Chapter 3: Time Domain Analysis of LTIC Systems

Problem 3.1

Linearity: For $x_3(t) = \alpha\, x_1(t) + \beta\, x_2(t)$ applied as the input, the output $y_3(t)$ is given by

$$\frac{d^n y_3}{dt^n} + a_{n-1}\frac{d^{n-1} y_3}{dt^{n-1}} + \cdots + a_1\frac{dy_3}{dt} + a_0 y_3(t) = b_m\frac{d^m(\alpha x_1(t) + \beta x_2(t))}{dt^m} + b_{m-1}\frac{d^{m-1}(\alpha x_1(t) + \beta x_2(t))}{dt^{m-1}}$$
$$+ \cdots + b_1\frac{d(\alpha x_1(t) + \beta x_2(t))}{dt} + b_0(\alpha x_1(t)\beta x_2(t)).$$

Rearranging the terms on the right hand side of the equation, we get

$$\frac{d^n y_3}{dt^n} + a_{n-1}\frac{d^{n-1} y_3}{dt^{n-1}} + \cdots + a_1\frac{dy_3}{dt} + a_0 y_3(t) = \alpha\left[b_m\frac{d^m x_1}{dt^m} + b_{m-1}\frac{d^{m-1}x_1}{dt^{m-1}} + \cdots + b_1\frac{dx_1}{dt} + b_0 x_1(t)\right]$$
$$+\beta\left[b_m\frac{d^m x_2}{dt^m} + b_{m-1}\frac{d^{m-1}x_2}{dt^{m-1}} + \cdots + b_1\frac{dx_2}{dt} + b_0 x_2(t)\right]$$

Expressing the higher order derivatives of $x_1(t)$ and $x_2(t)$ in terms of $y_1(t)$ and $y_2(t)$, we get

$$\frac{d^n y_3}{dt^n} + a_{n-1}\frac{d^{n-1} y_3}{dt^{n-1}} + \cdots + a_1\frac{dy_3}{dt} + a_0 y_3(t) = \alpha\left[\frac{d^n y_1}{dt^n} + a_{n-1}\frac{d^{n-1}y_1}{dt^{n-1}} + \cdots + a_1\frac{dy_1}{dt} + a_0 y_1(t)\right]$$
$$+\beta\left[\frac{d^n y_1}{dt^n} + a_{n-1}\frac{d^{n-1}y_1}{dt^{n-1}} + \cdots + a_1\frac{dy_1}{dt} + a_0 y_1(t)\right]$$

or,

$$\frac{d^n y_3}{dt^n} + a_{n-1}\frac{d^n y_3}{dt^n} + \cdots + a_1\frac{dy_3}{dt} + a_0 y_3(t) = \frac{d^n(\alpha y_1 + \beta y_2)}{dt^n} + a_{n-1}\frac{d^{n-1}(\alpha y_1 + \beta y_2)}{dt^{n-1}}$$
$$+ \cdots + a_1\frac{d(\alpha y_1 + \beta y_2)}{dt} + a_0\big(\alpha y_1(t) + \beta y_2(t)\big)$$

which implies that $$y_3(t) = \alpha y_1(t) + \beta y_2(t).$$

The system is therefore linear.

Time-invariance: For $x(t - t_0)$ applied as the input, the output $y_1(t)$ is given by

$$\frac{d^n y_1}{dt^n} + a_{n-1}\frac{d^{n-1} y_1}{dt^{n-1}} + \cdots + a_1\frac{dy_1}{dt} + a_0 y_1(t) = b_m\frac{d^m x(t-t_0)}{dt^m} + b_{m-1}\frac{d^{m-1}x(t-t_0)}{dt^{m-1}}$$
$$+ \cdots + b_1\frac{dx(t-t_0)}{dt} + b_0 x(t-t_0)$$

Substituting $\tau = t - t_0$ (which implies that $dt = d\tau$), we get

$$\frac{d^n y_1(\tau+t_0)}{d\tau^n} + a_{n-1}\frac{d^{n-1} y_1(\tau+t_0)}{d\tau^{n-1}} + \cdots + a_1\frac{dy_1(\tau+t_0)}{d\tau} + a_0 y_1(\tau+t_0) = b_m\frac{d^m x(\tau)}{d\tau^m} + b_{m-1}\frac{d^{m-1}x(\tau)}{d\tau^{m-1}}$$
$$+ \cdots + b_1\frac{dx(\tau)}{d\tau} + b_0 x(\tau)$$

Comparing with the original differential equation representation of the system, we get

$$y(\tau) = y_1(\tau + t_0) \quad \text{or,} \quad y_1(\tau) = y(\tau - t_0),$$

proving that the system is time-invariant. Note that the time invariance property is only valid if the coefficients a_r's and b_r's are constants. If a_r's and b_r's are functions of time, then the substitution ($\tau = t - t_0$) will also affect them. Clearly, $y(\tau) \neq y_1(\tau + t_0)$ in such a case and the system will NOT be time-invariant. ▌

Problem 3.2

(i) $\ddot{y}(t) + 4\dot{y}(t) + 8y(t) = \dot{x}(t) + x(t)$ with $x(t) = e^{-4t}u(t), y(0) = 0,$ and $\dot{y}(0) = 0.$

(a) Zero-input response of the system: The characteristic equation of the LTIC system (i) is

$$s^2 + 4s + 8 = 0,$$

which has roots at $s = -2 \pm j2$. The zero-input response is given by

$$y_{zi}(t) = Ae^{-2t}\cos(2t) + Be^{-2t}\sin(2t)$$

for $t \geq 0$, with A and B being constants. To calculate their values, we substitute the initial conditions $y(0^-) = 0$ and $\dot{y}(0^-) = 0$ in the above equation. The resulting simultaneous equations are

$$\begin{aligned} A &= 0 \\ -2A + 2B &= 0 \end{aligned}$$

that has the solution, $A = 0$ and $B = 0$. The zero-input response is therefore given by

$$y_{zi}(t) = 0.$$

Because of the zero initial conditions, the zero-input response is also zero.

(b) Zero-state response of the system: To calculate the zero-state response of the system, the initial conditions are assumed to be zero. Hence, the zero state response $y_{zs}(t)$ can be calculated by solving the differential equation

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 8 = \frac{dx}{dt} + x(t)$$

with initial conditions, $y(0^-) = 0$ and $\dot{y}(0^-) = 0$, and input $x(t) = \exp(-4t)u(t)$. The homogenous solution of system (i) has the same form as the zero-input response and is given by

$$y_{zs}^{(h)}(t) = C_1e^{-2t}\cos(2t) + C_2e^{-2t}\sin(2t)$$

for $t \geq 0$, with C_1 and C_2 being constants. The particular solution for input $x(t) = \exp(-4t)u(t)$ is of the form

$$y_{zs}^{(p)}(t) = Ke^{-4t}u(t).$$

Substituting the particular solution in the differential equation for system (i) and solving the resulting equation gives $K = -3/8$. The zero-state response of the system is, therefore, given by

$$y_{zs}(t) = \left(C_1e^{-2t}\cos(2t) + C_2e^{-2t}\sin(2t) - \tfrac{3}{8}e^{-4t}\right)u(t).$$

To compute the values of constants C_1 and C_2, we use the initial conditions, $y(0^-) = 0$ and $\dot{y}(0^-) = 0$ assumed for the zero-state response. Substituting the initial conditions in $y_{zs}(t)$ leads to the following simultaneous equations

$$\begin{aligned} C_1 - \tfrac{3}{8} &= 0 \\ -2C_1 + 2C_2 + \tfrac{3}{2} &= 0 \end{aligned}$$

with solution $C_1 = 3/8$ and $C_2 = -3/8$. The zero-state solution is given by

$$y_{zs}(t) = \tfrac{3}{8}\left(e^{-2t}\cos(2t) - e^{-2t}\sin(2t) - e^{-4t}\right)u(t).$$

(c) Overall response of the system: The overall response of the system is obtained by summing up the zero-input and zero-state responses, and is given by

$$y(t) = \tfrac{3}{8}\left(e^{-2t}\cos(2t) - e^{-2t}\sin(2t) - e^{-4t}\right)u(t).$$

(d) Steady state response of the system: The steady state response of the system is obtained by applying the limit, $t \to \infty$, to $y(t)$ and is given by

$$y(t) = \lim_{t\to\infty} \tfrac{3}{8}\left(e^{-2t}\cos(2t) - e^{-2t}\sin(2t) - e^{-4t}\right)u(t) = 0.$$

(ii) $\ddot{y}(t) + 6\dot{y}(t) + 4y(t) = \dot{x}(t) + x(t)$ with $x(t) = \cos(6t)u(t), y(0) = 2,$ and $\dot{y}(0) = 0.$

(a) Zero-input response of the system: The characteristic equation of the LTIC system (ii) is

$$s^2 + 6s + 4 = 0,$$

which has roots at $s = -3 \pm 2.2361 = -5.2361$ and -0.7639. The zero-input response is given by

$$y_{zi}(t) = Ae^{-5.2361t} + Be^{-0.7639t}$$

for $t \geq 0$ with A and B being constants. To calculate their values, we substitute the initial conditions $y(0^-) = 2$ and $\dot{y}(0^-) = 0$ in the above equation. The resulting simultaneous equations are

$$\begin{aligned} A + B &= 2 \\ -5.2361A - 0.7639B &= 0 \end{aligned}$$

that has a solution, $A = -0.3416$ and $B = 2.3416$. The zero-input response is therefore given by

$$y_{zi}(t) = \left(-0.3416e^{-5.2361t} + 2.3416e^{-0.7639t}\right)u(t).$$

(b) Zero-state response of the system: To calculate the zero-state response of the system, the initial conditions are assumed to be zero. Hence, the zero state response $y_{zs}(t)$ can be calculated by solving the differential equation

