,\ B = \frac{1}{4T}\int_0^T \sin\tfrac{2\pi t}{T}dt = \frac{1}{4T}\times\frac{-1}{2\pi/T}\left[\cos\tfrac{2\pi t}{T}\right]_0^T = \frac{1}{8\pi}\left[1 - \cos 2\pi\right] = 0.$$

In other words, $$B = \begin{cases} 0 & n = odd \\ -\frac{1}{\pi(n^2-1)} & n = even \end{cases}$$

which implies that

$$a_n = A - B = \begin{cases} 0 & n = odd \\ \frac{1}{\pi(n^2-1)} & n = even \end{cases}$$

Using Eq. (4.32), the CTFS sine coefficients b_n's are given by

$$b_n = \frac{2}{2T}\int_0^T \left[1 - 0.5\sin\left(\frac{\pi t}{T}\right)\right]\sin(n\omega_0 t)dt = \underbrace{\frac{1}{T}\int_0^T \sin(n\omega_0 t)dt}_{=C} - \underbrace{\frac{1}{2T}\int_0^T \sin\left(\frac{\pi t}{T}\right)\sin(n\omega_0 t)dt}_{=D}$$

where Integrals C and D are simplified as

$$C = \frac{1}{n\omega_0 T}\left[-\cos(n\omega_0 t)\right]_0^T = \frac{1}{n\pi}\left[-\cos(n\omega_0 T) + \cos(0)\right] = \frac{1}{n\pi}\left[1 - \cos(n\pi)\right] = \begin{cases} 0 & n = even \\ \frac{2}{n\pi} & n = odd \end{cases}$$

and

$$\begin{aligned} D &= \frac{1}{2T}\int_0^T \sin\left(\frac{\pi t}{T}\right)\sin\left(\frac{n\pi t}{T}\right)dt = \frac{1}{4T}\int_0^T \left[\cos\frac{\pi t}{T}(n-1) - \cos\left(\frac{\pi t}{T}(n+1)\right)\right]dt \\ &= \frac{1}{4T}\times\frac{1}{\pi(n-1)/T}\left[\sin\frac{\pi t}{T}(n-1)\right]_0^T - \frac{1}{4T}\times\frac{1}{\pi(n+1)/T}\left[\sin\frac{\pi t}{T}(n+1)\right]_0^T && [\text{for } n \neq 1] \\ &= \frac{1}{4\pi(n-1)}\left[\sin\pi(n-1) - \sin(0)\right] - \frac{1}{4\pi(n+1)}\left[\sin\pi(n+1) - \sin(0)\right] \\ &= 0 && [\text{for } n \neq 1] \end{aligned}$$

For (n = 1),

$$D = \frac{1}{2T}\int_0^T \sin^2\left(\frac{\pi t}{T}\right)dt = \frac{1}{4T}\int_0^T \left[1 - \cos\left(\frac{2\pi t}{T}\right)\right]dt = \left(\frac{1}{4} - \frac{1}{4T\times 2\pi/T}\underbrace{\left[\sin\frac{2\pi t}{T}\right]_0^T}_{=0}\right) = \frac{1}{4}.$$

In other words, $$D = \begin{cases} \frac{1}{4} & n = 1 \\ 0 & n > 1 \end{cases}.$$

Therefore, $$b_n = C - D = \begin{cases} 0 & n = even \\ \frac{2}{\pi} - \frac{1}{4} & n = 1 \\ \frac{2}{n\pi} & 1 \neq n = odd. \end{cases}$$

Problem 4.7

By inspection, we note that the time period $T_0 = T$, which implies that the fundamental frequency $\omega_0 = 2\pi/T$.

Using Eq. (4.30), the CTFS coefficient a_0 is given by

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)dt = \frac{1}{T}.$$

Using Eq. (4.31), the CTFS cosine coefficients a_n's are given by

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} \delta(t)\cos(n\omega_0 t)dt = \frac{2}{T}\cos(n\omega_0 t)\Big|_{t=0} = \frac{2}{T}..$$

Using Eq. (4.31), the CTFS sine coefficients b_n's are given by

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} \delta(t)\sin(n\omega_0 t)dt = \frac{2}{T}\sin(n\omega_0 t)\Big|_{t=0} = 0.$$

The value for b_n can also be derived by noting that $x(t)$ is an even function. For such functions, the CTFS coefficient $b_n = 0$. ▮

Problem 4.8

(i) $x_1(t) = \cos(7t) + \sin(15t + \pi/2) = \cos(7t) + \cos(15t)$.

The fundamental frequency of $\cos(7t)$ is given by $\omega_1 = 7$, which implies that the time period of this term is $T_1 = 2\pi/7$. The fundamental frequency of $\cos(15t)$ is given by $\omega_2 = 15$, which implies that the time period of this term is $T_2 = 2\pi/15$.

Since the ratio $$\frac{T_1}{T_2} = \frac{15}{7}$$

is a rational number, $x_1(t)$ is periodic with the overall period $T_0 = mT_1 = nT_2 = 2\pi$. The fundamental frequency is given by $\omega_0 = 1$.

The CTFS expansion $$x(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n\cos(nt) + b_n\sin(nt)\right)$$

we note that $$a_0 = 0, \quad a_7 = 1, \quad \text{and} \quad a_{15} = 1.$$

The remaining coefficients are all zero.

In other words,

$$a_0 = 0, \quad a_n = \begin{cases} 1 & n = 7,15 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad b_n = 0,$$

with the fundamental frequency $\omega_0 = 1$.

(ii) The fundamental frequency of $\sin(2t)$ is given by $\omega_1 = 2$, which implies that the time period of this term is $T_1 = \pi$.

The fundamental frequency of $\cos(4t + \pi/4)$ is given by $\omega_2 = 4$, which implies that the time period of this term is $T_2 = \pi/2$.

Since the ratio $$\frac{T_1}{T_2} = 2$$

is a rational number, therefore, $x_2(t)$ is periodic with the overall period $T_0 = mT_1 = nT_2 = \pi$. The fundamental frequency is given by $\omega_0 = 2$.

Comparing $x_2(t) = 3 + \sin(2t) + \cos(4t + \pi/4) = 3 + \sin(2t) + 0.707\cos(4t) - 0.707\sin(4t)$

with the CTFS expansion $x(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos(2nt) + b_n \sin(2nt)\right)$

we note that $a_0 = 3, \quad b_1 = 1, \quad a_2 = 0.707, \quad \text{and} \quad b_2 = -0.707$

The remaining coefficients are all zero.

In other words, $a_0 = 3, \quad a_n = \begin{cases} 0.707 & n = 2 \\ 0 & \text{otherwise} \end{cases}$ and $b_n = \begin{cases} 1 & n = 1 \\ -0.707 & n = 2 \\ 0 & \text{otherwise,} \end{cases}$

with the fundamental frequency $\omega_0 = 1$.

(iii) The fundamental frequency of $\exp(j2t)$ is given by $\omega_1 = 2$, which implies that the time period of this term is $T_1 = \pi$.

The fundamental frequency of $\exp(j5t)$ is given by $\omega_2 = 5$, which implies that the time period of this term is $T_2 = 2\pi/5$.

The fundamental frequency of $\exp(-j3t)$ is given by $\omega_3 = 3$, which implies that the time period of this term is $T_3 = 2\pi/3$.

Since the ratios $\frac{T_1}{T_2} = \frac{5}{2}, \ \frac{T_1}{T_3} = \frac{3}{2}, \text{ and } \frac{T_2}{T_3} = \frac{3}{5}$

are all rational numbers, therefore, $x_3(t)$ is periodic with the overall period $T_0 = mT_1 = nT_2 = pT_3 = 2\pi$. The fundamental frequency is given by $\omega_0 = 1$.

Comparing

$$\begin{aligned} x_3(t) &= 1.2 + e \times e^{j2t} + e^{j2} \times e^{j5t} + e^{-j2} \times e^{-j3t} \\ &= 1.2 + e \times \cos(2t) + je \times \sin(2t) + e^{j2} \times \cos(5t) + je^{j2} \times \sin(5t) + e^{-j2} \times \cos(3t) - je^{-j2} \times \sin(3t) \end{aligned}$$

with the CTFS expansion $x(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos(2nt) + b_n \sin(2nt)\right)$

we note that

$$a_0 = 1.2, \quad a_n = \begin{cases} e & n = 2 \\ e^{-j2} & n = 3 \\ e^{j2} & n = 5 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad b_n = \begin{cases} je & n = 2 \\ -je^{-j2} & n = 3 \\ je^{j2} & n = 5 \\ 0 & \text{otherwise} \end{cases}$$

with the fundamental frequency $\omega_0 = 1$.

(iv) Because of the $\exp(t + 1)$ term, the signal $x_4(t)$ is not periodic. Therefore, the CTFS expansion cannot be obtained. ▮

Problem 4.9

By definition,
$$D_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)e^{-jn\omega_0 t}dt,$$

which is expressed as
$$D_n = \underbrace{\frac{1}{T_0}\int_{-T_0/2}^{0} x(t)e^{-jn\omega_0 t}dt}_{A} + \underbrace{\frac{1}{T_0}\int_{0}^{T_0/2} x(t)e^{-jn\omega_0 t}dt}_{B}.$$

Substituting $t = -\alpha$ in Integral A, we get
$$A = \frac{1}{T_0}\int_{T_0/2}^{0} x(-\alpha)e^{jn\omega_0\alpha}(-d\alpha) = \frac{1}{T_0}\int_{0}^{T_0/2} x(-\alpha)e^{jn\omega_0\alpha}d\alpha.$$

Since $x(t)$ is an even function, therefore, $x(-\alpha) = x(\alpha)$ and the above integral reduces to
$$A = \frac{1}{T_0}\int_{0}^{T_0/2} x(\alpha)e^{jn\omega_0\alpha}d\alpha.$$

Substituting the value of Integral A from the above expression, the exponential CTFS coefficients are given by
$$D_n = \frac{1}{T_0}\int_{0}^{T_0/2} x(t)e^{jn\omega_0 t}dt + \frac{1}{T_0}\int_{0}^{T_0/2} x(t)e^{-jn\omega_0 t}dt$$

or,
$$D_n = \frac{1}{T_0}\int_{0}^{T_0/2} x(t)\left[e^{jn\omega_0 t} + e^{-jn\omega_0 t}\right]dt = \frac{2}{T_0}\int_{0}^{T_0/2} x(t)\cos(n\omega_0 t)dt.$$
▌

Problem 4.10

By definition,
$$D_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)e^{-jn\omega_0 t}dt,$$

which is expressed as
$$D_n = \underbrace{\frac{1}{T_0}\int_{-T_0/2}^{0} x(t)e^{-jn\omega_0 t}dt}_{A} + \underbrace{\frac{1}{T_0}\int_{0}^{T_0/2} x(t)e^{-jn\omega_0 t}dt}_{B}.$$

Substituting $t = -\alpha$ in Integral A, we get
$$A = \frac{1}{T_0}\int_{T_0/2}^{0} x(-\alpha)e^{jn\omega_0\alpha}(-d\alpha) = \frac{1}{T_0}\int_{0}^{T_0/2} x(-\alpha)e^{jn\omega_0\alpha}d\alpha.$$

