Chapter 8: Case Studies for CT Systems

Problem 8.1:

(a) The AM signal is given by

$$s(t) = A[1 + 3k\sin(2\pi f_1 t) + 2k\cos(2\pi f_2 t)]\cos(2\pi f_c t) .$$

To ensure that the envelope of $s(t) \geq 0$ for all t

$$(1 + 3k\sin(2\pi f_1 t) + 2k\cos(2\pi f_2 t)) \geq 0 .$$

Taking the worst case scenario, i.e., both $\sin(2\pi f_1 t)$ and $\cos(2\pi f_2 t)$ take their minimum values of -1 at the same time, we get

$$(1 - 3k - 2k) \geq 0$$

or,

$$k \leq 0.2 .$$

(b) Expressing $s(t) = A\cos(2\pi f_c t) + [3k\sin(2\pi f_1 t) + 2k\cos(2\pi f_2 t)]\cos(2\pi f_c t)$,

or,

$$s(t) = A\cos(2\pi f_c t) + \tfrac{3Ak}{2}\sin(2\pi(f_c + f_1)t) + \tfrac{3Ak}{2}\sin(2\pi(f_c - f_1)t) + Ak\cos(2\pi(f_c + f_2)t) + Ak\cos(2\pi(f_c - f_2)t).$$

Assuming $f_1 \neq f_2 \neq f_c$, the power in the modulated signal is composed of

$$\text{Power in the carrier} = \tfrac{1}{2}A^2$$

and

$$\text{Power in the modulating signal} = 2 \times \frac{(3Ak/2)^2}{2} + 2 \times \frac{(Ak)^2}{2} = \tfrac{13}{4}(Ak)^2 .$$

Hence, the ratio of power lost in the carrier and the total power = $\frac{1}{1+6.5k^2}$.

(c) The power spectrum of $s(t)$ is given by

$$\begin{aligned} S(f) = {} & A\pi\left[\delta(\omega - 2\pi f_c) + \delta(\omega + 2\pi f_c)\right] \\ & + j\tfrac{3Ak\pi}{2}\left[\delta(\omega + 2\pi(f_c + f_1)) - \delta(\omega - 2\pi(f_c + f_1))\right] \\ & + j\tfrac{3Ak\pi}{2}\left[\delta(\omega + 2\pi(f_c - f_1)) - \delta(\omega - 2\pi(f_c - f_1))\right] \\ & + Ak\pi\left[\delta(\omega + 2\pi(f_c + f_2)) + \delta(\omega - 2\pi(f_c + f_2))\right] \\ & + Ak\pi\left[\delta(\omega + 2\pi(f_c - f_2)) + \delta(\omega - 2\pi(f_c - f_2))\right]. \end{aligned}$$

For f_1 = 10 kHz, f_2 = 20 kHz, and f_c= 50 kHz, the power spectrum is shown in Fig. S8.1 (a).

(d) Signal $x(t)$ can be reconstructed using the synchronous detector shown in Fig. S8.1(b).

The information signal $x(t)$ can be extracted from the output of the above system by using an amplifier with a gain of $2/A^2$ and removing the dc offset. ▌