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 4 = \frac{dx}{dt} + x(t)$$

with initial conditions, $y(0^-) = 0$ and $\dot{y}(0^-) = 0$, and input $x(t) = \cos(6t)u(t)$. The homogenous solution of system (ii) has the same form as its zero-input response and is given by

$$y_{zs}^{(h)}(t) = C_1e^{-5.2361t} + C_2e^{-0.7639t}$$

for $t \geq 0$, with C_1 and C_2 being constants. The particular solution for input $x(t) = \cos(6t)u(t)$ is of the form

$$y_{zs}^{(p)}(t) = K_1\cos(6t) + K_2\sin(6t).$$

Substituting the particular solution in the differential equation for system (ii) and solving the resulting equation gives

$$\left(-36K_1\cos(6t)-36K_2\sin(6t)\right)+6\left(-6K_1\sin(6t)+6K_2\cos(6t)\right)$$
$$+4\left(K_1\cos(6t)+K_2\sin(6t)\right)=-6\sin(6t)+\cos(6t)$$

Collecting the coefficients of the cosine and sine terms, we get

$$\left(-36K_1+36K_2+4K_1-1\right)\cos(6t)+\left(-36K_2-36K_1+4K_2+6\right)\sin(6t)=0$$

or,

$$-32K_1+36K_2=1$$
$$-36K_1-32K_2=-6$$

which has the solution, K_1 = 0.0793 and K_2 = 0.0983. The zero-state response of the system is

$$y_{zs}(t)=\left(C_1e^{-5.2361t}+C_2e^{-0.7639t}+0.0793\cos(6t)+0.0983\sin(6t)\right)u(t).$$

To compute the values of constants C_1 and C_2, we use the zero initial conditions, $y(0^-)=0$ and $\dot{y}(0^-)=0$ assumed for the zero-state response. Substituting the initial conditions in $y_{zs}(t)$ leads to the following simultaneous equations

$$C_1+C_2+0.0793=0$$
$$-5.2361C_1-0.7639C_2+6(0.0983)=0$$

with solution C_1 = 0.1454 and C_2 = –0.2247. The zero-state solution is given by

$$y_{zs}(t)=\left(0.1454e^{-5.2361t}-0.2247e^{-0.7639t}+0.0793\cos(6t)+0.0983\sin(6t)\right)u(t).$$

(c) Overall response of the system: The overall response of the system is obtained by summing up the zero-input and zero-state responses, and is given by

$$y(t)=\left(-0.3416e^{-5.2361t}+2.3416e^{-0.7639t}\right)u(t)$$
$$+\left(0.1454e^{-5.2361t}-0.2247e^{-0.7639t}e^{-0.7639t}+0.0793\cos(6t)+0.0983\sin(6t)\right)u(t)$$

or, $$y(t)=\left(-0.1962e^{-5.2361t}+2.1169e^{-0.7639t}+0.0793\cos(6t)+0.0983\sin(6t)\right)u(t).$$

(d) Steady state response of the system: The steady state response of the system is obtained by applying the limit, $t\to\infty$, to $y(t)$ and is given by

$$y(t)=\left(0.0793\cos(6t)+0.0983\sin(6t)\right)u(t).$$

(iii) $\ddot{y}(t)+2\dot{y}(t)+y(t)=\ddot{x}(t)$ with $x(t)=\left[\cos(t)+\sin(2t)\right]u(t), y(0)=3,$ and $\dot{y}(0)=1.$

(a) Zero-input response of the system: The characteristic equation of the LTIC system (iii) is

$$s^2+2s+1=0,$$

which has roots at $s=-1,-1$. The zero-input response is given by

$$y_{zi}(t)=Ae^{-t}+Bte^{-t}$$

for $t \geq 0$, with A and B being constants. To calculate their values, we substitute the initial conditions $y(0^-) = 3$ and $\dot{y}(0^-) = 1$ in the above equation. The resulting simultaneous equations are

$$\begin{aligned} A &= 3 \\ -A + B &= 1 \end{aligned}$$

that has a solution, $A = 3$ and $B = 4$. The zero-input response is therefore given by

$$y_{zi}(t) = \left(3e^{-t} + 4te^{-t}\right)u(t).$$

(b) Zero-state response of the system: To calculate the zero-state response of the system, the initial conditions are assumed to be zero. Hence, the zero state response $y_{zs}(t)$ can be calculated by solving the differential equation

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 1 = \ddot{x}(t)$$

with initial conditions, $y(0^-) = 0$ and $\dot{y}(0^-) = 0$, and input $x(t) = [\cos(t) + \sin(t)]u(t)$. The homogenous solution of system (iii) has the same form as the zero-input response and is given by

$$y_{zs}^{(h)}(t) = C_1e^{-t} + C_2te^{-t}$$

for $t \geq 0$, with C_1 and C_2 being constants. The particular solution for input $x(t) = [\cos(t) + \sin(t)]u(t)$ is of the form

$$y_{zs}^{(p)}(t) = K_1\cos(t) + K_2\sin(t) + K_3\cos(2t) + K_4\sin(2t).$$

Substituting the particular solution in the differential equation for system (iii) and solving the resulting equation gives

$$\begin{aligned} &\left(-K_1\cos(t) - K_2\sin(t) - 4K_3\cos(2t) - 4K_4\sin(2t)\right) + 2\left(-K_1\sin(t) + K_2\cos(t) - 2K_3\sin(2t)\right. \\ &\quad \left. + 2K_4\cos(2t)\right) + 1\left(K_1\cos(t) + K_2\sin(t) + K_3\cos(2t) + K_4\sin(2t)\right) = -\cos(t) - 4\sin(2t) \end{aligned}$$

Collecting the coefficients of the cosine and sine terms, we get

$$\begin{aligned} &\left(-K_1 + 2K_2 + K_1 + 1\right)\cos(t) + \left(-K_2 - 2K_1 + K_2\right)\sin(t) + \\ &\quad \left(-4K_3 + 4K_4 + K_3\right)\cos(2t) + \left(-4K_4 - 4K_3 + K_4 + 4\right)\sin(2t) = 0 \end{aligned}$$

which gives $K_1 = 0$, $K_2 = -0.5$, $K_3 = 0.64$, and $K_4 = 0.48$. The zero-state response of the system is

$$y_{zs}(t) = \left(C_1e^{-t} + C_2te^{-t} - 0.5\sin(t) + 0.64\cos(2t) + 0.48\sin(2t)\right)u(t).$$

To compute the values of constants C_1 and C_2, we use the initial conditions, $y(0^-) = 0$ and $\dot{y}(0^-) = 0$. Substituting the initial conditions in $y_{zs}(t)$ leads to the following simultaneous equations

$$\begin{aligned} C_1 + 0.64 &= 0 \\ -C_1 + C_2 - 0.5 + 0.48 &= 0 \end{aligned}$$

with solution $C_1 = -0.64$ and $C_2 = -1.1$. The zero-state solution is given by

$$y_{zs}(t) = \left(-0.64e^{-t} - 1.1te^{-t} - 0.5\sin(t) + 0.64\cos(2t) + 0.48\sin(2t)\right)u(t).$$

(c) Overall response of the system: The overall response of the system is obtained by summing up the zero-input and zero-state responses, and is given by

or, $y(t) = \left(3e^{-t} + 4te^{-t}\right)u(t) + \left(-0.64e^{-t} - 1.1te^{-t} - 0.5\sin(t) + 0.64\cos(2t) + 0.48\sin(2t)\right)u(t)$

or, $$y(t) = \left(2.36e^{-t} + 2.9te^{-t} - 0.5\sin(t) + 0.64\cos(2t) + 0.48\sin(2t)\right)u(t).$$

(d) Steady state response of the system: The steady state response of the system is obtained by applying the limit, $t \to \infty$, to $y(t)$ and is given by

$$y(t) = \lim_{t\to\infty}\left(2.36e^{-t} + 2.9te^{-t} - 0.5\sin(t) + 0.64\cos(2t) + 0.48\sin(2t)\right)\ u(t)$$

or, $$y(t) = \left(-0.5\sin(t) + 0.64\cos(2t) + 0.48\sin(2t)\right)\ u(t).$$

(iv) $\ddot{y}(t) + 4y(t) = 5x(t)$ with $x(t) = 4te^{-t}u(t), y(0) = -2,$ and $\dot{y}(0) = 0.$

(a) Zero-input response of the system: The characteristic equation of the LTIC system (iv) is

$$s^2 + 4 = 0,$$

which has roots at $s = \pm j2$. The zero-input response is given by

$$y_{zi}(t) = A\cos(2t) + B\sin(2t)$$

for $t \geq 0$, with A and B being constants. To calculate their values, we substitute the initial conditions $y(0^-) = -2$ and $\dot{y}(0^-) = 0$ in the above equation. The resulting simultaneous equations are

$$\begin{aligned} A &= -2 \\ 2B &= 0 \end{aligned}$$

that has a solution, $A = -2$ and $B = 0$. The zero-input response is therefore given by

$$y_{zi}(t) = -2\cos(2t)u(t)$$

(b) Zero-state response of the system: To calculate the zero-state response of the system, the initial conditions are assumed to be zero. Hence, the zero state response $y_{zs}(t)$ can be calculated by solving the differential equation

$$\frac{d^2y}{dt^2} + 4 = 5x(t)$$

with initial conditions, $y(0^-) = 0$ and $\dot{y}(0^-) = 0$, and input $x(t) = 4t\exp(-t)\,u(t)$. The homogenous solution of system (iv) has the same form as the zero-input response and is given by

$$y_{zs}^{(h)}(t) = C_1\cos(2t) + C_2\sin(2t)$$

where C_1 and C_2 are constants. The particular solution for input $x(t) = 4t\exp(-t)\,u(t)$ is of the form

$$y_{zs}^{(p)}(t) = K_1e^{-t} + K_2te^{-t}.$$

Substituting the particular solution in the differential equation for system (iv) and solving the resulting equation gives

$$\left(K_1e^{-t} - K_2e^{-t} - K_2e^{-t} + K_2te^{-t}\right) + 4\left(K_1e^{-t} + K_2te^{-t}\right) = 20te^{-t}$$

