Chapter 9: Sampling and Quantization

Problem 9.1

(a) $x_1(t) = 5\mathrm{sinc}(200t) = \dfrac{1}{40}\dfrac{200\pi}{\pi}\mathrm{sinc}\left(\dfrac{200\pi \times t}{\pi}\right)$. Therefore

$$X_1(\omega) = \frac{1}{40}\mathrm{rect}\left(\tfrac{\omega}{400\pi}\right) = \begin{cases} 1/40 & |\omega| \le 200\pi \\ 0 & |\omega| > 200\pi. \end{cases}$$

The maximum frequency is given by 100 Hz. Based on the Nyquist theorem, the maximum sampling period is $T_s = \dfrac{1}{200} s = 5\ ms$.

(b) $x_2(t) = \underbrace{\underbrace{5\mathrm{sinc}(200t)}_{\max\ freq=100Hz} + \underbrace{8\sin(100\pi t)}_{\max\ freq=50Hz}}_{\max\ freq=100Hz}$.

Therefore, maximum sampling period is given by

$$T_s = \frac{1}{200} s = 5\ ms.$$

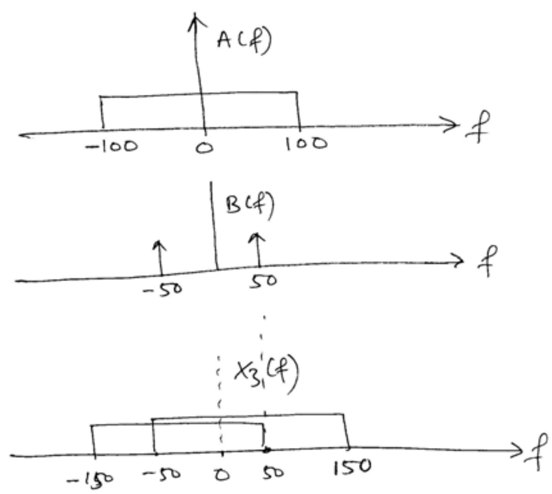


Fig. S9.1: $A(f)$ denotes the spectrum of sinc($200t$) and $B(f)$ denotes the spectrum of sin($100\pi t$). $X_3(f)$ is the sum of two shifted replicas of $A(f)$, and is non-zero within the band [–150,150] Hz.

(c) Since $x_3(t)$ is a product of sinc($200t$) and sin($100\pi t$), the spectrum of $x_3(t)$ can be obtained by convolving the spectrums of sinc($200t$) and sin($100\pi t$). From the theory of CT convolution, it can easily be seen that (see Fig. S9.1), the maximum frequency present in $x_3(t)$ is 150 Hz. Therefore, the maximum sampling period is given by

$$T_s = \frac{1}{2\times 150} s = 3.33\ ms.$$

(d) Since $x_4(t)$ is a convolution product of sinc(200t) and sin(100πt), the spectrum of $x_4(t)$ can be obtained by multiplying the spectrums of sinc(200t) and sin(100πt). As the spectrum of sin(100πt) includes two impulses at 50 Hz and −50 Hz, the spectrum of $x_4(t)$ will only include two impulses at 50 Hz and −50 Hz. As the maximum frequency present in $x_4(t)$ is 50 Hz, the maximum sampling period is given by

$$T_s = \frac{1}{2\times 50} s = 10\ ms.$$

Problem 9.2

(a) From Table 5.2, we know that $\frac{W}{\pi}\operatorname{sinc}\left(\frac{Wt}{\pi}\right) \leftrightarrow \operatorname{rect}\left(\frac{\omega}{2W}\right)$.

Applying the time-shifting property of the CTFT (see Table 5.4), we obtain

$$\frac{W}{\pi}\operatorname{sinc}\left(\frac{W(t-2)}{\pi}\right) \leftrightarrow \operatorname{rect}\left(\frac{\omega}{2W}\right)e^{-j2\omega}.$$

The uncertainty principle is satisfied as $x_1(t)$ is a bandlimited signal but NOT a time limited signal since a sinc function has infinite length.

(b) From Table 5.2, we know that $x_2(t) = \frac{W}{\pi}\operatorname{sinc}\left(\frac{Wt}{\pi}\right)$.

The uncertainty principle is satisfied as $x_2(t)$ is a bandlimited signal but NOT a time limited signal since a sinc function has infinite length.

(c) From Table 5.2, we know that $\frac{W}{\pi}\operatorname{sinc}\left(\frac{Wt}{\pi}\right) \leftrightarrow \operatorname{rect}\left(\frac{\omega}{2W}\right)$.

Applying the frequency-shifting property of the CTFT (see Table 5.4), we obtain

$$\frac{W}{\pi}\operatorname{sinc}\left(\frac{Wt}{\pi}\right)e^{j\omega_0 t} \leftrightarrow \operatorname{rect}\left(\frac{\omega-\omega_0}{2W}\right)$$

and $$\frac{W}{\pi}\operatorname{sinc}\left(\frac{Wt}{\pi}\right)e^{-j\omega_0 t} \leftrightarrow \operatorname{rect}\left(\frac{\omega+\omega_0}{2W}\right).$$

In other words, $x_3(t) = \frac{W}{\pi}\operatorname{sinc}\left(\frac{Wt}{\pi}\right)e^{j\omega_0 t} + \frac{W}{\pi}\operatorname{sinc}\left(\frac{Wt}{\pi}\right)e^{-j\omega_0 t} = \frac{2W}{\pi}\operatorname{sinc}\left(\frac{Wt}{\pi}\right)\cos(\omega_0 t)$,

which is not a time-limited signal.

The uncertainty principle is satisfied as $x_3(t)$ is a bandlimited signal but NOT a time limited signal since a sinc function has infinite length.

(d) Express $X_4(\omega) = u(\omega-\omega_0) - u(\omega-2\omega_0) = \begin{cases} 1 & \omega_0 < \omega < 2\omega_0 \\ 0 & \text{elsewhere} \end{cases} = \operatorname{rect}\left(\frac{\omega-1.5\omega_0}{\omega_0}\right)$.

From Table 5.2, we know that

$$\frac{(0.5\omega_0)}{\pi}\operatorname{sinc}\left(\frac{(0.5\omega_0)t}{\pi}\right) \leftrightarrow \operatorname{rect}\left(\frac{\omega}{\omega_0}\right).$$

Applying the frequency-shifting property of the CTFT (see Table 5.4), we obtain

$$\frac{(0.5\omega_0)}{\pi}\operatorname{sinc}\left(\frac{(0.5\omega_0)t}{\pi}\right)e^{j1.5\omega_0 t} \leftrightarrow \operatorname{rect}\left(\frac{\omega-1.5\omega_0}{\omega_0}\right).$$

In other words, $x_4(t) = \frac{(0.5\omega_0)}{\pi}\,\mathrm{sinc}\left(\frac{(0.5\omega_0)t}{\pi}\right)e^{j1.5\omega_0 t} = \frac{\omega_0}{2\pi}\,\mathrm{sinc}\left(\frac{\omega_0 t}{2\pi}\right)e^{j1.5\omega_0 t}$,

which is not a time-limited signal. The uncertainty principle is satisfied as $x_4(t)$ is a bandlimited signal but NOT a time limited signal since a sinc function has infinite length ▌

Problem 9.3

(a) Express $x_1(t) = \cos(\omega_0 t)\left[u(t+T) - u(t-T)\right] = \cos(\omega_0 t)\mathrm{rect}\left(\frac{t}{2T}\right)$.

From Table 5.2, we know that $\mathrm{rect}\left(\frac{t}{2T}\right) \leftrightarrow 2T\,\mathrm{sinc}\left(\frac{2\omega T}{2\pi}\right)$

and $\cos(\omega_0 t) \leftrightarrow \pi\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right]$.

Applying the frequency-convolution property of the CTFT (see Table 5.4), we obtain

$$\begin{aligned}\cos(\omega_0 t)\mathrm{rect}\left(\tfrac{t}{2T}\right) &\leftrightarrow \tfrac{1}{2\pi}\left\{2\mathrm{T}\mathrm{sinc}\left(\tfrac{2\omega T}{2\pi}\right) * \pi\left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right]\right\}\\ &= \mathrm{T}\left\{\mathrm{sinc}\left(\tfrac{\omega T}{\pi}\right) * \left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right]\right\}\\ &= \mathrm{T}\left\{\mathrm{sinc}\left(\tfrac{(\omega-\omega_0)T}{\pi}\right) + \mathrm{sinc}\left(\tfrac{(\omega+\omega_0)T}{\pi}\right)\right\}\end{aligned} .$$

We note that $X_1(\omega)$ is a summation of two sinc functions and is, therefore, not a band-limited signal. As $x_1(t)$ is a time-limited signal, the converse of the uncertainty principle is satisfied.

(b) From Table 5.2, we know that $\mathrm{rect}\left(\frac{t}{\tau}\right) \leftrightarrow \tau\,\mathrm{sinc}\left(\frac{\omega\tau}{2\pi}\right)$.

Applying the time-convolution property of the CTFT (see Table 5.4), we obtain

$$\mathrm{rect}\left(\tfrac{t}{\tau}\right) * \mathrm{rect}\left(\tfrac{t}{\tau}\right) \leftrightarrow \left\{\tau\,\mathrm{sinc}\left(\tfrac{\omega\tau}{2\pi}\right) \times \tau\,\mathrm{sinc}\left(\tfrac{\omega\tau}{2\pi}\right)\right\} = \tau^2\,\mathrm{sinc}^2\left(\tfrac{\omega\tau}{2\pi}\right).$$

We note that $X_2(\omega)$ is a product of two sinc functions and is, therefore, not a band-limited signal. As $x_2(t)$ is a time-limited signal, the converse of the uncertainty principle is satisfied.

(c) $x_3(t) = e^{-\alpha|t|}\mathrm{rect}\left(\frac{t}{\tau}\right) = \begin{cases} e^{-\alpha|t|} & -\tau/2 < t < \tau/2 \\ 0 & otherwise \end{cases}$.

