Chapter 2: Introduction to Systems

Problem 2.1

(i) The currents flowing out of node 1 along resistors R_1, R_2, and capacitor C, are given by

$$i_{R1} = \frac{y(t) - v(t)}{R_1}, \qquad i_{R2} = \frac{y(t)}{R_2}, \qquad i_C = C\frac{dy}{dt}$$

Applying the Kirchoff's current law to node 1 and summing up all the currents, gives

$$\frac{y(t) - v(t)}{R_1} + \frac{y(t)}{R_2} + C\frac{dy}{dt} = 0$$

or,

$$C\frac{dy}{dt} + \left(\frac{1}{R_1} + \frac{1}{R_2}\right)y(t) = \frac{v(t)}{R_1}$$

or,

$$\frac{dy}{dt} + \frac{R_1 + R_2}{CR_1R_2}y(t) = \frac{1}{CR_1}v(t). \tag{S2.1}$$

(ii)

(a) **Linear:** For $v_1(t)$ applied as the input, the output $y_1(t)$ is given by

$$\frac{dy_1}{dt} + \frac{R_1 + R_2}{CR_1R_2}y_1(t) = \frac{1}{CR_1}v_1(t) \quad \Rightarrow \quad v_1(t) = CR_1\frac{dy_1}{dt} + \frac{R_1 + R_2}{R_2}y_1(t)$$

For $v_2(t)$ applied as the input, the output $y_2(t)$ is given by

$$\frac{dy_2}{dt} + \frac{R_1 + R_2}{CR_1R_2}y_2(t) = \frac{1}{CR_1}v_2(t) \quad \Rightarrow \quad v_2(t) = CR_1\frac{dy_2}{dt} + \frac{R_1 + R_2}{R_2}y_2(t).$$

For $v_3(t) = \alpha\, v_1(t) + \beta\, v_2(t)$ applied as the input, the output $y_3(t)$ is given by

$$\frac{dy_3}{dt} + \frac{R_1 + R_2}{CR_1R_2}y_3(t) = \frac{1}{CR_1}\left(\alpha v_1(t) + \beta v_2(t)\right).$$

Substituting the values of $v_1(t)$ and $v_2(t)$ from the earlier equations, we get

$$\frac{dy_3}{dt} + \frac{R_1 + R_2}{CR_1R_2}y_3(t) = \alpha\left[\frac{dy_1}{dt} + \frac{R_1 + R_2}{CR_1R_2}y_1(t)\right] + \beta\left[\frac{dy_2}{dt} + \frac{R_1 + R_2}{CR_1R_2}y_2(t)\right].$$

Rearranging the terms on the right hand side of the equation, we get

$$\frac{dy_3}{dt} + \frac{R_1 + R_2}{CR_1R_2}y_3(t) = \frac{d\left(\alpha y_1 + \beta y_2\right)}{dt} + \frac{R_1 + R_2}{CR_1R_2}\left(\alpha y_1(t) + \beta y_2(t)\right),$$

which implies that

$$y_3(t) = \alpha y_1(t) + \beta y_2(t).$$

The system is, therefore, linear.

(b) **Time-invariance:** For $v(t - t_0)$ applied as the input, the output $y_1(t)$ is given by

$$\frac{dy_1}{dt} + \frac{R_1 + R_2}{CR_1R_2} y_1(t) = \frac{1}{CR_1} v(t - t_0) .$$

Substituting $\tau = t - t_0$ (which implies that $dt = d\tau$), we get

$$\frac{dy_1(\tau + t_0)}{d\tau} + \frac{R_1 + R_2}{CR_1R_2} y_1(\tau + t_0) = \frac{1}{CR_1} v(\tau) .$$

Comparing with Eq. (S2.1), we get

$$y(\tau) = y_1(\tau + t_0) \quad \text{or,} \quad y_1(\tau) = y(\tau - t_0) ,$$

proving that the system is time-invariant.

(c) **Memoryless**: Express Eq. (S2.1) as

$$y(t) = \frac{1}{CR_1} \int_{-\infty}^{t} v(\tau) d\tau - \frac{R_1 + R_2}{CR_1R_2} \int_{-\infty}^{t} y(\tau) d\tau .$$

The output $y(t)$ at $t = t_0$ is given by

$$y(t)\big|_{t=t_0} = \frac{1}{CR_1} \int_{-\infty}^{t_0} v(\tau) d\tau - \frac{R_1 + R_2}{CR_1R_2} \int_{-\infty}^{t_0} y(\tau) d\tau$$

From the first integral on the right hand side of the equation, it is clear that all previous values of the input $v(t)$, for $-\infty \le t \le t_0$, are needed to calculate the output $y(t)$ at $t = t_0$. The system has, therefore, a memory and is not memoryless.

(d) **Causal**: From the previous result, we deduce that the system is causal since only the past values of the input $v(t)$, for $-\infty \le t \le t_0$, are needed to calculate the output $y(t)$ at $t = t_0$.

(e) **Invertible:** The system is invertible as $v(t)$ can be determined from the following relationship

$$v(t) = CR_1 \frac{dy}{dt} + \frac{R_1 + R_2}{R_2} y(t) .$$

(f) **Stable:** The system is BIBO stable since a bounded input will always produce a bounded output as shown below.

Using Theorem 3.1, the output of the system defined by Eq. (S2.1) is given by

$$y(t) = e^{-p} \int_{-\infty}^{t} e^{p} \times \tfrac{1}{CR_1} v(\tau) d\tau$$

where

$$p(t) = \exp\left[-\tfrac{R_1+R_2}{CR_1R_2} t\right] .$$

From the above solution, it is clear that the output y(t) is bounded in the input v(t) is bounded. ∎

Problem 2.2

(a) The currents flowing in resistor R, inductor L, and capacitor C, are given by

$$i_R = \frac{y(t)}{R}, \quad i_L = \frac{1}{L} \int_{-\infty}^{t} y(\tau)\, d\tau, \quad i_C = C\frac{dy}{dt}.$$

Applying the Kirchoff's current law, we obtain

$$\frac{y(t)}{R} + \frac{1}{L}\int_{-\infty}^{t} y(\tau)d\tau + C\frac{dy}{dt} = i(t),$$

or, by differentiating, $$C\frac{d^2y}{dt^2} + \frac{1}{R}\frac{dy}{dt} + \frac{1}{L}y(t) = \frac{di}{dt}$$

or, $$\frac{d^2y}{dt^2} + \frac{1}{RC}\frac{dy}{dt} + \frac{1}{LC}y(t) = \frac{1}{C}\frac{di}{dt}. \tag{S2.2}$$

(ii)

(a) **Linear:** For $i_1(t)$ applied as the input, the output $y_1(t)$ is given by

$$\frac{d^2y_1}{dt^2} + \frac{1}{RC}\frac{dy_1}{dt} + \frac{1}{LC}y_1(t) = \frac{1}{C}\frac{di_1}{dt} \quad \Rightarrow \quad \frac{di_1}{dt} = C\frac{d^2y_1}{dt^2} + \frac{1}{R}\frac{dy_1}{dt} + \frac{1}{L}y_1(t)$$

For $i_2(t)$ applied as the input, the output $y_2(t)$ is given by

$$\frac{d^2y_2}{dt^2} + \frac{1}{RC}\frac{dy_2}{dt} + \frac{1}{LC}y_2(t) = \frac{1}{C}\frac{di_2}{dt} \quad \Rightarrow \quad \frac{di_2}{dt} = C\frac{d^2y_2}{dt^2} + \frac{1}{R}\frac{dy_2}{dt} + \frac{1}{L}y_2(t).$$

For $i_3(t) = \alpha\, i_1(t) + \beta\, i_2(t)$ applied as the input, the output $y_3(t)$ is given by

$$\frac{d^2y_3}{dt^2} + \frac{1}{RC}\frac{dy_3}{dt} + \frac{1}{LC}y_3(t) = \frac{1}{C}\frac{d(\alpha i_1 + \beta i_2)}{dt},$$

or, $$\frac{d^2y_3}{dt^2} + \frac{1}{RC}\frac{dy_3}{dt} + \frac{1}{LC}y_3(t) = \frac{\alpha}{C}\frac{di_1}{dt} + \frac{\beta}{C}\frac{di_2}{dt}$$

Substituting the values of di_1/dt and di_2/dt from the earlier equations, we get

$$\frac{d^2y_3}{dt^2} + \frac{1}{RC}\frac{dy_3}{dt} + \frac{1}{LC}y_3(t) = \alpha\left[\frac{d^2y_1}{dt^2} + \frac{1}{RC}\frac{dy_1}{dt} + \frac{1}{LC}y_1(t)\right] + \beta\left[\frac{d^2y_2}{dt^2} + \frac{1}{RC}\frac{dy_2}{dt} + \frac{1}{LC}y_2(t)\right].$$

Rearranging the terms on the right hand side of the equation, we get

$$\frac{d^2y_3}{dt^2} + \frac{1}{RC}\frac{dy_3}{dt} + \frac{1}{LC}y_3(t) = \frac{d^2(\alpha y_1 + \beta y_2)}{dt^2} + \frac{1}{RC}\frac{d(\alpha y_1 + \beta y_2)}{dt} + \frac{1}{LC}(\alpha y_1 + \beta y_2).$$

Comparing with (S2.2) implies that $y_3(t) = \alpha y_1(t) + \beta y_2(t)$.

The system is therefore linear.

(b) **Time-invariance:** For $i_1(t - t_0)$ applied as the input, the output $y_1(t)$ is given by

$$\frac{d^2y_1}{dt^2} + \frac{1}{RC}\frac{dy_1}{dt} + \frac{1}{LC}y_1(t) = \frac{1}{C}\frac{di_1(t - t_0)}{dt}.$$

Substituting $\tau = t - t_0$ (which implies that $dt = d\tau$), we obtain

$$\frac{d^2y_1(\tau + t_0)}{d\tau^2} + \frac{1}{RC}\frac{dy_1(\tau + t_0)}{d\tau} + \frac{1}{LC}y_1(\tau + t_0) = \frac{1}{C}\frac{di_1(\tau)}{d\tau}.$$

Comparing with Eq. (S2.2.2), we obtain

$$y(\tau) = y_1(\tau + t_0) \quad \text{or,} \quad y_1(\tau) = y(\tau - t_0),$$

proving that the system is time-invariant.

(c) **Memoryless**: Express Eq. (S2.2) as

$$y(t) = \frac{1}{C}\int_{-\infty}^{t} i(\alpha)d\alpha - \frac{1}{LC}\int_{-\infty}^{t}\int_{-\infty}^{\tau} y(\alpha)d\alpha d\tau - \frac{1}{RC}\int_{-\infty}^{t} y(\alpha)d\alpha .$$

The output $y(t)$ at $t = t_0$ is given by

$$y(t)\Big|_{t=t_0} = \frac{1}{C}\int_{-\infty}^{t_0} i(\alpha)d\alpha - \frac{1}{LC}\int_{-\infty}^{t_0}\int_{-\infty}^{\tau} y(\alpha)d\alpha d\tau - \frac{1}{RC}\int_{-\infty}^{t_0} y(\alpha)d\alpha$$

From the first integral on the right hand side of the equation, it is clear that all previous values of the input $i(t)$, for $-\infty \le t \le t_0$, are needed to calculate the output $y(t)$ at $t = t_0$. The system has, therefore, memory and is not memoryless.

(d) **Causal**: From the previous result, we deduce that the system is causal since only the past values of the input $i(t)$, for $-\infty \le t \le t_0$, are needed to calculate the output $y(t)$ at $t = t_0$.

(e) **Invertible:** The system is invertible as $i(t)$ can be determined from the following relationship

$$\frac{di}{dt} = C\frac{d^2y}{dt^2} + \frac{1}{R}\frac{dy}{dt} + \frac{1}{L}y(t) .$$

(f) **Stable:** The system is BIBO stable since a bounded input will always produce a bounded output. ▌

Problem 2.3

(i) For input $x_1(t)$, $x_2(t)$, and $\alpha x_1(t) + \beta x_2(t)$, the respective outputs are given by

$$x_1(t) \to c_1x_1(t) + c_2x_1^2(t) = y_1(t)$$
$$x_2(t) \to c_1x_2(t) + c_2x_2^2(t) = y_2(t)$$
$$\alpha x_1(t) + \beta x_2(t) \to c_1\left[\alpha x_1(t) + \beta x_2(t)\right] + c_2\left[\alpha x_1(t) + \beta x_2(t)\right]^2 = y(t)$$

Because $y(t) \ne \alpha y_1(t) + \beta y_2(t)$, the demodulator is a non-linear device.

(ii)

(a) **Time Invariance**: For input $x_1(t)$ and $x_2(t) = x_1(t - T)$, the respective outputs are given by

$$x_1(t) \to c_1x_1(t) + c_2x_1^2(t) = y_1(t)$$
$$x_2(t) = x_1(t-T) \to c_1x_2(t) + c_2x_2^2(t) = c_1x_1(t-T) + c_2x_1^2(t-T) = y_2(t)$$

Since $y_1(t-T) = y_2(t)$, the system is time invariant.