Collecting the coefficients of $\exp(-t)$ and $t\exp(-t)$, we get

$$\left(K_1e^{-t} - K_2e^{-t} - K_2e^{-t} + 4K_1e^{-t}\right) + \left(K_2te^{-t} + 4K_2te^{-t}\right) = 20te^{-t}$$

which gives $K_1 = 1.6$ and $K_2 = 4$. The zero-state response of the system is given by

$$y_{zs}(t) = \left(C_1 \cos(2t) + C_2 \sin(2t) + 1.6e^{-t} + 4te^{-t}\right).$$

To compute the values of constants C_1 and C_2, we use the initial conditions, $y(0^-) = 0$ and $\dot{y}(0^-) = 0$. Substituting the initial conditions in $y_{zs}(t)$ leads to the following simultaneous equations

$$\begin{aligned} C_1 + 1.6 &= 0 \\ 2C_2 - 1.6 + 4 &= 0 \end{aligned}$$

with solution $C_1 = -1.6$ and $C_2 = -1.2$. The zero-state solution is given by

$$y_{zs}(t) = \left(-1.6\cos(2t) - 1.2\sin(2t) + 1.6e^{-t} + 4te^{-t}\right)u(t).$$

(c) Overall response of the system: The overall response of the system is obtained by summing up the zero-input and zero-state responses, and is given by

or, $$y(t) = -2\cos(2t)\,u(t) + \left(-1.6\cos(2t) - 1.2\sin(2t) + 1.6e^{-t} + 4te^{-t}\right)u(t)$$

or, $$y(t) = \left(-3.6\cos(2t) - 1.2\sin(2t) + 1.6e^{-t} + 4te^{-t}\right)u(t).$$

(d) Steady state response of the system: The steady state response of the system is obtained by applying the limit, $t \to \infty$, to $y(t)$ and is given by

$$y(t) = \lim_{t\to\infty}\left(-2.32\cos(2t) - 0.24\sin(2t) + 0.32e^{-t} + 0.8te^{-t}\right)u(t) = \left(-2.32\cos(2t) - 0.24\sin(2t)\right)u(t).$$

(v) $\frac{d^4y}{dt^4} + 2\frac{d^2y}{dt^2} + y(t) = x(t) \quad with \quad x(t) = 2u(t), y(0) = \ddot{y}(0) = \dddot{y}(0) = 0, \quad and \ \dot{y}(0) = 1.$

(a) Zero-input response of the system: The characteristic equation of the LTIC system (v) is

$$s^4 + 2s + 1 = 0,$$

which has roots at $s = \pm j1, \pm j1$. The zero-input response is given by

$$y_{zi}(t) = Ae^{jt} + Bte^{jt} + Ce^{-jt} + Dte^{-jt},$$

for $t \geq 0$, with A and B being constants. To calculate their values, we substitute the initial conditions in the above equation. The resulting simultaneous equations are

$$\begin{array}{lllllll} A & & +C & & = & 0 \\ jA & +B & -jC & +D & = & 1 \\ -A & +j2B & -C & -j2D & = & 0 \\ -jA & -3B & +jC & -3D & = & 0 \end{array}$$

that has a solution, $A = -j0.75$ $B = -0.25$, $C = j0.75$ and $D = -0.25$. The zero-input response is

$$y_{zi}(t) = \left(-j0.75e^{jt} - 0.25te^{jt} + j0.75e^{-jt} - 0.25te^{-jt}\right)u(t),$$

which reduces to

$$y_{zi}(t) = \left(1.5\sin t - 0.5t\cos t\right)u(t).$$

(b) Zero-state response of the system: To calculate the zero-state response of the system, the initial conditions are assumed to be zero. Hence, the zero state response $y_{zs}(t)$ can be calculated by solving the differential equation

$$\frac{d^4y}{dt^4} + 2\frac{d^2y}{dt^2} + y(t) = x(t)$$

with all initial conditions set to 0 and input $x(t) = 2u(t)$. The homogenous solution of system (v) has the same form as its zero-input response and is given by

$$y_{zs}^{(h)}(t) = C_1e^{jt} + C_2te^{jt} + C_3e^{-jt} + C_4te^{-jt}$$

where C_i's are constants. The particular solution for input $x(t) = 2\,u(t)$ is of the form

$$y_{zs}^{(p)}(t) = Ku(t).$$

Substituting the particular solution in the differential equation for system (v) and solving the resulting equation gives

$$0 + 2(0) + K = 2\text{, or, } K = 2.$$

The zero-state response of the system is given by

$$y_{zs}(t) = C_1e^{jt} + C_2te^{jt} + C_3e^{-jt} + C_4te^{-jt} + 2,$$

for $(t \geq 0)$. To compute the values of constants C_i's, we use zero initial conditions. Substituting the initial conditions in $y_{zs}(t)$ leads to the following simultaneous equations

$$\begin{array}{lllllll} A & & +C & & = & -2 \\ jA & +B & -jC & +D & = & 0 \\ -A & +j2B & -C & -j2D & = & 0 \\ -jA & -3B & +jC & -3D & = & 0 \end{array}$$

with solution $C_1 = -1$, $C_2 = j0.5$, $C_3 = -1$, and $C_4 = -j0.5$. The zero-state solution is given by

$$y_{zs}(t) = \left(-e^{jt} + j0.5te^{jt} - e^{-jt} - j0.5te^{-jt}\right)u(t),$$

which reduces to

$$y_{zi}(t) = \left(-2\cos t - t\sin t\right)u(t).$$

(c) Overall response of the system: The overall response of the system is obtained by summing up the zero-input and zero-state responses, and is given by

or, $$y(t) = \left(1.5\sin t - 0.5t\cos t\right)u(t) + \left(-2\cos t - t\sin t\right)u(t)$$

or, $$y(t) = \left(1.5\sin t - 2\cos t - t\sin t - 0.5t\cos t + 2\right)\,u(t).$$

(d) Steady state response of the system: The steady state response of the system is obtained by applying the limit, $t \to \infty$, to $y(t)$ and is given by

$$y(t) = \lim_{t\to\infty}\left(1.5\sin t - 2\cos t - t\sin t - 0.5t\cos t + 2\right)u(t) \to \infty.$$

∎

Problem 3.3

(i) To evaluate the impulse response, set $x(t) = \delta(t)$. The resulting equation is

$$\dot{h}(t) = 2\delta(t) \quad \Rightarrow \quad h(t) = 2\int_{-\infty}^{t} \delta(t)dt + C = 2u(t) + C$$

where C is a constant that can be evaluated from the initial condition. Since the initial condition is 0, then $C = 0$.

To solve parts (ii)-(vi), we make use of the following theorem.

Theorem S2.1: The impulse response of an LTIC system initially at rest and described by the differential equation

$$\sum_{p=0}^{n} a_p \frac{d^p y}{dt^p} = x(t)$$

is given by

$$\sum_{p=0}^{n} a_p \frac{d^p h}{dt^p} = 0$$

with initial condition $\frac{d^{n-1}h}{dt^{n-1}}(0^+) = \frac{1}{a_n}$. The remaining lower order initial conditions are all zero.

(ii) Based on Theorem S2.1, the impulse response of system (ii) is given by

$$\dot{h}(t) + 6h(t) = 0$$

with initial condition $h(0^+) = 1$. The characteristic equation for the homogenous equation is

$$s + 6 = 0$$

which has a root at $s = -6$. The impulse response is given by

$$h(t) = C_1 e^{-6t}$$

for $t \geq 0$, with C_1 being a constant. Use the initial condition $h(0^+) = 1$, the value of $C_1 = 1$. The impulse response of system (ii) is given by

$$h(t) = e^{-6t} u(t) .$$

(iii) Assume $w(t) = dx/dt$. System (iii) can, therefore, be represented as a cascaded combination of two systems

System S1(iii): $$w(t) = \frac{dx(t)}{dt}$$

System S2(iii): $$2\dot{y}(t) + 5y(t) = w(t)$$

Based on Theorem S2.1, the impulse response of system S2(iii) is given by

$$2\dot{h}_2(t) + 5h_2(t) = 0$$

with initial condition $h_2(0^+) = 1/2$. The characteristic equation for the homogenous equation is

$$(2s + 5) = 0$$

which has a root at $s = -5/2$. The impulse response is given by

$$h_2(t) = C_1 e^{-5t/2}$$

for $t \geq 0$, with C_1 being a constant. Use the initial condition $h_2(0^+) = 1/2$, the value of $C_1 = 1/2$. The impulse response of system S2(iii) is

$$h_2(t) = \tfrac{1}{2}e^{-5t/2}u(t).$$

for $t \geq 0$. Combining the cascaded configuration, the impulse response of the overall system is

$$h(t) = \tfrac{dh_2(t)}{dt} = \tfrac{d}{dt}\left(\tfrac{1}{2}e^{-5t/2}u(t)\right) = \tfrac{1}{2}e^{-5t/2}\delta(t) - \tfrac{1}{2}\times\tfrac{5}{2}e^{-5t/2}u(t) = \tfrac{1}{2}\delta(t) - \tfrac{5}{4}e^{-5t/2}u(t).$$