Calculating the CTFT, we obtain

$$\begin{aligned}X_3(\omega) &= \int_{-\tau/2}^{\tau/2} e^{-\alpha|t|}\cdot e^{-j\omega t}dt = 2\int_0^{\tau/2} e^{-\alpha t}\cos(\omega t)dt = 2\left[\tfrac{e^{-\alpha t}}{\alpha^2+\omega^2}\left(-\alpha\cos\omega t + \omega\sin\omega t\right)\right]_0^{\tau/2}\\ &= 2\left[\tfrac{e^{-\alpha\tau/2}}{\alpha^2+\omega^2}\left(-\alpha\cos(\omega\tau/2) + \omega\sin(\omega\tau/2)\right) + \tfrac{\alpha}{\alpha^2+\omega^2}\right]\\ &= \tfrac{2}{\alpha^2+\omega^2}\left[\alpha + e^{-\alpha\tau/2}\left(-\alpha\cos(\omega\tau/2) + \omega\sin(\omega\tau/2)\right)\right]\end{aligned}$$

We note that $X_3(\omega)$ is not a band-limited signal. As $x_3(t)$ is a time-limited signal, the converse of the uncertainty principle is satisfied.

(d) Taking the CTFT, we get

$$X_4(\omega) = \Im\{\delta(t-5)\} + \Im\{\delta(t+5)\} = e^{-j5\omega}\Im\{\delta(t)\} + e^{j5\omega}\Im\{\delta(t)\}$$
$$= e^{-j5\omega} + e^{j5\omega} = 2\cos(5\omega)$$

We note that $X_4(\omega)$ is not a band-limited signal. As $x_4(t)$ is a time-limited signal, the converse of the uncertainty principle is satisfied. ▌

Problem 9.4

(a) Applying the width property (in frequency domain) of the CT convolution, we know that $x(t)$ is a bandlimited signal with maximum frequency 650 Hz. The minimum value of the sampling rate f_s that does not introduce any aliasing is 1300 samples/sec or 1300 Hz.

(b) The signals $v_1(t)$ and $v_2(t)$ are bandlimited to 300 Hz and 500 Hz, respectively. The spectrum of the produce $x(t) = v_1(t)v_2(t)$ is shown in Fig. S9.4(a). It is observed that $x(t)$ is a bandlimited signal with maximum frequency of 800 Hz. The minimum value of the sampling rate f_s that does not introduce any aliasing is 1600 samples/sec or 1600 Hz.

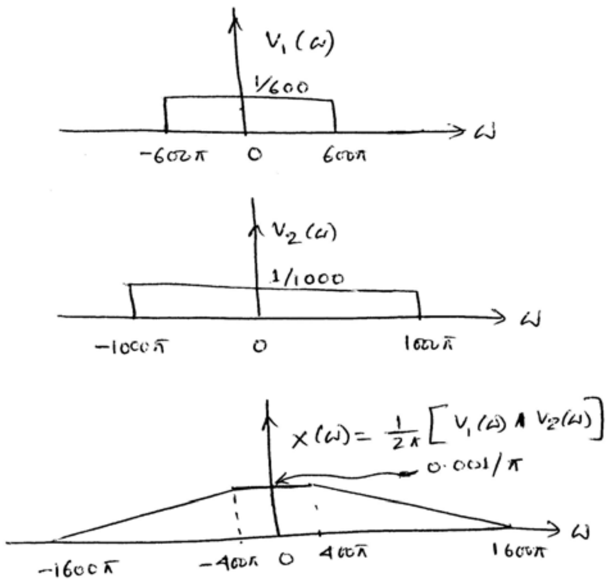
(c) A sampling interval of 2 ms implies a sampling frequency is 500 Hz. The spectrum of the sampled signal is a periodic function with a period of 1000π rad/s, and is shown in Fig. S9.4(b). It is observed that the spectrum is aliased, and hence $x(t)$ cannot be recovered from the sampled signal. The spectrum can be plotted by hand or the following Matlab code may be used to generate Fig. S9.4(b).

```
>> % MATLAB Program for Problem 9.4(c)
>> X_1p =[0:1/1200:1,  ones(1,799), 1:-1/1200:0];
>> kX_1p = [-1600:1600];
>> plot([-1600:1600],X_1p)
>> ylabel('Spectrum (x 0.001/PI)');
>> xlabel('radian frequency')
>> axis([-1700 1700 0  1.2])
>> print -dtiff plot.tiff
>> [kX, X] = shiftadd(kX_1p, X_1p, -5, 5, 1000);
>> plot(kX,X);
>> ylabel('Spectrum (x 0.001/PI)');
>> xlabel('radian frequency');
>> %axis([-1600 1600 0  2]);
>> print -dtiff plot.tiff
```

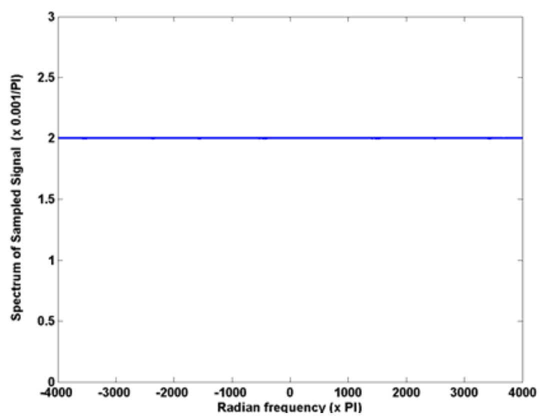
(d) A sampling interval of 0.1 ms implies a sampling frequency is 10,000 Hz. The spectrum of the sampled signal will be a periodic function with a period of 20,000π rad/s, and is shown in Fig. S9.4(c). It is observed that there is no aliasing in the spectrum, and hence $x(t)$ can easily be recovered from the sampled signal using a lowpass filter.

```
>> % MATLAB Program for Problem 9.4(d)
>> X_1p =[0:1/1200:1,  ones(1,799), 1:-1/1200:0];
>> kX_1p = [-1600:1600];
>> plot([-1600:1600],X_1p)
>> ylabel('Spectrum (x 0.001/PI)');
>> xlabel('radian frequency')
>> axis([-1700 1700 0  1.2])
>> print -dtiff plot.tiff
>> [kX, X] = shiftadd(kX_1p, X_1p, -1, 1, 20000);
>> plot(kX,X);
>> ylabel('Spectrum (x 0.001/PI)');
>> xlabel('radian frequency');
>> axis([-25000 25000 0  1.2]);
>> print -dtiff plot.tiff
```

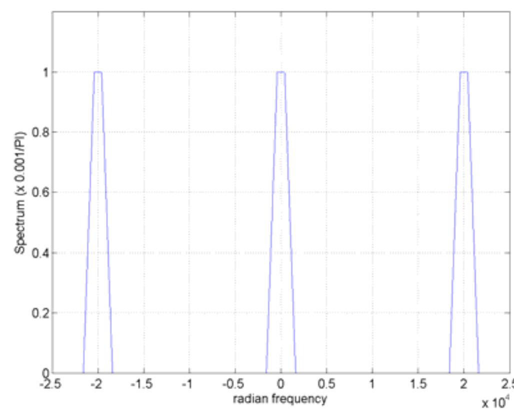
▌



(a)



(b)



(c)

Fig. S9.4: Plots for Problem 9.4.

<u>Problem 9.5</u>

The CTFT of $x(t)$ is given by

$$\begin{aligned} X(\omega) &= \Im\left\{\sin(400\pi t) + 2\cos(150\pi t)\right\} = \Im\left\{\sin(400\pi t)\right\} + 2\Im\left\{\cos(150\pi t)\right\} \\ &= j\pi\left[\delta(\omega+400\pi) - \delta(\omega-400\pi)\right] + 2\pi\left[\delta(\omega+150\pi) + \delta(\omega-150\pi)\right] \end{aligned}$$

and is shown in Fig. S9.5.

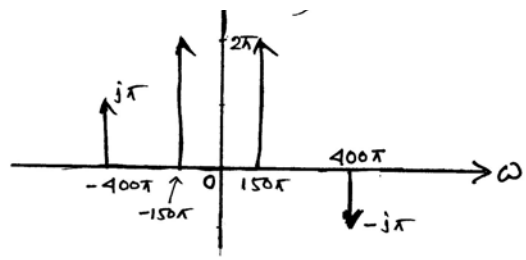


Fig. S9.5. Fourier transform $X(\omega)$ of the original CT signal.

Note that the maximum frequency in $x(t)$ is 400π rad/s, or, 200 Hz. Therefore, the Nyquist sampling rate is given by 400 samples/second. A lower sampling rate will cause aliasing and it will not be possible to reconstruct the signal perfectly.

(a) $f_s = 100$ samples/s:

The sampling rate us given by 200π rad/s and the sampling interval by 0.01s. The CTFT of the sampled signal is given by

$$X_s(\omega) = \tfrac{1}{T_s}\sum_{m=-\infty}^{\infty} X\left(\omega - m\omega_s\right) = \tfrac{1}{T_s}\sum_{m=-\infty}^{\infty} X\left(\omega - 200m\pi\right).$$

The CTFT $X_s(\omega)$ is shown in Fig. S9.5a(i). It is observed that the impulses located at $\omega = \pm 50\pi$, $\pm 150\pi$, $\pm 250\pi$, $\pm 350\pi$, ... have an area of $2\pi/T_s$, while the impulses located at $\omega = 0$, $\pm 200\pi$, $\pm 400\pi$, $\pm 600\pi$, $\pm 800\pi$, ... have an area with alternating values of $j\pi/T_s$ and $-j\pi/T_s$. Note that the impulses with area $j\pi/T_s$ will cancel the impulses with area $-j\pi/T_s$, and therefore, there will not be any impulses at $\omega = 0$, $\pm 200\pi$, $\pm 400\pi$, $\pm 600\pi$, $\pm 800\pi$, ... The revised plot of CTFT $X_s(\omega)$ is shown in Fig. S9.5a(ii).