(b) **Memory**: The output $v_2(t)$ depends only on the current value of the input $v_1(t)$. Therefore, the system is memoryless. All memoryless systems are causal. Therefore, the system is also causal.

(c) **Invertible**: The input to the system, $v_1(t)$ can be calculated using the following equation

$$v_1(t) = \frac{-c_1 \pm \sqrt{c_1^2 + 4c_2 v_2(t)}}{2c_2}.$$

For a given value of $v_2(t)$, two possible values of $v_1(t)$ exist As the input $v_1(t)$ cannot be uniquely determined from the output $v_2(t)$, the system is NOT invertible.

(d) **Stable**: Assuming that $|v_1(t)| \le M < \infty$, the output $v_2(t)$ is bounded by

$$\left|v_2(t)\right| = \left|c_1 v_1(t) + c_2 v_1^2(t)\right| < c_1\left|v_1(t)\right| + c_2\left|v_1^2(t)\right| < c_1 M + c_2 M^2 < \infty.$$

Therefore, the system is stable. ▮

Problem 2.4

(i) The modulated signal is given by

$$s(t) = A[1 + km(t)]\cos(2\pi f_c t) = 5[1 + 2k\sin(200\pi t)\ ]\cos(2000000\pi t).$$

$1 + 2k\sin(200\pi t) \ge 0$ implies $2k\sin(200\pi t) \ge -1$, or, $k \le 0.5$.

(ii) Any value of k in the range ($0 \le k \le 0.5$) can be used. We use $k = 0.5$ in the rest of the problem. The AM signal is given by

$$s(t) = A[1 + km(t)]\cos(2\pi f_c t) = 5[1 + 0.8\sin(200\pi t)\ ]\cos(2000000\pi t).$$

(iii) Expanding the above equation, we get

$$\begin{aligned} s(t) &= 5\cos(2000000\pi t) + 4\sin(200\pi t)\cos(2000000\pi t) \\ &= 5\cos(2000000\pi t) + 2\sin(2000200\pi t) + 2\sin(1999800\pi t) \end{aligned}$$

It is observed that the AM signal has three frequency components at 1,000,000 Hz, 1,000,200 Hz, and 999,900 Hz. The frequency component at 1,000,000 Hz represents the carrier signal, while the remaining two frequency components at 1,000,200 Hz and 999,900 Hz represent the sinusoidal tone. Therefore, the frequency of the sinusoidal tone is shifted to a higher frequency range. ▮

Problem 2.5

(i) Dividing both sides by M, Eq. (2.16) can be expressed as

$$\frac{d^2 y}{dt} + \frac{r}{M}\frac{dy}{dt} + \frac{k}{M}y(t) = \frac{1}{M}x(t).$$

Comparing with the given expression, the coefficients are given by

$$\omega_n^2 = \frac{k}{M} \Rightarrow \omega_n = \sqrt{\frac{k}{M}}$$

$$\text{and} \quad \frac{\omega_n}{Q} = \frac{r}{M} \Rightarrow Q = \frac{\sqrt{kM}}{r}.$$

(ii) Since $\omega_n = \sqrt{\frac{k}{M}}$, the natural frequency ω_n can be increased either by: (a) increasing the value of the spring constant k, or (b) by decreasing the value of the mass M.

Since $Q = \frac{\sqrt{kM}}{r}$, the value of Q can be reduced either by: (a) reducing the value of the spring constant k, (b) reducing the value of mass M, or (c) by increasing the value of r.

(iii) $$\frac{d^2y}{dt^2} + \delta\frac{dy}{dt} + \varepsilon y(t) = \gamma x(t), \quad \text{with } \delta = \frac{r}{M}, \varepsilon = \frac{k}{M}, \gamma = \frac{1}{M} \tag{S2.5.1}$$

(a) **Linear:** For $x_1(t)$ applied as the input, the output $y_1(t)$ is given by

$$\frac{d^2y_1}{dt^2} + \delta\frac{dy_1}{dt} + \varepsilon y_1(t) = \gamma x_1(t) \quad \Rightarrow \quad x_1(t) = \tfrac{1}{\gamma}\frac{d^2y_1}{dt^2} + \tfrac{\delta}{\gamma}\frac{dy_1}{dt} + \tfrac{\varepsilon}{\gamma}y_1(t)$$

For $x_2(t)$ applied as the input, the output $y_2(t)$ is given by

$$\frac{d^2y_2}{dt^2} + \delta\frac{dy_2}{dt} + \varepsilon y_2(t) = \gamma x_2(t) \quad \Rightarrow \quad x_2(t) = \tfrac{1}{\gamma}\frac{d^2y_2}{dt^2} + \tfrac{\delta}{\gamma}\frac{dy_2}{dt} + \tfrac{\varepsilon}{\gamma}y_2(t).$$

For $x_3(t) = \alpha x_1(t) + \beta x_2(t)$ applied as the input, the output $y_3(t)$ is given by

$$\frac{d^2y_3}{dt^2} + \delta\frac{dy_3}{dt} + \varepsilon y_3(t) = \gamma x_3(t) = \gamma\left(\alpha x_1(t) + \beta x_2(t)\right).$$

Substituting the values of $x_1(t), x_2(t)$ from the earlier equations, we obtain

$$\begin{aligned}\frac{d^2y_3}{dt^2} + \delta\frac{dy_3}{dt} + \varepsilon y_3(t) &= \alpha\left[\frac{d^2y_1}{dt^2} + \delta\frac{dy_1}{dt} + \varepsilon y_1(t)\right] + \beta\left[\frac{d^2y_2}{dt^2} + \delta\frac{dy_2}{dt} + \varepsilon y_2(t)\right] \\ &= \frac{d^2(\alpha y_1 + \beta y_2)}{dt^2} + \delta\frac{d(\alpha y_1 + \beta y_2)}{dt} + \varepsilon\left(\alpha y_1(t) + \beta y_2(t)\right)\end{aligned}$$

Comparing the left-hand and right-hand sides, we obtain

$$y_3(t) = \alpha y_1(t) + \beta y_2(t).$$

The system is therefore linear.

(b) **Time-invariance:** For $x_1(t)$ applied as the input, the output $y_1(t)$ is given by

$$\frac{d^2y_1}{dt^2} + \delta\frac{dy_1}{dt} + \varepsilon y_1(t) = \gamma x_1(t) \tag{S2.5.2}$$

For $x_2(t) = x_1(t - t_0)$ applied as the input, the output $y_2(t)$ is given by

$$\frac{d^2y_2}{dt^2} + \delta\frac{dy_2}{dt} + \varepsilon y_2(t) = \gamma x_2(t) = \gamma x_1(t - t_0). \tag{S2.5.3}$$

Substituting $\tau = t + t_0$ (which implies that $dt = d\tau$) in Eq. (S2.5.2), we obtain

$$\frac{d^2y_1(\tau - t_0)}{d\tau^2} + \delta\frac{dy_1(\tau - t_0)}{d\tau} + \varepsilon y_1(\tau - t_0) = \gamma x_1(\tau - t_0)$$

Substituting $t = \tau$, we obtain

$$\frac{d^2y_1(t - t_0)}{dt^2} + \delta\frac{dy_1(t - t_0)}{dt} + \varepsilon y_1(t - t_0) = \gamma x_1(t - t_0)$$

Comparing with Eq. (S2.5.2), we obtain

$$y_2(t) = y_1(t - t_0),$$

proving that the system is time-invariant.

(c) **Memoryless**: Express Eq. (S2.5.1) as

$$y(t) = \gamma \int_{-\infty}^{t}\int_{-\infty}^{\tau} x(\alpha)d\alpha d\tau - \varepsilon \int_{-\infty}^{t}\int_{-\infty}^{\tau} x(\alpha)d\alpha d\tau - \delta \int_{-\infty}^{t} y(\alpha)d\alpha .$$

The output $y(t)$ at $t = t_0$ is given by

$$y(t)\big|_{t=t_0} = \gamma \int_{-\infty}^{t_0}\int_{-\infty}^{\tau} x(\alpha)d\alpha d\tau - \varepsilon \int_{-\infty}^{t_0}\int_{-\infty}^{\tau} x(\alpha)d\alpha d\tau - \delta \int_{-\infty}^{t_0} y(\alpha)d\alpha$$

From the first integral on the right hand side of the equation, it is clear that all previous values of the input $x(t)$, for $-\infty \le t \le t_0$, are needed to calculate the output $y(t)$ at $t = t_0$. The system has, therefore, memory and is not memoryless.

(d) **Causal**: From the previous result, we deduce that the system is causal since only the past values of the input $x(t)$, for $-\infty \le t \le t_0$, are needed to calculate the output $y(t)$ at $t = t_0$.

(e) **Invertible:** The system is invertible as $i(t)$ can be determined from the following relationship

$$x(t) = \tfrac{1}{\gamma}\frac{d^2y}{dt^2} + \tfrac{\delta}{\gamma}\frac{dy}{dt} + \tfrac{\varepsilon}{\gamma}y(t) .$$

(f) **Stable:** The system is BIBO stable since a bounded input will always produce a bounded output. ▌

<u>Problem 2.6</u>

(i) Substituting, $t = k\Delta t$ in $\dfrac{d^2y}{dt} + 5\dfrac{dy}{dt} + 6y(t) = 0$, yields

$$\left.\frac{d^2y}{dt}\right|_{t=k\Delta t} + 5\left.\frac{dy}{dt}\right|_{t=k\Delta t} + 6y(t)\big|_{t=k\Delta t} = 0 .$$

Substituting the values of the first and second derivative from the backward finite difference scheme, we get

$$\frac{y(k\Delta t) - 2y((k-1)\Delta t) + y((k-2)\Delta t)}{(\Delta t)^2} + 5\frac{y(k\Delta t) - y((k-1)\Delta t)}{\Delta t} + 6y(k\Delta t) = 0,$$

or, $y(k\Delta t) - 2y((k-1)\Delta t) + y((k-2)\Delta t) + 5\left[y(k\Delta t) - y((k-1)\Delta t)\right]\Delta t + 6y(k\Delta t)(\Delta t)^2 = 0$.

Substituting $y(k\Delta t) = y[k]$, $y((k-1)\Delta t) = y[k-1]$, and $y((k-2)\Delta t) = y[k-2]$, the above equation reduces to

$$y[k] - 2y[k-1] + y[k-2] + 5\{y[k] - y[k-1]\}\Delta t + 6y[k](\Delta t)^2 = 0,$$

which simplifies to

$$\left(1+5\Delta t+6(\Delta t)^2\right)y[k]+\left(-2-5\Delta t\right)y[k-1]+y[k-2]=0\,.$$

(ii) Substituting the value of the first CT initial condition $y(0) = 3$ in

$$y[k] = y(k\Delta t)$$

for $k = 0$, we obtain $\qquad y[0] = y(0) = 3.$

Similarly, substituting the value of the second CT initial condition $\dot{y}(0) = -7$ in

$$\left.\frac{dy}{dt}\right|_{t=k\Delta t} \approx \frac{y(k\Delta t)-y((k-1)\Delta t)}{\Delta t}$$

for $k = 0$, we get $\qquad \dot{y}(0) = \dfrac{y[0]-y[-1]}{\Delta t} = -7\,.$

Simplifying the above, we obtain

$$y[-1] = y[0]-\Delta t\dot{y}(0) = 3+7\Delta t\,.$$

(iii) By substituting $\Delta t = 0.02$s in the difference equation,

$$\left(1+5\Delta t+6(\Delta t)^2\right)y[k]+\left(-2-5\Delta t\right)y[k-1]+y[k-2]=0\,,$$

we obtain $\qquad (1.1024)y[k]+(-2.1)y[k-1]+y[k-2]=0\,,$

or, $\qquad y[k]=1.90493y[k-1]-0.90711y[k-2]$

with $y[0] = 3$ and $y[-1] = 3 + 7(0.02) = 3.14$.

The Matlab code used to compute and plot the DT solution is given in Program 2.6. For comparison, we also plot the CT solution. The two plots are included in Fig. S2.6.

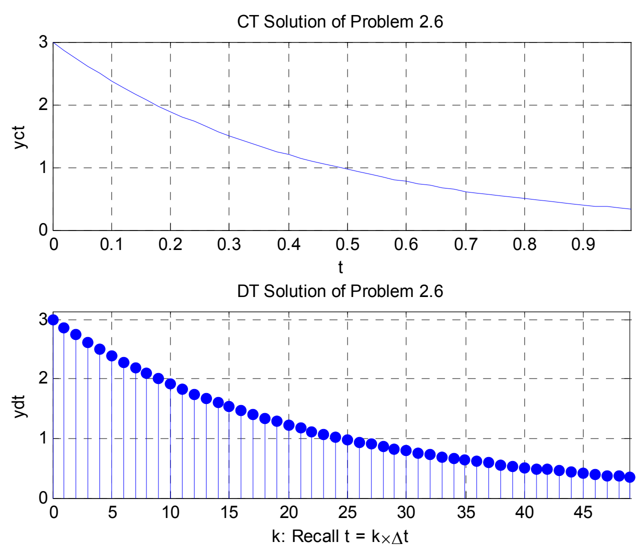

Fig. S2.6: Plots obtained from the CT (top) and DT (bottom) solution of the 2nd order differential equation in Problem 2.6.