(iv) System (iv) is represented as a cascaded combination of two systems

System S1(iv): $$w(t) = 2\tfrac{dx(t)}{dt} + 3x(t)$$

System S2(iv): $$\dot{y}(t) + 3y(t) = w(t)$$

Based on Theorem S2.1, the impulse response of system S2(iv) is given by

$$\dot{h}_2(t) + 3h_2(t) = 0$$

with initial condition $h_2(0^+) = 1$. The characteristic equation for the homogenous equation is

$$(s+3) = 0$$

which has a root at $s = -3$. The impulse response is given by

$$h_2(t) = C_1e^{-3t}$$

for $t \geq 0$, with C_1 being a constant. Use the initial condition $h_2(0^+) = 1$, the value of $C_1 = 1$. The impulse response of system S2(ii) is

$$h_2(t) = e^{-3t}u(t).$$

for $t \geq 0$. Combining the cascaded configuration, the impulse response of the overall system is

$$\begin{aligned} h(t) &= 2\tfrac{dh_2(t)}{dt} + 3h_2(t) = 2\tfrac{d}{dt}\left(e^{-3t}u(t)\right) + 3e^{-3t}u(t) \\ &= 2e^{-3t}\delta(t) - 6e^{-3t}u(t) + 3e^{-3t}u(t) = 2\delta(t) - 3e^{-3t}u(t) \end{aligned}.$$

(v) Based on Theorem S2.1, the impulse response of system (v) is given by

$$\ddot{h}(t) + 5\dot{h}(t) + 4h(t) = 0$$

with initial conditions $dh(0^+)/dt = 1$ and $h(0^+) = 0$. The characteristic equation for the homogenous equation is

$$(s^2 + 5s + 4) = 0$$

which has roots at $s = -1$ and -4. The impulse response is given by

$$h_2(t) = C_1e^{-t} + C_2e^{-4t}$$

for $t \geq 0$, with C_1 and C_2 being constants. Using the initial conditions

$$\begin{aligned} C_1 + C_2 &= 0 \\ -C_1 - 4C_2 &= 1 \end{aligned}$$

which has the solution $C_1 = 1/3$, $C_2 = -1/3$. The impulse response of system (v) is given by

$$h(t) = \left(\tfrac{1}{3}e^{-t} - \tfrac{1}{3}e^{-4t}\right)u(t).$$

(vi) Based on Theorem S2.1, the impulse response of system (vi) is given by

$$\ddot{h}(t) + 2\dot{h}(t) + h(t) = 0$$

with initial conditions $dh(0^+)/dt = 1$ and $h(0^+) = 0$. The characteristic equation for the homogenous equation is

$$(s^2 + 2s + 1) = 0$$

which has two roots at $s = -1$. The impulse response is given by

$$h_2(t) = C_1 e^{-t} + C_2 t e^{-t}$$

for $t \geq 0$, with C_1 and C_2 being constants. Using the initial conditions

$$\begin{aligned} C_1 &= 0 \\ -C_1 + C_2 &= 1 \end{aligned}$$

which has the solution $C_1 = 0$, $C_2 = 1$. The impulse response of system (vi) is given by

$$h(t) = te^{-t}\, u(t).$$

■

Problem 3.4

(i) Functions $x(\tau) = \exp(-\alpha\tau)u(\tau)$, $h(\tau) = \exp(-\beta\tau)u(\tau)$, and is $h(-\tau) = \exp(\beta\tau)u(-\tau)$ are plotted, respectively, in Fig. S3.4(a)-(c). The function $h(t-\tau) = h(-(\tau - t)$ is obtained by shifting $h(-\tau)$ by time t in Fig. S3.4(d). We consider the following two cases of t.

Case 1: For $t < 0$, the waveform $h(t-\tau)$ is on the left hand side of the vertical axis. As apparent in the subplot for step 5a in Fig. 3.7, waveforms for $h(t-\tau)$ and $x(\tau)$ do not overlap. In other words, $x(\tau)h(t-\tau) = 0$ for all τ, hence, $y(t) = 0$.

Case 2: For $t \geq 0$, we see from the subplot for step 5b in Fig. 3.7 that the nonzero parts of $h(t-\tau)$ and $x(\tau)$ overlap over the duration $t = [0, t]$. Therefore,

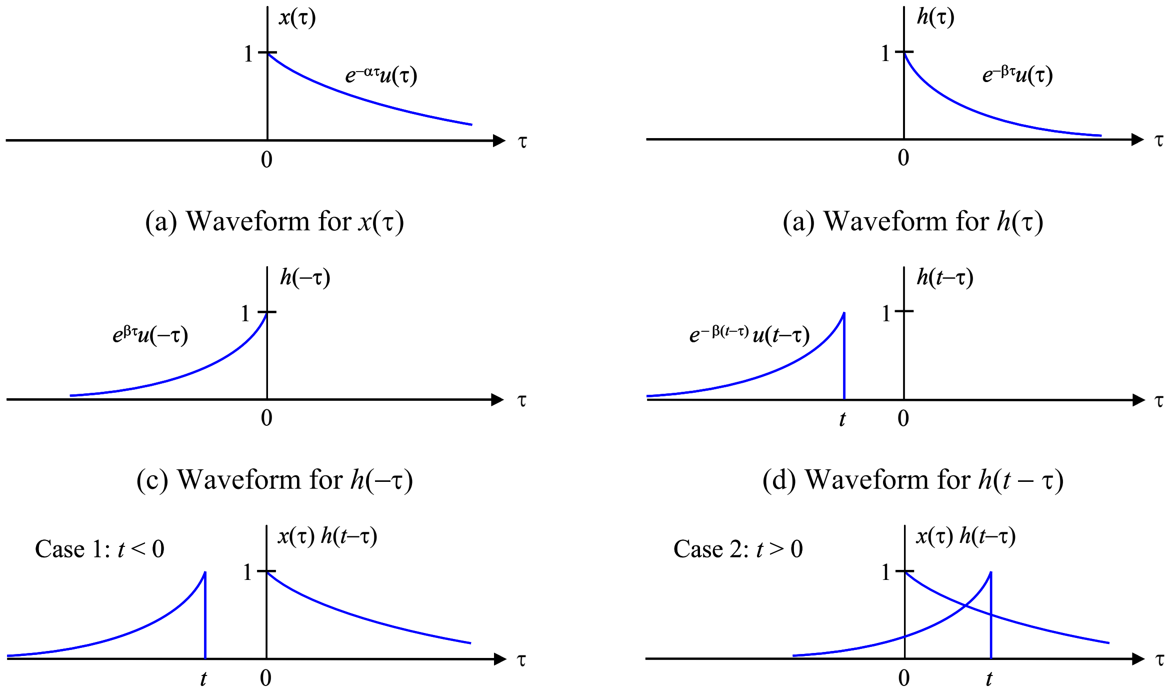
$$y(t) = \int_0^t e^{-\alpha\tau} e^{-\beta(t-\tau)}\, d\tau = e^{-\beta t} \int_0^t e^{-(\alpha-\beta)\tau}\, d\tau. \qquad \text{(S3.4.1)}$$

Since $(\alpha \neq \beta)$, therefore, the exponential term can be integrated as

$$y(t) = e^{-\beta t}\left[\frac{e^{-(\alpha-\beta)\tau}}{-(\alpha-\beta)}\right]_0^t = \tfrac{1}{(\beta-\alpha)} e^{-\beta t}\left[e^{-(\alpha-\beta)t} - 1\right] = \tfrac{1}{(\beta-\alpha)}\left[e^{-\alpha t} - e^{-\beta t}\right].$$

Combining the two cases, we get $y(t) = \begin{cases} 0 & t < 0 \\ \tfrac{1}{(\beta-\alpha)}\left[e^{-\beta t} - e^{-\alpha t}\right] & t \geq 0 \end{cases}$,

which is equivalent to $\quad y(t) = \tfrac{1}{(\beta-\alpha)}\left[e^{-\beta t} - e^{-\alpha t}\right] u(t).$



(a) Waveform for $x(\tau)$ (a) Waveform for $h(\tau)$

(c) Waveform for $h(-\tau)$ (d) Waveform for $h(t-\tau)$

(e) Overlap between $x(\tau)$ and $h(t-\tau)$ for $t < 0$ (f) Overlap between $x(\tau)$ and $h(t-\tau)$ for $t \geq 0$

Fig. S3.4.1: Convolution between $x(\tau) = \exp(-\alpha\tau)u(\tau)$ and $h(\tau) = \exp(-\beta\tau)u(\tau)$ in Problem 3.4.

(ii) For $\alpha = \beta$, Eq. (S3.4.1) reduces to

$$y(t) = e^{-\beta t}\int_0^t e^{-(\alpha-\beta)\tau}\,d\tau = e^{-\beta t}\int_0^t 1\,d\tau = te^{-\beta t}.$$

The output $y(t)$ is therefore given by

$$y(t) = te^{-\beta t}u(t) = te^{-\alpha t}u(t).$$

(iii) Part (ii) is a special case of part (i) as the result for part (ii) can be obtained by applying the limit, $\alpha \to \beta$, to the solution of part (i). Since applying the limit results in a 0/0 case, we apply the L'Hopital's rule to get

$$y(t) = \lim_{\alpha\to\beta}\tfrac{1}{(\beta-\alpha)}\left[e^{-\alpha t} - e^{-\beta t}\right]u(t) = \lim_{\alpha\to\beta}\tfrac{1}{(-1)}\left[-te^{-\alpha t} - 0\right]u(t) = te^{-\alpha t}u(t).$$ ▌

Problem 3.5

(i) The output $y(t)$ is given by

$$y(t) = u(t) * u(t) = \int_{-\infty}^{\infty} u(\tau)u(t-\tau)\,d\tau = \int_0^{\infty} u(t-\tau)\,d\tau.$$

Recall that
$$u(t-\tau)=\begin{cases}1 & \text{if}\,(\tau\le t)\\ 0 & \text{if}\,(\tau>t).\end{cases}$$

Therefore, the output $y(t)$ is given by
$$y(t)=\begin{cases}t & \text{if}\,(t\ge 0)\\ 0 & \text{if}\,(t<0)\end{cases}=tu(t)=r(t).$$

The aforementioned convolution can also be computed graphically.