When the sampled signal is passed through a lowpass filter with transfer function

$$H(\omega) = \begin{cases} T_s & |\omega| \le 100\pi \\ 0 & \text{elsewhere} \end{cases},$$

a CT signal $y(t)$ will be reconstructed. The spectrum $Y(\omega)$ of the signal $y(t)$ is shown in Fig. S9.5a(iii) and can be expressed as

$$Y(\omega) = 2\pi\left[\delta(\omega + 50\pi) + \delta(\omega - 50\pi)\right].$$

Calculating the inverse CTFT, we obtain the reconstructed signal

$$y(t) = 2\cos(50\pi t).$$

Since the sampling rate does not satisfy the Nyquist criterion, the reconstructed signal is different from the original signal.

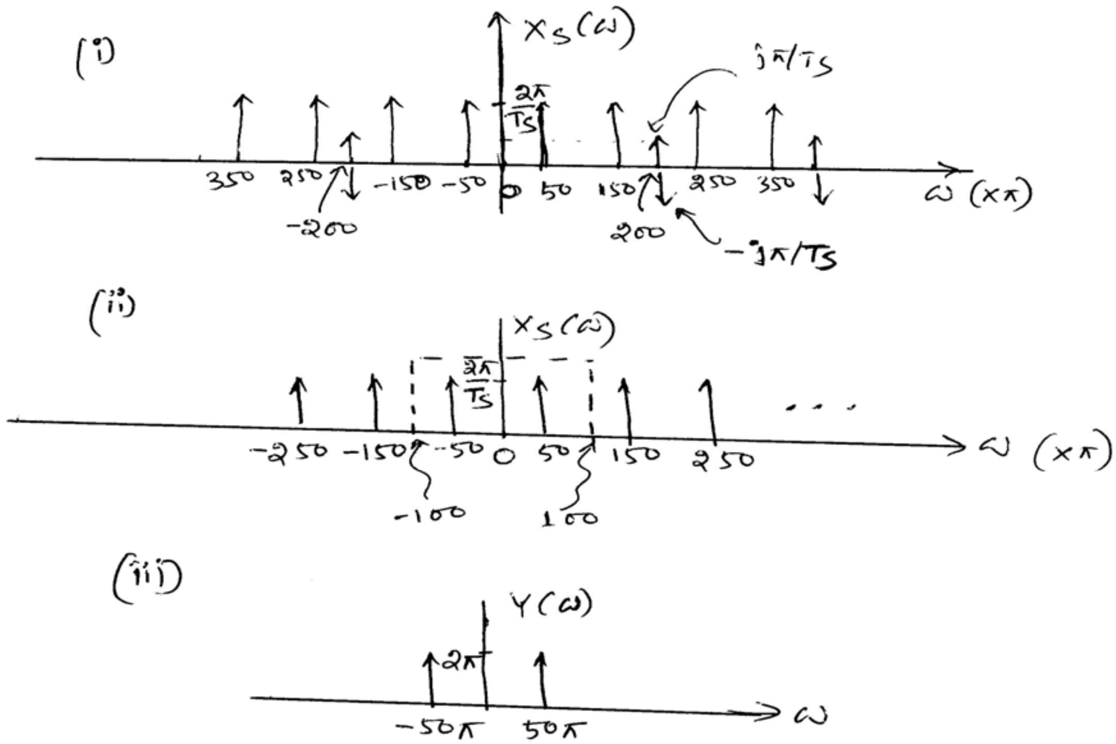


Fig. S9.5a. Spectrum of the sampled signal and the reconstructed signal in Problem 9.5(a).

You may be wondering – why does the sin(400π*t*) term totally disappear when sampled at 100Hz? Calculate the sampled signal to see what happens.

$$F_s = 100 \quad \Rightarrow \quad T_s = \tfrac{1}{100} = 0.01.$$

The sampled signal is then given by

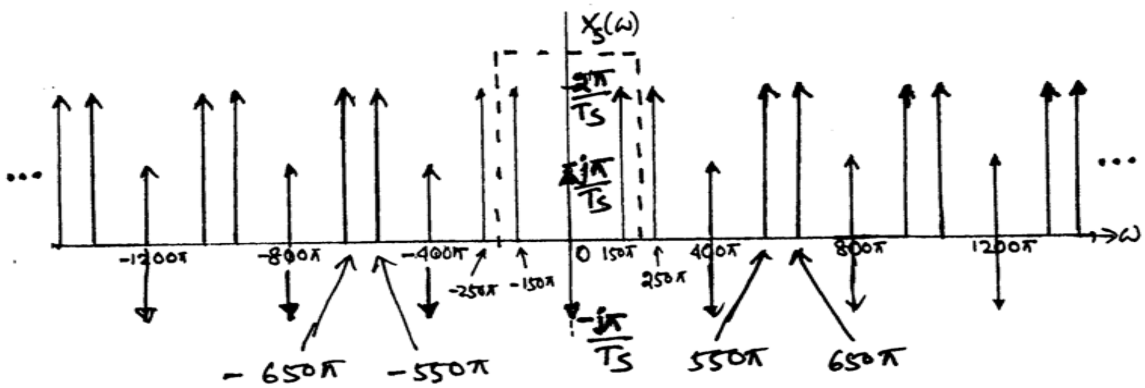
$$\begin{aligned} x(kT_s) &= \sin(400\pi kT_s) + 2\cos(150\pi kT_s) = \sin(400\pi k \times 0.01) + 2\cos(150\pi k \times 0.01) \\ &= \underbrace{\sin(4\pi k)}_{=0} + 2\cos(1.5\pi k) \\ &= 2\cos(0.5\pi k) \end{aligned}$$

If you sample, sin(400π*t*) with a sampling frequency 100 Hz (at time instances kT_s), the sampled output or the sin(400π*t*) component is always zero.

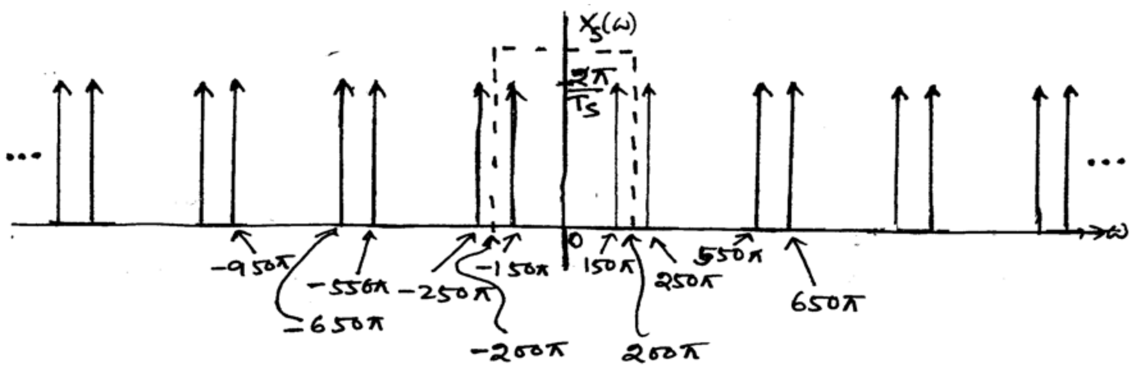
(b) $f_s = 200$ samples/s:

The sampling rate us given by 400π rad/s and the sampling interval by 0.005s. The CTFT of the sampled signal is given by

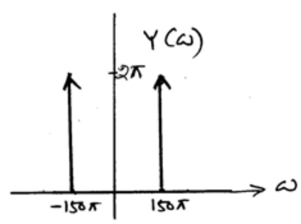
$$X_s(\omega) = \tfrac{1}{T_s}\sum_{m=-\infty}^{\infty} X(\omega - m\omega_s) = \tfrac{1}{T_s}\sum_{m=-\infty}^{\infty} X(\omega - 400m\pi).$$



(i)



(ii)



(iii)

Fig. S9.5b. Spectrum of the sampled signal and the reconstructed signal in Problem 9.5(b).

The CTFT $X_s(\omega)$ is shown in Fig. S9.5b(i). It is observed that the impulses located at $\omega = \pm 150\pi$, $\pm 250\pi$, $\pm 550\pi$, $\pm 650\pi$, ... have an area of $2\pi/T_s$, while the impulses located at $\omega = 0$, $\pm 400\pi$, $\pm 800\pi$, ... have an area with alternating values of $j\pi/T_s$ and $-j\pi/T_s$. Note that the impulses with area $j\pi/T_s$ will cancel the impulses with area $-j\pi/T_s$, and therefore, there will not be any impulses at $\omega = 0$, $\pm 400\pi$, $\pm 800\pi$, ... The revised plot of CTFT $X_s(\omega)$ is shown in Fig. S9.5b(ii).

When the sampled signal is passed through a lowpass filter with transfer function

$$H(\omega) = \begin{cases} T_s & |\omega| \le 200\pi \\ 0 & \text{elsewhere} \end{cases},$$

a CT signal $y(t)$ will be reconstructed. The spectrum $Y(\omega)$ of the signal $y(t)$ is shown in Fig. S9.5b(iii) and can be expressed as

$$Y(\omega) = 2\pi\left[\delta(\omega + 150\pi) + \delta(\omega - 150\pi)\right].$$

Calculating the inverse CTFT, we obtain the reconstructed signal

$$y(t) = 2\cos(150\pi t).$$

Note that the sampling rate does not satisfy Nyquist criterion in this case. As a result, the reconstructed signal is different from the original signal.