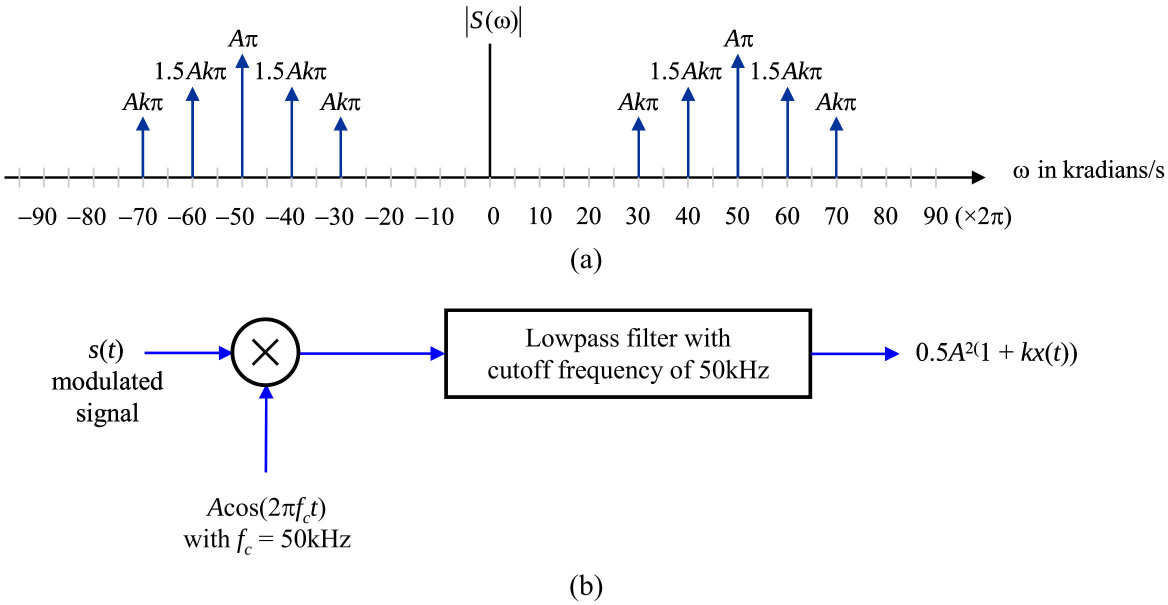


Figure S8.1: (a) Spectrum for the AM modulated signal $s(t)$ in Problem 8.1(c). (b) synchronous detector in Problem 8.1(d).

Problem 8.2:

(a) The AM signal is given by

$$s(t) = A[1 + k\,\mathrm{sinc}(5\times10^3 t)]\cos(2\pi f_c t)\,.$$

To ensure that the envelope of $s(t) \geq 0$ for all t

$$(1 + k\,\mathrm{sinc}(5\times10^3 t)) \geq 0\,.$$

The minimum value of $\mathrm{sinc}(\alpha t) = \sin(\pi\alpha t)/\pi\alpha t$ occurs at $\pi\alpha t = 3\pi/2$, or $t = 3/2\alpha$ and is given by $-2/3\pi$.

Therefore, $$(1 - 2k/3\pi) \geq 0\,,\ \text{or,}\ \ k \leq 3\pi/2\,.$$

(b) Expressing $$s(t) = A\cos(2\pi f_c t) + Ak\,\mathrm{sinc}(5\times10^3 t)\cos(2\pi f_c t)\,,$$

Assuming $f_c \neq 5 \times 10^3$, Power in the carrier $= \frac{1}{2}A^2$.

Because the information signal is a sinc function that decays with time, therefore, the average power in the modulating signal is zero. Instead, we compare the energy used to transmit the carrier and the information signal.

For simplicity, we assume that the sinc function has a duration of five side lobes on each side of the main lobe. The duration of the main lobe is 2×10^{-4} seconds, hence, the approximated duration of the sinc function is $12 \times 2 \times 10^{-4} = 2.4 \times 10^{-3}$ seconds. The energy consumed in transmitting the carrier is, therefore, given by $1.2 \times 10^{-3}A^2$.

To compute the energy in the modulating signal, we use the Parseval's theorem

$$\text{Energy in the modulating signal} = \int_{-\infty}^{\infty} g^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|G^2(\omega)\right|d\omega,$$

where $$g(t) = Ak\sin c(5\times 10^3 t)\cos(2\pi f_c t).$$

Calculating the CTFT, we obtain

$$G(\omega) = \frac{Ak}{2\pi}\frac{1}{5\times10^3}\text{rect}\left[\frac{\omega}{2\times5\times10^3\pi}\right] * \pi\left[\delta(\omega-2\pi f_c)+\delta(\omega+2\pi f_c)\right],$$

or, $$G(\omega) = 10^{-4}Ak\ \text{rect}\left[\frac{\omega-2\pi f_c}{10^4\pi}\right] + 10^{-4}Ak\ \text{rect}\left[\frac{\omega+2\pi f_c}{10^4\pi}\right].$$

Hence, $$\text{Energy in the modulating signal} = 10^{-4}(Ak)^2.$$

Ratio of energy lost in the carrier and the total energy = $\frac{1}{1+k^2/12}$.