Program 2.6: MATLAB code for Problem 2.6.

```
% MATLAB code for Problem 2.6
% plot CT result
N = 50;                              % No of points to be plotted
k = 0:N-1;                           % time index
dt = 0.02;                           % discretization step
t = k*dt;                            % time instants
yct = exp(-3*t)+2*exp(-2*t);         % CT answer
subplot(2,1,1)
plot(t,yct), grid on
xlabel('t')                          % Label of X-axis
ylabel('yct')                        % Label of Y-axis
title('CT Solution of Problem 2.6');
axis ([0 max(t) 0 max(yct)])

% compute finite difference approximation
ydt(1) = 3+7*dt;                     % this is actually y[-1]
ydt(2)= 3;                           % this is actually y[0]
for i = 3:N+1
   ydt(i) = 1/(1+5*dt+6*dt^2)*((2+5*dt)*ydt(i-1)- ydt(i-2));
end
subplot(2,1,2)
stem(k,ydt(2:N+1),'fill'), grid on
xlabel('k: Recall t = k\times\Deltat');% Label of X-axis
ylabel('ydt')                        % Label of Y-axis
title('DT Solution of Problem 2.6');
axis ([0 max(k) 0 max(ydt)])
```

Problem 2.7

Starting from the initial value $x(0) = 0$V, The output of the delta modulator receiver is computed from the recursive expression

$$\hat{x}(kT) = \hat{x}((k-1)T) + b_k \Delta$$

where b_k is 1 if bit 1 is received, while b_k is -1 if bit 0 is received. The computed values are shown in Table S2.7 with the waveform plotted in Fig. S2.7.

Table S2.7: Decoded output of the delta modulator in Problem 2.7

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
t	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	1.0
b_k		1	1	1	1	1	0	1	1	1	1	1	1	0	0	0	0	0	0	0	0
$\hat{x}(t)$	0	0.1	0.2	0.3	0.4	0.5	0.4	0.5	0.6	0.7	0.8	0.9	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2

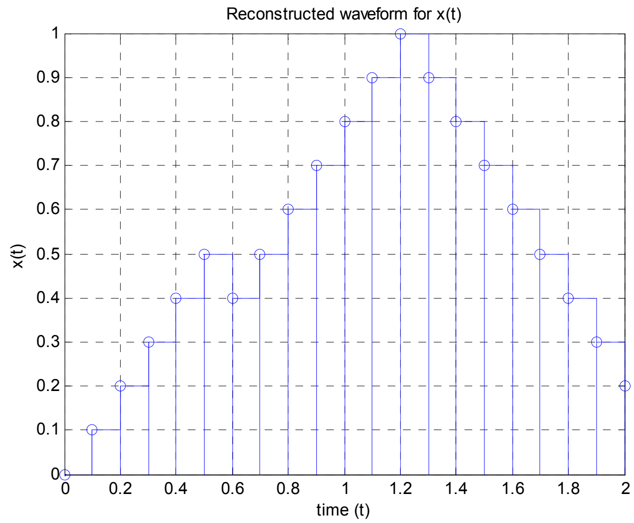

Fig. S2.7:Waveform reconstructed from the delta-modulated bit stream in Problem 2.7.

Problem 2.8

Eq. (2.27) represents a lowpass filter that averages the four neighboring pixels to compute the value of the reference pixel located at (m, n). Specifically,

$$y[m,n] = \frac{1}{4}\left(x[m,n] + x[m,n-1] + x[m-1,n] + x[m-1,n-1]\right).$$

The system is invertible as the input $x[m, n]$ can be constructed in a recursive way from the current value of the input and the previously reconstructed values of the input pixels based on the following relationship

$$x[m,n] = 4y[m,n] - x[m,n-1] - x[m-1,n] + x[m-1,n-1].$$

Since both the original filter and the inverse system use the values of the past pixels to compute the reference pixel, both systems are causal in nature. ▌

Problem 2.9

(i) $y(t) = x(t-2)$

(a) Linearity: Since

$$x_1(t) \to x_1(t-2) = y_1(t)$$
$$x_2(t) \to x_2(t-2) = y_2(t)$$
$$\alpha x_1(t) + \beta x_2(t) \to \alpha x_1(t-2) + \beta x_2(t-2) = \alpha y_1(t) + \beta y_2(t)$$

therefore, the system is a linear system.

(b) Time Invariance: For inputs $x_1(t)$ and $x_2(t) = x_1(t-T)$, the outputs are given by

$$x_1(t) \to x_1(t-2) = y_1(t)$$
$$x_2(t) = x_1(t-T) \to x_2(t-2) = x_1(t-T-2) = y_2(t)$$

and $y_1(t-T) = x_1(t-T-2) = y_2(t)$, the system is time invariant.

(c) Stability: Assume that the input is bounded $|x(t)| \le M$. Then, the output

$$|y(t)| = |x(t-2)| \le M$$

is also bounded proving that the system is BIBO stable.

(d) Causality: Since the output depends only on the past input and does not depend on the future values of the input, therefore, the system is causal.

(ii) $y(t) = x(2t-5)$

(a) Linearity: Since

$$x_1(t) \to x_1(2t-5) = y_1(t)$$
$$x_2(t) \to x_2(2t-5) = y_2(t)$$
$$\alpha x_1(t) + \beta x_2(t) \to \alpha x_1(2t-5) + \beta x_2(2t-5) = \alpha y_1(t) + \beta y_2(t)$$

therefore, the system is a linear system.

(b) Time Invariance: For inputs $x_1(t)$ and $x_2(t) = x_1(t-T)$, the outputs are given by

$$x_1(t) \to x_1(2t-5) = y_1(t)$$
$$x_2(t) = x_1(t-T) \to x_2(2t-5) = y_2(t)$$

which implies that

$$x_2(t) = x_1(t-T) \to y_2(t) = x_2(2t-5)\big|_{x_2(t)=x_1(t-T)} = x_1(2t-5-T).$$

On the other hand,

$$y_1(t-T) = x_1(2t-5)\big|_{t=t-T} = x_1(2(t-T)-5) = x_1(2t-2T-5),$$

and $y_1(t-T) \ne y_2(t)$. Therefore, the system is NOT time invariant.

(c) Stability: Assume that the input is bounded $|x(t)| \le M$. Then, the output

$$|y(t)| = |x(2t-5)| \le M$$

is also bounded proving that the system is BIBO stable.

(d) Causality: For $(t > 5)$, the output depends on the future values of the input, therefore, the system is NOT causal.

(iii) $y(t) = x(2t) - 5$

(a) Linearity: Since

$$x_1(t) \to x_1(2t) - 5 = y_1(t)$$
$$x_2(t) \to x_2(2t) - 5 = y_2(t)$$
$$\alpha x_1(t) + \beta x_2(t) \to \alpha x_1(2t) + \beta x_2(2t) - 5 \ne \alpha y_1(t) + \beta y_2(t)$$

because $\alpha y_1(t) + \beta y_2(t) = \alpha x_1(2t) + \beta x_2(2t) - 5(\alpha + \beta)$. Therefore, the system is NOT linear.

(b) Time Invariance: For inputs $x_1(t)$ and $x_2(t) = x_1(t-T)$, the outputs are given by

$$x_1(t) \to x_1(2t) - 5 = y_1(t)$$
$$x_2(t) = x_1(t-T) \to x_2(2t) - 5 = y_2(t)$$

which implies that

$$x_2(t) = x_1(t-T) \to y_2(t) = x_2(2t) - 5\big|_{x_2(t)=x_1(t-T)} = x_1(2t-T) - 5 .$$

On the other hand,

$$y_1(t-T) = x_1(2t)\big|_{t=t-T} - 5 = x_1(2(t-T)) - 5 = x_1(2t-2T) - 5 .$$

Clearly, $y_1(t-T) \neq y_2(t)$, therefore, the system is NOT time invariant.

(c) Stability: Assume that the input is bounded $|x(t)| \leq M$. Then, the output

$$|y(t)| = |x(2t) - 5| \leq |x(2t)| + 5 \leq M + 5$$

is also bounded proving that the system is BIBO stable.

(d) Causality: For ($t > 0$), the system requires future values of the input to calculate the current value of the input. Therefore, the system is NOT causal.

(iv) $y(t) = tx(t+10)$

(a) Linearity: Since

$$x_1(t) \to tx_1(t+10) = y_1(t)$$
$$x_2(t) \to tx_2(t+10) = y_2(t)$$
$$\alpha x_1(t) + \beta x_2(t) \to \alpha t x_1(t+10) + \beta t x_2(t+10) = \alpha y_1(t) + \beta y_2(t)$$

therefore, the system is a linear system.

(b) Time Invariance: For inputs $x_1(t)$ and $x_2(t) = x_1(t - T)$, the outputs are given by

$$x_1(t) \to tx_1(t+10) = y_1(t)$$
$$x_2(t) = x_1(t-T) \to tx_2(t+10) = tx_1(t-T+10) = y_2(t)$$

We also note that $y_1(t-T) = (t-T)x_1(t-T+10) \neq y_2(t)$,

therefore, the system is NOT time invariant.

(c) Stability: Assume that the input is bounded $|x(t)| \leq M$. Then, the output

$$|y(t)| = |tx(t+10)| = |t||x(t+10)| \leq M|t|$$

is unbounded as $t \to \infty$. Therefore, the system is NOT BIBO stable.

(d) Causality: Since the output depends on the future values of the input, and therefore the system is NOT causal.

(v) $y(t) = 2u(x(t)) = \begin{cases} 2 & x(t) \geq 0 \\ 0 & x(t) < 0 \end{cases}$

(a) Linearity: Since

$$\begin{aligned}
&x_1(t) \to 2u\big(x_1(t)\big) = y_1(t)\\
&x_2(t) \to 2u\big(x_2(t)\big) = y_2(t)\\
&\alpha x_1(t) + \beta x_2(t) \to 2u\big(\alpha x_1(t) + \beta x_2(t)\big) = y(t)
\end{aligned}$$

and $\alpha y_1(t) + \beta y_2(t) = 2\alpha u\big(x_1(t)\big) + 2\beta u\big(x_2(t)\big) \neq y(t)$. Therefore, the system is NOT linear.

Also, we note that

$$y_2(t) - y_1(t) = 2u\big(x_2(t)\big) - 2u\big(x_1(t)\big) \neq \lambda\big[x_2(t) - x_1(t)\big].$$

Therefore, the system is NOT an incrementally linear system either.

(b) Time Invariance: For inputs $x_1(t)$ and $x_2(t) = x_1(t - T)$, the outputs are given by

$$\begin{aligned}
&x_1(t) \to 2u\big(x_1(t)\big) = y_1(t)\\
&x_2(t) = x_1(t-T) \to 2u\big(x_2(t)\big) = 2u\big(x_1(t-T)\big) = y_2(t)
\end{aligned}$$

We note that $y_1(t-T) = 2u\big(x_1(t-T)\big) = y_2(t)$, therefore, the system is time invariant.

(c) Stability: Since $|y(t)| \leq 2$, therefore, the system is BIBO stable.

(d) Causality: The output at any time instant does not depend on future value of the input. The system is, therefore, causal.

(vi) $$y(t) = \begin{cases} 0 & \text{t<0} \\ x(t) - x(t-5) & t \geq 0 \end{cases} \qquad = \big[x(t) - x(t-5)\big]u(t)$$

(a) Linearity: Since

$$\begin{aligned}
&x_1(t) \to \big[x_1(t) - x_1(t-5)\big]u(t) = y_1(t)\\
&x_2(t) \to \big[x_2(t) - x_2(t-5)\big]u(t) = y_2(t)\\
&\alpha x_1(t) + \beta x_2(t) \to \big[\{\alpha x_1(t) + \beta x_2(t)\} - \{\alpha x_1(t-5) + \beta x_2(t-5)\}\big]u(t) = y(t)\\
&\qquad\qquad = \alpha\big[x_1(t) - x_1(t-5)\big]u(t) + \beta\big[x_1(t) - x_1(t-5)\big]u(t)\\
&\qquad\qquad = \alpha y_1(t) + \beta y_2(t),
\end{aligned}$$

the system is linear.

(b) Time Invariance: For inputs $x_1(t)$ and $x_2(t) = x_1(t - T)$, the outputs are given by

$$\begin{aligned}
&x_1(t) \to \big[x_1(t) - x_1(t-5)\big]u(t) = y_1(t)\\
&x_2(t) = x_1(t-T) \to \big[x_2(t) - x_2(t-5)\big]u(t) = y_2(t)
\end{aligned}$$

We note that $y_1(t-T) \neq y_2(t)$ since

$$y_1(t-T) = y_1(t)\big|_{t=t-T} = \big[x_1(t) - x_1(t-5)\big]u(t)\big|_{t=t-T} = \big[x_1(t-T) - x_1(t-T-5)\big]u(t-T)$$

and $$y_2(t) = \big[x_2(t) - x_2(t-5)\big]u(t) == \big[x_1(t-T) - x_1(t-T-5)\big]u(t).$$

Therefore, the system is NOT time invariant.