Since $x(t)$ is an odd function, $x(-\alpha) = -x(\alpha)$ and the above integral reduces to
$$A = -\frac{1}{T_0}\int_{0}^{T_0/2} x(\alpha)e^{jn\omega_0\alpha}d\alpha.$$

Substituting the value of Integral A from the above expression, the exponential CTFS coefficients are given by

$$D_n = -\frac{1}{T_0}\int_0^{T_0/2} x(t)e^{jn\omega_0 t}dt + \frac{1}{T_0}\int_0^{T_0/2} x(t)e^{-jn\omega_0 t}dt$$

or,
$$D_n = -\frac{1}{T_0}\int_0^{T_0/2} x(t)\left[e^{jn\omega_0 t} - e^{-jn\omega_0 t}\right] dt = \frac{-2j}{T_0}\int_0^{T_0/2} x(t)\sin(n\omega_0 t)dt .$$

▮

Problem 4.11

(a) By inspection, we note that the time period $T_0 = 2\pi$, which implies that the fundamental frequency $\omega_0 = 1$. Using Eq. (4.44), the DTFS coefficients D_n's are given by

$$D_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)e^{-jn\omega_0 t}dt = \frac{1}{2\pi}\int_0^{\pi} 3e^{-jnt}dt = \begin{cases} \frac{3}{2}, & n = 0 \\ \frac{3}{j2n\pi}\left(1 - e^{-jn\pi}\right) & n \neq 0. \end{cases}$$

or,
$$D_n = \frac{3}{j2n\pi}\left(1-(-1)^n\right) = \begin{cases} \frac{3}{2} & n = 0 \\ 0 & \text{even } n, n \neq 0. \\ \frac{3}{jn\pi} & \text{odd } n \end{cases}$$

The magnitude and phase spectra are given by

Magnitude Spectrum:
$$|D_n| = \begin{cases} \frac{3}{2}, & n = 0 \\ 0, & \text{even } n, n \neq 0. \\ \frac{3}{n\pi}, & \text{odd } n. \end{cases}$$

Phase Spectrum:
$$\angle D_n = \begin{cases} 0, & \text{even } n \\ -\frac{\pi}{2}, & \text{odd } n, n > 0 \\ \frac{\pi}{2}, & \text{odd } n, n < 0. \end{cases}$$

The magnitude and phase spectra are shown in row 1 of the subplots included in Fig. S4.11.

(b) By inspection, we note that the time period $T_0 = 2T$, which implies that the fundamental frequency $\omega_0 = \pi/T$. Since $x(t)$ is an even function, therefore, the DTFS coefficients D_n's are given by

$$D_n = \frac{2}{T_0}\int_0^{T_0/2} x(t)\cos(n\omega_0 t)dt = \begin{cases} \frac{1}{T}\int_0^{0.5T} 0.5dt + \frac{1}{T}\int_{0.5T}^{T} dt = 0.25 + 0.5 = 0.75 & n = 0 \\ \frac{1}{T}\int_0^{0.5T} 0.5\cos(n\omega_0 t)dt + \frac{1}{T}\int_{0.5T}^{T}\cos(n\omega_0 t)dt, & n \neq 0. \end{cases}$$

For ($n \neq 0$), the DTFS coefficients are given by or, $D_n = \frac{0.5}{T}\left[\frac{\sin(n\omega_0 t)}{n\omega_0}\right]_0^{0.5T} + \frac{1}{T}\left[\frac{\sin(n\omega_0 t)}{n\omega_0}\right]_{0.5T}^{T} = \frac{0.5}{n\pi}\left[\sin(0.5n\pi) - 2\sin(0.5n\pi)\right] = -\frac{0.5}{n\pi}\sin(0.5n\pi)$.

Combining the above results, we get

$$D_n = \begin{cases} \frac{3}{4} & n = 0 \\ 0 & \text{even } n, n \neq 0. \\ -\frac{1}{2|n|\pi} & \text{odd } n, n = (4k+1) \\ \frac{1}{2|n|\pi} & \text{odd } n, n = (4k+3). \end{cases}$$

The magnitude and phase spectra are given by

Magnitude Spectrum: $$|D_n| = \begin{cases} \frac{3}{4}, & n = 0 \\ 0, & \text{even } n, n \neq 0. \\ \frac{1}{2|n|\pi}, & \text{odd } n. \end{cases}$$

Phase Spectrum: $$\prec D_n = \begin{cases} 0, & \text{even } n \\ \pi, & \text{odd } n, n = (4k+1) \\ 0, & \text{odd } n, n = (4k+3). \end{cases}$$

The magnitude and phase spectra are shown in row 2 of the subplots included in Fig. S4.11.

(c) By inspection, we note that the time period $T_0 = T$, which implies that the fundamental frequency $\omega_0 = 2\pi/T$. Using Eq. (4.44), the DTFS coefficients D_n's are given by

$$D_n = \frac{1}{T}\int_0^T x(t)e^{-jn\omega_0 t}dt = \frac{1}{T}\int_0^T \left(1 - \tfrac{t}{T}\right)e^{-jn\omega_0 t}dt = \begin{cases} \frac{1}{T}\times\frac{T}{2} = \frac{1}{2}, & n = 0 \\ \frac{1}{T}\int_0^T \left(1 - \tfrac{t}{T}\right)e^{-jn\omega_0 t}dt & n \neq 0. \end{cases}$$

For ($n \neq 0$), the DTFS coefficients are given by

$$D_n = \frac{1}{T}\int_0^T \left(1 - \tfrac{t}{T}\right) e^{-jn\omega_0 t}dt = \left[\left(1 - \tfrac{t}{T}\right)\frac{e^{-jn\omega_0 t}}{(-jn\omega_0)} - \left(-\tfrac{1}{T}\right)\frac{e^{-jn\omega_0 t}}{(-jn\omega_0)^2}\right]_0^T,$$

which reduces to

$$D_n = \left[0 - \tfrac{1}{T}\frac{1}{(-jn\omega_0)} + \tfrac{1}{T}\frac{e^{-jn\omega_0 T}}{(-jn\omega_0)^2} - \tfrac{1}{T}\frac{1}{(-jn\omega_0)^2}\right]_0^T = \frac{1}{j2n\pi}.$$

Combining the two cases, we get

$$D_n = \begin{cases} \frac{1}{2}, & n = 0 \\ \frac{1}{j2n\pi}, & n \neq 0. \end{cases}$$

The magnitude and phase spectra are given by

Magnitude Spectrum: $$|D_n| = \begin{cases} \frac{1}{2}, & n = 0 \\ \frac{1}{2|n|\pi}, & n \neq 0. \end{cases}$$

Phase Spectrum: $$\prec D_n = \begin{cases} 0, & n = 0 \\ 0.5\pi, & n < 0 \\ -0.5\pi, & n > 0. \end{cases}$$

The magnitude and phase spectra are shown in row 3 of the subplots included in Fig. S4.11.

(d) By inspection, we note that the time period $T_0 = 2T$, which implies that the fundamental frequency $\omega_0 = \pi/T$. Since $x(t)$ is an even function, the DTFS coefficients D_n's are given by

$$D_n = \frac{2}{T_0}\int_0^{T_0/2} x(t)\cos(n\omega_0 t)dt = \begin{cases} \frac{1}{T}\int_0^T \left(1 - \frac{t}{T}\right)dt = 0.50, & n = 0 \\ \frac{1}{T}\int_0^T \left(1 - \frac{t}{T}\right)\cos(n\omega_0 t)dt, & n \neq 0. \end{cases}$$

For ($n \neq 0$), the DTFS coefficients are given by

$$D_n = \frac{1}{T}\int_0^T \left(1 - \frac{t}{T}\right)\cos(n\omega_0 t)dt = \frac{1}{T}\left[\left(1 - \frac{t}{T}\right)\frac{\sin(n\omega_0 t)}{(n\omega_0)} - \left(-\frac{1}{T}\right)\frac{-\cos(n\omega_0 t)}{(n\omega_0)^2}\right]_0^T,$$

which reduces to

$$D_n = \left[0 - 0 - \frac{\cos(n\pi)}{(n\pi)^2} + \frac{1}{(n\pi)^2}\right] = \frac{1 - (-1)^n}{(n\pi)^2}.$$

Combining the two cases, we get

$$D_n = \begin{cases} \frac{1}{2}, & n = 0 \\ 0, & \text{even } n, n \neq 0 \\ \frac{2}{(n\pi)^2} & \text{odd } n, n \neq 0. \end{cases}$$

Since D_n is always positive, its phase spectrum is 0.

The magnitude and phase spectra are shown in row 4 of the subplots included in Fig. S4.11.

(e) By inspection, we note that the time period $T_0 = 2T$, which implies that the fundamental frequency $\omega_0 = \pi/T$. For ($n = 0$), the exponential DTFS coefficients is given by

$$D_0 = \frac{1}{2T}\int_0^{2T} x(t)dt = \frac{1}{2T}\int_0^T \left[1 - 0.5\sin\left(\frac{\pi t}{T}\right)\right]dt = \frac{1}{2T}\int_0^T dt - \frac{1}{4T}\int_0^T \sin\left(\frac{\pi t}{T}\right)dt$$
$$= \frac{1}{2} + \frac{1}{4T} \times \frac{1}{\pi/T}\left[\cos\left(\frac{\pi t}{T}\right)\right]_0^T = \frac{1}{2} + \frac{1}{4\pi}\left[\cos(\pi) - \cos(0)\right] = \frac{1}{2} - \frac{1}{2\pi}$$

For ($n = 0$), the exponential DTFS coefficients is given by

$$D_n = \frac{1}{2T}\int_0^T \left[1 - 0.5\sin\left(\tfrac{\pi t}{T}\right)\right] e^{-jn\omega_0 t}\,dt = \underbrace{\frac{1}{2T}\int_0^T e^{-jn\omega_0 t}\,dt}_{=A} - \underbrace{\frac{1}{4T}\int_0^T \sin\left(\tfrac{\pi t}{T}\right) e^{-jn\omega_0 t}\,dt}_{=B} \cdot$$