(c) The power spectrum of $s(t)$ is given by

$$S(f) = A\pi\left[\delta(\omega-2\pi f_c)+\delta(\omega+2\pi f_c)\right]$$
$$+10^{-4}Ak\ \text{rect}\left[\frac{\omega-2\pi f_c}{10^4\pi}\right] + 10^{-4}Ak\ \text{rect}\left[\frac{\omega+2\pi f_c}{10^4\pi}\right].$$

For f_c= 20 kHz, the power spectrum is shown in Fig. S8.2(a).

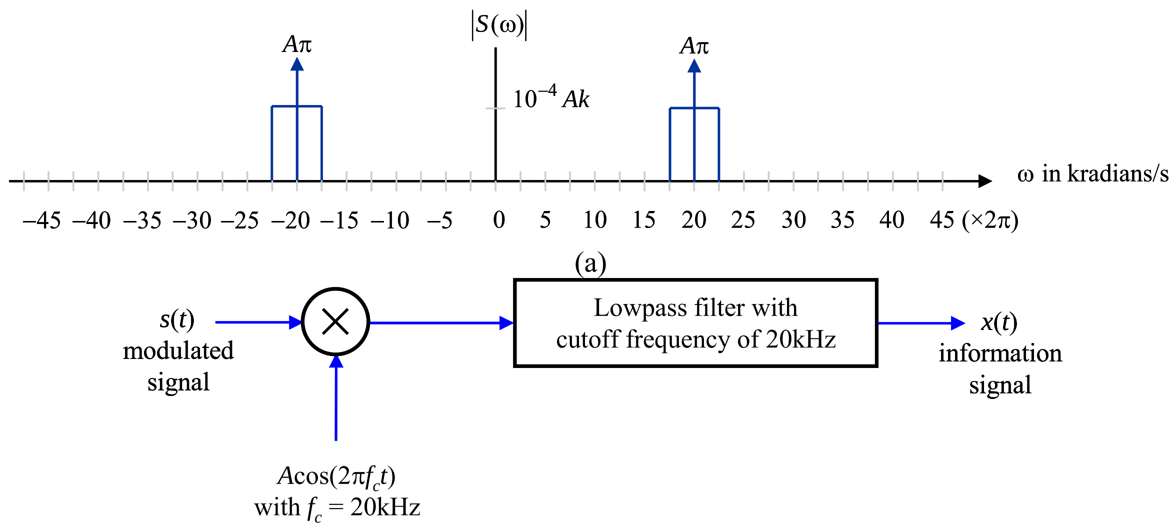


Figure S8.2: (a) Spectrum for the AM modulated signal $s(t)$ in Problem 8.2(c), (b) synchronous detector in Problem 8.2(d).

(d) Signal $x(t)$ can be reconstructed using the synchronous detector shown in Fig. S8.2(b).

The information signal $x(t)$ can be extracted from the output of the above system by using an amplifier with a gain of $2/A^2$ and removing the dc offset. ▮

Problem 8.3

For a sinusoidal tone as the information signal, the AM signal is given by

$$\begin{aligned} s(t) &= A[1 + k\sin(2\pi f_0 t)]\cos(2\pi f_c t) \\ &= A\cos(2\pi f_c t) + \tfrac{1}{2}Ak\sin(2\pi(f_0 + f_c)t) + \tfrac{1}{2}Ak\sin(2\pi(f_0 - f_c)t). \end{aligned}$$

The power used to transmit the carrier is given by $0.5A^2$, while the power used to transmit the modulating signal is given by $2 \times (Ak/2)^2/2 = 0.25\,A^2k^2$. The fraction of power in the information signal is given by

$$\eta = \frac{0.25A^2k^2}{0.5A^2 + 0.25A^2k^2} = \frac{k^2}{2 + k^2}.$$

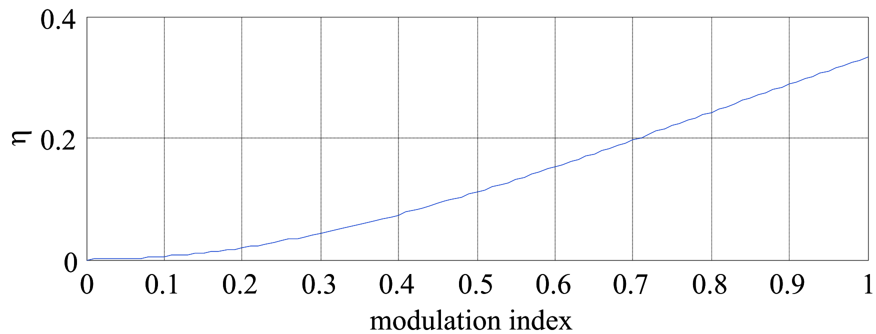


Figure S8.3: Relative power in the information signal as a function of the modulation index k .