(c) Stablity: Assume that the input is bounded $|x(t)| \le M$. Then, the output

$$|y(t)| = |x(t) - x(t-5)| \le |x(t)| + |x(t-5)| \le 2M$$

is also bounded. Therefore, the system is BIBO stable.

(d) Causality: The output does not depend on the future values of the input, therefore, the system is causal.

(vii) $y(t) = 7x^2(t) + 5x(t) + 3$

(a) Linearity: For $x_1(t)$ applied as the input, the output $y_1(t)$ is given by

$$y_1(t) = 7x_1^2(t) + 5x_1(t) + 3$$

For $x_2(t)$ applied as the input, the output $y_2(t)$ is given by

$$y_2(t) = 7x_2^2(t) + 5x_2(t) + 3.$$

For $x_3(t) = \alpha\, x_1(t) + \beta\, x_2(t)$ applied as the input, the output $y_3(t)$ is given by

$$y_3(t) = 7(\alpha x_1(t) + \beta x_2(t))^2 + 5(\alpha x_1(t) + \beta x_2(t)) + 3,$$

or, $$y_3(t) = \alpha\underbrace{(7x_1^2(t) + 5x_1(t) + 3)}_{y_1(t)} + \beta\underbrace{(7x_1^2(t) + 5x_2(t) + 3)}_{y_2(t)} + 14\alpha\beta x_1(t)x_2(t) + 3(1-\alpha-\beta)$$

The above result implies that

$$y_3(t) \ne \alpha y_1(t) + \beta y_2(t),$$

And hence the system is not linear.

(b) Time Invariance: For $x_1(t)$ and $x_2(t)$ applied as the inputs, the outputs are given by

$$x_1(t) \to 7x_1^2(t) + 5x_1(t) + 3 = y_1(t)$$
$$x_2(t) = x_1(t-T) \to 7x_2^2(t) + 5x_2(t) + 3 = y_2(t).$$

Substituting $x_2(t) = x_1(t-T)$ we obtain,

$$y_2(t) = 7x_1^2(t-T) + 5x_2(t-T) + 3.$$

We also note that $$y_1(t-T) = 7x_1^2(t-T) + 5x_2(t-T) + 3,$$

implying that $y_1(t-T) = y_2(t)$. The system is, therefore, time invariant.

(c) Stablity: Assume that the input is bounded $|x(t)| \le M$. Then, the output

$$|y(t)| = \left|7x^2(t) + 5x(t) + 3\right| \le 7|x(t)||x(t)| + 5|x(t)| + 3 \le 7M^2 + 5M + 3$$

is also bounded. Therefore, the system is BIBO stable.

(d) Causality: The output $y(t)$ at $t = t_0$ requires only one value of the input $y(t)$ at $(t = t_0)$. Therefore, the system is causal.

(viii) $y(t) = \mathrm{sgn}(x(t))$

(a) Linearity: For $x_1(t)$ applied as the input, the output $y_1(t) = \mathrm{sgn}(x_1(t))$.

For $x_2(t)$ applied as the input, the output $y_2(t) = \mathrm{sgn}(x_2(t))$.

For $x_3(t) = \alpha\, x_1(t) + \beta\, x_2(t)$ applied as the input, the output $y_3(t)$ is given by

$$y_3(t) = \mathrm{sgn}(\alpha x_1(t) + \beta x_2(t)) \neq \alpha\, \mathrm{sgn}(x_1(t)) + \beta\, \mathrm{sgn}(x_2(t)).$$

The above result implies that

$$y_3(t) \neq \alpha y_1(t) + \beta y_2(t),$$

And hence the system is not linear.

(b) Time Invariance: For $x_1(t)$ and $x_2(t) = x_1(t-T)$ applied as the inputs, the outputs are given by

$$x_1(t) \to \mathrm{sgn}(x_1(t)) = y_1(t)$$
$$x_2(t) = x_1(t-T) \to \mathrm{sgn}(x_2(t)) = y_2(t)$$

Substituting $x_2(t) = x_1(t-T)$ we obtain,

$$y_2(t) = \mathrm{sgn}(x_1(t-T)).$$

We also note that $$y_1(t-T) = \mathrm{sgn}(x_1(t-T)),$$

implying that $y_1(t-T) = y_2(t)$. The system is, therefore, time invariant.

(c) Stablity: The system is stable as the output is always bounded between the values of -1 and 1.

(d) Causality: The output $y(t)$ at $(t = t_0)$ requires only one value of the input $x(t)$ at $(t = t_0)$, therefore, the system is causal.

(ix) $y(t) = \int_{-t_0}^{t_0} x(\lambda)d\lambda + 2x(t)$

(a) Linearity: For $x_1(t)$ and $x_2(t)$ applied as the inputs, the outputs are given by

$$y_1(t) = \int_{-t_0}^{t_0} x_1(\lambda)d\lambda + 2x_1(t),$$

$$y_2(t) = \int_{-t_0}^{t_0} x_2(\lambda)d\lambda + 2x_2(t).$$

For $x_3(t) = \alpha\, x_1(t) + \beta\, x_2(t)$ applied as the input, the output $y_3(t)$ is given by

$$\begin{aligned} y_3(t) &= \int_{-t_0}^{t_0} \left(\alpha x_1(\lambda) + \beta x_2(\lambda)\right) d\lambda + 2\left(\alpha x_1(t) + \beta x_2(t)\right) \\ &= \alpha \underbrace{\left[\int_{-t_0}^{t_0} x_1(\lambda)d\lambda + 2x_1(t)\right]}_{y_1(t)} + \beta \underbrace{\left[\int_{-t_0}^{t_0} x_2(\lambda)d\lambda + 2x_2(t)\right]}_{y_2(t)}, \\ &= \alpha y_1(t) + \beta y_2(t) \end{aligned}$$

Therefore, the system is linear.

(b) Time Invariance: For $x_1(t)$ and $x_2(t) = x_1(t-T)$ applied as the inputs, the outputs are given by

$$x_1(t) \rightarrow y_1(t) = \int_{-t_0}^{t_0} x_1(\lambda)d\lambda + 2x_1(t)$$

$$x_2(t) = x_1(t-T) \rightarrow y_2(t) = \int_{-t_0}^{t_0} x_2(\lambda)d\lambda + 2x_2(t),$$

Substituting $x_2(t) = x_1(t-T)$ we obtain,

$$y_2(t) = \int_{-t_0}^{t_0} x_1(\lambda - T)d\lambda + 2x_1(t-T).$$

By substituting $\lambda' = \lambda - T$, we get $y_2(t) = \int_{-t_0-T}^{t_0-T} x_1(\lambda')d\lambda' + 2x_1(t-T)$.

We also note that $$y_1(t-T) = \int_{-t_0}^{t_0} x_1(\lambda)d\lambda + 2x_1(t-T),$$

implying that $y_1(t-T) \neq y_2(t)$. The system is, therefore, NOT time invariant.

(c) Stablity: Assume that the input is bounded $|x(t)| \leq M$. Then, the output

$$|y(t)| = \left| \int_{-t_0}^{t_0} x(\lambda)d\lambda + 2x(t) \right| \leq \int_{-t_0}^{t_0} |x(\lambda)|d\lambda + 2|x(t)| \leq 2Mt_0 + 2M$$

is also bounded. Therefore, the system is BIBO stable.

(d) Causality: To solve the integral, the output $y(t)$ always requires the values of the input $x(t)$ within the range $(-t_0 \leq t \leq t_0)$ no matter when $y(t)$ (even for $t < -t_0$) is being determined. Therefore, the system is NOT causal.

(x) $$y(t) = \int_{-\infty}^{t_0} x(\lambda)d\lambda + \frac{dx}{dt}$$

(a) Linearity: For $x_1(t)$ and $x_2(t)$ applied as the inputs, the outputs are given by

$$y_1(t) = \int_{-\infty}^{t_0} x_1(\lambda)d\lambda + \frac{dx_1}{dt},$$

$$y_2(t) = \int_{-\infty}^{t_0} x_2(\lambda)d\lambda + \frac{dx_2}{dt}.$$

For $x_3(t) = \alpha\, x_1(t) + \beta\, x_2(t)$ applied as the input, the output $y_3(t)$ is given by

$$y_3(t) = \int_{-\infty}^{t_0} \left(\alpha x_1(\lambda) + \beta x_2(\lambda)\right) d\lambda + \frac{d\left(\alpha x_1(t) + \beta x_2(t)\right)}{dt}$$

$$= \alpha \underbrace{\left[\int_{-\infty}^{t_0} x_1(\lambda) d\lambda + \frac{d\left(x_1(t)\right)}{dt} \right]}_{y_1(t)} + \beta \underbrace{\left[\int_{-\infty}^{t_0} x_2(\lambda) d\lambda + \frac{d\left(x_2(t)\right)}{dt} \right]}_{y_2(t)},$$

$$= \alpha y_1(t) + \beta y_2(t)$$

Therefore, the system is linear.

(b) Time Invariance: For $x_1(t)$ and $x_2(t) = x_1(t-T)$ applied as the inputs, the outputs are given by

$$x_1(t) \to y_1(t) = \int_{-\infty}^{t_0} x_1(\lambda) d\lambda + \frac{dx_1}{dt}$$

$$x_2(t) = x_1(t-T) \to y_2(t) = \int_{-\infty}^{t_0} x_2(\lambda) d\lambda + \frac{dx_2}{dt}.$$

Substituting $x_2(t) = x_1(t-T)$ we obtain,

$$y_2(t) = \int_{-\infty}^{t_0} x_1(\lambda - T) d\lambda + \frac{dx_1(t-T)}{dt}.$$

By substituting $\lambda' = \lambda - T$, we get $y_2(t) = \int_{-\infty}^{t_0 - T} x_1(\lambda') d\lambda' + \frac{dx_1(t-T)}{dt}$.

We also note that $$y_1(t-T) = \int_{-\infty}^{t_0} x_1(\lambda) d\lambda + \frac{dx_1(t-T)}{dt},$$

implying that $y_1(t-T) \neq y_2(t)$. The system is, therefore, NOT time invariant.

(c) Stablity: Assume that the input is bounded $|x(t)| \leq M$. Then, the output

$$|y(t)| = \left| \int_{-\infty}^{t_0} x(\lambda) d\lambda + 2\frac{dx(t)}{dt} \right| \leq \int_{-\infty}^{t_0} |x(\lambda)| d\lambda + 2\left|\frac{dx(t)}{dt}\right|$$

is unbounded because of the integral which integrates $x(t)$ from $(-\infty \leq t \leq t_0)$. Therefore, the system is NOT stable.

(d) Causality: To solve the integral, the output $y(t)$ always requires only the values of the input $x(t)$ within the range $(-\infty \leq t \leq t_0)$ no matter when $y(t)$ (even for $t < -t_0$) is being determined. Therefore, the system is NOT causal.

(xi) $$\frac{d^4 y}{dt^4} + 3\frac{d^3 y}{dt^3} + 5\frac{d^2 y}{dt^2} + 3\frac{dy}{dt} + y(t) = \frac{d^2 x}{dt^2} + 2x(t) + 1$$

(a) Linearity: For $x_1(t)$ applied as the input, the output $y_1(t)$ is given by

$$\frac{d^4 y_1}{dt^4} + 3\frac{d^3 y_1}{dt^3} + 5\frac{d^2 y_1}{dt^2} + 3\frac{dy_1}{dt} + y_1(t) = \frac{d^2 x_1}{dt^2} + 2x_1(t) + 1. \tag{S2.9.1}$$

For $x_2(t)$ applied as the input, the output $y_2(t)$ is given by

$$\frac{d^4 y_2}{dt^4} + 3\frac{d^3 y_2}{dt^3} + 5\frac{d^2 y_2}{dt^2} + 3\frac{dy_2}{dt} + y_2(t) = \frac{d^2 x_2}{dt^2} + 2x_2(t) + 1. \tag{S2.9.2}$$

For $y_3(t) = \alpha\, x_1(t) + \beta\, x_2(t)$ applied as the input, the output $y_3(t)$ is given by

$$\frac{d^4 y_3}{dt^4} + 3\frac{d^3 y_3}{dt^3} + 5\frac{d^2 y_3}{dt^2} + 3\frac{dy_3}{dt} + y_2(t) = \frac{d^2\left(\alpha x_1(t) + \beta x_2(t)\right)}{dt^2} + 2\left(\alpha y_1(t) + \beta y_2(t)\right) + 1,$$

or, $$\frac{d^4 y_3}{dt^4} + 3\frac{d^3 y_3}{dt^3} + 5\frac{d^2 y_3}{dt^2} + 3\frac{dy_3}{dt} + y_2(t) = \alpha\underbrace{\left[\frac{d^2 x_1}{dt^2} + 2x_1(t) + 1\right]}_{\text{Term I}} + \beta\underbrace{\left[\frac{d^2 x_2}{dt^2} + 2x_2(t) + 1\right]}_{\text{Term II}} + [1 - \alpha - \beta].$$

Substituting the values of the derivative terms (Terms I and II) from Eqs. (S2.9.1) ans (2.9.2), we obtain

$$\frac{d^4 y_3}{dt^4} + 3\frac{d^3 y_3}{dt^3} + 5\frac{d^2 y_3}{dt^2} + 3\frac{dy_3}{dt} + y_2(t) = \alpha\underbrace{\left[\frac{d^4 y_1}{dt^4} + 3\frac{d^3 y_1}{dt^3} + 5\frac{d^2 y_1}{dt^2} + 3\frac{dy_1}{dt} + y_1(t)\right]}_{\text{Term I}}$$

$$+ \beta\underbrace{\left[\frac{d^4 y_2}{dt^4} + 3\frac{d^3 y_2}{dt^3} + 5\frac{d^2 y_2}{dt^2} + 3\frac{dy_2}{dt} + y_2(t)\right]}_{\text{Term II}} + [1 - \alpha - \beta]$$

which implies that $$y_3(t) \neq \alpha y_1(t) + \beta y_2(t).$$

The system is, therefore, NOT linear. Note that the dc term of (+ 1) on the right hand side of the differential equation contributes to the nonlinearity of the system

(b) Time-invariance: The system is time-invariant. The proof is similar to Problem 2.1.