Solving for Integrals A and B, we get

$$A = \frac{1}{2T}\int_0^T e^{-jn\omega_0 t}\,dt = \frac{1}{-j2n\omega_0 T}\left[e^{-jn\omega_0 t}\right]_0^T = \frac{1}{-j2n\pi}\left[e^{-jn\pi} - 1\right] = \frac{1}{j2n\pi}\left[1 - (-1)^n\right]$$

and

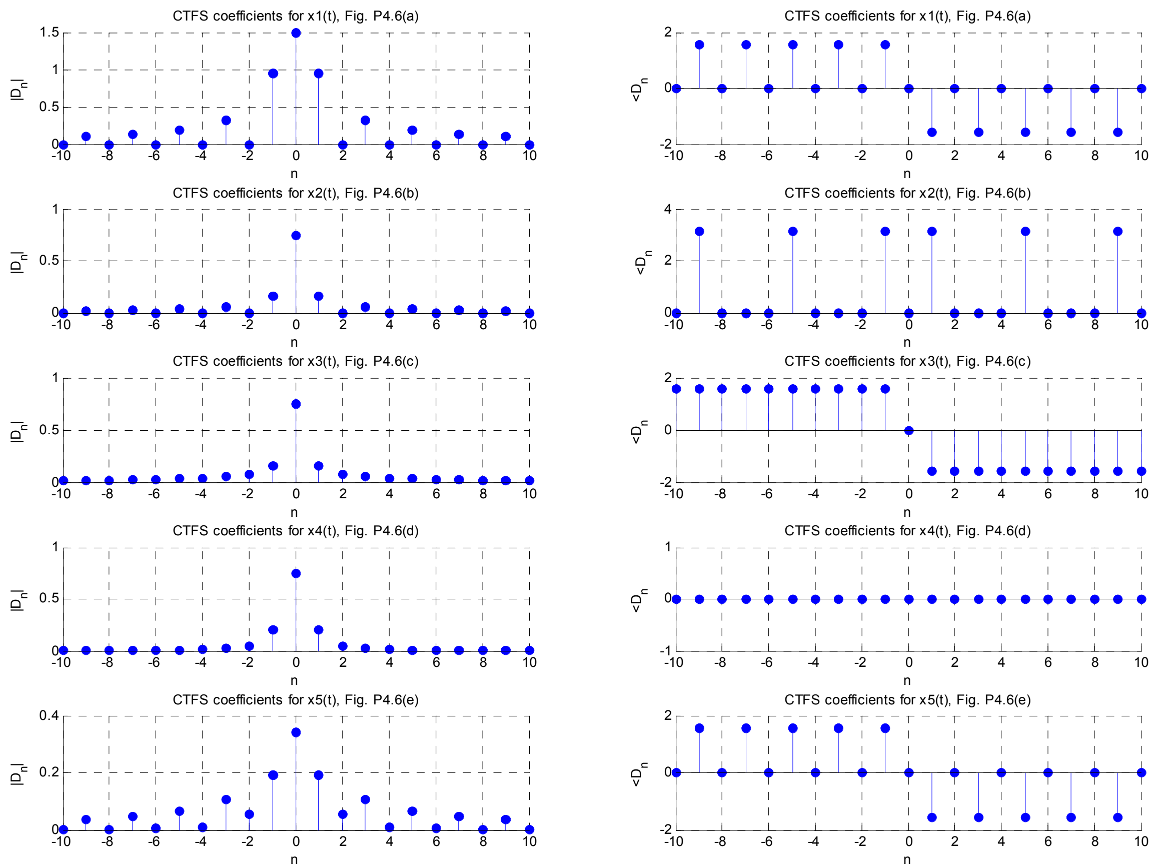


Fig. S4.11: Magnitude and phase spectra calculated in P4.11 for the periodic functions shown in Fig. P4.6.

$$B = \frac{1}{4T}\int_0^T \sin\left(\tfrac{\pi t}{T}\right) e^{\frac{-jn\pi t}{T}} dt = \frac{1}{j8T}\int_0^T \left[e^{\frac{j\pi t}{T}} - e^{\frac{-j\pi t}{T}} \right] e^{\frac{-jn\pi t}{T}} dt = \frac{1}{j8T}\int_0^T \left[e^{\frac{-j(n-1)\pi t}{T}} - e^{\frac{-j(n+1)\pi t}{T}} \right] dt$$

$$= \frac{1}{j8T}\left[\frac{T}{-j(n-1)\pi} e^{\frac{-j(n-1)\pi t}{T}} + \frac{T}{j(n+1)\pi} e^{\frac{-j(n+1)\pi t}{T}} \right]_0^T \qquad \text{for } n \neq \pm 1$$

$$= \frac{1}{8}\left[\tfrac{1}{(n-1)\pi}\left(e^{-j(n-1)\pi} - 1\right) - \tfrac{1}{(n+1)\pi}\left(e^{-j(n+1)\pi} - 1\right) \right]$$

$$= \frac{1}{8\pi}\left[\tfrac{1}{(n-1)} - \tfrac{1}{(n+1)} \right]\left[(-1)^{(n-1)} - 1 \right] = \tfrac{-1}{4\pi(n^2-1)}\left[1 - (-1)^{(n-1)} \right]$$

$$= \tfrac{-1}{4\pi(n^2-1)}\left[1 + (-1)^n \right]$$

For $n = \pm 1$, Integral B reduces to

$$\text{For } n = 1, \;\; B = \frac{1}{j8T}\int_0^T \left[1 - e^{\frac{-j2\pi t}{T}} \right] dt = \frac{1}{j8T}\left[t + \tfrac{T}{j2\pi} e^{\frac{-j2\pi t}{T}} \right]_0^T = \frac{T}{j8T} = \frac{1}{j8}$$

and

$$\text{For } n = -1, \;\; B = \frac{1}{j8T}\int_0^T \left[e^{\frac{-j2\pi t}{T}} - 1 \right] dt = \frac{1}{j8T}\left[-\tfrac{T}{j2\pi} e^{\frac{-j2\pi t}{T}} - t \right]_0^T = \frac{-T}{j8T} = -\frac{1}{j8}.$$

In other words,

$$B = \begin{cases} \pm\frac{1}{j8} & n = \pm 1 \\ \tfrac{-1}{4\pi(n^2-1)}\left[1 + (-1)^n \right] & otherwise \end{cases}$$

Combining, the above cases, the CTFS coefficients can be expressed as

$$D_n = \begin{cases} \frac{1}{2} - \frac{1}{2\pi} & n = 0 \\ \frac{1}{j2n\pi}\left[1 - (-1)^n \right] \mp \frac{1}{j8} & n = \pm 1 \\ \frac{1}{j2n\pi}\left[1 - (-1)^n \right] + \tfrac{1}{4\pi(n^2-1)}\left[1 + (-1)^n \right] & otherwise \end{cases}$$

$$= \begin{cases} \frac{1}{2}\left(1 - \frac{1}{\pi}\right) & n = 0 \\ \mp j\left(\frac{1}{\pi} - \frac{1}{8}\right) & n = \pm 1 \\ \tfrac{1}{4\pi(n^2-1)}\left[1 + (-1)^n \right] + \frac{1}{j2n\pi}\left[1 - (-1)^n \right] & otherwise \end{cases}$$

$$= \begin{cases} \frac{1}{2}\left(1 - \frac{1}{\pi}\right) & n = 0 \\ \mp j\left(\frac{1}{\pi} - \frac{1}{8}\right) & n = \pm 1 \\ \frac{1}{2\pi(n^2-1)} & 0 \neq n = even \\ \frac{1}{jn\pi} & \pm 1 \neq n = odd \end{cases}$$

The expressions for the magnitude and phase spectra are given by

Magnitude Spectrum:
$$|D_n| = \begin{cases} \frac{1}{2}\left(1 - \frac{1}{\pi}\right) & n = 0 \\ \frac{1}{\pi} - \frac{1}{8} & n = \pm 1 \\ \frac{1}{2\pi(n^2-1)} & 0 \neq n = even \\ \left|\frac{1}{n\pi}\right| & \pm 1 \neq n = odd \end{cases} = \begin{cases} \approx 0.3408 & n = 0 \\ \approx 0.1933 & n = \pm 1 \\ \approx \frac{0.1592}{n^2-1} & 0 \neq n = even \\ \approx \frac{0.3183}{|n|} & \pm 1 \neq n = odd \end{cases}$$

Phase Spectrum: $$\measuredangle D_n = \begin{cases} 0 & n = even \\ \measuredangle(\mp j) & n = \pm 1 \\ \measuredangle(\frac{1}{jn}) & \pm 1 \neq n = odd \end{cases} = \begin{cases} 0 & n = even \\ -\frac{\pi}{2} & n = odd, n > 0 \\ \frac{\pi}{2} & n = odd, n < 0 \end{cases}$$

The magnitude and phase spectra are shown in row 5 of the subplots included in Fig. S4.11.

▌

Problem 4.12

By inspection, we note that the time period $T_0 = T$, which implies that the fundamental frequency $\omega_0 = \pi/T$.

Using Eq. (4.44), the exponential CTFS coefficient D_n's are given by

$$D_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)e^{-jn\omega_0 t}dt = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)e^{-jnt}dt = \frac{1}{T}$$

The magnitude spectrum $|D_n|$ is constant at $1/T$ for all values of n. The phase spectrum $<D_n$ is always 0.

▌

Problem 4.13

In each case, we show that the exponential CTFS coefficients obtained directly from Eq. (4.44) are identical to those obtained from the trigonometric CTFS coefficients.

(a) From the solution of Problem P4.6(a), we know that

$$a_0 = \tfrac{1}{2},\ a_n = 0, \text{ and } b_n = \begin{cases} \frac{6}{n\pi} & n = odd \\ 0 & n = even \end{cases}.$$

Using Eq. (4.45), the exponential CTFS coefficients for $x1(t)$ are given by

$$D_n = \begin{cases} a_0 & n = 0 \\ \frac{1}{2}(a_n - jb_n) & n > 0 \\ \frac{1}{2}(a_{-n} + jb_{-n}) & n < 0 \end{cases} = \begin{cases} a_0 & n = 0 \\ -\frac{1}{2}jb_n & n > 0 \\ \frac{1}{2}jb_{-n} & n < 0 \end{cases} \qquad [\because\ a_n = a_{-n} = 0]$$

$$= \begin{cases} \frac{3}{2} & n = 0 \\ 0 & n = even \\ -j\frac{3}{n\pi} & n\text{=odd}, n > 0 \\ -j\frac{3}{n\pi} & n\text{=odd}, n < 0 \end{cases} = \begin{cases} \frac{3}{2} & n = 0 \\ 0 & n = even \\ \frac{3}{jn\pi} & n = odd \end{cases}$$

(b) From the solution of Problem P4.6(b), we know that

$$a_0 = \tfrac{3}{4},\ a_n = -\tfrac{1}{n\pi}\sin(n\pi/2) = \begin{cases} 0 & n = even \\ -\frac{1}{n\pi} & n = 4k+1 \\ \frac{1}{n\pi} & n = 4k+3 \end{cases}, \text{ and } b_n = 0.$$

Using Eq. (4.45), the exponential CTFS coefficients for $x2(t)$ are given by

$$D_n = \begin{cases} a_0 & n=0 \\ \frac{1}{2}(a_n - jb_n) & n>0 \\ \frac{1}{2}(a_{-n} + jb_{-n}) & n<0 \end{cases} = \begin{cases} a_0 & n=0 \\ \frac{1}{2}a_n & n>0 \\ \frac{1}{2}a_{-n} & n<0 \end{cases} \qquad [\because\ b_n = b_{-n} = 0]$$

$$= \begin{cases} \frac{3}{4} & n=0 \\ 0 & 0 \neq n = even \\ -\frac{1}{2n\pi} & n = 4k+1 \\ \frac{1}{2n\pi} & n = 4k+3 \end{cases} = \begin{cases} \frac{3}{4} & n=0 \\ 0 & 0 \neq n = even \\ -\left|\frac{1}{2n\pi}\right| & n = \pm 1, \pm 5, \ldots \\ \left|\frac{1}{2n\pi}\right| & n = \pm 3, \pm 7, \ldots \end{cases}$$

(c) From the solution of Problem P4.6(c), we know that

$$a_0 = \tfrac{1}{2}, \quad a_n = 0, \quad \text{and} \quad b_n = \tfrac{1}{n\pi}.$$