(ii) The output $y(t)$ is given by
$$y(t)=u(-t)*u(-t)=\int_{-\infty}^{\infty}u(-\tau)u(\tau-t)\,d\tau=\int_{-\infty}^{0}u(\tau-t)\,d\tau\,.$$

The output $y(t)$ is given by
$$y(t)=\int_{-\infty}^{0}u(\tau-t)\,d\tau=\begin{cases}0 & \text{if}\,(t\ge 0)\\ \int_{t}^{0}u(\tau-t)\,d\tau & \text{if}\,(t<0)\end{cases}=\begin{cases}0 & \text{if}\,(t\ge 0)\\ -t & \text{if}\,(t<0)\end{cases}=-tu(-t).$$

The aforementioned convolution can also be computed graphically.

(iii) The output $y(t)$ is given by
$$y(t)=[u(t)-2u(t-1)+u(t-2)]*[u(t+1)-u(t-1)]$$

Using the properties of the convolution integral, the output is expressed as
$$\begin{aligned}y(t)=&[u(t)*u(t+1)]-[u(t)*u(t-1)]-2[u(t-1)*u(t+1)]\\&+2[u(t-1)*u(t-1)]+[u(t-2)*u(t+1)]-[u(t-2)*u(t-1)]\end{aligned}$$

Based on the results of part (i), i.e., $u(t) * u(t) = r(t)$, the overall output is given by
$$y(t)=r(t+1)-r(t-1)-2r(t)+2r(t-2)+r(t-1)-r(t-3)\,.$$

(iv) The output $y(t)$ is given by
$$y(t)=e^{2t}u(-t)*e^{-3t}u(t)=\int_{-\infty}^{\infty}e^{2\tau}u(-\tau)e^{-3(t-\tau)}u(t-\tau)\,d\tau=e^{-3t}\int_{-\infty}^{0}e^{5\tau}u(t-\tau)\,d\tau\,.$$

Solving for the two cases ($t\ge 0$) and ($t<0$), we get
$$y(t)=e^{-3t}\int_{-\infty}^{0}e^{5\tau}u(t-\tau)\,d\tau=\begin{cases}e^{-3t}\int_{-\infty}^{t}e^{5\tau}\,d\tau & (t<0)\\ e^{-3t}\int_{-\infty}^{0}e^{5\tau}\,d\tau & (t\ge 0)\end{cases}=\begin{cases}\frac{1}{5}e^{2t} & (t<0)\\ \frac{1}{5}e^{-3t} & (t\ge 0).\end{cases}$$

Therefore, the output $y(t)$ is given by
$$y(t)=\tfrac{1}{5}e^{2t}u(-t)+\tfrac{1}{5}e^{-3t}u(t).$$

(v) The output $y(t)$ is given by

$$y(t)=x(t)*h(t)=\int_{-\infty}^{\infty}\sin(2\pi\tau)\left[\underbrace{u(\tau-2)-u(\tau-5)}_{=0\ for\ \tau<2,\tau>5}\right]\left[u(t-\tau)-u(t-\tau-2)\right]d\tau$$

$$=\int_{2}^{5}\sin(2\pi\tau)\left[u(t-\tau)-u(t-\tau-2)\right]d\tau=\underbrace{\int_{2}^{5}\sin(2\pi\tau)u(t-\tau)d\tau}_{=A}-\underbrace{\int_{2}^{5}\sin(2\pi\tau)u(t-\tau-2)d\tau}_{=B}$$

Calculating Term A and Term B separately, we get

$$A=\begin{cases}0 & t\le 2\\ \int_{2}^{t}\sin(2\pi\tau)d\tau & 2\le t\le 5\\ \int_{2}^{5}\sin(2\pi\tau)d\tau & t\ge 5\end{cases}=\begin{cases}0 & t\le 2\\ \frac{1-\cos 2\pi t}{2\pi} & 2\le t\le 5\\ 0 & t\ge 5\end{cases}$$

$$B=\begin{cases}0 & t-2\le 2\\ \int_{2}^{t-2}\sin(2\pi\tau)d\tau & 2\le t-2\le 5\\ \int_{2}^{5}\sin(2\pi\tau)d\tau & t-2\ge 5\end{cases}=\begin{cases}0 & t\le 4\\ \frac{1-\cos 2\pi(t-2)}{2\pi} & 4\le t\le 7\\ 0 & t\ge 7\end{cases}=\begin{cases}0 & t\le 4\\ \frac{1-\cos 2\pi t}{2\pi} & 4\le t\le 7\\ 0 & t\ge 7\end{cases}$$

The overall output is given by

$$\text{Therefore, } y(t)=A-B=\begin{cases}0 & t\le 2, t\ge 7\\ \frac{1}{2\pi}(1-\cos 2\pi t) & 2\le t\le 4\\ 0 & 4\le t\le 5\\ -\frac{1}{2\pi}(1-\cos 2\pi t) & 5\le t\le 7\end{cases}$$

(vi) Considering the two cases ($t<0$) and ($t\ge 0$) separately

Case I ($t<0$): $$y(t)=\int_{-\infty}^{t}e^{2\tau}e^{-5(t-\tau)}d\tau+\int_{t}^{0}e^{2\tau}e^{5(t-\tau)}d\tau+\int_{0}^{\infty}e^{-2\tau}e^{5(t-\tau)}d\tau$$

which reduces to $$y(t)=e^{-5t}\int_{-\infty}^{t}e^{7\tau}d\tau+e^{5t}\int_{t}^{0}e^{-3\tau}d\tau+e^{-5t}\int_{0}^{\infty}e^{-7\tau}d\tau$$

or, $$y(t)=e^{-5t}\times\tfrac{1}{7}e^{7t}+e^{5t}\times\tfrac{1}{3}\left(e^{-3t}-1\right)+e^{-5t}\times\tfrac{1}{7}=\tfrac{1}{7}e^{2t}-\tfrac{4}{21}e^{-5t}+\tfrac{1}{3}e^{-8t}$$

Case II ($t\ge 0$): $$y(t)=\int_{-\infty}^{0}e^{2\tau}e^{-5(t-\tau)}d\tau+\int_{0}^{t}e^{-2\tau}e^{-5(t-\tau)}d\tau+\int_{t}^{\infty}e^{-2\tau}e^{5(t-\tau)}d\tau$$

which reduces to $$y(t)=e^{-5t}\int_{-\infty}^{0}e^{7\tau}d\tau+e^{-5t}\int_{0}^{t}e^{3\tau}d\tau+e^{-5t}\int_{t}^{\infty}e^{-7\tau}d\tau$$

or, $\quad y(t) = \frac{1}{7}e^{5t} + e^{-5t} \times \frac{1}{3}\left(e^{3t} - 1\right) + e^{-5t} \times \frac{1}{7}e^{-7t} = \frac{1}{7}e^{5t} + \frac{1}{3}e^{-2t} - \frac{1}{3}e^{-5t} + \frac{1}{7}e^{-12t}$.

Hence, the overall expression for $y(t)$ is given by

$$y(t) == \begin{cases} \frac{1}{5}e^{2t} & (t < 0) \\ \frac{1}{5}e^{-3t} & (t \geq 0). \end{cases}$$

(vii) Note that

$$\begin{aligned}
\sin(t)u(t) * \cos(t)u(t) &= \tfrac{1}{2j}\left(e^{jt} - e^{-jt}\right)u(t) * \tfrac{1}{2}\left(e^{jt} + e^{-jt}\right)u(t) \\
&= \tfrac{1}{4j}\left[e^{jt}u(t) * e^{jt}u(t)\right] - \tfrac{1}{4j}\left[e^{-jt}u(t) * e^{jt}u(t)\right] \\
&\quad + \tfrac{1}{4j}\left[e^{jt}u(t) * e^{-jt}u(t)\right] - \tfrac{1}{4j}\left[e^{-jt}u(t) * e^{-jt}u(t)\right] \\
&= \tfrac{1}{4j}\left[e^{jt}u(t) * e^{jt}u(t)\right] - \tfrac{1}{4j}\left[e^{-jt}u(t) * e^{-jt}u(t)\right]
\end{aligned}$$

Based on the result of Problem 3.4, we know that

$$e^{-jt}u(t) * e^{-jt}u(t) = te^{-jt}u(t)$$

and $\quad e^{jt}u(t) * e^{jt}u(t) = te^{jt}u(t)$.