> You may be wondering – why does the sin(400πt) term totally disappear when sampled at 200Hz? Calculate the sampled signal to see what happens.
>
> $$F_s = 200 \quad \Rightarrow \quad \mathrm{T}_s = \tfrac{1}{200} = 0.005.$$
>
> The sampled signal is then given by
>
> $$\begin{aligned} x(kT_s) &= \sin\left(400\pi kT_s\right) + 2\cos\left(150\pi kT_s\right) = \sin\left(400\pi k \times 0.005\right) + 2\cos\left(150\pi k \times 0.005\right) \\ &= \underbrace{\sin\left(2\pi k\right)}_{=0} + 2\cos\left(0.75\pi k\right) \\ &= 2\cos\left(0.75\pi k\right) \end{aligned}$$
>
> If you sample, sin(400πt) with a sampling frequency 200 Hz (at time instances kT_s), the sampled output for the the sin(400πt) component is always zero.

(c) f_s = 400 samples/s:

The sampling rate us given by 800π rad/s and the sampling interval by 0.0025s. The CTFT of the sampled signal is given by

$$X_s(\omega) = \tfrac{1}{T_s}\sum_{m=-\infty}^{\infty} X\left(\omega - m\omega_s\right) = \tfrac{1}{T_s}\sum_{m=-\infty}^{\infty} X\left(\omega - 800m\pi\right).$$

The CTFT $X_s(\omega)$ is shown in Fig. S9.5c(i). It is observed that the impulses located at ω = ±150π, ±650π, ±950π, ... have an area of $2\pi/T_s$, while the impulses located at ω = 0, ±400π, ±800π, ... have an area with alternating values of $j\pi/T_s$ and $-j\pi/T_s$. Note that the impulses with area $j\pi/T_s$ will cancel the impulses with area $-j\pi/T_s$, and therefore, there will not be any impulses at ω = 0, ±400π, ±800π, ... The revised plot of CTFT $X_s(\omega)$ is shown in Fig. S9.5c(ii).

When the sampled signal is passed through a lowpass filter with transfer function

$$H(\omega) = \begin{cases} T_s & |\omega| \le 400\pi \\ 0 & \text{elsewhere} \end{cases},$$

a CT signal $y(t)$ will be reconstructed. The spectrum $Y(\omega)$ of the signal $y(t)$ is shown in Fig. S9.5b(iii) and can be expressed as

$$Y(\omega) = 2\pi\left[\delta(\omega + 150\pi) + \delta(\omega - 150\pi)\right].$$

Calculating the inverse CTFT, we obtain the reconstructed signal

$$y(t) = 2\cos(150\pi t).$$

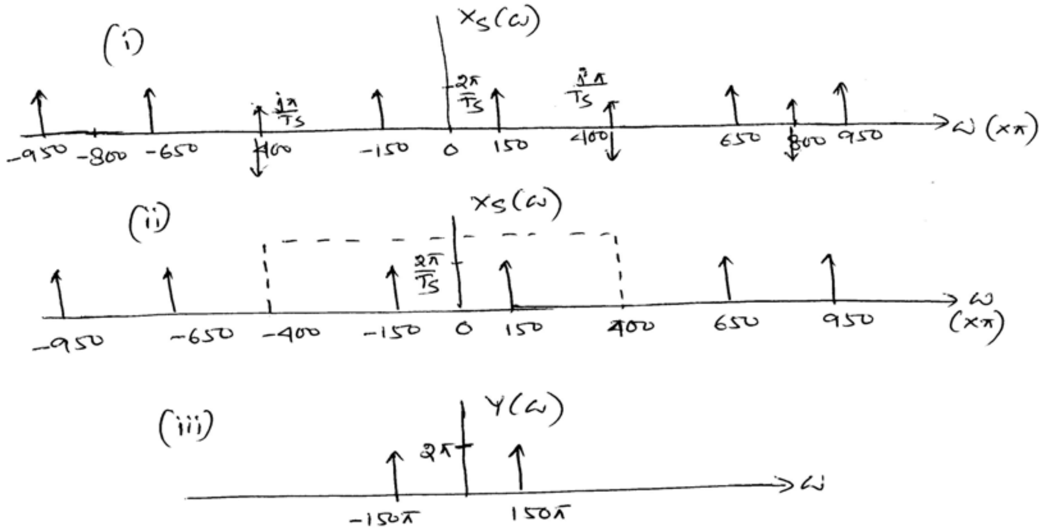


Fig. S9.5c. Spectrum of the sampled signal and the reconstructed signal in Problem 9.5(c).

Note that the sampling rate does not satisfy Nyquist criterion in this case. As a result, the reconstructed signal is different from the original signal.

(d) $f_s = 500$ samples/s:

The sampling rate us given by 1000π rad/s and the sampling interval by 0.002s. The CTFT of the sampled signal is given by

$$X_s(\omega) = \tfrac{1}{T_s}\sum_{m=-\infty}^{\infty} X\left(\omega - m\omega_s\right) = \tfrac{1}{T_s}\sum_{m=-\infty}^{\infty} X\left(\omega - 1000m\pi\right).$$

The CTFT $X_s(\omega)$ is shown in Fig. S9.5d(i). Note that none of the impulses overlap and hence there is no cancellation of impulses. The CTFT$X_s(\omega)$ is shown in Fig. S9.5d(ii).

When the sampled signal is passed through a lowpass filter with transfer function

$$H(\omega) = \begin{cases} T_s & |\omega| \le 500\pi \\ 0 & \text{elsewhere} \end{cases},$$

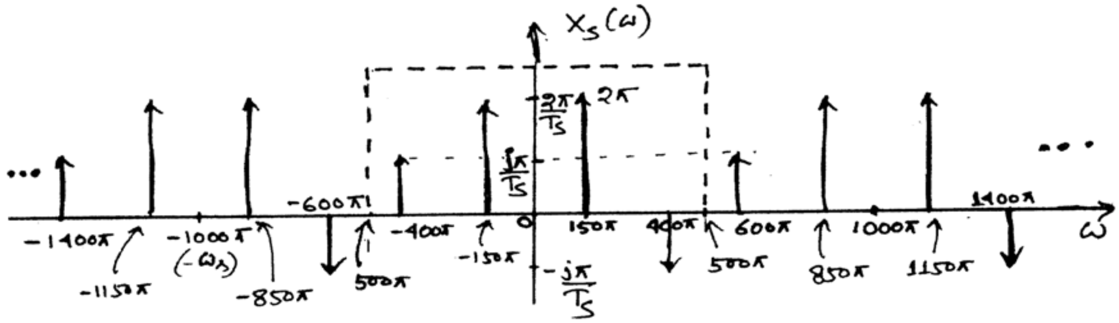
a CT signal $y(t)$ will be reconstructed. The spectrum $Y(\omega)$ of the signal $y(t)$ is shown in Fig. S9.d(ii), and can be expressed as

$$Y(\omega) = 2\pi\left[\delta(\omega+150\pi) + \delta(\omega-150\pi)\right] + j\pi\left[\delta(\omega+400\pi) - \delta(\omega-400\pi)\right].$$

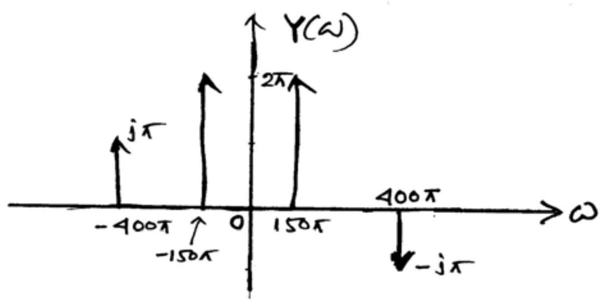
Calculating the inverse CTFT, we obtain the reconstructed signal

$$y(t) = 2\cos(150\pi t) + \sin(400\pi t).$$

Note that because the sampling rate satisfies the Nyquist sampling criterion in this case, the reconstructed signal is identical to the original signal. ▌



(i)



(ii)

Fig. S9.5d. Spectrum of the (i) sampled signal and the (ii) reconstructed signal in Problem 9.5(d).

<u>Problem 9.6</u>

The CT signal $x(t)$ is sketched in Fig. S9.6a.

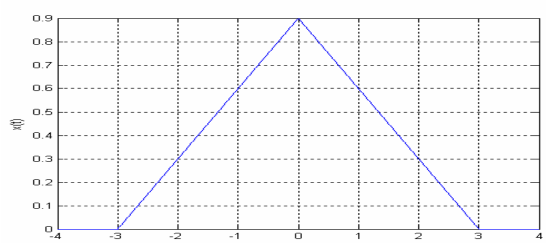


Figure S9.6a: The CT signal $x(t)$ for Problem 9.6.

(a) From Table 5.2, entry (18), we know the following CTFT pair

$$\Delta(\tfrac{t}{\tau}) = \begin{cases} 1 - \frac{|t|}{\tau} & |t| \le \tau \\ 0 & \text{otherwise} \end{cases} \xleftrightarrow{\text{CTFT}} \tau \operatorname{sinc}^2(\tfrac{\omega t}{2\pi}) .$$

The input signal can be expressed in terms of the function $\Delta(t)$ as follows

$$x(t) = \begin{cases} 0.25(3 - |t|) & 0 \le |t| \le 3 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 0.75\left(1 - \frac{|t|}{3}\right) & |t| \le 3 \\ 0 & \text{otherwise} \end{cases} = 0.75\Delta(\tfrac{t}{3})$$

Therefore, $$X(\omega) = 0.75 \times 3\text{sinc}^2\left(\tfrac{3\omega}{2\pi}\right) = 2.25\text{sinc}^2\left(\tfrac{3\omega}{2\pi}\right).$$

Note that the signal is not band limited. In other words, the maximum frequency present in this signal is infinite. Therefore, the Nyquist sampling frequency is also infinite.