Using Eq. (4.45), the exponential CTFS coefficients for $x3(t)$ are given by

$$D_n = \begin{cases} a_0 & n=0 \\ \frac{1}{2}(a_n - jb_n) & n>0 \\ \frac{1}{2}(a_{-n} + jb_{-n}) & n<0 \end{cases} = \begin{cases} a_0 & n=0 \\ -\frac{1}{2}jb_n & n>0 \\ \frac{1}{2}jb_{-n} & n<0 \end{cases} \qquad [\because\ a_n = a_{-n} = 0]$$

$$= \begin{cases} \frac{1}{2} & n=0 \\ -j\frac{1}{2n\pi} & n>0 \\ -j\frac{1}{2n\pi} & n<0 \end{cases} = \begin{cases} \frac{1}{2} & n=0 \\ \frac{-j}{2n\pi} & n \neq 0 \end{cases}$$

(d) From the solution of Problem P4.6(d), we know that

$$a_0 = \tfrac{1}{2},\ a_n = \tfrac{2}{n^2\pi^2}\left[1-(-1)^n\right] = \begin{cases} 0 & n = even \\ \frac{4}{n^2\pi^2} & n = odd \end{cases}, \text{ and } b_n = 0.$$

Using Eq. (4.45), the exponential CTFS coefficients for $x4(t)$ are given by

$$D_n = \begin{cases} a_0 & n=0 \\ \frac{1}{2}(a_n - jb_n) = \frac{1}{2}a_n & n>0 \\ \frac{1}{2}(a_{-n} + jb_{-n}) = \frac{1}{2}a_{-n} & n<0 \end{cases} = \begin{cases} a_0 & n=0 \\ \frac{1}{2}a_n & n>0 \\ \frac{1}{2}a_{-n} & n<0 \end{cases} \qquad [\because\ b_n = b_{-n} = 0]$$

$$= \begin{cases} \frac{1}{2} & n=0 \\ 0 & 0 \neq n = even \\ \frac{1}{2}\frac{4}{n^2\pi^2} = \frac{2}{n^2\pi^2} & n = odd, n>0 \\ \frac{1}{2}\frac{4}{(-n)^2\pi^2} = \frac{2}{n^2\pi^2} & n = odd, n<0 \end{cases} = \begin{cases} \frac{1}{2} & n=0 \\ 0 & 0 \neq n = even. \\ \frac{2}{n^2\pi^2} & n = odd \end{cases}$$

(e) From the solution of Problem P4.6(e), we know that

$$a_0 = \tfrac{1}{2} - \tfrac{1}{2\pi},\ a_n = \begin{cases} 0 & n = odd \\ \frac{1}{\pi(n^2-1)} & n = even \end{cases}, \text{ and } b_n = \begin{cases} 0 & n = even \\ \frac{2}{n\pi} - \frac{1}{4} & n = 1 \\ \frac{2}{n\pi} & 1 \neq n = odd \end{cases}$$

Using Eq. (4.45), the exponential CTFS coefficients for $x5(t)$ are given by

($n = 0$): $$D_0 = \tfrac{1}{2} - \tfrac{1}{2\pi}$$

($n = 1$): $$D_1 = \tfrac{1}{2}(a_1 - jb_1) = -\tfrac{j}{2}\left(\tfrac{2}{\pi} - \tfrac{1}{4}\right) = j\left(\tfrac{1}{8} - \tfrac{1}{\pi}\right)$$

($n = -1$): $$D_{-1} = \tfrac{1}{2}(a_1 + jb_1) = \tfrac{j}{2}\left(\tfrac{2}{\pi} - \tfrac{1}{4}\right) = -j\left(\tfrac{1}{8} - \tfrac{1}{\pi}\right)$$

($n > 1$): $$D_n = \tfrac{1}{2}(a_n - jb_n) = \begin{cases} -\frac{j}{n\pi} & n = odd \\ \frac{1}{2\pi(n^2-1)} & n = even \end{cases} = \begin{cases} \frac{1}{jn\pi} & n = odd \\ \frac{1}{2\pi(n^2-1)} & n = even \end{cases}$$

($n < -1$): $$D_n = \tfrac{1}{2}(a_{-n} + jb_{-n}) = \begin{cases} \frac{j}{2}\left(\frac{2}{-n\pi}\right) & n = odd \\ \frac{1}{2\pi(n^2-1)} & n = even \end{cases} = \begin{cases} \frac{1}{jn\pi} & n = odd \\ \frac{1}{2\pi(n^2-1)} & n = even \end{cases}$$

Combining the above results, we obtain

$$D_n = \begin{cases} \frac{1}{2}\left(1 - \frac{1}{\pi}\right) & n = 0 \\ \pm j\left(\frac{1}{8} - \frac{1}{\pi}\right) & n = \pm 1 \\ \frac{1}{2\pi(n^2-1)} & 0 \neq n = even \\ \frac{1}{jn\pi} & \pm 1 \neq n = odd. \end{cases}$$

Problem 4.14

Problem 4.11(b) computes the exponential DTFS coefficients of $x2(t)$ as

$$x2(t) \xleftrightarrow{\text{CTFS}} D_n^x = -\frac{0.5}{n\pi}\sin(0.5n\pi)$$

with fundamental frequency $\omega_0 = \pi/T$. Differentiating $x2(t)$ with respect to t, we get

$$\frac{dx2(t)}{dt} = \underbrace{\sum_{k=-\infty}^{\infty} 0.5\delta(t - 0.5T - 2kT)}_{0.5g(t)} - \underbrace{\sum_{m=-\infty}^{\infty} 0.5\delta(t + 0.5T - 2kT)}_{0.5g(t+T)},$$

where the first term $g(t)$ represents an impulse train with period $T_0 = 2T$ and with impulses located at $(T/2 + 2kT)$. Using the time differentiation property,

$$\frac{dx2(t)}{dt} \xleftrightarrow{\text{CTFS}} jn\omega_0 D_n^x = \frac{jn\pi}{T} \times -\frac{0.5}{n\pi}\sin(0.5n\pi) = -j\frac{1}{2T}\sin(0.5n\pi)$$

implying that $$0.5g(t) - 0.5g(t+T) \xleftrightarrow{\text{CTFS}} -j\frac{1}{2T}\sin(0.5n\pi).$$

Using the time shifting property,

$$g(t) - g(t+T) \xleftrightarrow{\text{CTFS}} D_n^g\left(1 - e^{jn\omega_0 T}\right) = D_n^g\left(1 - e^{jn\pi}\right)$$

with D_n^g representing the exponential CTFS coefficients of $g(t)$. Hence,

$$D_n^g\left(1 - e^{jn\pi}\right) = -j\frac{1}{T}\sin(0.5n\pi)$$

or, $$D_n^g = -j\frac{1}{2T}\frac{\sin(0.5n\pi)}{1-e^{jn\pi}} = -j\frac{1}{T}\frac{\sin(0.5n\pi)}{e^{jn\pi/2}(-2j\sin(0.5n\pi))} = \frac{e^{-jn\pi/2}}{2T}.$$ ▌

Problem 4.15

(i) As shown in Problem P4.8(i), $x_1(t)$ is periodic with the overall period $T_0 = 2\pi$ and fundamental frequency $\omega_0 = 1$. The function $x_1(t)$ can be expressed as follows:

$$x_1(t) = \cos(7t) + \cos(15t) = \tfrac{1}{2}e^{j7t} + \tfrac{1}{2}e^{-j7t} + \tfrac{1}{2}e^{j15t} + \tfrac{1}{2}e^{-j15t}.$$

Comparing with the exponential CTFS expansion with $\omega_0 = 1$,

$$x(t) = \sum_{n=1}^{\infty} D_n \exp(jnt),$$

we note that $$D_7 = D_{-7} = 0.5 \quad \text{and} \quad D_{15} = D_{-15} = 0.5.$$

The remaining coefficients are all zero.

(ii) As shown in Problem P4.8(ii), $x_2(t)$ is periodic with the overall period $T_0 = \pi$ and fundamental frequency $\omega_0 = 2$. Expanding

$$x_2(t) = 3 + \sin(2t) + \cos(4t + \pi/4)$$

as $$x_2(t) = 3 + \frac{1}{j2}e^{j2t} - \tfrac{1}{2}e^{-j2t} + \tfrac{1}{2}e^{j\pi/4}e^{j4t} + \tfrac{1}{2}e^{-j\pi/4}e^{-j4t}$$

Comparing with the exponential CTFS expansion with $\omega_0 = 2$,

$$x(t) = \sum_{n=1}^{\infty} D_n \exp(j2nt),$$

we note that

$$D_{-2} = \tfrac{1}{2}e^{-j\pi/4}, \quad D_{-1} = j\tfrac{1}{2}, \quad D_0 = 3, \quad D_1 = -j\tfrac{1}{2} \quad \text{and} \quad D_2 = \tfrac{1}{2}e^{j\pi/4}.$$

The remaining coefficients are all zero.

(iii) As shown in Problem P4.8(iii), $x_3(t)$ is periodic with the overall period $T_0 = 2\pi$ and fundamental frequency $\omega_0 = 1$. Expanding

$$x_3(t) = 1.2 + e^{j2t+1} + e^{j(5t+2)} + + e^{-j(3t+2)}$$

as $$x_3(t) = 1.2 + e \times e^{j2t} + e^{j2} \times e^{j5t} + e^{-j2} \times e^{-j3t}.$$

Comparing with the exponential CTFS expansion with $\omega_0 = 1$,

$$x(t) = \sum_{n=1}^{\infty} D_n \exp(jnt),$$

we note that

$$D_{-3} = e^{-j2}, \quad D_0 = 1.2, \quad D_2 = e, \quad \text{and} \quad D_5 = e^{j2}.$$

The remaining coefficients are all zero.

(i) Since the signal is not periodic because of the exp(t + 1) term, the exponential CTFS expansion cannot be obtained. ▮

Problem 4.16

For the impulse train $$p(t) = \sum_{k=-\infty}^{\infty} \delta(t - 2k\pi) \xleftrightarrow{\text{CTFS}} E_n = \frac{1}{2\pi}$$

with period $T_0 = 2\pi$ and fundamental frequency $\omega_0 = 1$.