Fig. S8.3 plots the fractional power in the information signal as a function of the modulation index k. For $k = 0.7$, $\eta = 0.1968$. The value of η increases as the modulation index k is increased.

Problem 8.4

The modulated signal is given by $s(t) = [1 + 2k\sin(2\pi f_1 t)]\cos(2\pi f_c t)$.

Demodulating with $\cos[2\pi(f_c + \Delta f)t]$ gives

$$s(t) \times c(t) = [1 + 2k\sin(2\pi f_1 t)]\cos(2\pi f_c t)\cos(2\pi(f_c + \Delta f)t),$$

which can be expressed as

$$s(t) \times c(t) = \tfrac{1}{2}[1 + 2k\sin(2\pi f_1 t)]\cos(2\pi\Delta f t) + \tfrac{1}{2}[1 + 2k\sin(2\pi f_1 t)]\cos(2\pi(2f_c + \Delta f)t).$$

The output $y(t)$ of the lowpass filter with a cutoff frequency of $2\,f_c$ Hz is given by

$$s(t) \times c(t) = \tfrac{1}{2}[1 + 2k\sin(2\pi f_1 t)]\cos(2\pi\Delta f t).$$

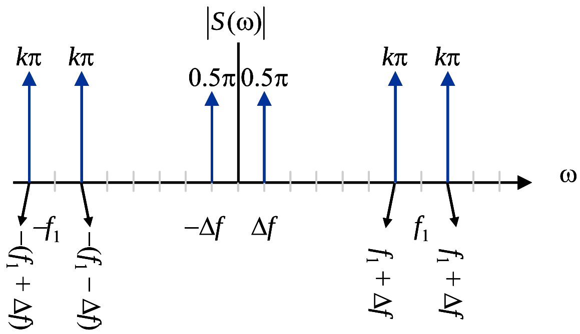
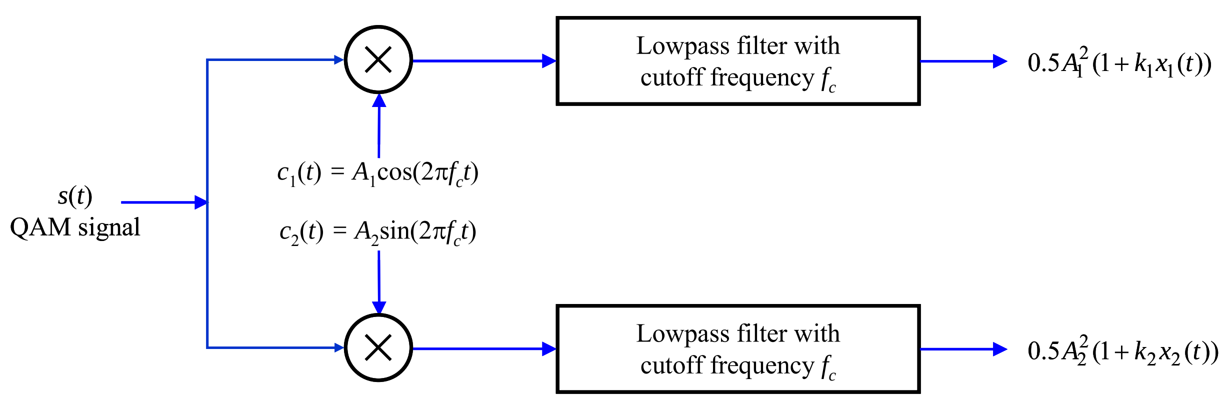


Figure S8.4: Spectrum for the demodulated signal $s(t)$ in Problem 8.4.