Hence, the output is given by

$$y(t) = \tfrac{1}{4j}te^{jt}u(t) - \tfrac{1}{4j}te^{-jt}u(t) = \tfrac{1}{2}t\sin(t)u(t).$$

Note that the above convolution can also be performed directly as follows:

$$y(t) = [\sin(t)u(t)] * [\cos(t)u(t)] = \int_{-\infty}^{\infty} \sin(\tau)u(\tau)\cos(t-\tau)u(t-\tau)d\tau = \int_{0}^{\infty} \sin(\tau)\cos(t-\tau)u(t-\tau)d\tau$$

$$= \int_0^t \sin(\tau)\cos(t-\tau)d\tau \qquad t > 0 \qquad [y(t) = 0, t < 0]$$

$$= \int_0^t \sin(\tau)\left[\cos(t)\cos(\tau) + \sin(t)\sin(\tau)\right]d\tau = \cos(t)\int_0^t \sin(\tau)\cos(\tau)d\tau + \sin(t)\int_0^t \sin^2(\tau)d\tau$$

$$= 0.5\cos(t)\int_0^t \sin(2\tau)d\tau + 0.5\sin(t)\int_0^t [1-\cos(2\tau)]d\tau = 0.5\cos(t)\left[\tfrac{-\cos(2\tau)}{2}\right]_0^t + 0.5\sin(t)\left[\tau - \tfrac{\sin(2\tau)}{2}\right]_0^t$$

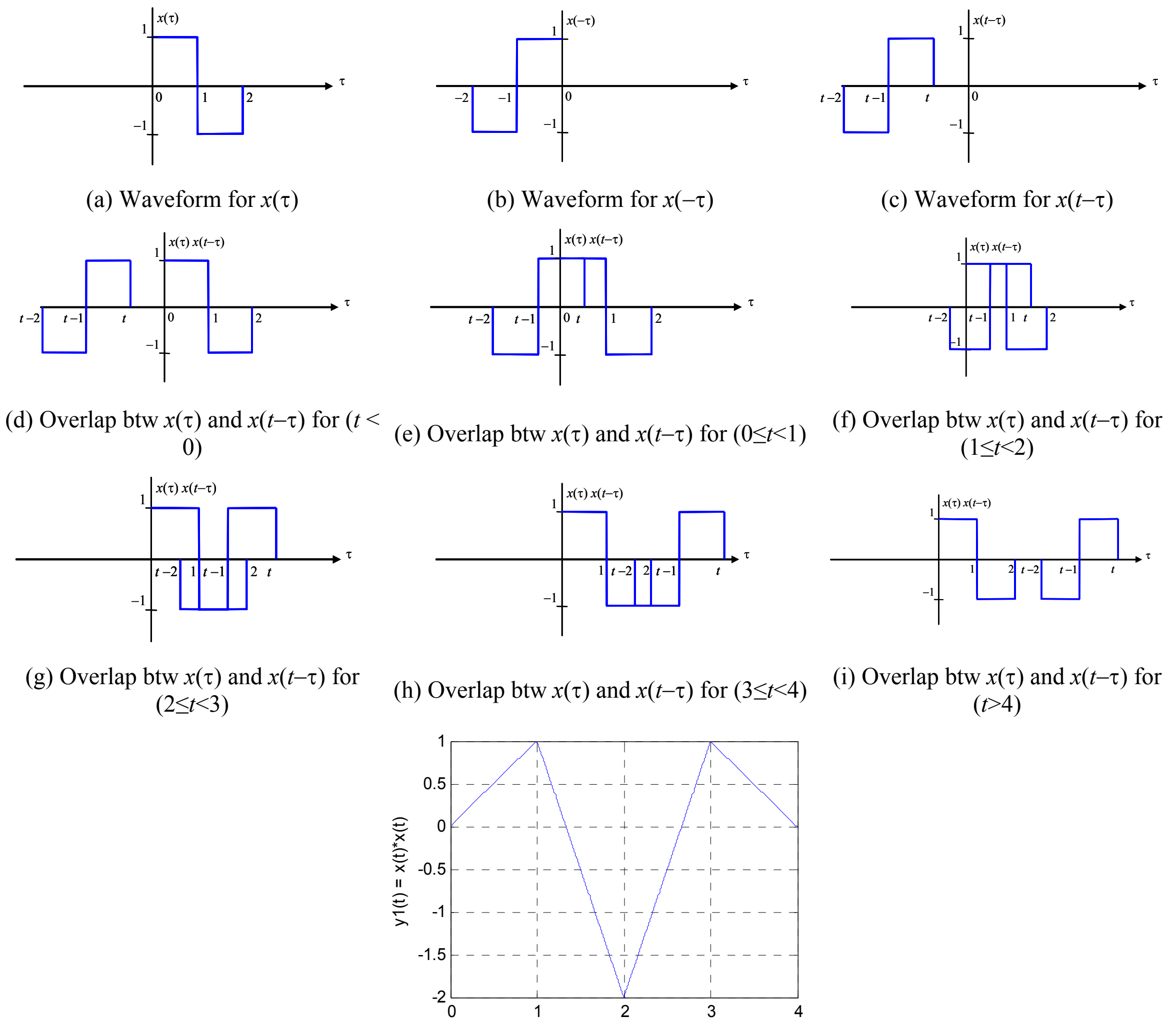
$$= 0.5\cos(t)\left[\tfrac{1}{2} - \tfrac{1}{2}\cos(2t)\right] + 0.5\sin(t)\left[t - \tfrac{1}{2}\sin(2t)\right] = \tfrac{1}{4}\cos(t) + \tfrac{1}{2}t\sin(t) - \tfrac{1}{4}\left[\underbrace{\cos(t)\cos(2t) + \sin(t)\sin(2t)}_{=\cos(2t-t)=\cos(t)}\right]$$

$$= \tfrac{1}{2}t\sin(t) + \tfrac{1}{4}\cos(t) - \tfrac{1}{4}\cos(t) = \tfrac{1}{2}t\sin(t)$$

∎

Problem 3.6

(i) Using the graphical approach, the convolution of $x(t)$ with itself is shown in Fig. S3.6.1, where we consider six different cases for different values of t.



(a) Waveform for $x(\tau)$ (b) Waveform for $x(-\tau)$ (c) Waveform for $x(t-\tau)$

(d) Overlap btw $x(\tau)$ and $x(t-\tau)$ for ($t<0$) (e) Overlap btw $x(\tau)$ and $x(t-\tau)$ for ($0\leq t<1$) (f) Overlap btw $x(\tau)$ and $x(t-\tau)$ for ($1\leq t<2$)

(g) Overlap btw $x(\tau)$ and $x(t-\tau)$ for ($2\leq t<3$) (h) Overlap btw $x(\tau)$ and $x(t-\tau)$ for ($3\leq t<4$) (i) Overlap btw $x(\tau)$ and $x(t-\tau)$ for ($t>4$)

(j) Convolution output $y_1(t)$

Fig. S3.6.1: Convolution of $x(t)$ with $x(t)$ in Problem 3.6(i).

Case I ($t<0$): Since there is no overlap, $y_1(t)=0$.

Case II ($0\leq t<1$):
$$y_1(t)=\int_0^t 1.1d\tau = t.$$

Case III ($1\leq t<2$):
$$y_1(t)=\int_0^{t-1}(-1).1d\tau+\int_{t-1}^{1}1.1d\tau+\int_1^t 1.(-1)d\tau$$
$$=-(t-1)+(1-t+1)-(t-1)=4-3t.$$

Case IV ($2\leq t<3$):
$$y_1(t)=\int_{t-2}^{1}(-1).1d\tau+\int_1^{t-1}(-1).(-1)d\tau+\int_{t-1}^{2}1.(-1)d\tau$$
$$=-(1-t+2)+(t-1-1)-(2-t+1)=3t-8.$$

Case V ($3 \le t < 4$): $$y_1(t) = \int_{t-2}^{2} (-1).(-1) d\tau = (2 - t + 2) = 4 - t\,.$$

Case VI ($t > 4$): Since there is no overlap, $y_1(t) = 0$.

Combining all the cases, the result of the convolution $y_1(t) = x(t) * x(t)$ is given by

$$y_1(t) = \begin{cases} t & (0 \le t < 1) \\ (4 - 3t) & (1 \le t < 2) \\ (3t - 8) & (2 \le t < 3) \\ (4 - t) & (3 \le t < 4) \\ 0 & \text{elsewhere.} \end{cases}$$

The output is $y_1(t)$ plotted in Fig. S3.6.1(j).

(ii) Using the graphical approach, the convolution of $x(t)$ with $z(t)$ is shown in Fig. S3.6.2, where we consider six different cases for different values of t.

Case I ($t < -1$): Since there is no overlap, $y_2(t) = 0$.

Case II ($-1 \le t < 0$): $$y_2(t) = \int_{-1}^{t} 1.\tau d\tau = \tfrac{t^2}{2} - \tfrac{1}{2}.$$

Case III ($0 \le t < 1$):
$$\begin{aligned} y_2(t) &= \int_{-1}^{t-1} (-1).\tau d\tau + \int_{t-1}^{t} 1.\tau d\tau \\ &= -\left(\tfrac{(t-1)^2}{2} - \tfrac{1}{2}\right) + \left(\tfrac{t^2}{2} - \tfrac{(t-1)^2}{2}\right) = -\tfrac{t^2}{2} + 2t - \tfrac{1}{2}. \end{aligned}$$

Case IV ($1 \le t < 2$):
$$\begin{aligned} y_2(t) &= \int_{t-2}^{t-1} (-1).\tau d\tau + \int_{t-1}^{1} 1.\tau d\tau \\ &= -\left(\tfrac{(t-1)^2}{2} - \tfrac{(t-2)^2}{2}\right) + \left(\tfrac{1}{2} - \tfrac{(t-1)^2}{2}\right) = -\tfrac{t^2}{2} + \tfrac{3}{2}. \end{aligned}$$