Expressing $$\frac{dx(t)}{dt} = \underbrace{\sum_{k=-\infty}^{\infty} \delta\left(t + \tfrac{\pi}{4} - 2k\pi\right)}_{p(t+\pi/4)} - \underbrace{\sum_{k=-\infty}^{\infty} \delta\left(t - \tfrac{\pi}{4} - 2k\pi\right)}_{p(t-\pi/4)},$$

and using the time shifting property, we observe that

$$\frac{dx(t)}{dt} \xleftrightarrow{\text{CTFS}} e^{jn\omega_0\pi/4} E_n - e^{-jn\omega_0\pi/4} E_n \, .$$

Substituting $\omega_0 = 1$, we get $$\frac{dx(t)}{dt} \xleftrightarrow{\text{CTFS}} 2j\sin(0.25n\pi)E_n \, .$$

Using the time differentiation property,

$$jn\omega_0 D_n = 2j\sin(0.25n\pi)E_n \, ,$$

or, $$D_n = \tfrac{2}{n}\sin(0.25n\pi)E_n \, .$$

Substituting $E_n = 1/2\pi$, we get $D_n = \frac{1}{\pi n}\sin(0.25n\pi) = \frac{1}{4} \times \frac{\sin(0.25n\pi)}{0.25\pi n} = \frac{1}{4}\text{sinc}(0.25n)$. ▮

Problem 4.17

Example 4.14 derived the exponential DTFS coefficients of the square wave with the duty cycle (τ/T) as

$$D_n = \tfrac{\tau}{T}\text{sinc}\left(\tfrac{n\tau}{T}\right) \, .$$

(i) For T = 5 ms, the fundamental frequency is $f_0 = 1/T$ = 1/5ms = 200 Hz, while the fundamental angular frequency is $\omega_0 = 2\pi f_0 = 400\pi$ radians/s. With τ = 1ms, the exponential CTFS coefficients are given by

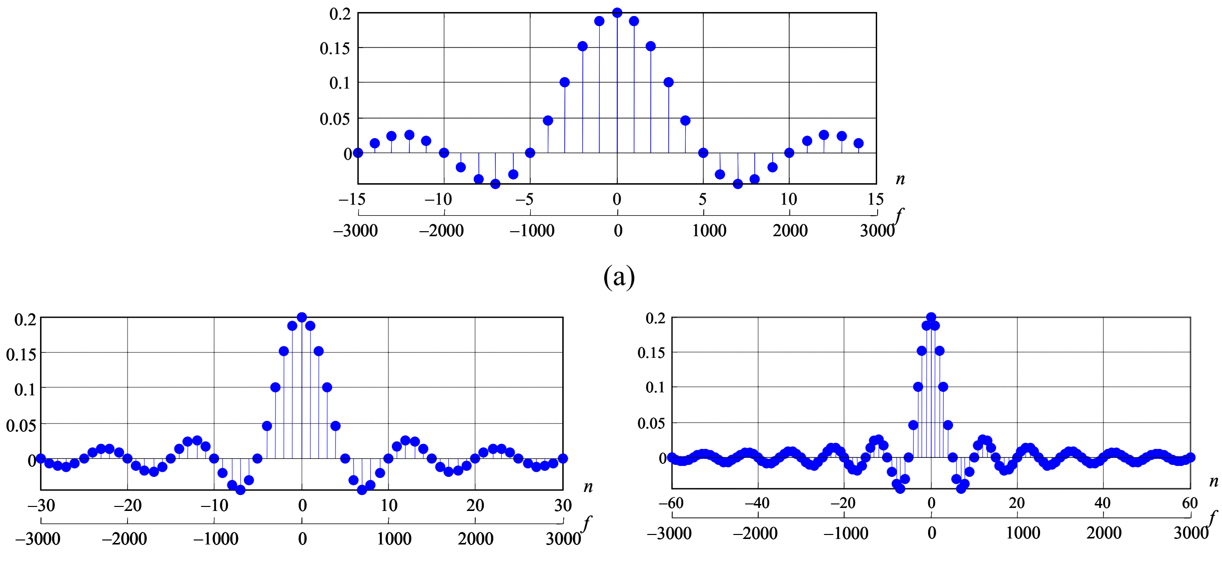
$$D_n = \tfrac{1}{5}\text{sinc}\left(\tfrac{n}{5}\right) \, ,$$

which are plotted in Fig. S4.17(a) in terms of two scales: (a) number n of the CTFS coefficients; and (b) the corresponding frequency $f = nf_0$ in Hz.

(ii) For T = 10 ms, the fundamental frequency is $f_0 = 1/T$ = 1/10ms = 100 Hz, while the fundamental angular frequency is $\omega_0 = 2\pi f_0 = 200\pi$ radians/s. With τ = 2ms, the expression for the exponential CTFS coefficients stay the same as in part (i) and is given by

$$D_n = \tfrac{1}{5}\text{sinc}\left(\tfrac{n}{5}\right) \, ,$$

which are plotted in Fig. S4.17(b) in terms of two scales: (a) number n of the CTFS coefficients; and (b) the corresponding frequency $f = nf_0$ in Hz.



(b) (c)

Fig. S4.17: DTFS coefficients for Problem 4.17.

(iii) Finally, for T = 20 ms, the fundamental frequency is $f_0 = 1/T = 1/20\text{ms} = 50$ Hz, while the fundamental angular frequency is $\omega_0 = 2\pi f_0 = 100\pi$ radians/s. With $\tau = 4$ms, the expression for the exponential CTFS coefficients stay the same as in parts (i) and (ii) and is given by

$$D_n = \tfrac{1}{5}\operatorname{sinc}\left(\tfrac{n}{5}\right),$$

which are plotted in Fig. S4.17(b) in terms of two scales: (a) number n of the CTFS coefficients; and (b) the corresponding frequency $f = nf_0$ in Hz.

From Fig. S4.17, we make the following observations.

DC Coefficient: Keeping the duty cycle (τ/T) of the square wave constant maintains the same dc or average value of the signal. Therefore, the dc coefficient D_0 stays the same for the three representations.

Zero Crossings: Since the duty cycle (τ/T) is kept constant, the width of the main lobe and side lobes of the discrete sinc function stay the same in the discrete (n) domain. A change in the fundamental frequency causes the widths to be different in Hertz. ▮

Problem 4.18

(a) In time domain, the average power of $x1(t)$ is given by

$$P_{x1} = \frac{1}{T}\int_0^T |x1(t)|^2\,dt = \frac{1}{2\pi}\int_0^{\pi} 9\,dt = \frac{9}{2}.$$

Using the Parseval's theorem, the average power of $x1(t)$ is given by

$$P_{x1} = \sum_{n=-\infty}^{\infty}|D_n|^2 = |D_0|^2 + 2\sum_{n=1,3,5,\ldots}^{\infty}|D_n|^2 = 2.25 + \frac{18}{\pi^2}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n^2}.$$

Using the results of Problem 4.21, we know that

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \ldots,$$

which gives
$$P_{x1} = 2.25 + \frac{18}{\pi^2} \times \frac{\pi^2}{8} = 4.5.$$

(b) In time domain, the average power of $x2(t)$ is given by

$$P_{x2} = \frac{1}{2T}\int_0^T |x2(t)|^2 dt = \frac{1}{2T}\left[0.25T + T\right] = 0.625.$$

Using the Parseval's theorem, the average power of $x2(t)$ is given by

$$P_{x2} = \sum_{n=-\infty}^{\infty} |D_n|^2 = |D_0|^2 + 2\sum_{n=1,3,5,\ldots}^{\infty} |D_n|^2 = \frac{9}{16} + \frac{2}{4\pi^2}\sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^2}.$$

Using the results of Problem 4.21, we know that

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \ldots$$

which gives
$$P_{x2} = \frac{9}{16} + \frac{2}{4\pi^2} \times \frac{\pi^2}{8} = \frac{10}{16} = 0.625.$$

(c) In time domain, the average power of $x3(t)$ is given by

$$P_{x3} = \frac{1}{T}\int_0^T |x3(t)|^2 dt = -\frac{T}{3T}\left[(1 - t/T)^3\right]_0^T = \frac{1}{3}.$$

Using the Parseval's theorem, the average power of $x3(t)$ is given by

$$P_{x3} = \sum_{n=-\infty}^{\infty} |D_n|^2 = |D_0|^2 + 2\sum_{n=1,2,3,\ldots}^{\infty} |D_n|^2 = \frac{1}{4} + \frac{2}{4\pi^2}\sum_{n=1,2,3,\ldots}^{\infty} \frac{1}{n^2}.$$

Using the results of Problem 4.23, we know that

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \ldots$$

which gives
$$P_{x3} = \frac{1}{4} + \frac{2}{4\pi^2} \times \frac{\pi^2}{6} = \frac{1}{3}.$$

(d) In time domain, the average power of $x4(t)$ is given by

$$P_{x4} = \frac{1}{2T}\int_{-T}^{T} |x4(t)|^2 dt = \frac{1}{T}\int_0^T (1 - t/T)^2 dt = -\frac{T}{3T}\left[(1 - t/T)^3\right]_0^T = \frac{1}{3}.$$

Using the Parseval's theorem, the average power of $x2(t)$ is given by

$$P_{x4} = \sum_{n=-\infty}^{\infty} |D_n|^2 = |D_0|^2 + 2\sum_{n=1,3,5,\ldots}^{\infty} |D_n|^2 = \frac{1}{4} + \frac{8}{\pi^4}\sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^4}.$$

Using the result $$1+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\frac{1}{9^4}+\ldots=1.0147=\frac{\pi^2}{96}$$

which gives $$P_{x4}=\frac{1}{4}+\frac{8}{\pi^2}\times\frac{\pi^2}{96}=\frac{1}{3}.$$ ▮

Problem 4.19

(a) Within one period $t = [0, 2\pi]$, function $x1(t)$ is absolutely integrable as

$$\int_0^{2\pi}|x1(t)|dt=\int_0^{\pi}3dt=3\pi.$$

Function $x1(t)$ has only one maxima and one minima within one period, hence, has bounded variations.

Finally, there are only two discontinuities within one period.

Function $x1(t)$ satisfies the Dirichlet conditions.

(b) Within one period $t = [0, 2T]$, function $x2(t)$ is absolutely integrable as

$$\int_0^{2T}|x2(t)|dt=1.5T.$$

Function $x2(t)$ has only one maxima at and two minimas within one period $t = [0, T]$, hence, has bounded variations.

Finally, there are only two discontinuities $t = T/2$ and $3T/2$ within one period $t = [0, T]$.

Function $x2(t)$ satisfies the Dirichlet conditions.

(c) Within one period $t = [0, T]$, function $x3(t)$ is absolutely integrable as

$$\int_0^{T}|x3(t)|dt=\frac{T}{2}.$$

Function $x3(t)$ has only one minima and one maxima within one period $t = [0, T]$, hence, has bounded variations.

Finally, there are only one discontinuity at $t = 0$ within one period $t = [0, T]$.

Function $x3(t)$ satisfies the Dirichlet conditions.

(d) Within one period $t = [0, 2T]$, function $x4(t)$ is absolutely integrable as

$$\int_0^{2T}|x4(t)|dt=T.$$

Function $x4(t)$ has only one minima and one maxima within one period $t = [0, 2T]$, hence, has bounded variations.

Finally, there is no discontinuity within one period $t = [0, 2T]$.

Function $x4(t)$ satisfies the Dirichlet conditions.

(e) Within one period $t = [0, 2T]$, function $x5(t)$ is absolutely integrable as

$$\int_0^{2T} |x5(t)| dt = \frac{(\pi - 1)T}{2\pi}.$$

Function $x5(t)$ has only one minima and two maximas within one period t = [0, 2 T], hence, has bounded variations.

Finally, there is no discontinuity within one period t = [0, 2 T].

Function $x5(t)$ satisfies the Dirichlet conditions. ▌

Problem 4.20

Determine if the following functions satisfy the Dirichlet conditions and have CTFS representation.

(i) $x(t) = 1/t$, $t = (0, 2]$ and $x(t) = x(t+2)$;

(ii) $g(t) = \cos(\pi / 2t)$, $t = (0, 1]$ and $g(t) = g(t+1)$;

(iii) $h(t) = \sin(\ln(t))$, $t = (0, 1]$ and $h(t) = h(t+1)$.