(b) Time-invariance: For $x_1(t)$ and $x_2(t) = x_1(t - T)$ applied as the inputs, the outputs are given by

$$\frac{d^4 y_1}{dt^4} + 3\frac{d^3 y_1}{dt^3} + 5\frac{d^2 y_1}{dt^2} + 3\frac{dy_1}{dt} + y_1(t) = \frac{d^2 x_1}{dt^2} + 2x_1(t) + 1 \tag{S2.9.3}$$

$$\frac{d^4 y_2}{dt^4} + 3\frac{d^3 y_2}{dt^3} + 5\frac{d^2 y_2}{dt^2} + 3\frac{dy_2}{dt} + y_2(t) = \frac{d^2 x_2}{dt^2} + 2x_2(t) + 1.$$

Substituting $x_2(t) = x_1(t - T)$ we obtain,

$$\frac{d^4 y_2}{dt^4} + 3\frac{d^3 y_2}{dt^3} + 5\frac{d^2 y_2}{dt^2} + 3\frac{dy_2}{dt} + y_2(t) = \frac{d^2 x_1(t - T)}{dt^2} + 2x_1(t - T) + 1. \tag{S2.9.4}$$

Substituting $\tau = t + T$ (which implies that $dt = d\tau$) in Eq. (S2.9.3), we obtain

$$\frac{d^4 y_1(\tau - T)}{d\tau^4} + 3\frac{d^3 y_1(\tau - T)}{d\tau^3} + 5\frac{d^2 y_1(\tau - T)}{d\tau^2} + 3\frac{dy_1(\tau - T)}{d\tau} + y_1(\tau - T) = \frac{d^2 x_1(\tau - T)}{d\tau^2} + 2x_1(\tau - T) + 1.$$

Or, $$\frac{d^4y_1(t-T)}{dt^4}+3\frac{d^3y_1(t-T)}{dt^3}+5\frac{d^2y_1(t-T)}{dt^2}+3\frac{dy_1(t-T)}{dt}+y_1(t-T)=\frac{d^2x_1(t-T)}{dt^2}+2x_1(t-T)+1.$$

Comparing with Eq. (S2.9.4), we obtain

$$y_2(t)=y_1(t-T),$$

proving that the system is time-invariant.

(c) Stablity: The system is BIBO stable since a bounded input will always produce a bounded output.

(d) Causality: Express Eq. (S2.2) as follows:

$$y(t)=-3\int_{-\infty}^{t}y(\alpha)d\alpha-5\int_{-\infty}^{t}\int_{-\infty}^{\tau}y(\alpha)d\alpha d\tau-3\int_{-\infty}^{t}\int_{-\infty}^{\tau}\int_{-\infty}^{\theta}y(\alpha)d\alpha d\tau d\theta-\int_{-\infty}^{t}\int_{-\infty}^{\tau}\int_{-\infty}^{\theta}\int_{-\infty}^{\phi}y(\alpha)d\alpha d\tau d\theta d\phi$$
$$+5\int_{-\infty}^{t}\int_{-\infty}^{\tau}x(\alpha)d\alpha d\tau+2\int_{-\infty}^{t}\int_{-\infty}^{\tau}\int_{-\infty}^{\theta}y(\alpha)d\alpha d\tau d\theta+\int_{-\infty}^{t}\int_{-\infty}^{\tau}\int_{-\infty}^{\theta}\int_{-\infty}^{\phi}d\alpha d\tau d\theta d\phi$$

The output $y(t)$ at $t = t_0$ is given by

$$y(t)\Big|_{t=t_0}=-3\int_{-\infty}^{t_0}y(\alpha)d\alpha-5\int_{-\infty}^{t_0}\int_{-\infty}^{\tau}y(\alpha)d\alpha d\tau-3\int_{-\infty}^{t_0}\int_{-\infty}^{\tau}\int_{-\infty}^{\theta}y(\alpha)d\alpha d\tau d\theta-\int_{-\infty}^{t_0}\int_{-\infty}^{\tau}\int_{-\infty}^{\theta}\int_{-\infty}^{\phi}y(\alpha)d\alpha d\tau d\theta d\phi$$
$$+5\int_{-\infty}^{t_0}\int_{-\infty}^{\tau}x(\alpha)d\alpha d\tau+2\int_{-\infty}^{t_0}\int_{-\infty}^{\tau}\int_{-\infty}^{\theta}y(\alpha)d\alpha d\tau d\theta+\int_{-\infty}^{t_0}\int_{-\infty}^{\tau}\int_{-\infty}^{\theta}\int_{-\infty}^{\phi}d\alpha d\tau d\theta d\phi$$

The system is causal since only the past values of the input $x(t)$, for $-\infty \le t \le t_0$, are needed to calculate the output $y(t)$ at $t = t_0$. ▌

Problem 2.10

(i) $y[k]=ax[k]+b$

(a) Linearity: Since

$$\begin{aligned}&x_1[k]\to ax_1[k]+b=y_1[k]\\&x_2[k]\to ax_2[k]+b=y_1[k]\\&\left(\alpha x_1[k]+\beta x_2[k]\right)\to a\left(\alpha x_1[k]+\beta x_2[k]\right)+b\neq\alpha y_1[k]+\beta y_2[k]\end{aligned}$$

the system is NOT a linear system.

(b) Time Invariance: For inputs $x_1[k]$ and $x_2[k] = x_1[k-K]$, the outputs are given by

$$\begin{aligned}&x_1[k]\to ax_1[k]+b=y_1[k]\\&x_2[k]=x_1[k-K]\to ax_2[k+2]+b=y_2[k]\end{aligned}$$

We note that $$y_2[k]=ax_2[k]+b=ax_1[k-K+2]+b=y_1[k-K]$$

which implies that the system is time invariant.

(c) Stability: Assume that the input is bounded $|x[k]| \le M$. Then, the output

$$|y[k]|=|ax[k]+b|\le|ax[k]|+|b|\le aM+|b|$$

is also bounded. Therefore, the system is BIBO stable.

(d) Causality: Since the output requires only the current value of input, the system is causal.

(ii) $y[k] = 5x[3k-2]$

(a) Linearity: Since

$$\begin{aligned} &x_1[k] \to y_1[k] = 5x_1[3k-2] \\ &x_2[k] \to y_2[k] = 5x_2[3k-2] \\ &(\alpha x_1[k] + \beta x_2[k]) \to 5(\alpha x_1[3k-2] + \beta x_2[3k-2]) = \alpha y_1[k] + \beta y_2[k] \end{aligned}$$

the system is a linear system.

(b) Time Invariance: For inputs $x_1[k]$ and $x_2[k] = x_1[k-K]$, the outputs are given by

$$\begin{aligned} &x_1[k] \to 5x_1[3k-2] = y_1[k] \\ &x_2[k] = x_1[k-K] \to 5x_2[3k-2]\big|_{x_2[k]=x_1[k-K]} = 5x_1[3k-2-K] = y_2[k] \end{aligned}$$

We also note that $\quad y_1[k-K] = 5x_1[3(k-K)-2] \neq y_2[k]$

Therefore, the system is NOT time invariant.

(c) Stability: Assume that the input is bounded $|x[k]| \le M$. Then, the output

$$|y[k]| = |5x[3k-2]| = 5|x[3k-2]| \le 5M$$

is also bounded. Therefore, the system is BIBO stable.

(d) Causality: When $k \ge 2$, the output requires the future value of the input. Therefore, the system is NOT causal.

(iii) $y[k] = 2^{x[k]}$

(a) Linearity: Since

$$\begin{aligned} &x_1[k] \to 2^{x_1[k]} = y_1[k] \\ &x_2[k] \to 2^{x_2[k]} = y_2[k] \\ &(\alpha x_1[k] + \beta x_2[k]) \to 2^{\alpha x_1[k] + \beta x_2[k]} = 2^{\alpha x_1[k]} 2^{\beta x_2[k]} = \left(2^{x_1[k]}\right)^{\alpha} \left(2^{x_2[k]}\right)^{\beta} = (y_1[k])^{\alpha} (y_2[k])^{\beta} \\ &\qquad \neq \alpha y_1[k] + \beta y_2[k] \end{aligned}$$

the system is NOT a linear system.

(b) Time Invariance: For inputs $x_1[k]$ and $x_2[k] = x_1[k-K]$, the outputs are given by

$$\begin{aligned} &x_1[k] \qquad\qquad \to 2^{x_1[k]} = y_1[k] \\ &x_2[k] = x_1[k-K] \to 2^{x_2[k]} = y_2[k] \end{aligned}$$

Note that $\quad y_2[k] = 2^{x_2[k]} = 2^{x_1[k-K]}$

and $\quad y_1[k-K] = 2^{x_1[k]}\Big|_{k=k-K} = 2^{x_1[k-K]}.$

Since $y_2[k] = y_1[k-K]$, the system is time invariant.

(c) Stability: Assume that the input is bounded $|x[k]| \le M$. Then, the output

$$|y[k]| = \left|2^{x[k]}\right| \le 2^{|x[k]|} \le 2^{M}$$

is also bounded. Therefore, the system is BIBO stable.

(d) Causality: Since the output requires only the current value of the input, therefore, the system is causal.

(iv) $y[k] = \sum_{m=-\infty}^{k} x[m]$

(a) Linearity: Since

$$x_1[k] \to \sum_{m=-\infty}^{k} x_1[m] = y_1[k]$$

$$x_2[k] \to \sum_{m=-\infty}^{k} x_2[m] = y_2[k]$$

$$\begin{aligned}(\alpha x_1[k] + \beta x_2[k]) \to \sum_{m=-\infty}^{k} (\alpha x_1[m] + \beta x_2[m]) &= \alpha \sum_{m=-\infty}^{k} x_1[m] + \beta \sum_{m=-\infty}^{k} x_2[m] \\ &= \alpha y_1[k] + \beta y_2[k]\end{aligned}$$

the system is a linear system.

(b) Time Invariance: For inputs $x_1[k]$ and $x_2[k] = x_1[k-K]$, the outputs are given by

$$x_1[k] \to \sum_{m=-\infty}^{k} x_1[m] = y_1[k]$$

$$x_2[k] = x_1[k-K] \to \sum_{m=-\infty}^{k} x_2[m] = y_2[k]$$

Note that $$y_2[k] = \sum_{m=-\infty}^{k} x_2[m] = \sum_{m=-\infty}^{k} x_1[m-K] = \sum_{m=-\infty}^{k-K} x_1[m] = y_1[k-K]$$

And hence the system is time invariant.

(c) Stability: Assume that the input is bounded $|x[k]| \le M$. Then, the output

$$|y[k]| = \left|\sum_{m=-\infty}^{k} x[m]\right| \le \sum_{m=-\infty}^{k} |x[m]|$$

may become unbounded as an infinite number of absolute values of $x[k]$ are being added. Therefore, the system is NOT BIBO stable.

(d) Causality: Since the output does not depend on the future values of the input, therefore, the system is causal.

(v) $y[k] = \sum_{m=k-2}^{k+2} x[m] - 2\,|\,x[k]\,|$

(a) Linearity: Since

$$x_1[k] \to \sum_{m=k-2}^{k+2} x_1[m] - 2|x_1[k]| = y_1[k]$$
$$x_2[k] \to \sum_{m=k-2}^{k+2} x_2[m] - 2|x_2[k]| = y_2[k]$$
$$(\alpha x_1[k] + \beta x_2[k]) \to \sum_{m=k-2}^{k+2} (\alpha x_1[m] + \beta x_2[m]) - 2|(\alpha x_1[k] + \beta x_2[k])|$$
$$\neq \alpha\left(\sum_{m=k-2}^{k+2} x_1[m] - 2|x_1[k]|\right) + \beta\alpha\left(\sum_{m=k-2}^{k+2} x_2[m] - 2|x_2[k]|\right)$$

the system is NOT a linear system. Note that the absolute term on the right hand side of the equation makes the system nonlinear.