(b) The CTFT $X(\omega)$ is sketched in Fig. S9.6b. It is observed that the maximum value of $X(\omega) = 2.25$ at $\omega = 0$. From Fig. S9.6a, it is clear that

$$|X(\omega)| < 0.0225 \quad \text{for} \quad \omega > 5.62 \text{ radians/s.}$$

The MATLAB code used to plot Fig. S. 9.6b is given below

```
>> clf ;
>> w =-10:0.01:10  ;
>> s = sinc(3*w/(2*pi)) ;
>> X = 2.5*s.^2 ;
>> subplot(3,1,1), plot(w, X), grid on
>> xlabel('w')                        % Label of X-axis
>> ylabel('X(w)')                     % Label of Y-axis
>> axis([-10, 10, 0, 3]) ;
>> subplot(3,1,2), plot(w, X), grid on
>> xlabel('w')                        % Label of X-axis
>> ylabel('X(w)')                     % Label of Y-axis
>> axis([-10, 10, 0, 0.03]) ;
>> subplot(3,1,3), plot(w, X), grid on
>> xlabel('w')                        % Label of X-axis
>> ylabel('X(w)')                     % Label of Y-axis
>> axis([5.5, 5.7, 0.02, 0.03]) ;print -dtiff plot.tiff ;% Save figure as a TIFF file
>> k= -5:5 ;
>> x = 0.25*(3-abs(k)).*((abs(k)<4)) ;
>> stem(k, x, 'filled'),grid
>> ylabel('x[k]');
>> xlabel('k')
>> print -dtiff plot.tiff
```

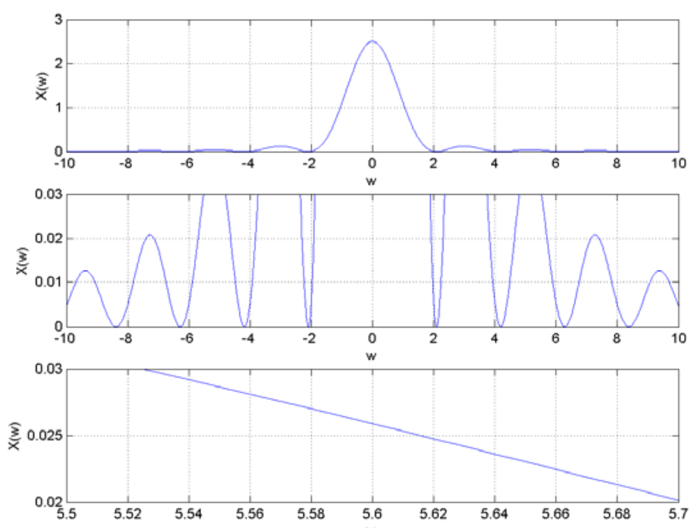


Figure S9.6b: CTFT X(ω) in Problem 9.6.

It is observed that $|X(\omega)| < 0.0225 \quad \text{for} \quad \omega > 5.62$ radians/s.

Expressed in Hz, the maximum effective frequency in the signal is given by $B = 5.62/2\pi = 0.89$ Hz. Therefore, the Nyquist sampling rate would be $2 \times 0.89 = 1.78$ samples/second.

(c) The discrete signal $x[k]$ for the interval $(-5 \le t \le 5)$ is plotted in Fig. S9.6(c).

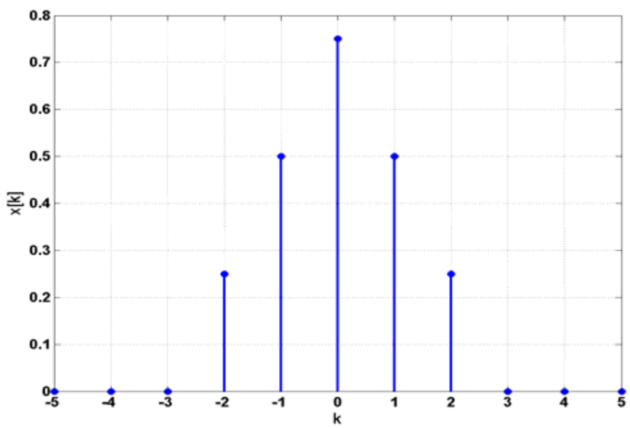


Figure S9.6c: DT signal $x[k]$ in Problem 9.6(c).

(d) The quantizer in Example 9.3 has eight reconstruction levels r_m as follows:

$$r_m = -0.875, -0.625 , -0.375 , -0.125, 0.125, 0.375, 0.625, 0.875.$$

The decision levels are given by

$$d_m = -1\text{V}, -0.75\text{V}, -0.5\text{V} , -0.25\text{V} , 0\text{V}, 0.25\text{V}, 0.50\text{V}, 0.75\text{V}, 1\text{V}.$$

Note that the quantizer generates

$$y[k] = r_m = \tfrac{1}{2}\left[d_m + d_{m+1}\right]$$

where $$d_m \le x[k] < d_{m+1} .$$

The quantization error is shown in Table S9.6a, and is plotted in Fig. S9.6d. The Matlab code used to generate Fig. S9.6d is given below:

```
>> % MATLAB program for Fig. 9.6d
>> k = [-5  -4  -3  -2  -1  0  1  2  3  4  5];
>> x = [0   0  0  0.25   0.50   0.75  0.50   0.25   0   0   0];
>> q = [0.125 0.125 0.125 0.375 0.625 0.875 0.625 0.375 0.125 0.125 0.125];
>> e = x-q;
>> subplot(4,1,1),plot(k, x),grid on
>> xlabel('t'), ylabel('x(t)')
>> %title('The Original Continuous Signal')
>> subplot(4,1,2),stem(k, x),grid on
>> xlabel('k'), ylabel('x[k]')
>> %title('The Original Discrete Signal')
>> subplot(4,1,3),stem(k, q),grid on
>> xlabel('k'), ylabel('q[k]')
>> %title('The Quantized Discrete Signal')
>> subplot(4,1,4),stem(k, e),grid on
>> xlabel('k'), ylabel('e[k]')
>> %title('The Error Signal')
>> print -dtiff plot.tiff
```

Table S9.6a: Quantized sample values and the quantization errors

t	*k*	*x*[*k*]	*Q*{*x*[*k*]}	*e*[*k*]
–5	–5	0	0.125	–0.125
–4	–4	0	0.125	–0.125
–3	–3	0	0.125	–0.125
–2	–2	0.25	0.375	–0.125
–1	–1	0.50	0.625	–0.125
0	0	0.75	0.875	–0.125
1	1	0.50	0.625	–0.125
2	2	0.25	0.375	–0.125
3	3	0	0.125	–0.125
4	4	0	0.125	–0.125
5	5	0	0.125	–0.125

The theoretical maximum quantization error for this quantizer is 0.25/2 = 0.125. It is observed that the maximum error occurs at all values of *k*.

(e) The decision and reconstruction levels for the 16-level quantizer are shown in Table S9.6b. The quantized sample values are shown in Table S9.6, while the quantization error values are plotted in Fig. S9.6e.

Table S9.9b: Decision and reconstruction levels for the 16-level quantizer

m	Decision levels	Reconstruction levels		*m*	Decision levels	Reconstruction levels
0	–1.000	–0.9375		8	0	0.0625
1	–0.875	–0.8125		9	0.125	0.1875
2	–0.750	–0.6875		10	0.25	0.3125
3	–0.625	–0.5625		11	0.375	0.4375
4	–0.500	–0.4375		12	0.50	0.5625
5	–0.375	–0.3125		13	0.625	0.6875
6	–0.250	–0.1875		14	0.75	0.8125
7	–0.125	–0.0625		15	0.875	0.9375

Table S9.9c: Quantized sample values

t	*k*	*x*[*k*]	Q{*x*[*k*]}	*e*[*k*]
–5	–5	0	0.0625	–0.0625
–4	–4	0	0.0625	–0.0625
–3	–3	0	0.0625	–0.0625
–2	–2	0.25	0.3125	–0.0625
–1	–1	0.50	0.5625	–0.0625
0	0	0.75	0.8125	–0.0625
1	1	0.50	0.5625	–0.0625
2	2	0.25	0.3125	–0.0625
3	3	0	0.0625	–0.0625
4	4	0	0.0625	–0.0625
5	5	0	0.0625	–0.0625

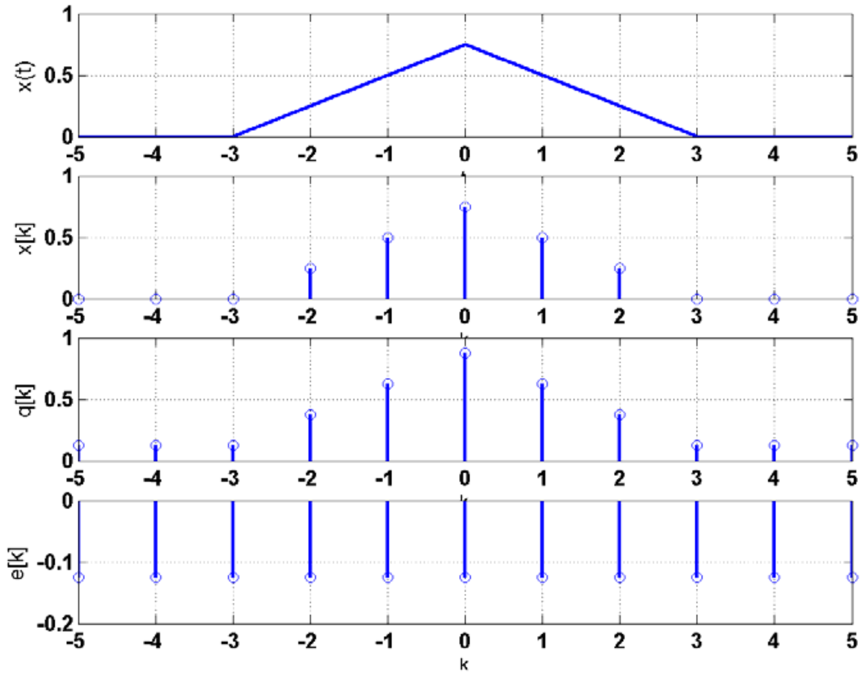


Figure S9.6d: Quantization error $e[k]$ in Problem 9.6(d).