Solution:

(i) $$\int_0^2 |x(t)| dt = \int_0^2 \frac{1}{t} dt = \left| -\frac{1}{2t^2} \right|_0^2 = \infty$$

As the function $x(t)$ is not absolutely integrable, $x(t)$ does not satisfy the Dirichlet conditions.

(ii) As shown in Fig. S4.20 (top plot), function $g(t)$ has an infinite number of maximas and minimas in one period. Therefore, $g(t)$ does not satisfy the Dirichlet conditions.

(ii) As shown in Fig. S4.20 (bottom plot), function $h(t)$ appears to satisfy the Dirichlet conditions. However, Matlab is not able to plot all the peaks because of its limited resolution. When $t = (0,1]$, $\ln(t) = (-\infty, 0]$ and is a CT function. The function $\sin(\ln(t))$ will have a maxima every 2π interval of $\ln(t)$ implying that the total number of maxima's are infinite. The function $h(t)$ therefore does not satisfy the Dirichlet conditions.

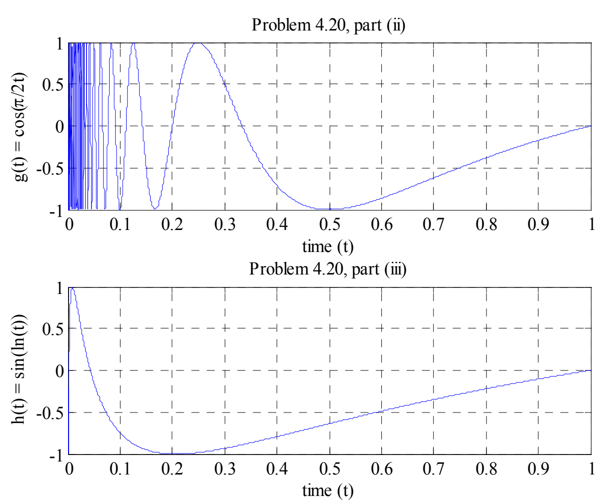


Fig. S4.20: One period of the functions $g(t)$ and $h(t)$ in Problem 4.20(ii) and (iii).

Problem 4.21

Example 4.9 derived the trigonometric CTFS coefficients of the triangular wave $f(t)$, shown in Fig. S4.21, as follows

$$f(t)=\sum_{n=1,3,5,\ldots}^{\infty}\frac{24}{(n\pi)^2}\cos(0.5n\pi t)=\frac{24}{\pi^2}\left[\cos(0.5\pi t)+\frac{1}{3^2}\cos(1.5\pi t)+\frac{1}{5^2}\cos(2.5\pi t)+\frac{1}{7^2}\cos(3.5\pi t)+\cdots\right]$$

Substituting ($t = 0$) on both sides, we get

$$f(0)=\frac{24}{\pi^2}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n^2}=\frac{24}{\pi^2}\left[1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{9^2}+\ldots\right].$$

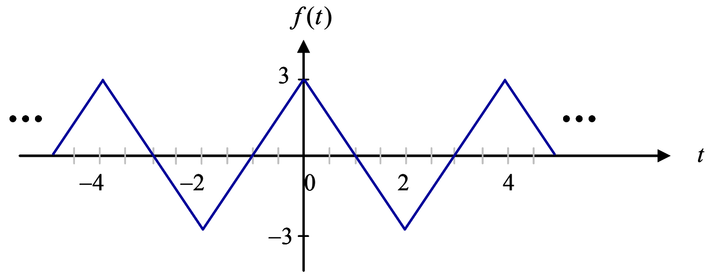


Figure S4.21: Periodic signal $f(t)$ considered in Problem 4.21.

From Fig. S4.21, we note that $f(0)=3$.

Equating the above two equations, we obtain

$$\frac{\pi^2}{8}=\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n^2}=1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{9^2}+\ldots$$ ▌

Problem 4.22

From the solution of Problem 4.6(c), we know that the trigonometric CTFS expansion of the half sawtooth wave is given by

$$x3(t)=\frac{1}{2}+\sum_{n=1}^{\infty}\frac{1}{n\pi}\sin(2n\pi t/T)$$

Substituting $t = T/4$, we get $x3(T/4)=\frac{1}{2}+\sum_{n=1}^{\infty}\frac{1}{n\pi}\sin(n\pi/2)=\frac{1}{2}+\frac{1}{\pi}\left[1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+-\ldots\right]$

Since $x3(T/4) = (1 - (T/4)/T) = 0.75$, therefore,

$$0.75=\frac{1}{2}+\frac{1}{\pi}\left[1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+-\ldots\right]$$

which implies that $\frac{\pi}{4}=\sum_{n=1}^{\infty}\frac{1}{n}\times(-1)^{n-1}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+-\ldots$ ▌

Problem 4.23

From the solution of Problem 4.11(c), we know that the exponential CTFS expansion of the half sawtooth wave is given by

$$D_n = \begin{cases} \frac{1}{2}, & n = 0 \\ \frac{1}{j2n\pi}, & n \neq 0. \end{cases}$$

Computing the power from the exponential CTFS coefficients, we get

$$P_x = \sum_{n=-\infty}^{\infty} |D_n|^2 = \frac{1}{4} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{4n^2\pi^2} = \frac{1}{4} + 2\sum_{n=1}^{\infty} \frac{1}{4n^2\pi^2} = \frac{1}{4} + \frac{1}{2\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Computing the power in the time domain, we obtain

$$P_x = \frac{1}{T}\int_0^T |x3(t)|^2 dt = \frac{1}{T}\int_0^T \left(1 - \frac{t}{T}\right)^2 dt = \frac{1}{T} \times (-T/3) \left|\left(1 - \frac{t}{T}\right)^3\right|_0^T = \frac{1}{T} \times (-T/3) \times (0-1) = \frac{1}{3}.$$

Equating the two expressions for the power

$$\frac{1}{3} = \frac{1}{4} + \frac{1}{2\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2}$$

or,

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \ldots$$

Problem 4.24

(i) The transfer function $H(\omega)$ is given by

$$\begin{aligned} H(\omega) &= \int_{-\infty}^{\infty} e^{-2|t|} e^{-j\omega t} dt = \int_0^{\infty} e^{-2t} e^{-j\omega t} dt + \int_{-\infty}^{0} e^{2t} e^{-j\omega t} dt \\ &= \frac{1}{(-2-j\omega)} e^{-(2+j\omega)t}\Big|_0^{\infty} + \frac{1}{(2+j\omega)} e^{(2-j\omega)t}\Big|_{-\infty}^{0} \\ &= \frac{1}{(-2-j\omega)} \times [0-1] + \frac{1}{(2+j\omega)} \times [1-0] = \frac{4}{4+\omega^2} \end{aligned}$$

(ii) Since the transfer function $H(\omega)$ is real valued, therefore, its magnitude spectrum

$$|H(\omega)| = H(\omega) = \frac{4}{4+\omega^2}.$$

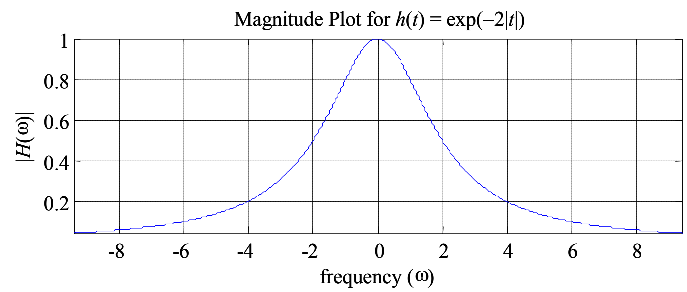


Fig. S4.24: Magnitude spectrum for $h(t) = \exp(-2|t|)$

The magnitude spectrum $|H(\omega)|$ is shown in Fig. S4.24.

(iii) The exponential CTFS coefficients E_n of the output signal $y(t)$ are given by $E_n = D_n H(\omega_0)$ where $\omega_0 = 2\pi/T$ and D_n are the exponential CTFS coefficients for the input signal. As found in P4.12, the exponential CTFS coefficients for the input impulse train are given by:

$$D_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)e^{-jn\omega_0 t}dt = \frac{1}{T}\int_{-T/2}^{T/2}\delta(t)e^{-jnt}dt = \frac{1}{T}$$

The exponential DTFS coefficients E_n are then given by

$$E_n = \frac{1}{T}\left[\frac{4}{4+\omega^2}\right]_{\omega=2n\pi/T} = \left[\frac{4T}{4T^2+(2n\pi)^2}\right].$$

In the time domain, the output signal is expressed as

$$y(t) = \sum_{n=-\infty}^{\infty} E_n e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty}\frac{T}{T^2+(n\pi)^2}e^{j2n\pi t/T}.$$

Problem 4.25

(i) The transfer function $H(\omega)$ is given by

$$H(\omega) = \int_0^{\infty}\left(e^{-2t}-e^{-4t}\right)e^{-j\omega t}dt = \int_0^{\infty} e^{-(2+j\omega)t}dt - \int_0^{\infty} e^{-(4+j\omega)t}dt$$

$$= \frac{-1}{(2+j\omega)}e^{-(2+j\omega)t}\Big|_0^{\infty} + \frac{-1}{(4+j\omega)}e^{-(4+j\omega)t}\Big|_0^{\infty}$$

$$= \frac{-1}{(2+j\omega)}\times[0-1] - \frac{-1}{(4+j\omega)}\times[0-1] = \frac{2}{(2+j\omega)(4+j\omega)}$$

(ii) The magnitude response is given by

$$|H(\omega)| = \frac{2}{\sqrt{\left(4+\omega^2\right)\left(16+\omega^2\right)}}.$$

The magnitude spectrum $|H(\omega)|$ is shown in Fig. S4.25.

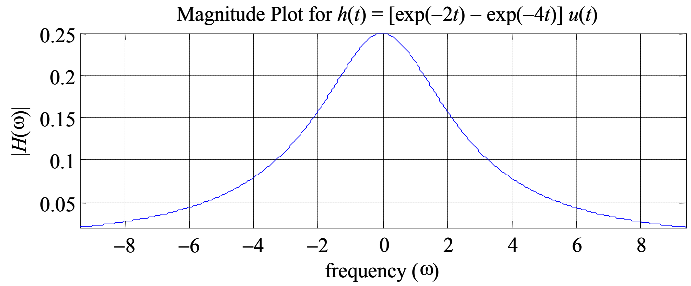


Fig. S4.25: Magnitude spectrum for $h(t) = [\exp(-2t) - \exp(-4t)]\, u(t)$.