(b) Time Invariance: For inputs $x_1[k]$ and $x_2[k] = x_1[k - K]$, the outputs are given by

$$x_1[k] \to \sum_{m=k-2}^{k+2} x_1[m] - 2|x_1[k]| = y_1[k]$$
$$x_2[k] = x_1[k-K] \to \sum_{m=k-2}^{k+2} x_2[m] - 2|x_2[k]| = y_2[k]$$

Note that

$$y_2[k] = \left[\sum_{m=k-2}^{k+2} x_2[m] - 2|x_2[k]|\right]_{x_2[k]=x_1[k-K]}$$
$$= \sum_{m=k-2}^{k+2} x_1[m-K] - 2|x_1[k-K]| = \sum_{p=k-K-2}^{k-K+2} x_1[p] - 2|x_1[k-K]|$$

which is the same as $y_1[k - K]$. Therefore, the system is time invariant.

(c) Stability: Assume that the input is bounded $|x[k]| \le M$. Then, the output

$$|y[k]| = \left|\sum_{m=k-2}^{k+2} x[m] - 2|x[k]|\right| \le \sum_{m=k-2}^{k+2} |x[m]| + 2|x[k]| \le 7M$$

is also bounded. Therefore, the system is BIBO stable.

(d) Causality: Since the output requires future values of the input, therefore, the system is NOT causal.

(vi) $y[k] + 5y[k-1] + 9y[k-2] + 5y[k-3] + y[k-4] = 2x[k] + 4x[k-1] + 2x[k-2]$

(a) Linearity: For $x_1[k]$ applied as the input, the output $y_1[k]$ is given by

$$y_1[k] + 5y_1[k-1] + 9y_1[k-2] + 5y_1[k-3] + y_1[k-4] = 2x_1[k] + 4x_1[k-1] + 2x_1[k-2].$$

For $x_2[k]$ applied as the input, the output $y_2[k]$ is given by

$$y_2[k] + 5y_2[k-1] + 9y_2[k-2] + 5y_2[k-3] + y_2[k-4] = 2x_2[k] + 4x_2[k-1] + 2x_2[k-2].$$

For $x_3[k] = \alpha\, x_1[k] + \beta\, x_2[k]$ applied as the input, the output $y_3[k]$ is given by

$$y_3[k]+5y_3[k-1]+9y_3[k-2]+5y_3[k-3]+y_3[k-4]$$
$$=2(\alpha x_1[k]+\beta x_2[k])+4(\alpha x_1[k-1]+\beta x_2[k-1])+2(\alpha x_1[k-2]+\beta x_2[k-2]),$$

which can be expressed as

$$y_3[k]+5y_3[k-1]+9y_3[k-2]+5y_3[k-3]+y_3[k-4]$$
$$=\alpha(2x_1[k]+4x_1[k-1]+2x_1[k-2])+\beta(2x_1[k]+4x_1[k-1]+2x_1[k-2]).$$

Substituting the value of the input terms, we get

$$y_3[k]+5y_3[k-1]+9y_3[k-2]+5y_3[k-3]+y_3[k-4]$$
$$=\alpha(y_1[k]+5y_1[k-1]+9y_1[k-2]+5y_1[k-3]+y_1[k-4])$$
$$+\beta(y_2[k]+5y_2[k-1]+9y_2[k-2]+5y_2[k-3]+y_2[k-4])$$

or,

$$y_3[k]+5y_3[k-1]+9y_3[k-2]+5y_3[k-3]+y_3[k-4]$$
$$=(\alpha y_1[k]+\beta y_2[k])+5(\alpha y_1[k-1]+\beta y_2[k-1])+9(\alpha y_1[k-2]+\beta y_2[k-2])$$
$$+5(\alpha y_1[k-3]+\beta y_2[k-3])+(\alpha y_1[k-4]+\beta y_2[k-4]),$$

which implies that $y_3[k] = \alpha\, y_1[k] + \beta\, y_2[k]$. Therefore, the system is linear.

(b) Time Invariance: For inputs $x_1[k]$ and $x_2[k] = x_1[k - K]$, the outputs are given by

$$x_1[k] \rightarrow\ y_1[k]+5y_1[k-1]+9y_1[k-2]+5y_1[k-3]+y_1[k-4]=2x_1[k]+4x_1[k-1]+2x_1[k-2]$$

$$x_2[k] = x_1[k-K] \rightarrow$$

$$y_2[k]+5y_2[k-1]+9y_2[k-2]+5y_2[k-3]+y_2[k-4]=2x_2[k]+4x_2[k-1]+2x_2[k-2]$$

From the above equations, we can prove that $y_2[k] = y_1[k - K]$ implying that the system is time invariant.

(c) Stability: Assume that the input is bounded $|x[k]| \le M$. Taking the absolute value of both sides of the difference equation, we get

$$\left|y_1[k]+5y_1[k-1]+9y_1[k-2]+5y_1[k-3]+y_1[k-4]\right|=\left|2x_1[k]+4x_1[k-1]+2x_1[k-2]\right|$$

implying that $\left|y_1[k]+5y_1[k-1]+9y_1[k-2]+5y_1[k-3]+y_1[k-4]\right|\le 8M$.

Since sum of several output samples are bounded, $y[k]$ itself must be bounded. Therefore, the system is BIBO stable.

(d) Causality: Since the output can be computed iteratively from

$$y[k]=(2x[k]+4x[k-1]+2x[k-2])-(5y[k-1]+9y[k-2]+5y[k-3]+y[k-4]),$$

which requires only the previous values of the input, therefore, the system is causal.

(vii) $y[k]=0.5x[6k-2]+0.5x[6k+2]$

(a) Linearity: For $x_1[k]$ applied as the input, the output $y_1[k]$ is given by

$$y_1[k]=0.5x_1[6k-2]+0.5x_1[6k+2].$$

For $x_2[k]$ applied as the input, the output $y_2[k]$ is given by

$$y_2[k]=0.5x_2[6k-2]+0.5x_2[6k+2].$$

For $x_3[k] = \alpha\, x_1[k] + \beta\, x_2[k]$ applied as the input, the output $y_3[k]$ is given by

$$y_3[k] = 0.5\big(\alpha x_1[6k-2] + \beta x_2[6k-2]\big) + 0.5\big(\alpha x_1[6k+2] + \beta x_2[6k+2]\big)$$

which can be expressed as

$$y_3[k] = \alpha\underbrace{\big(0.5x_1[6k-2] + 0.5x_1[6k+2]\big)}_{y_1[k]} + \beta\underbrace{\big(0.5x_2[6k-2] + 0.5x_2[6k+2]\big)}_{y_2[k]},$$

implying that $y_3[k] = \alpha\, y_1[k] + \beta\, y_2[k]$. Therefore, the system is linear.

(b) Time Invariance: For inputs $x_1[k]$ and $x_2[k] = x_1[k-K]$, the outputs are given by

$$x_1[k] \to y_1[k] = 0.5x_1[6k-2] + 0.5x_1[6k+2]$$
$$x_2[k] = x_1[k-K] \to 0.5x_2[6k-2] + 0.5x_2[6k+2] = y_2[k]$$

The second equation implies that

$$\begin{aligned} y_2[k] &= \big[0.5x_2[6k-2] + 0.5x_2[6k+2]\big]_{x_2[k]=x_1[k-K]} \\ &= \big[0.5x_1[6k-2-K] + 0.5x_2[6k+2-K]\big] \end{aligned}$$

We also note that $\quad y_1[k-K] = 0.5x_1[6k-6K-2] + 0.5x_1[6k-6K+2]$

Since $y_2[k] \neq y_1[k-K]$, the system is NOT time invariant.

(c) Stability: Assuming that the input is bounded $|x[k]| \le M$, the output

$$|y_1[k]| = |0.5x_1[6k-2] + 0.5x_1[6k+2]| \le |0.5x_1[6k-2]| + |0.5x_1[6k-2]| \le M$$

is also bounded. Therefore, the system is BIBO stable.

(d) Causality: Since the output requires future values of the input, the system is NOT causal. ▌

Problem 2.11

(i) Using the linearity property

$$5x(t) \to 5y(t).$$

The output waveform is drawn in Fig. S2.11(a).

(ii) Using the time invariance property

$$x(t-1) \to y(t-1)$$

and $$x(t+1) \to y(t+1).$$

Using the above result and the linearity property

$$\big[0.5x(t-1) + 0.5x(t+1)\big] \to \big[0.5y(t-1) + 0.5y(t+1)\big] \;.$$

The output waveform is drawn in Fig. S2.11(b).

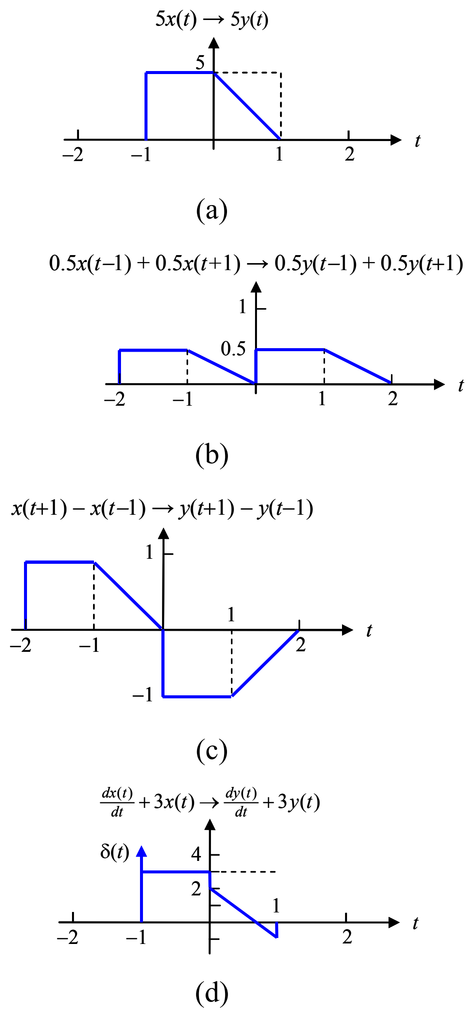

Fig. S2.11: Output waveforms for CT transformations in Problem 2.11.

(iii) Using the time invariance property, we obtain

$$x(t-1) \to y(t-1)$$

and $$x(t+1) \to y(t+1).$$

Using the above result and the linearity property, we obtain

$$\left[x(t+1) - x(t-1)\right] \to \left[y(t+1) + y(t-1)\right].$$

The output waveform is drawn in Fig. S2.11(c).

(iv) Since differentiation is a linear operation,

$$\frac{dx(t)}{dt} \to \frac{dy(t)}{dt}$$

Using the above result and the linearity property

$$\left[\frac{dx(t)}{dt}+3x(t)\right] \rightarrow \left[\frac{dy(t)}{dt}+3y(t)\right].$$

The output waveform is drawn in Fig. S2.11(d). ∎

Problem 2.12

In terms of the DT impulse function $\delta[k]$, the output can be represented by

$$y[k]=4\delta[k+2]-2\delta[k+1]+4\delta[k]+2\delta[k-1]+4\delta[k-2].$$

(i) Using the linearity and time invariance properties, we obtain

$$4x[k-1]\rightarrow 4y[k-1]=16\delta[k+1]-8\delta[k]+16\delta[k-1]+8\delta[k-2]+16\delta[k-3].$$

The output waveform is drawn in Fig. S2.12(a).

(ii) Using the time invariance property, we obtain

$$x[k-2]\rightarrow y[k-2]=4\delta[k]-2\delta[k-1]+4\delta[k-2]+2\delta[k-3]+4\delta[k-4]$$

and $$x[k+2]\rightarrow y[k+2]=4\delta[k+4]-2\delta[k+3]+4\delta[k+2]+2\delta[k+1]+4\delta[k].$$

Using the above result and the linearity property, we obtain

$$[0.5x[k-2]+0.5x[k+2]]\rightarrow[0.5y[k-2]+0.5y[k+2]]$$

or, $$[0.5x[k-2]+0.5x[k+2]]\rightarrow 2\delta[k+4]-\delta[k+3]+2\delta[k+2]+\delta[k+1]$$
$$+4\delta[k]-\delta[k-1]+2\delta[k-2]+\delta[k-3]+2\delta[k-4].$$

The output waveform is drawn in Fig. S2.12(b).

(iii) Using the time invariance property, we obtain

$$x[k+1]\rightarrow y[k+1]=4\delta[k+3]-2\delta[k+2]+4\delta[k+1]+2\delta[k]+4\delta[k-1]$$

and $$x[k-1]\rightarrow y[k-1]=4\delta[k+1]-2\delta[k]+4\delta[k-1]+2\delta[k-2]+4\delta[k-3].$$

Using the above result and the linearity property, we obtain

$$[x[k-1]-2x[k]+x[k+1]]\rightarrow[y[k-1]-2y[k]+y[k+1]]$$

or, $$[x[k-1]-2x[k]+x[k+1]]\rightarrow 4\delta[k+3]-10\delta[k+2]+12\delta[k+1]-8\delta[k]+4\delta[k-1]-6\delta[k-2]+4\delta[k-3].$$

The output waveform is drawn in Fig. S2.12(c).