The theoretical maximum quantization error for this quantizer is 0.125/2 = 0.0625. It is observed that the maximum error occurs at all values of k. The MATLAB code for Fig. S9.6e is given below:

```
>> % MATLAB program for Figure S9.6e
>> k = [-5  -4  -3  -2  -1  0  1  2  3  4  5];
>> x = [0   0  0  0.25   0.50   0.75  0.50   0.25   0   0   0];
>> q = [0.0625 0.0625 0.0625 0.3125 0.5625 0.8125 0.5625 0.3125 0.0625 0.0625
0.0625];
>> e = x-q;
>> subplot(2,1,1),stem(k, q),grid on
>> xlabel('k'), ylabel('q[k]')
>> %title('The Quantized Discrete Signal')
>> subplot(2,1,2),stem(k, e),grid on
>> xlabel('k'), ylabel('e[k]')
>> %title('The Error Signal')
>> print -dtiff plot.tiff
```

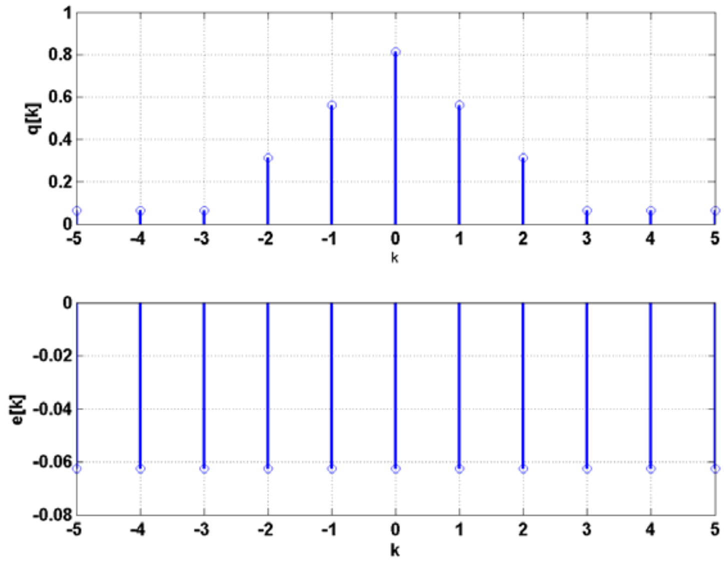


Figure S9.6e: The quantization error signal $e[k]$ in Problem 9.6(e).

Problem 9.7

Recall that
$$r(t) = \text{rect}(\tfrac{t}{\tau}) * \sum_{k=-\infty}^{\infty} \delta(t - kT_s)$$

is a periodic signal with rect(t/τ) repeated. The individual replicas are separated by a duration of T_s. Applying CTFS, the periodic signal $r(t)$ can be represented as

$$r(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_s t}$$

where $\omega_s = 2\pi/T_s$. The CTFS coefficient is given by

$$D_n = \frac{1}{T_s} \int_{-T_s/2}^{T_s/2} \text{rect}(\tfrac{t}{\tau}) e^{-jn\omega_s t} dt = \frac{1}{T_s} \int_{-\tau/2}^{\tau/2} e^{-jn\omega_s t} dt,$$

which simplifies to
$$D_n = \frac{1}{T_s}\left[\frac{e^{-jn\omega_s \tau/2} - e^{jn\omega_s \tau/2}}{-jn\omega_s}\right] = \frac{\omega_s \tau}{2\pi}\text{sinc}\left(\frac{n\omega_s \tau}{2\pi}\right).$$

Problem 9.8

(a) Consider the bandpass signal $x(t)$ with CTFT

$$X(\omega) = 0 \quad \text{for} \quad |\omega| < \omega_1 \text{ and } |\omega| > \omega_2$$

The spectrum of $x(t)$ is shown in Fig. P9.8(a) with an additional constraint that $\omega_1 > (\omega_2 - \omega_1)$. Sampling by a periodic impulse train

$$\sum_{k=-\infty}^{\infty} \delta(t - kT_s) \xleftrightarrow{\text{CTFT}} \frac{2\pi}{T_s} \sum_{m=-\infty}^{\infty} \delta(\omega - \tfrac{2m\pi}{T_s}),$$

the spectrum of the periodic signal $x_s(t)$ is given by

$$\underbrace{x(t)\times\sum_{k=-\infty}^{\infty}\delta(t-kT_s)}_{x_s(t)}\xleftrightarrow{\text{CTFT}}\underbrace{\frac{1}{T_s}\sum_{m=-\infty}^{\infty}X(\omega-\tfrac{2m\pi}{T_s})}_{X_s(\omega)}.$$

shown in Fig. P9.8(b). Note that the original signal $x(t)$ can be perfectly reconstructed by filtering with a bandpass signal provided there is no overlap with (m = 0) replica. This would require

Right corner of ($m = -1$) replica: $\quad -\frac{2\pi}{T_s}+\omega_2\le\omega_1,\quad \text{or,}\quad \frac{2\pi}{T_s}\ge(\omega_2-\omega_1)$.

and

Left corner of ($m = 1$) replica: $\quad -\frac{2\pi}{T_s}+\omega_1\le-\omega_1,\quad \text{or,}\quad \frac{2\pi}{T_s}\le 2\omega_1$.

Combining the two conditions, we get $\quad (\omega_2-\omega_1)\le\frac{2\pi}{T_s}\le 2\omega_1$.

Note that this rate is less than the Nyquist baseband sampling rate of $2\omega_2$ radians/s.

(b) The minimum sampling rate is $(\omega_2 - \omega_1)$ radians/s.

(c) For the signal to be constructed, the transfer function of the bandpass filter is given by

$$H_{bp}(\omega)=\begin{cases}T_s & \omega_1\le|\omega|\le\omega_2\\ 0 & \text{elsewhere.}\end{cases}$$

▌

Problem 9.9

(a) Using the CTFT $\quad \exp(-j\tfrac{1}{2}(\omega_1+\omega_2)t)\xleftrightarrow{\text{CTFT}}2\pi\delta(\omega+\tfrac{\omega_1+\omega_2}{2}t)$.

Hence, $\quad x(t)*\exp(-j\tfrac{1}{2}(\omega_1+\omega_2)t)\xleftrightarrow{\text{CTFT}}X(\omega+\tfrac{\omega_1+\omega_2}{2}t)$

Fig. S9.9 plots the spectra of the signals at the output of the multiplier $q(t)$ (Fig. S9.9(a)), at the output of the lowpass filter (Fig. 9.9(b)), and at the output of the impulse train multiplier (Fig. S9.9(c)).

(b) From Fig. 9.9(b), it is clear that the output of the lowpass filter is a baseband signal with the highest frequency of $\omega_{max} = 0.5(\omega_1 + \omega_2)$. Using Nyquist sampling theorem, the sampling frequency is given by $\quad f_s\ge\frac{1}{2\pi}(\omega_1+\omega_2)$.

The sampling interval is limited by $T_s\le 2\pi/(\omega_1+\omega_2)$.

(c) If the sampling frequency is set to the Nyquist rate, i.e.,

$$f_s=\frac{1}{2\pi}(\omega_1+\omega_2),$$

then the original signal $x(t)$ is obtained from $x_s(t)$ by filtering with a bandpass filter with the transfer function

$$H(\omega)=\begin{cases}1 & \omega_1\le|\omega|\le\omega_2\\ 0 & \text{elsewhere.}\end{cases}$$

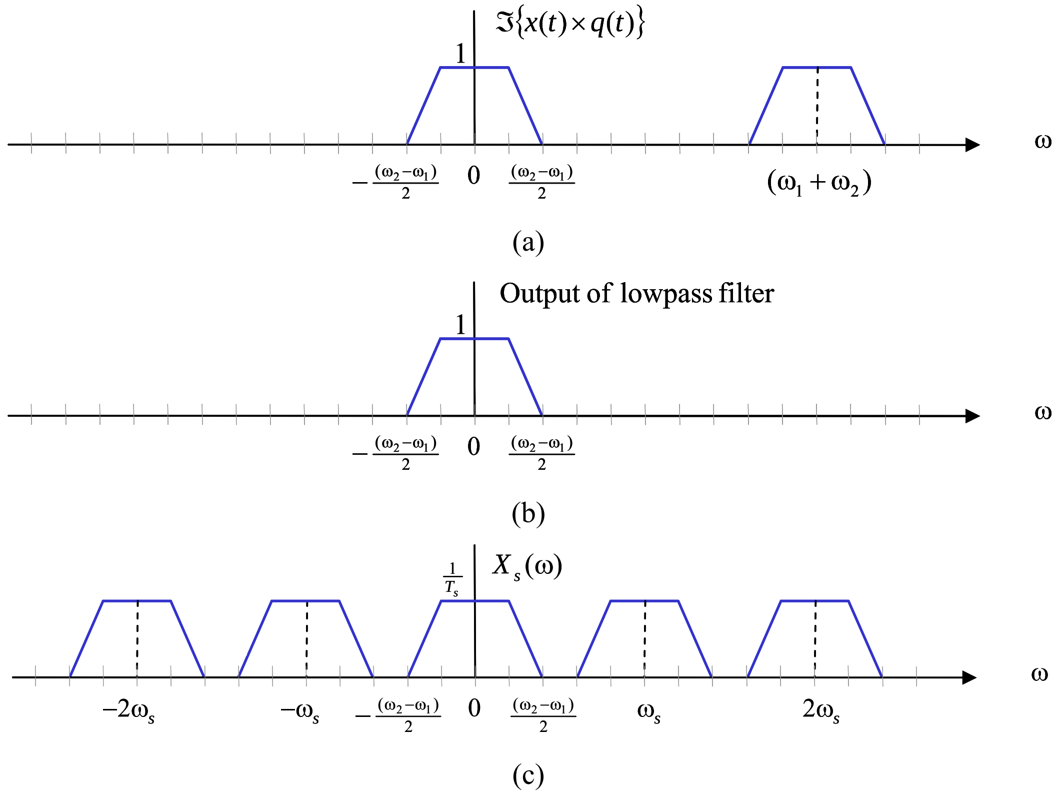


Fig. S9.9: Spectra of the signals: (a) at the output of the multiplier $q(t)$; (b) at the output of the lowpass filter; and (c) at the output of the impulse train multiplier.