(iii) The exponential CTFS coefficients E_n of the output signal $y(t)$ are given by $E_n = D_n H(\omega_0)$ with $\omega_0 = \pi/T$. For the raised cosine wave, the exponential CTFS coefficients D_n are given by

$$D_n = \begin{cases} 0.75 & n = 0 \\ -\dfrac{0.5}{n\pi}\sin(0.5n\pi) & n \neq 0 \end{cases}.$$

Therefore, the CTFS coefficients E_n of the output signal $y(t)$ are given by

$$E_n = \left[\frac{2}{(2+j\omega)(4+j\omega)}\right]_{\omega = n\pi/T} \times \begin{cases} 0.75 & n = 0 \\ -\dfrac{0.5}{n\pi}\sin(n\pi/2) & n \neq 0 \end{cases}$$

$$= \begin{cases} 1/4 & n = 0 \\ \dfrac{2T^2}{(2T + jn\pi)(4T + jn\pi)} & n \neq 0 \end{cases} \times \begin{cases} 3/4 & n = 0 \\ -\dfrac{0.5}{n\pi}\sin(n\pi/2) & n \neq 0 \end{cases}$$

$$= \begin{cases} 3/16 & n = 0 \\ -\dfrac{T^2 \sin(n\pi/2)}{n\pi(2T + jn\pi)(4T + jn\pi)} & n \neq 0. \end{cases}$$

In the time domain, the output signal is expressed as

$$y(t) = \sum_{n=-\infty}^{\infty} E_n e^{jn\omega_0 t} = \frac{3}{16} - \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{T^2 \sin(0.5n\pi)}{n\pi(2T + jn\pi)(4T + jn\pi)} e^{jn\pi t/T}.$$

Problem 4.26

(i) The transfer function $H(\omega)$ is given by

$$H(\omega) = \int_0^\infty t e^{-4t} e^{-j\omega t} dt = \int_0^\infty t e^{-(4+j\omega)t} dt = t \left.\frac{e^{-(4+j\omega)t}}{-(4+j\omega)}\right|_0^\infty - \left.\frac{e^{-(4+j\omega)t}}{(4+j\omega)^2}\right|_0^\infty$$

$$= \frac{-1}{(4+j\omega)} \times [0-0] - \frac{1}{(4+j\omega)^2} \times [0-1] = \frac{1}{(4+j\omega)^2}.$$

(ii) The magnitude response is given by

$$|H(\omega)| = \frac{1}{\sqrt{(16+\omega^2)(16+\omega^2)}} = \frac{1}{16+\omega^2}.$$

The magnitude spectrum $|H(\omega)|$ is shown in Fig. S4.26.

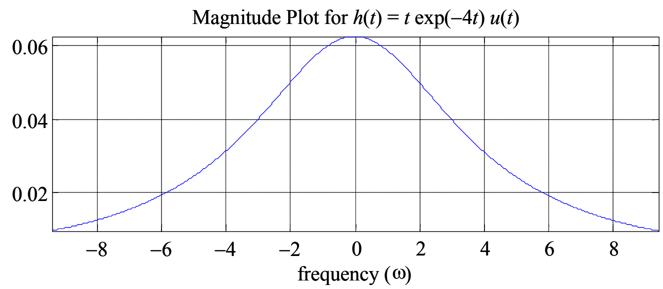


Fig. S4.26: Magnitude spectrum for $h(t) = t\exp(-4t)\,u(t)$.

(iii) The exponential CTFS coefficients E_n of the output signal $y(t)$ are given by $E_n = D_n H(\omega_0)$ with $\omega_0 = \pi/T$. For the sawtooth wave, the exponential CTFS coefficients D_n are given by

$$D_n = \begin{cases} \frac{1}{2}, & n = 0 \\ 0, & \text{even } n, n \neq 0 \\ \frac{2}{(n\pi)^2} & \text{odd } n, n \neq 0. \end{cases}$$

Therefore, the CTFS coefficients E_n of the output signal $y(t)$ are given by

$$E_n = \left[\frac{1}{(4+j\omega)^2}\right]_{\omega=2n\pi/T} \times \begin{cases} \frac{1}{2}, & n=0 \\ 0, & \text{even } n \\ \frac{2}{(n\pi)^2} & \text{odd } n \end{cases} = \begin{cases} \frac{1}{32}, & n=0 \\ 0, & \text{even } n \\ \dfrac{2T^2}{(n\pi)^2(4T+j2n\pi)^2} & \text{odd } n. \end{cases}$$

In the time domain, the output signal is expressed as

$$y(t) = \sum_{n=-\infty}^{\infty} E_n e^{jn\omega_0 t} = \frac{1}{32} + \sum_{\substack{n=-\infty \\ n=odd}}^{\infty} \frac{2T^2}{(n\pi)^2(4T+j2n\pi)^2} e^{jn\pi t/T}.$$ ▌

Problem 4.27

(i) (a) Expressing

$$x_1(t) = \frac{7}{\pi}\sum_{m=0}^{\infty}\frac{1}{2m+1}\sin[8\pi(2m+1)t] = \frac{7}{\pi}\left[\sin(8\pi t) + \frac{1}{3}\sin(24\pi t) + \frac{1}{5}\sin(40\pi t) + \ldots\right]$$

we note that the signal $x_1(t)$ contains the fundamental component $\sin(8\pi t)$ and its harmonics. Therefore, the signal is periodic, and the fundamental frequency for $x_1(t)$ is given by $\omega_0 = 8\pi$ radian/sec. The fundamental period is $T_0 = 2\pi/\omega_0 = 0.25$ sec.

(b) Since $x_1(-t) = \frac{7}{\pi}\sum_{m=0}^{\infty}\frac{1}{2m+1}\sin[-8\pi(2m+1)t] = -\frac{7}{\pi}\sum_{m=0}^{\infty}\frac{1}{2m+1}\sin[8\pi(2m+1)t] = -x_1(t)$,

the signal is odd.

(c) The following MATLAB code is used to reconstruct the function in the time domain. The number n of harmonics is set to 4000.

```
% initializing CTFS parameters
nterms = 4000;
w0 = 8*pi;
t = -1:0.001:1;
a0 = 0;
an = zeros(1,nterms);
nnz = 1:2:nterms;
bn2d = zeros(2,nterms/2);
bn2d(1,:) = 1./nnz;
bn = reshape(bn2d,1,nterms);
% calculating time-domain function
x1 = (7/pi)* ictfs(w0,t, a0,an,bn);
plot(t,x1);
xlabel('t');
ylabel('x1(t)');
axis([-1  1  -3  3]), grid on;
title ('Reconstruction from CTFS')
```

(d) The resulting waveform is shown in Fig. S4.27.

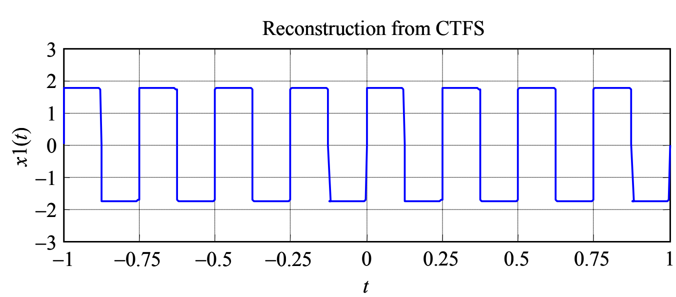


Fig. S4.27: Signal $x1(t)$ reconstructed from the first 4000 trigonometric CTFS coefficients in Problem 4.27(a).

(ii) (a) Expressing

$$x_2(t) = 1.5 + \sum_{m=0}^{\infty}\frac{1}{4m+1}\cos[2\pi(4m+1)t] = 1.5 + \left[\cos(2\pi t) + \frac{1}{5}\cos(10\pi t) + \frac{1}{9}\sin(18\pi t) + \ldots\right]$$

we note that the signal $x_2(t)$ contains the fundamental component $\cos(2\pi t)$ and its harmonics. Therefore, the signal is periodic, and the fundamental frequency for $x_2(t)$ is given by $\omega_0 = 2\pi$ radian/sec. The fundamental period is $T_0 = 2\pi/\omega_0 = 1$ sec.

(b) Since $x_2(-t) = 1.5 + \sum_{m=0}^{\infty} \frac{1}{4m+1}\cos[-2\pi(4m+1)t] = 1.5 + \sum_{m=0}^{\infty} \frac{1}{4m+1}\cos[2\pi(4m+1)t] = x_2(t)$,

the signal is even.

(c) The following MATLAB code is used to reconstruct the function in the time domain. The number n of harmonics is set to 4000.

```
% initializing CTFS parameters
nterms = 4000 ;
w0 = 2*pi ;
t = -4:0.001:4 ;
a0 = 1.5 ;
nnz = 1:4:nterms;
an2d = zeros(4,nterms/4);
an2d(1,:) = 1./nnz ;
an = reshape(an2d,1,nterms)  ;
bn = zeros(1,nterms) ;
% calculating time-domain function
x2 = ictfs(w0,t,a0,an,bn);
plot(t,x2)
xlabel('t');
ylabel('x2(t)');
axis([-2  2  -2  5]), grid on
title ('Signal Reconstruction from CTFS')
```

(d) The resulting waveform is shown in Fig. S4.27.

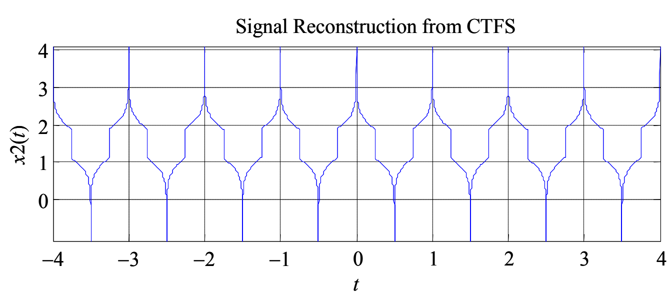


Fig. S4.27: Signal $x2(t)$ reconstructed from the first 4000 trigonometric CTFS coefficients in Problem 4.27(b).

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Problem 4.28

From Example 4.8, the CTFS coefficients are given by

$$a_0 = 1.7079\,,\ a_n = \frac{3.4157}{1+25n^2}\,,\ \text{and}\ b_n = \frac{17.0787n}{1+25n^2}\,.$$

The periodic signal $g(t)$ is, therefore, given by

$$g(t) = 1.7079 + \sum_{n=1}^{\infty} \frac{3.4157}{1+25n^2}\cos(nt) + \sum_{n=1}^{\infty} \frac{17.0787}{1+25n^2} n\sin(nt)$$

with the fundamental frequency $\omega_o = 1$ radians/s.

The following MATLAB code is used to reconstruct the function in the time domain. The number n of harmonics is set to 4000.

```
% initializing CTFS parameters
nterms = 2000 ;
n = 1:nterms;
w0 = 1 ;
t = -12:0.01:12 ;
a0=1.7079 ;
an = 3.4157./(1+25*n.*n) ;
bn = 17.0787*n./(1+25*n.*n) ;
% calculating time-domain function
g = ictfs(w0,t, a0,an,bn) ;
% plotting the function
plot(t,g)
xlabel('t');
ylabel('g(t)');
axis([-12  12  0  4]), grid on
title ('Reconstruction of g(t) from CTFS')
```

The resulting waveform is shown in Fig. S4.28. It is observed that the plot is identical to that of Fig. 4.10.

▌

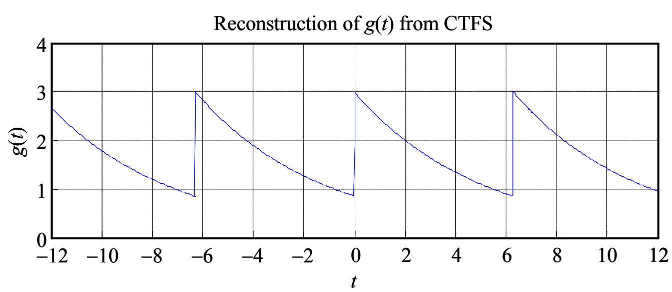

Fig. S4.28: Signal $g(t)$ reconstructed from the first 2000 trigonometric CTFS coefficients in Problem 4.28.