(iv) Using the time invariance property, we obtain

$$x[-k]\rightarrow y[-k]=4\delta[-k+2]-2\delta[-k+1]+4\delta[-k]+2\delta[-k-1]+4\delta[-k-2].$$

Using the property $\delta[-k+m]=\delta[k-m]$, the above equation reduces to

$$x[-k]\rightarrow y[-k]=4\delta[k-2]-2\delta[k-1]+4\delta[k]+2\delta[k+1]+4\delta[k+2]$$

The output waveform is drawn in Fig. S2.12(d). ∎

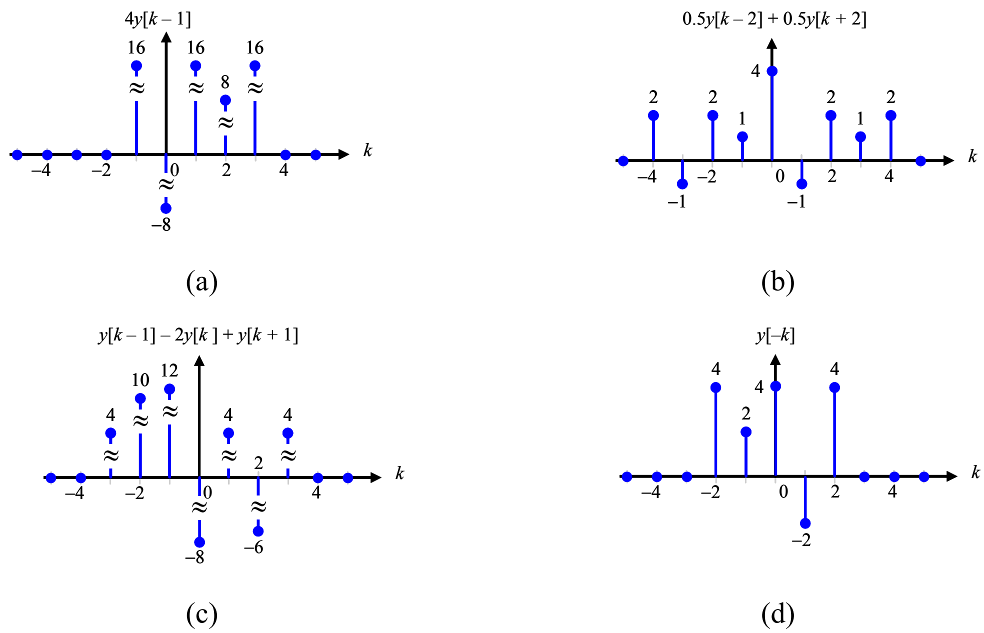

Fig. S2.12: Output waveforms for DT transformations in Problem 2.12.

Problem 2.13

(i) The system is invertible with the inverse system given by

$$x(t) = \tfrac{1}{3}y(t-2).$$

(ii) To calculate the inverse system, we differentiate the integral to get

$$\frac{dy(t)}{dt} = x(t-10).$$

The inverse system is obtained through two steps. Step 1 compute $z(t) = dy/dt$, while Step 2 computes $x(t)$ from the relationship $x(t) = z(t + 10)$.

(iii) The system $y(t) = |x(t)|$ is not invertible as $x(t) = \pm a$ produces the same output $y(t) = a$.

(iv) If $y(t)$ is differentiable then $x(t)$ can always be calculated uniquely from the expression

$$x(t) = \frac{dy(t)}{dt} + y(t)$$

and the system is invertible. However, if $y(t)$ is not differentiable (for example, it contains a discontinuity), then $x(t)$ cannot always be calculated uniquely and the system is not invertible.

(v) System represented by $y(t) = \cos(2\pi x(t))$ is not invertible as different values of $x(t) = (\theta + 2m\pi)$, where m is an integer, produce the same output.

Problem 2.14

(i) Not invertible as $y[-1]$ is always 0 irrespective of the value of $x[k]$.

(ii) The input-output relationship can be expressed as:

$$y[0] = x[2]$$
$$y[-1] = y[1] = x[2] + x[3]$$
$$y[-2] = y[2] = x[2] + x[3] + x[4]$$
....

It is observed that the output $y[k],\ -\infty < k < \infty$ depends on the $x[k],\ k \ge 2$. From the output values, the value of $x[k],\ k \ge 2$ can be uniquely calculated. However, the input value $x[k],\ k < 2$ cannot be calculated from $y[k]$. Therefore the system is not invertible.

(iii) The system is not invertible as

$$y[k] = \begin{cases} x[k] & k = 0, \pm 2, \pm 4, ... \\ 0 & \text{elsewhere} \end{cases}$$

All odd values of $x[k]$ are lost and can not be recovered from $y[k]$.

(iv) System is invertible with the inverse system given by

$$x[k+2] = y[k] - \left(2x[k+1] - 6x[k] + 2x[k-1] + x[k-2]\right),$$
$$\text{Or, } x[k] = y[k-2] - 2x[k-1] + 6x[k-2] - 2x[k-3] - x[k-4].$$

(v) System is invertible with the inverse system given by

$$x[k] = y[k] + 2y[k-1] + y[k-2].$$

<u>Problem 2.15</u>

The proof can be made as follows.

$$x(t) \to y(t)$$
$$\text{or, } x(t+\Delta t) \to y(t+\Delta t) \qquad [\text{time invariance property}]$$
$$\text{or, } \frac{x(t+\Delta t) - x(t)}{\Delta t} \to \frac{y(t+\Delta t) - y(t)}{\Delta t} \qquad [\text{linearity property}]$$
$$\text{or, } \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t} \to \lim_{\Delta t \to 0} \frac{y(t+\Delta t) - y(t)}{\Delta t}$$
$$\text{or, } \frac{dx}{dt} \to \frac{dy}{dt}$$

<u>Problem 2.16</u>

The periodic signal $x_p(t)$ can be expressed in terms of $x(t)$ as follows

$$x_p(t) = \sum_{m=-\infty}^{\infty} x(t-2m), \qquad m = \text{integers}.$$

Applying the linearity property, the output for the periodic signal $x_p(t)$ is given by

$$x_p(t) = \sum_{m=-\infty}^{\infty} x(t-2m) \to \sum_{m=-\infty}^{\infty} y(t-2m) = y_p(t)$$

which is shown in Fig. S2.16.

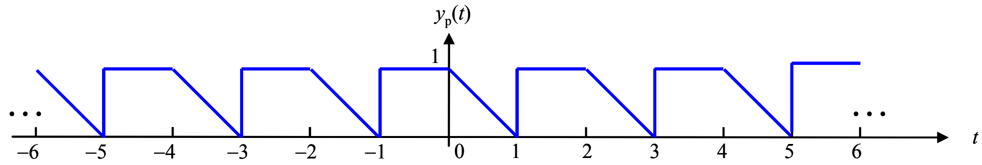

Fig. S2.16: Output $y_p(t)$ for the periodic signal $x_p(t)$ in Problem 2.16.

Problem 2.17

The impulse response of the system is defined in the problem as follows:

$$\delta(t) \to h(t) \text{ or, } h(t) = y(t)\big|_{x(t)=\delta(t)}.$$

(i) Substituting $x(t) = \delta(t)$, we obtain

$$h(t) = \delta(t+2) - 2\delta(t) + 2\delta(t-2).$$

(ii) Substituting $x(t) = \delta(t)$, we obtain

$$h(t) = \int_{t-t_o}^{t+t_o} \delta(\tau-4)d\tau = \int_{t-t_o-4}^{t+t_o-4} \delta(\lambda)d\lambda = \begin{cases} 1 & t-t_o-4 < 0 < t+t_o-4 \\ 0 & otherwise \end{cases}.$$

$$= \begin{cases} 1 & 4-t_o < t < 4+t_o \\ 0 & otherwise. \end{cases}$$

(iii) Substituting $x(t) = \delta(t)$, we obtain

$$h(t) = \int_{-\infty}^{t} e^{-2(t-\tau)}\delta(\tau-4)d\tau = \int_{-\infty}^{t} e^{-2(t-4)}\delta(\tau-4)d\tau = e^{-2(t-4)}\int_{-\infty}^{t} \delta(\tau-4)d\tau$$

Noting that $\int_{-\infty}^{t} \delta(\tau-4)d\tau = \begin{cases} 1 & t>4 \\ 0 & otherwise \end{cases} = u(t-4)$, the impulse response is obtained as follows:

$$h(t) = e^{-2(t-4)}u(t-4).$$

(iv) Substituting $x(t) = \delta(t)$, we obtain

$$h(t) = \int_{-\infty}^{\infty} f(T-\tau)\delta(t-\tau)d\tau = \int_{-\infty}^{\infty} f(T-t)\delta(t-\tau)d\tau = f(T-t)\underbrace{\int_{-\infty}^{\infty} \delta(t-\tau)d\tau}_{=1}$$

$$= f(T-t)$$

Problem 2.18

The output $h[k]$ of a DT LTI system to an unit impulse function $\delta[k]$ is shown in Fig. P2.18. Find the output for the following set of inputs.

(i) $x[k] = \delta[k+1] + \delta[k] + \delta[k-1]$

(ii) $x[k] = \sum_{m=-\infty}^{\infty} \delta[k-4m]$

(iii) $x[k] = u[k]$

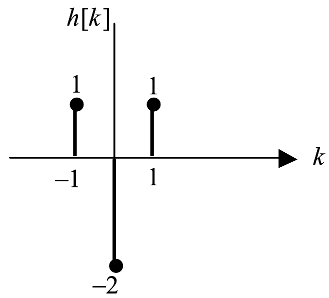

Fig. P2.18: Output $h[k]$ for input $x[k] = \delta[k]$ in Problem 2.18.

Solution:

In terms of the DT impulse function $\delta[k]$, the impulse response $h[k]$ is expressed as

$$\delta[k] \to h[k] = \delta[k+1] - 2\delta[k] + \delta[k-1].$$

(i) Using the linearity and time invariance property

$$\begin{aligned} \delta[k] &\to h[k] = \delta[k+1] - 2\delta[k] + \delta[k-1] \\ \delta[k+1] &\to h[k+1] = \delta[k+2] - 2\delta[k+1] + \delta[k] \\ \text{and} \quad \delta[k-1] &\to h[k-1] = \delta[k] - 2\delta[k-1] + \delta[k-2] \end{aligned}$$

Using the linearity property

$$\begin{aligned} &\delta[k+1] + \delta[k] + \delta[k-1] \to h[k+1] + h[k] + h[k-1] \\ \text{or,} \quad &\delta[k+1] + \delta[k] + \delta[k-1] \to \delta[k+2] - \delta[k+1] - \delta[k-1] + \delta[k-2]. \end{aligned}$$

(ii) Using the linearity and time invariance property

$$\sum_{m=-\infty}^{\infty} \delta[k-4m] \to \sum_{m=-\infty}^{\infty} h[k-4m] = y[k]$$

which will be a periodic signal (with period $K = 4$). One period of the output signal $y[k]$ is given below.

$$y[k] = \begin{cases} 1 & k = -1 \\ -2 & k = 0 \\ 1 & k = 1 \\ 0 & k = 2. \end{cases}$$

(iii) Recall that

$$u[k] = \sum_{m=0}^{\infty} \delta[k-m] \to y[k] = \sum_{m=0}^{\infty} h[k-m],$$

which results in the output

$$y[k] = h[k] + h[k-1] + h[k-2] + h[k-3] + \cdots$$

Substituting different values of k, we get

($k = -2$): $y[-2] = h[-2] + h[-3] + h[-4] + h[-5] + \ldots = 0.$

($k = -1$): $y[-1] = h[-1] + h[-2] + h[-3] + h[-4] + \ldots = 1.$

($k = 0$): $y[0] = h[0] + h[-1] + h[-2] + h[-3] + \ldots = -1.$

($k = 1$): $y[1] = h[1] + h[0] + h[-1] + h[-2] + \ldots = 0.$

By expanding for other values of k, it can be shown that $y[k] = 0$ for $|k| \geq 2$. In other words,

$$y[k] = \delta[k+1] - \delta[k].$$

Problem 2.19

(i) Substituting $x[k] = \delta[k]$, we obtain

$$y[k] = \delta[k] - 2\delta[k-1] + \delta[k-2].$$

(ii) Substituting $x[k] = \delta[k-1] + \delta[k+1]$, we get

$$\begin{aligned} y[k] &= \left(\delta[k-1] + \delta[k+1]\right) - 2\left(\delta[k-2] + \delta[k]\right) + \left(\delta[k-3] + \delta[k-1]\right) \\ &= \delta[k-3] - 2\delta[k-2] + 2\delta[k-1] - 2\delta[k] + \delta[k+1]. \end{aligned}$$

(iii) Express $x[k] = 3\delta[k+3] + 2\delta[k+2] + \delta[k+1] + \delta[k-1] + 2\delta[k-2] + 3\delta[k-3].$

Substituting the above value of $x[k]$ in the difference equation, we get

$$\begin{aligned} y[k] = &\left(3\delta[k+3] + 2\delta[k+2] + \delta[k+1] + \delta[k-1] + 2\delta[k-2] + 3\delta[k-3]\right) \\ &-2\left(3\delta[k+2] + 2\delta[k+1] + \delta[k] + \delta[k-2] + 2\delta[k-3] + 3\delta[k-4]\right) \\ &+\left(3\delta[k+1] + 2\delta[k] + \delta[k-1] + \delta[k-3] + 2\delta[k-4] + 3\delta[k-5]\right) \end{aligned}$$

or, $$y[k] = 3\delta[k+3] - 4\delta[k+2] + 2\delta[k-1] - 4\delta[k-4] + 3\delta[k-5].$$

Problem 2.20

(i) Linearity: For $x_1[k]$ applied as the input, the output $y_1[k]$ is given by

$$y_1[k] = \frac{1}{5}\sum_{m=0}^{4} x_1[k-m].$$

For $x_2[k]$ applied as the input, the output $y_2[k]$ is given by

$$y_2[k] = \frac{1}{5}\sum_{m=0}^{4} x_2[k-m]$$

For $x_3[k] = \alpha\, x_1[k] + \beta\, x_2[k]$ applied as the input, the output $y_3[k]$ is given by

$$y_3[k] = \frac{1}{5}\sum_{m=0}^{4}\left(\alpha x_1[k-m] + \beta x_2[k-m]\right) = \alpha \times \frac{1}{5}\sum_{m=0}^{4} x_1[k-m] + \beta \times \frac{1}{5}\sum_{m=0}^{4} x_2[k-m]$$

which implies that $y_3[k] = \alpha\, y_1[k] + \beta\, y_2[k]$. Therefore, the system is linear.