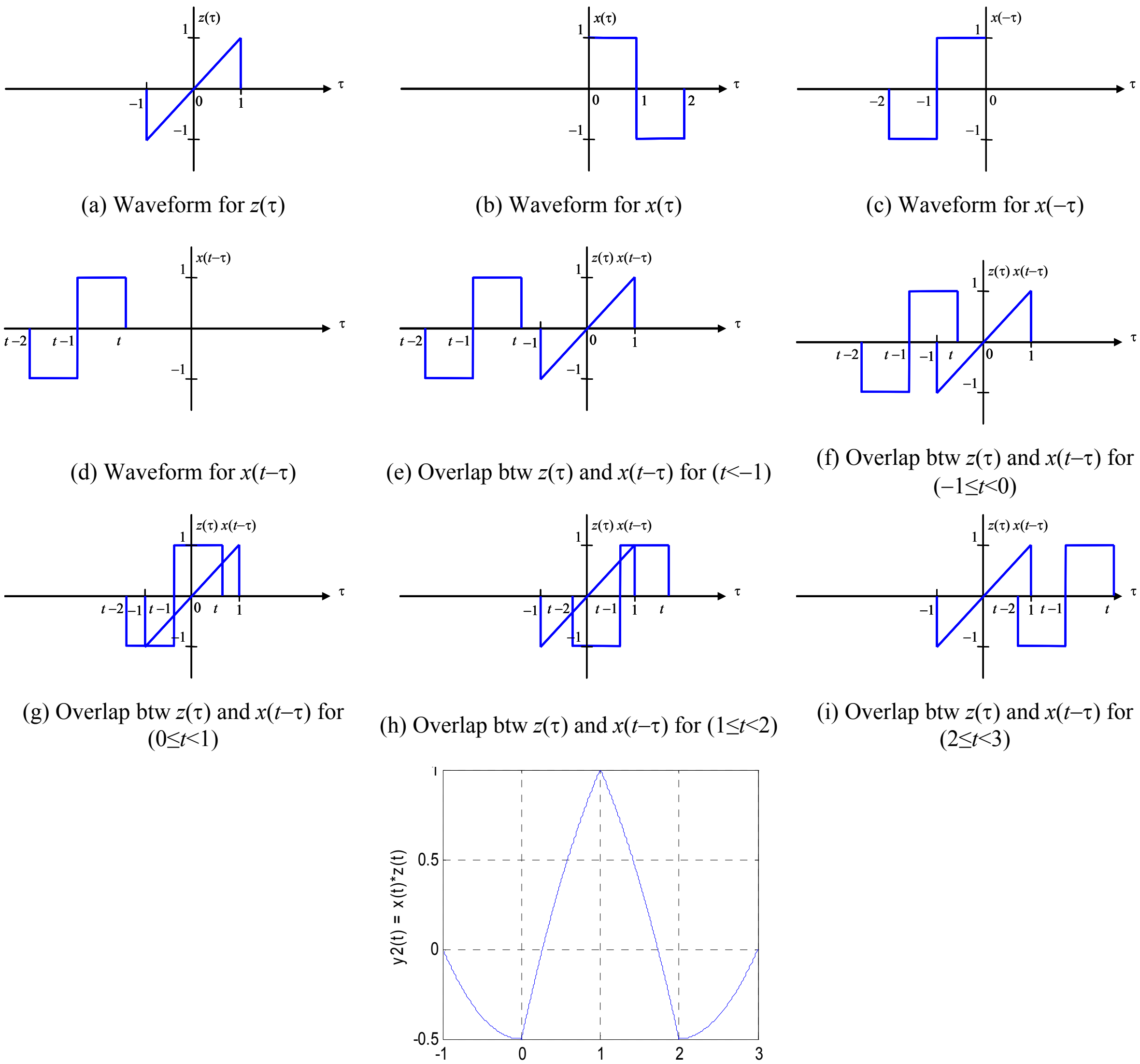
Case V ($2 \le t < 3$): $$y_2(t) = \int_{t-2}^{1} (-1).\tau d\tau = \tfrac{(t-2)^2}{2} - \tfrac{1}{2} = \tfrac{t^2}{2} - 2t + \tfrac{3}{2}\,.$$

Case VI ($t > 4$): Since there is no overlap, $y_2(t) = 0$.

Combining all the cases, the result of the convolution $y_2(t) = x(t) * z(t)$ is given by

$$y_2(t) = \begin{cases} \tfrac{t^2}{2} - \tfrac{1}{2} & (-1 \le t < 0) \\ -\tfrac{t^2}{2} + 2t - \tfrac{1}{2} & (0 \le t < 1) \\ -\tfrac{t^2}{2} + \tfrac{3}{2} & (1 \le t < 2) \\ \tfrac{t^2}{2} - 2t + \tfrac{3}{2} & (2 \le t < 3) \\ 0 & \text{elsewhere.} \end{cases}$$

The output is $y_2(t)$ plotted in Fig. S3.6.2(j).



(a) Waveform for $z(\tau)$ (b) Waveform for $x(\tau)$ (c) Waveform for $x(-\tau)$

(d) Waveform for $x(t-\tau)$ (e) Overlap btw $z(\tau)$ and $x(t-\tau)$ for $(t<-1)$ (f) Overlap btw $z(\tau)$ and $x(t-\tau)$ for $(-1\le t<0)$

(g) Overlap btw $z(\tau)$ and $x(t-\tau)$ for $(0\le t<1)$ (h) Overlap btw $z(\tau)$ and $x(t-\tau)$ for $(1\le t<2)$ (i) Overlap btw $z(\tau)$ and $x(t-\tau)$ for $(2\le t<3)$

(j) Convolution output $y_2(t)$

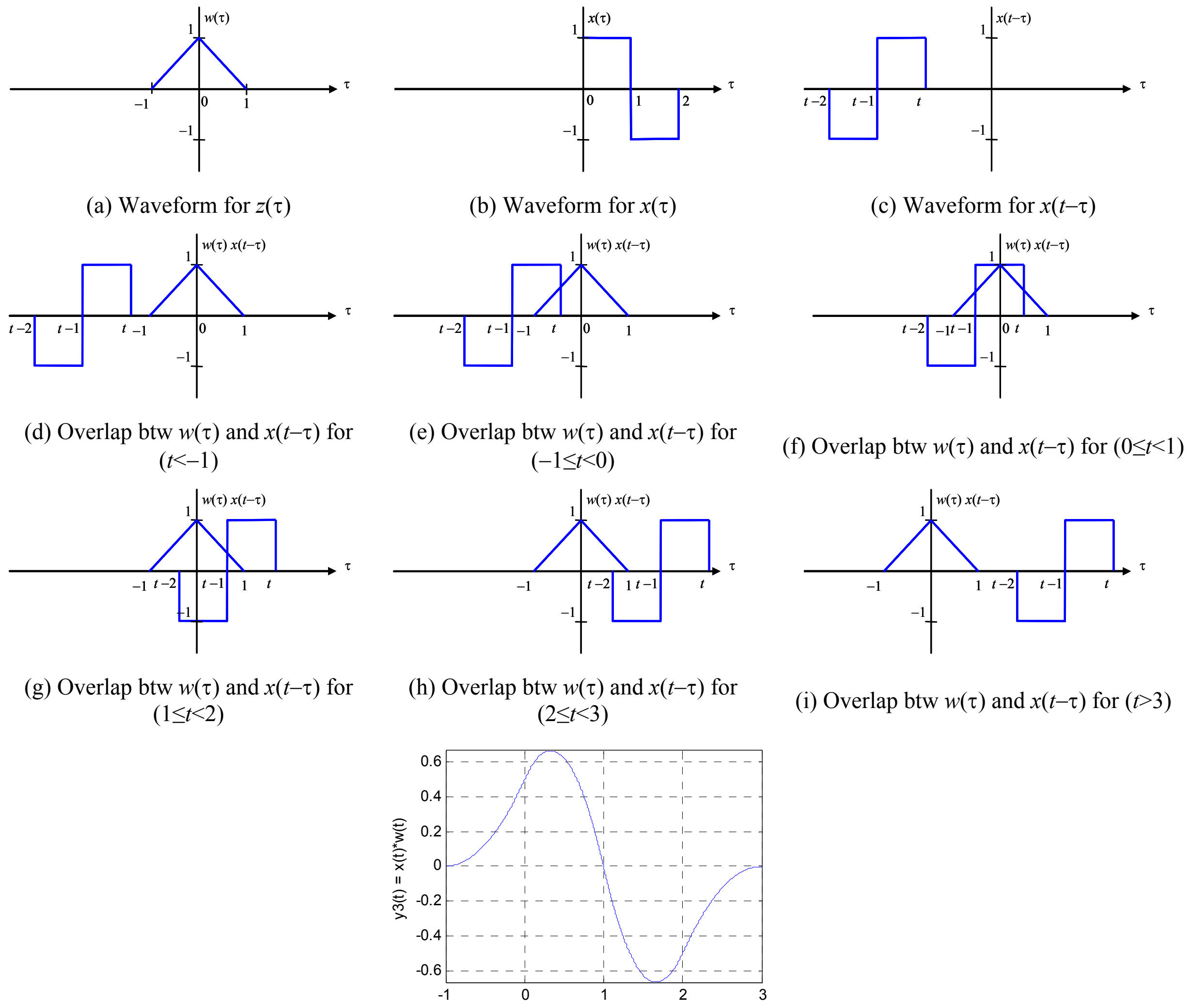
Fig. S3.6.2: Convolution of $x(t)$ with $z(t)$ in Problem 3.6(ii).

(iii) Using the graphical approach, the convolution of $x(t)$ with $w(t)$ is shown in Fig. S3.6.3, where we consider six different cases for different values of t.

Case I $(t<-1)$: Since there is no overlap, $y_3(t)=0$.