The magnitude spectrum of the demodulated signal is shown in Fig. S8.4. Distortion introduced by the difference in the frequencies of the carriers used at the modulator and demodulator is generally difficult to remove.

<u>Problem 8.5</u>

The following block diagram can be used to recover $x_1(t)$ and $x_2(t)$ directly from the QAM signal.



By removing the dc offset and adjusting the gain, the information signals $x_1(t)$ and $x_2(t)$ can be derived from the two outputs.

<u>Problem 8.6</u>

With $r = 0$, the input-output relationship for the spring damping system is given by

$$M\frac{d^2y}{dt} + ky(t) = x(t).$$

Taking the Laplace transform, we get

$$(Ms^2 + k)Y(s) = X(s),$$

which results in the transfer function

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{(Ms^2 + k)}.$$

The impulse response is obtained by expressing the transfer function as

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{M}\sqrt{M/k}\,\frac{\sqrt{k/M}}{(s^2 + k/M)},$$

and taking the inverse Laplace transform

$$H(s) = \frac{1}{\sqrt{kM}}\sin\left[\sqrt{k/M}\;t\right]u(t).$$

Note that the poles of the above system are located at the imaginary axis at $s = \pm j\sqrt{k/M}$. The system is, therefore, marginally stable. ▌

Problem 8.7

Integrating by parts, we obtain

$$\int k_m' t e^{-\omega_n t} dt = k_m' t \times \frac{e^{-\omega_n t}}{(-\omega_n)} - k_m' \times \frac{e^{-\omega_n t}}{(-\omega_n)^2} + C$$

which reduces to Eq. (8.42). ▌

Problem 8.8

With $L_a = 0$, the input-output relationship for the armature circuit is given by

$$L_a \frac{di_a}{dt} + R_a i_a + \underbrace{k_f \omega(t)}_{V_{\text{emf}}(t)} = V_a(t)\,.$$

The torque and load equations will remain same as Eqs. (8.30) and (8.34), and are given by

$$T_m = k_m i_a(t)$$

$$J\frac{d^2\theta}{dt^2} + r\frac{d\theta}{dt} = T_m - T_d \approx T_m \qquad [\text{assuming} T_d = 0]$$

Taking the Laplace transform of the above three equations, we obtain

$$R_a I_a(s) + k_f \Omega(s) = V_a(s)$$

$$T_m(s) = k_m I_a(s)$$

$$\left[Js^2 + rs\right]\theta(s) = T_m(s)\,.$$

Rearranging the terms in the above three equations, we obtain

$$H(s) = \frac{\theta(s)}{V_a(s)} = \frac{k_m}{R_a J s^2 + \left[R_a r + k_m k_f\right]s}$$

To obtain $h(t)$, we rearrange the terms of $H(s)$ as follows:

$$H(s) = \frac{k_m / R_a J}{s^2 + \frac{R_a r + k_m k_f}{R_a J} s} = \frac{\beta}{s^2 + \alpha s} \qquad \left[\alpha = \frac{R_a r + k_m k_f}{R_a J}, \beta = \frac{k_m}{R_a J} \right]$$

$$= \frac{\beta}{\alpha}\left[\frac{1}{s} - \frac{1}{s+\alpha}\right]$$

Calculating the inverse Laplace transform of $H(s)$, we obtain $h(t)$ as follows:

$$h(t) = \frac{\beta}{\alpha}\left(1 - e^{-\alpha t}\right)u(t).$$

The transfer function has two poles, at $s = 0$ and at $s = -\alpha$. Because, α is always positive, the second pole always lies in the left-half of the s-plane. As the first pole is located on the imaginary axis, the overall system is a marginally stable system. The system will be unstable only if a DC (constant) signal is applied at the input.