Time Invariance: For inputs $x_1[k]$ and $x_2[k] = x_1[k-K]$, the outputs are given by

$$x_1[k] \to y_1[k] = \frac{1}{5}\sum_{m=0}^{4} x_1[k-m]$$

and $$x_2[k]=x_1[k-K]\rightarrow y_2[k]=\frac{1}{5}\sum_{m=0}^{4}x_2[k-m]=\frac{1}{5}\sum_{m=0}^{4}x_1[k-K-m].$$

From the above equations, it is clear that $y_2[k] = y_1[k - K]$ implying that the system is time invariant.

(ii) The impulse response of the system is given by

$$h[k]=\frac{1}{5}\sum_{m=0}^{4}\delta[k-m]=0.2\left(\delta[k]+\delta[k-1]+\delta[k-2]+\delta[k-3]+\delta[k-4]\right)$$
$$=\begin{cases}0.2 & k=0,1,2,3,4\\ 0 & otherwise\end{cases}$$

(iii) By expressing

$$u[k]=\sum_{m=0}^{\infty}\delta[k-m]\rightarrow y[k]=\sum_{m=0}^{\infty}h[k-m],$$

the output $y[k]$ to the unit step function is given by

$$y[k]=h[k]+h[k-1]+h[k-2]+h[k-3]+h[k-4]+h[k-5]+\cdots$$

Substituting different values of k, we get

($k \le -1$): $y[k] = 0.$

($k = 0$): $y[0] = h[0] + h[-1] + h[-2] + h[-3] + h[-4] + h[-5] \ldots = 0.2.$

($k = 1$): $y[1] = h[1] + h[0] + h[-1] + h[-2] + h[-3] + h[-4] + \ldots = 0.4.$

($k = 2$): $y[2] = h[2] + h[1] + h[0] + h[-1] + h[-2] + h[-3] +\ldots = 0.6.$

($k = 3$): $y[3] = h[3] + h[2] + h[1] + h[0] + h[-1] + h[-2] + \ldots = 0.8.$

($k \ge 4$): $y[k] = 1$

(iv) Using the linearity and time invariance property

$$\delta[k]=\left(u[k]-u[k-1]\right)\rightarrow h[k]=\left(y[k]-y[k-1]\right)$$

which leads to

$$h[k]=\underbrace{\begin{cases}0 & k\le -1\\ 0.2 & k=0\\ 0.4 & k=1\\ 0.6 & k=2\\ 0.8 & k=3\\ 1.0 & k\ge 4\end{cases}}_{y[k]}-\underbrace{\begin{cases}0 & k\le 0\\ 0.2 & k=1\\ 0.4 & k=2\\ 0.6 & k=3\\ 0.8 & k=4\\ 1.0 & k\ge 5\end{cases}}_{y[k-1]}=\begin{cases}0 & k\le -1\\ 0.2 & 0\le k\le 4\\ 0 & k\ge 5.\end{cases}$$

Note that the above value of the impulse response is the same as the value obtained in part (ii). ▌

Problem 2.21

(i) The linearity and time invariance properties can be proved directly from the definitions. Instead, we use the property of linear, constant coefficient finite difference equations, which always represent linear and time invariant systems.

(ii) Series Configuration: Denoting the output of system S_1 by $w[k]$, we obtain

$$w[k] = x[k] - 2x[k-1] + x[k-2] .$$

For system S2, $w[k]$ is the input and $y[k]$ is the output. The output in given by

$$y[k] = w[k] + w[k-1] - 2w[k-2] .$$

Substituting the value of $w[k]$ from the earlier equation, we obtain

$$y[k] = (x[k] - 2x[k-1] + x[k-2]) + (x[k-1] - 2x[k-2] + x[k-3]) - 2(x[k-2] - 2x[k-3] + x[k-4])$$

which reduces to

$$y[k] = x[k] - x[k-1] - 3x[k-2] + 5x[k-3] - 2x[k-4] .$$

(iii) Parallel Configuration: Denoting the output of system S_1 by $w_1[k]$, we obtain

$$w_1[k] = x[k] - 2x[k-1] + x[k-2] .$$

For system S_2, $x[k]$ is the input and $w_2[k]$ is the output. The output is given by

$$w_2[k] = x[k] + x[k-1] - 2x[k-2] .$$

The overall output of the parallel configuration is given by

$$y[k] = w_1[k] + w_2[k] = (x[k] - 2x[k-1] + x[k-2]) + (x[k] + x[k-1] - 2x[k-2]),$$

which reduces to

$$y[k] = 2x[k] - x[k-1] - x[k-2] .$$

(iv) Since both series and parallel configurations are represented by linear, constant coefficient finite difference equations, both systems are linear and time invariant. ▌

Additional work:

Supplement to Problem 2.4 How do we demodulate the AM signal?

The modulated signal can be demodulated as follows:

Multiplying the modulated signal $s(t) = A[1 + km(t)]\cos(2\pi f_c t)$ with $\cos(2\pi f_c t)$, we get

$$s(t) = A[1 + km(t)]\cos^2(2\pi f_c t) = \underbrace{\tfrac{1}{2}A[1 + km(t)]}_{\text{Low Frequency Component}} + \underbrace{\tfrac{1}{2}A[1 + km(t)]\cos(4\pi f_c t)}_{\text{High Frequency Component}}.$$

By using a low pass filter, the high frequency component can be filtered out. The low frequency component can be used be extract the information bearing signal $m(t)$. ▮

Supplement to Problem 2.9 Determine if the systems are memoryless and invertible. If invertible, find the inverse system.

(i) $y(t) = x(t-2)$

Also, note that the system requires past memory and is not memoryless. Further, the system is invertible as the input can be reconstructed using the relationship $x(t) = y(t+2)$.

(ii) $y(t) = x(2t-5)$

Since all noncausal systems must have memory, the system is NOT memoryless. Furthermore, the system is invertible with the inverse system given by

$$y(t) = x(2t-5) \Rightarrow x(t) = y(\tfrac{t+5}{2}) = y(0.5t + 2.5).$$

(iii) $y(t) = x(2t) - 5$

Since all noncausal systems must have memory, the system is NOT memoryless. Furthermore the system is invertible with the inverse system given by

$$y(t) = x(2t) - 5 \quad \Rightarrow \quad x(2t) = y(t) + 5 \quad \Rightarrow \quad x(t) = y(0.5t) + 5.$$

(iv) $y(t) = tx(t+10)$

Since all noncausal systems must have memory, therefore, the system is NOT memoryless. Futhermore, the system is NOT invertible as its inverse

$$x(t) = \tfrac{y(t-10)}{t-10},$$

is not defined for $t = 10$. Therefore, the system is NOT invertible.

(v) $y(t) = 2u\left(x(t)\right) = \begin{cases} 2 & x(t) \geq 0 \\ 0 & x(t) < 0 \end{cases}$

Also note that the system is memoryless since the output at any time instant does not depend on the past or future values of the input. The system is not invertible since it is not possible to calculate the input $x(t)$ uniquely from $y(t)$. The output is always 0 for $x(t) < 0$. Likewise, the output is always 2 for $x(t) > 0$.

(vi) $y(t) = \begin{cases} 0 & t<0 \\ x(t) - x(t-5) & t \geq 0 \end{cases} = \left[x(t) - x(t-5)\right]u(t)$

The system is NOT memoryless since it requires past values of the input to compute the current value of the output. For ($t < 0$), the output is always 0. Hence, it is not possible to calculate the input $x(t)$ uniquely from $y(t)$ for ($t < 0$). Therefore, the system is NOT invertible.

(vii) $y(t) = 7x^2(t) + 5x(t) + 3$

The system is also memoryless since only the current value of the input is required to calculate the output. The system is NOT invertible as the inverse system given by

$$x(t) = \frac{-5 \pm \sqrt{25 - 28(3 - y(t)}}{14}$$

produces two possible values for inverting each value of $y(t)$.

(viii) $y(t) = \text{sgn}(x(t))$

The system is also memoryless since only the current value of the input is required to calculate the output. The system is NOT invertible because it is not possible to determine $x(t)$ from $y(t)$. All positive values of $x(t)$ produce the same output of +1, while all negative values of $x(t)$ produce the same output of −1.

(ix) $y(t) = \int_{-t_0}^{t_0} x(\lambda)d\lambda + 2x(t)$

The system is NOT memoryless as it is not causal. The system is also invertible.

(x) $y(t) = \int_{-\infty}^{t_0} x(\lambda)d\lambda + \frac{dx}{dt}$

The system is NOT memoryless as it is not causal.

The system is invertible.

(xi) $\frac{d^4y}{dt^4} + 3\frac{d^3y}{dt^3} + 5\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + y(t) = \frac{d^2x}{dt^2} + 2x(t) + 1$

The output $y(t)$ at $t = t_0$ is given by

$$y(t)\Big|_{t=t_0} = -3\int_{-\infty}^{t_0} y(\alpha)d\alpha - 5\int_{-\infty}^{t_0}\int_{-\infty}^{\tau} y(\alpha)d\alpha d\tau - 3\int_{-\infty}^{t_0}\int_{-\infty}^{\tau}\int_{-\infty}^{\theta} y(\alpha)d\alpha d\tau d\theta - \int_{-\infty}^{t_0}\int_{-\infty}^{\tau}\int_{-\infty}^{\theta}\int_{-\infty}^{\phi} y(\alpha)d\alpha d\tau d\theta d\phi$$
$$+5\int_{-\infty}^{t_0}\int_{-\infty}^{\tau} x(\alpha)d\alpha d\tau + 2\int_{-\infty}^{t_0}\int_{-\infty}^{\tau}\int_{-\infty}^{\theta} y(\alpha)d\alpha d\tau d\theta + \int_{-\infty}^{t_0}\int_{-\infty}^{\tau}\int_{-\infty}^{\theta}\int_{-\infty}^{\phi} d\alpha d\tau d\theta d\phi$$

The system has memory since the past values of the input $x(t)$, for $-\infty \le t \le t_0$, are needed to calculate the output $y(t)$ at $t = t_0$.

Invertiblity: The system is invertible. ▌

Supplement to Problem 2.10 Determine if the systems are memoryless and invertible.

(i) $y[k] = ax[k] + b$

Since the output requires only the current value of input, therefore, the system is both causal and memoryless.

Further, the system is invertible with the inverse system given by

$$x[k] = \tfrac{1}{a}\left(y[k] + b\right).$$

(ii) $y[k] = 5x[3k-2]$

The system is NOT memoryless as it is not causal. Further, the system is invertible with the inverse system given by

$$x[k] = \tfrac{1}{5}\, y\left[\tfrac{k+2}{3}\right].$$

(iii) $y[k] = 2^{x[k]}$

The system is memoryless as well as causal since the output does not depend on the past or future values of the input. Furthermore, the system is invertible with the inverse system given by

$$x[k] = \log_2\left(x[k]\right).$$

(iv) $y[k] = \sum_{m=-\infty}^{k} x[m]$

The system is NOT memoryless as the output requires past values of the input. Furthermore, the system is invertible with the inverse system given by

$$x[k] = y[k] - y[k-1].$$

(v) $y[k] = \sum_{m=k-2}^{k+2} x[m] - 2\,|\,x[k]\,|$

The system is NOT memoryless as it is not causal. Further, the system is NOT invertible because of the $|x[k]|$ term.

(vi) $y[k] + 5y[k-1] + 9y[k-2] + 5y[k-3] + y[k-4] = 2x[k] + 4x[k-1] + 2x[k-2]$

The system is NOT memoryless as it requires some past values of the input. Furthermore, the system is invertible.

(vii) $y[k] = 0.5x[6k-2] + 0.5x[6k+2]$

The system is NOT memoryless as it is not causal. Furthermore, the system is invertible. ▮