Problem 4.29

From Example 4.9, the CTFS coefficients are given by

$$a_0 = 0,\ a_n = \begin{cases} 0 & n \text{ is even} \\ \frac{24}{(n\pi)^2} & n \text{ is odd.} \end{cases},\ \text{and } b_n = 0.$$

The periodic signal $f(t)$ is, therefore, given by

$$f(t) = \sum_{n=1}^{\infty} \frac{24}{(n\pi)^2}\cos(0.5n\pi t) = \frac{24}{\pi^2}\left[\cos(0.5\pi t) + \frac{1}{9}\cos(1.5\pi t) + \frac{1}{25}\cos(2.5\pi t) + \cdots\right]$$

with the fundamental frequency $\omega_o = 0.5\pi$ radians/s.

The following MATLAB code is used to reconstruct the function in the time domain. The number n of harmonics is set to 2000.

```
% initializing CTFS parameters
nterms = 2000 ;
an = zeros(1,nterms);
nnz = 1:2:nterms;
w0 = 0.5*pi ;
t = -8:0.01:8 ;
a0=0 ;
an2d = zeros(2,nterms/2);
an2d(1,:) = 24./(pi*pi*nnz.*nnz) ;
an=reshape(an2d,1,nterms) ;
bn = zeros(1,nterms);
% calculating time-domain function
x = ictfs(w0,t, a0,an,bn) ;
% plotting the function
plot(t,x)
xlabel('t');
ylabel('f(t)');
axis([-8  8  -4  4]), grid on
title ('Reconstruction of f(t) from CTFS')
```

The resulting waveform is shown in Fig. S4.29. It is observed that the plot is identical to that of Fig. 4.11.

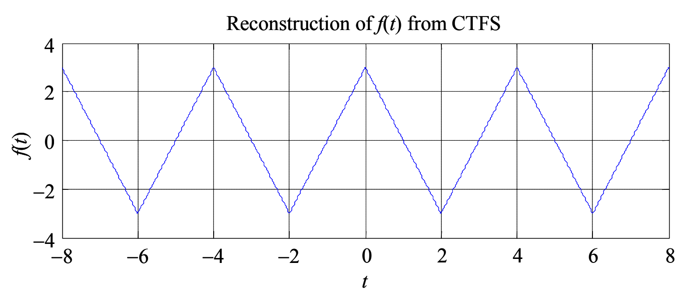

Fig. S4.29: Signal $f(t)$ reconstructed from the first 2000 trigonometric CTFS coefficients in Problem 4.29.

Problem 4.30

From Example 4.12, the CTFS coefficients are given by

$$D_n \approx \frac{0.3416}{0.2+jn}.$$

The periodic signal $g(t)$ is, therefore, given by

$$g(t) = \sum_{n=-\infty}^{\infty} \frac{0.3416}{0.2+jn} \exp\left(jn\omega_o t\right)$$

with the fundamental frequency $\omega_o = 1$ radians/s.

The following MATLAB code is used to reconstruct the function in the time domain. The number n of harmonics is set to 4000.

```
% initializing CTFS parameters
nterms = 4000 ;
n=(-nterms/2):nterms/2;
dn = 0.3416./(0.2+i*n);
nnz = 1:2:nterms;
w0 = 1;
t = -12:0.01:12 ;
% calculating time-domain function
g = ictfs(w0,t,dn) ;
% plotting the function
plot(t,g)
xlabel('t');
ylabel('g(t)');
axis([-12  12  0  4]), grid on
title ('Reconstruction of g(t) from CTFS')
```

The resulting waveform is shown in Fig. S4.30. It is observed that the plot is identical to that of Fig. 4.10.

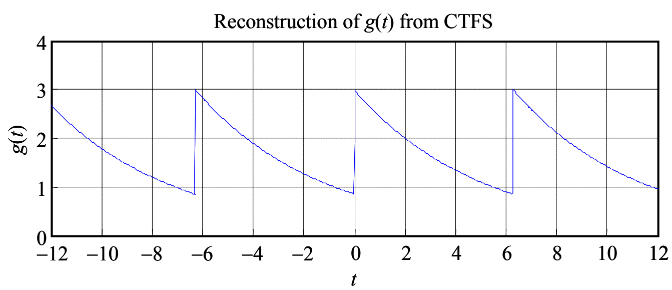

Fig. S4.30: Signal $g(t)$ reconstructed from the first 4000 exponential CTFS coefficients in Problem 4.30.

Problem 4.31

From Example 4.13, the CTFS coefficients are given by

$$D_n = \begin{cases} 0 & n = \text{even} \\ \frac{12}{(n\pi)^2} & n = \text{odd}. \end{cases}.$$

The periodic signal $f(t)$ is, therefore, given by

$$f(t) = \sum_{\substack{n=-\infty \\ n \text{ is odd}}}^{\infty} \frac{12}{(n\pi)^2} \exp(jn\omega_o t)$$

with the fundamental frequency $\omega_o = \pi/2$ radians/s.

The following MATLAB code is used to reconstruct the function in the time domain. The number n of harmonics is set to 4000.

```
% initializing CTFS parameters
nterms = 4000 ;
w0 = 0.5*pi ;
t = -8:0.01:8 ;
nnz = 1:2:nterms;
dn2d = zeros(2,nterms/2);
dn2d(2,:) = 12./(pi*pi*nnz.*nnz) ;
dn=reshape(dn2d,1,nterms) ;
dn = [fliplr(dn(2:length(dn))), dn];
% calculating time-domain function
f = ictfs(w0,t, dn) ;
% plotting the function
plot(t,f)
xlabel('t');
ylabel('f(t)');
axis([-8  8  -4  4]), grid on
title ('Signal Reconstruction from CTFS')
```

The resulting waveform is shown in Fig. S4.31. It is observed that the plot is identical to that of Fig. 4.11.

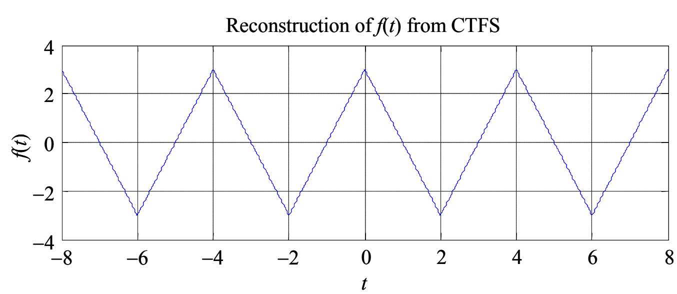

Fig. S4.31: Signal $f(t)$ reconstructed from the first 4000 exponential CTFS coefficients in Problem 4.31.

Problem 4.32

From the solution of Problem 4.24, the exponential CTFS coefficients are given by

$$E_n = \left[\frac{T}{T^2 + n^2\pi^2}\right]$$

with the time domain representation

$$y(t) = \sum_{n=-\infty}^{\infty} \frac{T}{T^2 + n^2\pi^2} e^{j2n\pi t/T}$$

where $\omega_0 = 2\pi/T = 2\pi$ radians/s

The following MATLAB code is used to reconstruct the function in the time domain. The number n of harmonics is set to 4000.

```
% initializing CTFS parameters
nterms = 4000 ;
T = 1;
w0 = 2*pi/T;
t = -6:0.01:6;
nnz = 0:nterms;
en = (4*T)./(4*T^2 + (nnz*pi).^2);
en = [fliplr(en(2:length(en))), en];
% calculating time-domain function
y = ictfs(w0,t, en) ;
% plotting the function
plot(t,y)
xlabel('t');
ylabel('y(t)');
axis([-6  6  0  2.5]), grid on
title ('Signal Reconstruction from CTFS')
print -dtiff plot.tiff;
```

The resulting waveform is shown in Fig. S4.32.

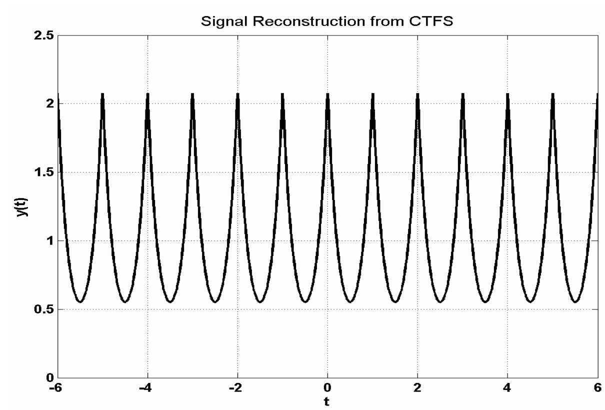

Fig. S4.32: Signal $y(t)$ reconstructed from the first 4000 exponential CTFS coefficients in Problem 4.32.

Problem 4.33

From the solution of Problem 4.25, the exponential CTFS coefficients are given by

$$E_n = \begin{cases} 3/16 & n = 0 \\ -\dfrac{T^2 \sin(n\pi/2)}{n\pi(2T + jn\pi)(4T + jn\pi)} & n \neq 0. \end{cases}$$

with the time domain representation

$$y(t) = \sum_{n=-\infty}^{\infty} E_n e^{jn\omega_0 t} = \frac{3}{16} - \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{T^2 \sin(n\pi/2)}{n\pi(2T + jn\pi)(4T + jn\pi)} e^{jn\pi t/T}$$

where $\omega_o = 2\pi/T = 2\pi$ radians/s

The following MATLAB code is used to reconstruct the function in the time domain. The number n of harmonics is set to 4000.

```
% initializing CTFS parameters
nterms = 4000 ;
T = 1;
w0 = 2*pi/T;
t = -6:0.01:6;
n = -nterms:nterms;
en = -(T^2*sin(0.5*n*pi))./((n+eps)*pi.*(2*T+j*n*pi).*(4*T+j*n*pi));
en(n == 0) = 3/16;
% calculating time-domain function
y = ictfs(w0,t, en) ;
% plotting the function
plot(t,real(y))     % imaginary part of y(t) is 0
xlabel('t');
ylabel('y(t)');
axis([-6  6  0.12  0.26]), grid on
title ('Signal Reconstruction from CTFS')
print -dtiff plot.tiff;
```

The resulting waveform is shown in Fig. S4.33.

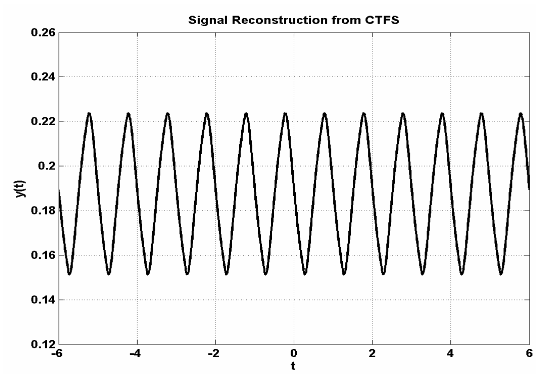

Fig. S4.33: Signal $y(t)$ reconstructed from the first 4000 exponential CTFS coefficients in Problem 4.33.