For any other sampling frequency, $f_s > \frac{1}{2\pi}(\omega_1+\omega_2)$,

the original signal $x(t)$ is obtained from $x_s(t)$ by filtering with a lowpass filter with the transfer function

$$H(\omega)=\begin{cases}1 & 0\le|\omega|\le(\omega_1+\omega_2)/2\\ 0 & \text{elsewhere}\end{cases}$$

and then multiplying the resulting baseband signal with $2\cos(\pi((\omega_1+\omega_2)t)$. ▌

Problem 9.10

(a) The CTFS of the sawtooth wave is given by

$$s(t)=\sum_{n=-\infty}^{\infty}D_n\exp(j2n\pi t/T_s)$$

where the CTFS coefficients are given by

$$D_n=\frac{1}{T_s}\int_{-T_s/2}^{0}(1+\tfrac{2}{T_s}t)e^{-j2n\pi t/T_s}dt+\frac{1}{T_s}\int_{0}^{T_s/2}(1-\tfrac{2}{T_s}t)e^{-j2n\pi t/T_s}dt\,,$$

which simplifies to
$$D_n = \begin{cases} \frac{1}{2} & n = 0 \\ 0 & \text{even } n, n \neq 0 \\ \frac{2}{(n\pi)^2} & \text{odd } n, n \neq 0. \end{cases}$$

The CTFT of $z(t) = x(t) \times s(t)$ is given by

$$x(t) \times s(t) \xleftrightarrow{\text{CTFT}} \tfrac{1}{2\pi} X(\omega) * 2\pi \sum_{n=-\infty}^{\infty} D_n \delta(\omega - 2n\pi/T_s),$$

or,
$$Z(\omega) = \sum_{n=-\infty}^{\infty} D_n X(\omega - 2n\pi/T_s).$$

(b) The spectrum of the CTFT of the sampled signal $z(t)$ is shown in Fig. S9.10.

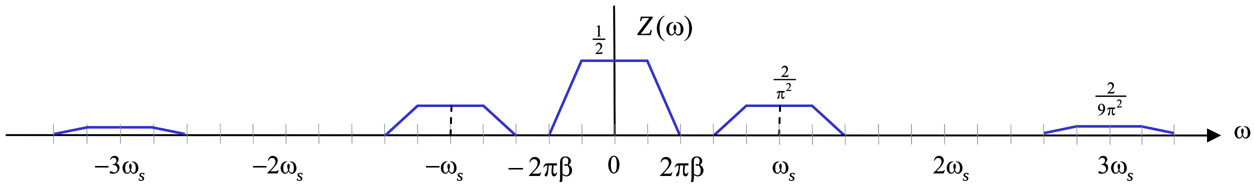


Fig S9.10: Spectrum of $z(t)$ sampled with a periodic sawtooth wave with $\omega_s = 2\pi/T_s$.

(c) From Fig. S9.10, the original signal $x(t)$ can be recovered from $z(t)$ if the main replica at $\omega = 0$ does not overlap with the replica at $\omega = \omega_s$. In other words, $\omega_s \geq 4\pi\beta$. Thus the fundamental frequency of the sawtooth saw wave is the same as the Nyquist sampling rate for $x(t)$.

The original signal $x(t)$ can be recovered from $z(t)$ by filtering $z(t)$ with a lowpass filter specified below

$$H(\omega) = \begin{cases} 2 & 0 \leq |\omega| \leq 0.5\omega_s \\ 0 & \text{otherwise.} \end{cases}$$

(d) Signal $z(t)$ is different from the sampled signal $x_s(t)$ obtained by ideal impulse train sampling in the following two ways. First, all replicas at even multiples of ω_s, (i.e., $\omega = 2k\omega_s$) are missing. Second, the magnitudes of the odd-numbered replicas are not constant and decrease with the increasing frequency. ∎

Problem 9.11

(a) The CTFS of the alternating sign impulse train with period $T = 2T_s$ is given by

$$s(t) = \sum_{n=-\infty}^{\infty} D_n \exp(jn\pi t/T_s)$$

where the CTFS coefficients are

$$D_n = \frac{1}{2T_s} \int_0^{2T_s} [\delta(t) - \delta(t - T_s)] e^{-jn\pi t/T_s} dt = \frac{1}{2T_s} \left[1 - e^{-jn\pi}\right],$$

or,
$$D_n = \begin{cases} 0 & \text{even } n \\ \frac{1}{T_s} & \text{odd } n. \end{cases}$$

The CTFT of $z(t) = x(t) \times s(t)$ is given by

$$x(t)\times s(t) \xleftrightarrow{\text{CTFT}} \tfrac{1}{2\pi}X(\omega) * 2\pi \sum_{n=-\infty}^{\infty} D_n \delta(\omega - n\pi/T_s),$$

or,

$$Z(\omega) = \sum_{n=-\infty}^{\infty} D_n X(\omega - n\pi/T_s).$$

(b) The spectrum of the CTFT of the sampled signal $z(t)$ is shown in Fig. S9.10.

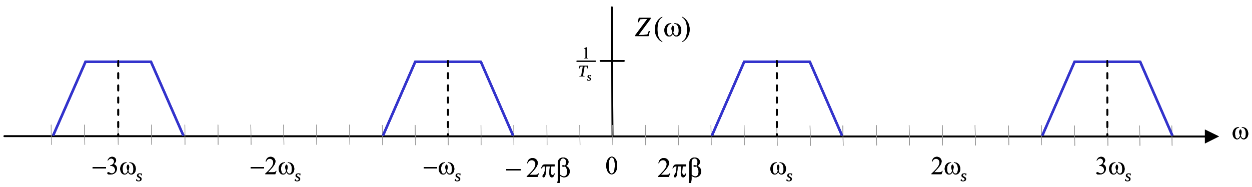


Fig S9.11: Spectrum of $z(t)$ sampled with an alternating sign impulse train with $\omega_s = \pi/T_s$.

(c) From Fig. S9.11, the original signal $x(t)$ can be recovered from $z(t)$ if the first replica at $\omega = \omega_s$ does not overlap with the replica at $\omega = 3\omega_s$. In other words, $\omega_s \geq 2\pi\beta$. Thus the fundamental frequency of the alternating sign impulse train is half the Nyquist sampling rate for $x(t)$.

The original signal $x(t)$ can be recovered from $z(t)$ by multiplying $z(t)$ with $\cos(\omega_s t)$ and then passing the resulting output through a lowpass filter specified below

$$H(\omega) = \begin{cases} T_s & 0 \leq |\omega| \leq 0.5\omega_s \\ 0 & \text{otherwise.} \end{cases}$$

(d) All replicas at even multiples of ω_s, (i.e., $\omega = 2k\omega_s$, for $k = 0, \pm1, \pm2, \pm3, \ldots$) are missing in the spectrum of $z(t)$. This is the only difference between the spectrum of the sampled signal $x_s(t)$ obtained by ideal impulse train and that of $z(t)$. ▌

Problem 9.12

(a) The CTFS of the alternating sign impulse train with period $T = T_s$ is given by

$$s(t) = \sum_{n=-\infty}^{\infty} D_n \exp(j2n\pi t/T_s)$$

where the CTFS coefficients are

$$D_n = \frac{1}{T_s}\int_0^{T_s} [\delta(t) + \delta(t-\Delta)] e^{-jn\pi t/T_s} dt = \frac{1}{T_s}\left[1 + e^{-jn\Delta/T_s}\right],$$

or,

$$D_n = \tfrac{2}{T_s} e^{-jn\Delta/2T_s} \cos(\tfrac{n\Delta}{T_s})$$

The CTFT of $z(t) = x(t) \times s(t)$ is given by

$$x(t)\times s(t) \xleftrightarrow{\text{CTFT}} \tfrac{1}{2\pi}X(\omega) * 2\pi \sum_{n=-\infty}^{\infty} D_n \delta(\omega - n\pi/T_s),$$

or,

$$Z(\omega) = \sum_{n=-\infty}^{\infty} D_n X(\omega - n\pi/T_s).$$

(b) The spectrum of the CTFT of the sampled signal $z(t)$ is shown in Fig. S9.10.

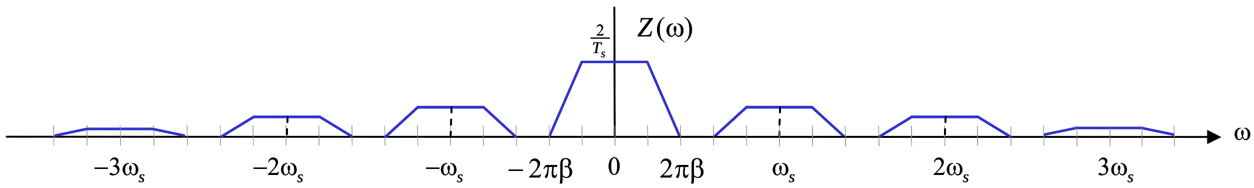


Fig S9.11: Spectrum of $z(t)$ sampled with an alternating sign impulse train with $\omega_s = \pi/T_s$.

(c) From Fig. S9.11, the original signal $x(t)$ can be recovered from $z(t)$ if the first replica at $\omega = 0$ does not overlap with the replica at $\omega = \omega_s$. In other words, $\omega_s \geq 4\pi\beta$. Thus the fundamental frequency of the sampling train is the same as the Nyquist sampling rate for $x(t)$.