▮

Problem 8.9

Case I ($\xi_n > 1$): Express the impulse response as

$$h(t) = \frac{k'_m}{2\omega_n\sqrt{\xi_n^2 - 1}} \int e^{-\xi_n\omega_n t}\left[e^{\omega_n\sqrt{\xi_n^2-1}\,t} - e^{-\omega_n\sqrt{\xi_n^2-1}\,t} \right] dt$$

$$= \frac{k'_m}{2\omega_n\sqrt{\xi_n^2 - 1}} \left[\int e^{-(\xi_n - \sqrt{\xi_n^2-1})\omega_n t} dt - \int e^{-(\xi_n + \sqrt{\xi_n^2-1})\omega_n t} dt \right]$$

which reduces to

$$h(t) = \frac{k'_m}{2\omega_n\sqrt{\xi_n^2-1}} \times \frac{e^{-(\xi_n - \sqrt{\xi_n^2-1})\omega_n t}}{-(\xi_n - \sqrt{\xi_n^2-1})\omega_n} - \frac{k'_m}{2\omega_n\sqrt{\xi_n^2-1}} \times \frac{e^{-(\xi_n + \sqrt{\xi_n^2-1})\omega_n t}}{-(\xi_n + \sqrt{\xi_n^2-1})\omega_n},$$

or,
$$h(t) = \frac{k'_m e^{-\xi_n\omega_n t}}{2\omega_n\sqrt{\xi_n^2-1}} \left[\frac{e^{-\omega_n\sqrt{\xi_n^2-1}\,t}}{\omega_n\xi_n + \omega_n\sqrt{\xi_n^2-1})} - \frac{e^{\omega_n\sqrt{\xi_n^2-1}\,t}}{\omega_n\xi_n - \omega_n\sqrt{\xi_n^2-1}} \right].$$

Case II ($\xi_n < 1$): Recall that the integral

$$\int e^{\alpha t}\sin(\beta t)dt = \frac{e^{\alpha t}}{\alpha^2 + \beta^2}\left[\alpha\sin(\beta t) - \beta\cos(\beta t)\right] + C.$$

With $\alpha = -\xi_n\omega_n$ and $\beta = \omega_n\sqrt{\xi_n^2 - 1}$, the integral

$$h(t) = \frac{k'_m}{\omega_n\sqrt{1-\xi_n^2}} \int e^{\alpha t}\sin(\beta t)dt = \frac{k'_m}{\omega_n\sqrt{1-\xi_n^2}} \times \frac{e^{\alpha t}}{\alpha^2+\beta^2}\left[\alpha\sin(\beta t) - \beta\cos(\beta t)\right] + C.$$

Substituting the value of α and β proves Eq. (8.44).

▮

Problem 8.10

Substituting Eq. (8.54) in Eq. (8.55), we get $P(s) = \beta \dfrac{e^{-\tau s}}{(s+\gamma)} A(s)$.

Substituting the above value of $P(s)$ in Eq. (8.56) gives

$$B(s) = \frac{\mu\beta e^{-\tau s} - \sigma(s+\gamma)}{(s+\lambda)(s+\gamma)} A(s).$$

Substituting the value of $A(s)$ from Eq. (8.53) in the above equation, we get

$$B(s) = \frac{\mu\beta e^{-\tau s} - \sigma(s+\gamma)}{(s+\lambda)(s+\gamma)(s-\alpha)} \left[G(s) - \eta B(s)\right].$$

Rearranging terms including $B(s)$ on the left side of the equation gives

$$B(s)\left[1 + \frac{\eta\,[\mu\beta e^{-\tau s} - \sigma(s+\gamma)]}{(s+\lambda)(s+\gamma)(s-\alpha)}\right] = \frac{\mu\beta e^{-\tau s} - \sigma(s+\gamma)}{(s+\lambda)(s+\gamma)(s-\alpha)} G(s),$$

which results in the transfer function

$$H(s) = \frac{B(s)}{G(s)} = \frac{\mu\beta e^{-\tau s} - \sigma(s+\gamma)}{(s+\lambda)(s+\gamma)(s-\alpha) + \eta\,[\mu\beta e^{-\tau s} - \sigma(s+\gamma)]}.$$

Problem 8.11

Note that $$V_1(s) = \phi_1(s) - \theta(s)$$

and $$V_2(s) = K_1 G(s) V_1(s) = K_1 G(s)\left[\phi_1(s) - \theta(s)\right].$$

Substituting $\theta(s) = V_2(s)/s$ in the above equation, we get

$$V_2(s) = K_1 G(s)\left[\phi_1(s) - V_2(s)/s\right]$$

or, $$V_2(s) = \frac{sK_1 G(s)}{s + K_1 G(s)} \phi_1(s).$$

The output $$V(s) = K_2 V_2(s) = K_2 \frac{sK_1 G(s)}{s + K_1 G(s)} \phi_1(s),$$

which results in the transfer function $H(s) = \dfrac{V(s)}{\phi(s)} = K_1 K_2 \dfrac{sG(s)}{s + K_1 G(s)}$.

Differentiator: For the PLL to behave as an ideal differentiator, its transfer function $H(s) = Ks$, i.e.,

$$H(s) = Ks = K_1 K_2 \frac{sG(s)}{s + K_1 G(s)},$$

or, $$K(s + K_1 G(s)) = K_1 K_2 G(s).$$

Solving in terms of $G(s)$, we obtain $$G(s) = \frac{K}{K_1(K_2 - K)} s.$$

Another way of obtaining an ideal differentiator is to set $K_1 \to \infty$ in the transfer function as shown below:

$$\lim_{K_1 \to \infty} H(s) = \lim_{K_1 \to \infty} K_2 \frac{sG(s)}{s/K_1 + G(s)} = K_2 s\,. \qquad \blacksquare$$

<u>Problem 8.12</u>

The simulink model for the simulation is shown in Fig. S8.12a. The results of the simulation are plotted in Fig. S8.12b. While the number of antigens rises at an alarming rate, the plasma cells are not produced to compensate for this increase in antigens. Consequently, the antibodies are destroyed by the antigens. ∎

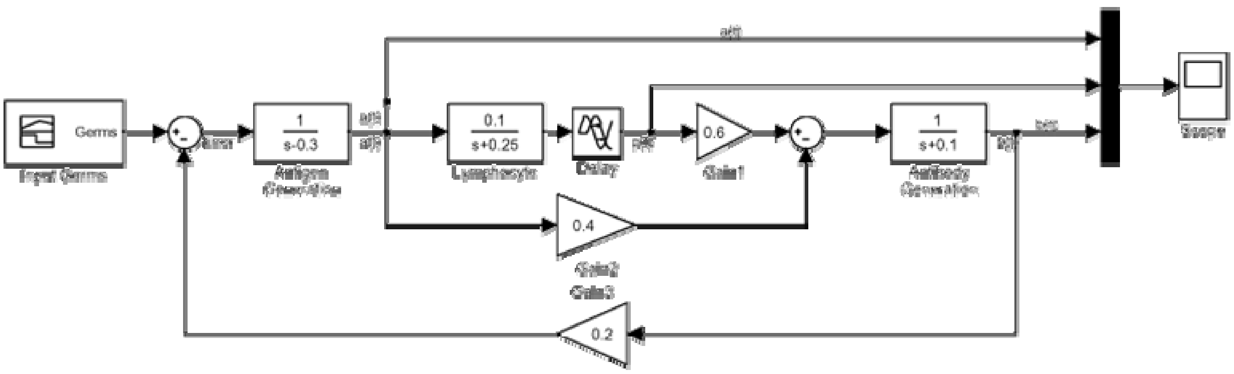


Fig. S8.12a: Simulink model for the immune response system of humans for Problem 8.12.

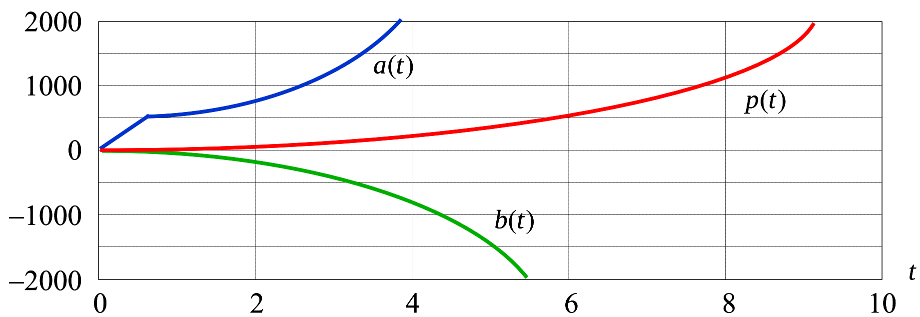


Fig. S8.12b: Time evolution of the number of antigens $a(t)$, plasma cells $p(t)$, and antibodies $b(t)$ in Problem 8.12.