The original signal $x(t)$ can be recovered from $z(t)$ by filtering $z(t)$ with a lowpass filter specified below

$$H(\omega) = \begin{cases} 0.5T_s & 0 \leq |\omega| \leq 0.5\omega_s \\ 0 & \text{otherwise.} \end{cases}$$

(d) Compared with the ideal impulse train sampling, the replicas are attenuated by a factor of $2\cos(\frac{n\Delta}{T_s})$. ▌

Problem 9.13

In terms of the sampling frequency f_s, the sampled signal is given by

$$X_s(\omega) = f_s \sum_{m=-\infty}^{\infty} X(\omega - 2m\pi f_s).$$

The output of the bandlimited channel is given by

$$X_{ch}(\omega) = f_s \left[X_{USB}(\omega \pm 4\pi f_s) + X(\omega \pm 6\pi f_s) + X_{LSB}(\omega \pm 8\pi f_s) \right]$$

where $X_{USB}(\omega)$ is the upper side band of $X(\omega)$ given by

$$X_{USB}(\omega_s) = \begin{cases} X(\omega) & 0 \leq \omega \leq \pi f_s \\ 0 & \text{elsewhere} \end{cases}$$

and $X_{LSB}(\omega)$ is the lower side band of $X(\omega)$ given by

$$X_{USB}(\omega_s) = \begin{cases} X(\omega) & 0 \leq \omega \leq \pi f_s \\ 0 & \text{elsewhere.} \end{cases}$$

Note that
$$X(\omega) = X_{LSB}(\omega) + X_{USB}(\omega)$$

and $X(\omega)$ is a baseband signal with a bandwidth of $B = \pi f_s$.

The original signal $X(\omega)$ can be recovered from $X_{ch}(\omega)$ by first multiplying with $\cos(6\pi f_s t)$ and then filtering the output with a lowpass filter whose transfer function is given by

$$H(\omega) = \begin{cases} 1/f_s & 0 \leq |\omega| \leq \pi f_s \\ 0 & \text{otherwise.} \end{cases}$$ ▌

Problem 9.14

Given that the maximum quantization noise ($\Delta/2$) is limited to $\pm p$ percent of the peak-to-peak value V_{pp}, therefore,

$$\frac{\Delta}{2} = 0.01p \times V_{pp}\text{, or, } \Delta = 0.02p \times V_{pp}\text{,}$$

implying that
$$\text{No. of Levels}\,(L) = \frac{V_{pp}}{0.02p \times V_{pp}} + 1 = \frac{50}{p} + 1,$$

and
$$\text{No. of bits per level} \geq \log_2\left[\tfrac{50}{p} + 1\right] = \frac{\log_{10}\left[\tfrac{50}{p} + 1\right]}{\log_{10}(2)} = 3.32\log_{10}\left[\tfrac{50}{p} + 1\right].$$

For $p << 1$, the above expression reduces to

$$\text{No. of bits per level} \geq 3.32\log_{10}\left[\tfrac{50}{p}\right].$$

∎

Problem 9.15

(a) Based on the Nyquist sampling theorem, the sampling frequency f_s is given by

$$f_s \geq 2B = 8000\ \text{samples/s.}$$

The maximum sampling period is given by $T_s \leq 1/8000 = 0.125$ms.

(b) Based on Problem 9.14,

$$\text{No. of bits per level} \geq 3.32\log_{10}\left[\tfrac{50}{5}\right] = 3.32.$$

Rounding off to the nearest whole number, length of the code word = 4 bits/sample.

(c) The data rate of PCM is 4 bits/sample × 8000 samples/s = 32 kbits/s.

∎

Problem 9.16

Sampling Rate = 2×10^3 samples/s.

Transmission Speed = 2×10^6 bits/s.

Maximum number of bits per sample = $\dfrac{2\times10^6}{2\times10^3} = 10$.

Number of Levels (L) = 10^{10} = 1024.

As the input amplitude range is ±1 V, the quantization step size is given by, $\Delta = \dfrac{2}{1024} \approx 0.00195$.

The maximum distortion is given by, $\Delta/2 \approx 0.000977$.

∎

Problem 9.17

The input-output relationship of the ideal sampling system is given by

$$x(t) \to y(t) = \sum_{k=-\infty}^{\infty} x(kT_s)\delta(t-kT_s)$$

(i) Linearity: Since

$$x_1(t) \to \sum_{k=-\infty}^{\infty} x_1(kT_s)\delta(t-kT_s) = y_1(t)$$

$$x_2(t) \to \sum_{k=-\infty}^{\infty} x_2(kT_s)\delta(t-kT_s) = y_2(t)$$

$$\begin{aligned}\alpha x_1(t) + \beta x_2(t) &\to \sum_{k=-\infty}^{\infty} \left[\alpha x_1(kT_s) + \beta x_2(kT_s)\right]\delta(t-kT_s) \\ &= \alpha \sum_{k=-\infty}^{\infty} x_1(kT_s)\delta(t-kT_s) + \beta \sum_{k=-\infty}^{\infty} x_2(kT_s)\delta(t-kT_s) \\ &= \alpha y_1(t) + \beta y_2(t)\end{aligned}$$

the system is a linear system.

(ii) Time Invariance: For inputs $x_1(t)$ and $x_2(t) = x_1(t - T)$, the outputs are given by

$$x_1(t) \to \sum_{k=-\infty}^{\infty} x_1(kT_s)\delta(t-kT_s) = y_1(t)$$

$$x_2(t) = x_1(t-T) \to \sum_{k=-\infty}^{\infty} x_2(kT_s)\delta(t-kT_s) = x_1(t-T-2) = y_2\left(t\right)$$

If $x_2(t) = x_1(t-T)$, then $y_2(t) = \sum_{k=-\infty}^{\infty} x_2(kT_s)\delta(t-kT_s) = \sum_{k=-\infty}^{\infty} x_1(kT_s - T)\delta(t-kT_s)$.

Since $y_1(t-T) = \sum_{k=-\infty}^{\infty} x_1(kT_s)\delta(t-T-kT_s) \neq y_2(t)$, the system is NOT time invariant.

(iii) Memoryless: The system output at time $t = kT_s$ depends only on the input value at $t = kT_s$. Therefore, the system is memoryless.

(iv) Causality: Since the system is memoryless, it is also causal.

(v) Stability: Assume that the input is bounded $|x(t)| \le M$. Then, the absolute output can be expressed as:

$$\left|y(t)\right| = \left|\sum_{k=-\infty}^{\infty} x(kT_s)\delta(t-kT_s)\right| = \sum_{k=-\infty}^{\infty} \left|x(kT_s)\right|\delta(t-kT_s) < M \sum_{k=-\infty}^{\infty} \delta(t-kT_s)$$

Note that because of the CT impulse functions, $|y(t)|$ is not strictly a bounded function. Therefore, the system is NOT BIBO stable.

(vi) As discussed in chapter 9, the $x(t)$ can be reconstructed from the samples, if the sampling rate is greater than Nyquist sampling rate. Therefore, the system is in general not invertible. However, if the sampling theorem is satisfied, the input can be reconstructed using the following equation:

$$x(t) = \mathrm{sinc}\left(f_s t\right) * y(t)$$

where f_s is the sampling frequency. ▮

Problem 9.18

The input-output relationship of the ideal sampling system is given by

$$x[k] \to Q\{x[k]\} = \tfrac{1}{2}\left[d_m + d_{m+1}\right] = y[k] \quad \text{for} \quad d_m \le x[k] < d_{m+1}\,.$$

(i) Linearity: In a linear system, there is one-to-one correspondence between input and output values. In other words, no two different input signal values will produce the same output value. In the above quantization system, this is not the case. Note that all signal values in the range $[d_m, d_{m+1})$ produce the same output value, which is $0.5[d_m + d_{m+1}]$. In other words, the quantization system is nonlinear.

(ii) Time Invariance: From the input-output relationship of the system, it is observed that the output value depends on the input value (and the decision levels), and does not depend anyway on the time factor, i.e., when the output is calculated. Therefore, if we have an input output pair, $x[k] \to y[k]$, and if we delay $x[k]$, the output $y[k]$ will also be delayed by the same amount. Hence, the system is time-invariant.

(iii) Memoryless: The system output at time k depends only on the input value at the same time k. Therefore, the system is memoryless.

(iv) Causality: Since the system is memoryless, it is also causal.

(v) Stability: In a quantizer, the output must be equal to one of the reconstruction levels, and the reconstruction levels are always finite valued. Therefore, the system is BIBO stable.

(vi) In an invertible system, there is unique correspondence between input and output values. In other words, no two different input signal values will produce the same output value. In the above quantization system, this is not the case. If the output of the system is $0.5[d_m + d_{m+1}]$, the input could be of any value in the range $[d_m, d_{m+1})$. So, it is not possible to uniquely determine the input values. In other words, the quantization system is NOT invertible. ▮

Problem 9.19

(a) The space required to store each audio sample (per channel) is 16 bits. As there are two channels,

Space needed to represent each stereo audio sample = 32 bits = 4 bytes

An audio clip has an average duration of 5 minutes (300 s). Therefore,

Number of audio samples/clip = $44100 \times 300 = 13.23 \times 10^6$ samples.

To store an audio clip,

Required storage space = $13.23 \times 10^6 \times 4 = 52.92 \times 10^6$ bytes.

(b) As the mp3 format reduces the file size to $1/8^{\text{th}}$ of its original size, to store an mp3 audio clip,

Required storage space = $52.92 \times 10^6 / 8 = 6.615 \times 10^6$ bytes.

(c) Number of mp3-compressed audio clips = $\dfrac{1024 \times 10^6}{6.615 \times 10^6} \approx 154.8$. ▮

Problem 9.20

(a) The space required to store each color (3 channels) pixel is 24 bits or 3 bytes. As there are 2560 × 1920 pixels in an image, therefore,

$$\text{Space needed to represent each image} = 2560 \times 1920 \times 3 = 14.746 \times 10^6 \text{ bytes.}$$

(b) The image in JPEG format requires one-tenth of the uncompressed storage space. Therefore,

$$\text{Space needed to represent each image (JPEG)} = 1.475 \times 10^6 \text{ bytes}$$

(c) The camera has 512×10^6 bytes of memory. Therefore,

$$\text{Number of JPEG-compressed that can be stored} = \frac{512 \times 10^6}{1.475 \times 10^6} \approx 347 \,.$$

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