Case II $(-1\le t<0)$:
$$y_3(t)=\int_{-1}^{t}1.(1+\tau)d\tau=\frac{(1+t)^2}{2}=\frac{t^2}{2}+t+\frac{1}{2}.$$

Case III $(0\le t<1)$:
$$y_3(t)=\int_{-1}^{t-1}(-1).(1+\tau)d\tau+\int_{t-1}^{0}1.(1+\tau)d\tau+\int_{0}^{t}1.(1-\tau)d\tau$$
$$=-\left(\frac{t^2}{2}-0\right)+\left(\frac{1}{2}-\frac{t^2}{2}\right)-\left(\frac{(1-t)^2}{2}-\frac{1}{2}\right)=-\frac{3t^2}{2}+t+\frac{1}{2}.$$



(a) Waveform for $z(\tau)$ (b) Waveform for $x(\tau)$ (c) Waveform for $x(t-\tau)$

(d) Overlap btw $w(\tau)$ and $x(t-\tau)$ for $(t<-1)$ (e) Overlap btw $w(\tau)$ and $x(t-\tau)$ for $(-1\le t<0)$ (f) Overlap btw $w(\tau)$ and $x(t-\tau)$ for $(0\le t<1)$

(g) Overlap btw $w(\tau)$ and $x(t-\tau)$ for $(1\le t<2)$ (h) Overlap btw $w(\tau)$ and $x(t-\tau)$ for $(2\le t<3)$ (i) Overlap btw $w(\tau)$ and $x(t-\tau)$ for $(t>3)$

(j) Convolution output $y_3(t)$

Fig. S3.6.3: Convolution of $x(t)$ with $w(t)$ in Problem 3.6(iii).

Case IV $(1 \le t < 2)$:

$$y_3(t) = \int_{t-2}^{0} (-1).(1+\tau)d\tau + \int_{0}^{t-1} (-1).(1-\tau)d\tau + \int_{t-1}^{1} 1.(1-\tau)d\tau$$

$$= -\left(\tfrac{1}{2} - \tfrac{(t-1)^2}{2}\right) + \left(\tfrac{(t-2)^2}{2} - \tfrac{1}{2}\right) - \left(0 - \tfrac{(2-t)^2}{2}\right) = \tfrac{3t^2}{2} - 5t + \tfrac{7}{2}.$$

Case V $(2 \le t < 3)$: $$y_3(t) = \int_{t-2}^{1} (-1).(1-\tau)d\tau = 0 - \tfrac{(3-t)^2}{2} = -\tfrac{t^2}{2} + 3t - \tfrac{9}{2}.$$

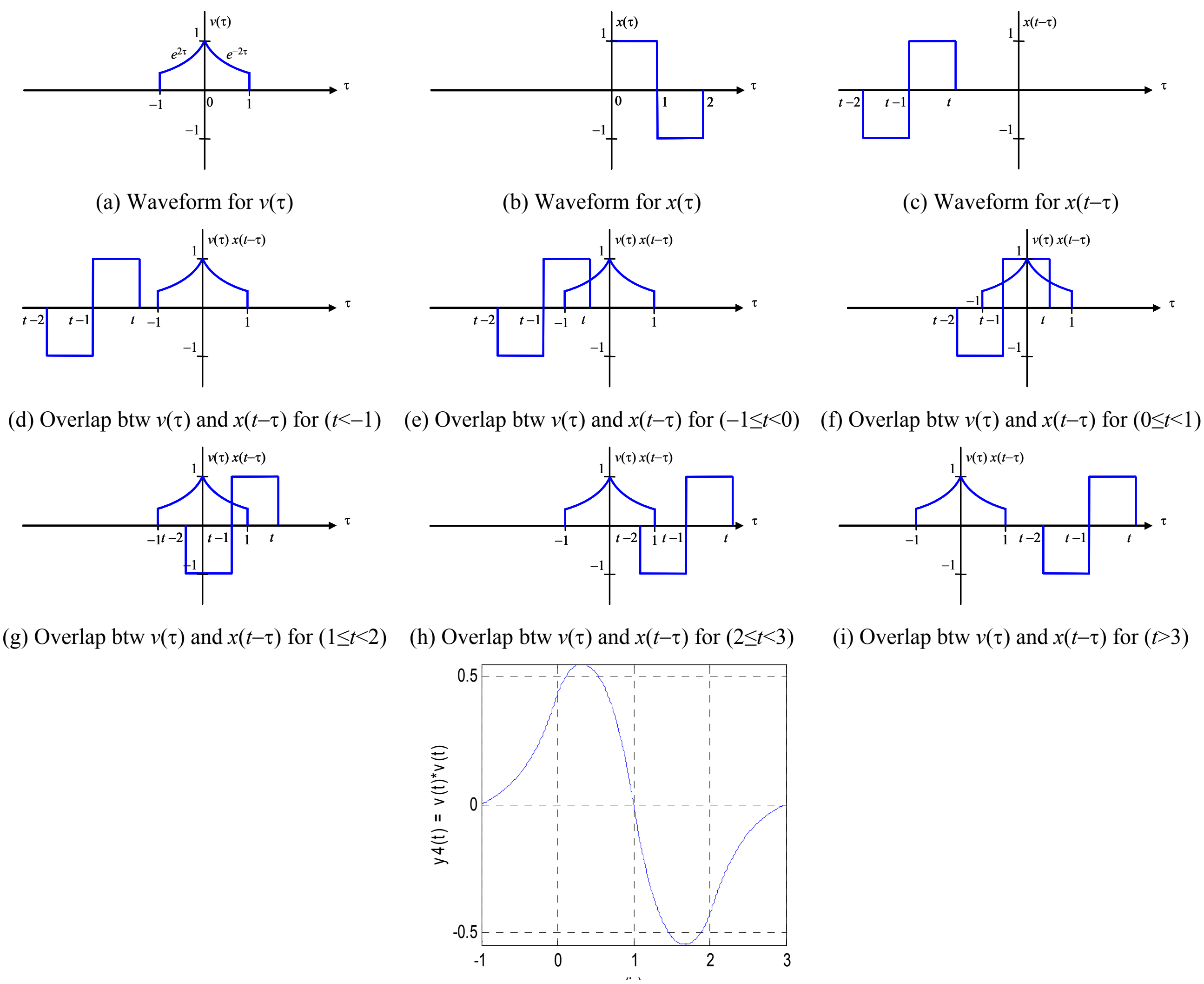
Case VI $(t > 4)$: Since there is no overlap, $y_3(t) = 0$.

Combining all the cases, the result of the convolution $y_3(t) = x(t) * w(t)$ is given by

$$y_3(t) = \begin{cases} \frac{t^2}{2} + t + \frac{1}{2} & (-1 \le t < 0) \\ -\frac{3t^2}{2} + t + \frac{1}{2} & (0 \le t < 1) \\ \frac{3t^2}{2} - 5t + \frac{7}{2} & (1 \le t < 2) \\ \frac{t^2}{2} + 3t - \frac{9}{2} & (2 \le t < 3) \\ 0 & \text{elsewhere.} \end{cases}$$

The output is $y_3(t)$ plotted in Fig. S3.6.3(j).

(iv) Using the graphical approach, the convolution of $x(t)$ with $v(t)$ is shown in Fig. 3.6.4, where we consider six different cases for different values of t.



(a) Waveform for $v(\tau)$ (b) Waveform for $x(\tau)$ (c) Waveform for $x(t-\tau)$

(d) Overlap btw $v(\tau)$ and $x(t-\tau)$ for $(t<-1)$ (e) Overlap btw $v(\tau)$ and $x(t-\tau)$ for $(-1\le t<0)$ (f) Overlap btw $v(\tau)$ and $x(t-\tau)$ for $(0\le t<1)$

(g) Overlap btw $v(\tau)$ and $x(t-\tau)$ for $(1\le t<2)$ (h) Overlap btw $v(\tau)$ and $x(t-\tau)$ for $(2\le t<3)$ (i) Overlap btw $v(\tau)$ and $x(t-\tau)$ for $(t>3)$

(j) Convolution output $y_4(t)$

Fig. S3.6.4: Convolution of $x(t)$ with $v(t)$ in Problem 3.6(iv).

Case I $(t < -1)$: Since there is no overlap, $y_4(t) = 0$.

Case II $(-1 \le t < 0)$: $$y_4(t) = \int_{-1}^{t} 1.e^{2\tau} d\tau = \tfrac{1}{2}\left(e^{2t} - e^{-2}\right).$$

Case III $(0 \le t < 1)$: $$\begin{cases} y_4(t) = \int_{-1}^{t-1} (-1).e^{2\tau} d\tau + \int_{t-1}^{0} 1.e^{2\tau} d\tau + \int_{0}^{t} 1.e^{-2\tau} d\tau \\ \quad = -\tfrac{1}{2}\left(e^{2(t-1)} - e^{-2}\right) + \tfrac{1}{2}\left(1 - e^{2(t-1)}\right) + \tfrac{1}{2}\left(1 - e^{-2t}\right) \\ \quad = -e^{2(t-1)} + \tfrac{1}{2}e^{-2} + 1 - \tfrac{1}{2}e^{-2t}. \end{cases}$$

Case IV $(1 \le t < 2)$: $$\begin{cases} y_4(t) = \int_{t-2}^{0} (-1).e^{2\tau} d\tau + \int_{0}^{t-1} (-1).e^{-2\tau} d\tau + \int_{t-1}^{1} 1.e^{-2\tau} d\tau \\ \quad = -\tfrac{1}{2}\left(1 - e^{2(t-2)}\right) + \tfrac{1}{2}\left(e^{-2(t-1)} - 1\right) + \tfrac{1}{(-2)}\left(e^{-2} - e^{-2(t-1)}\right) \\ \quad = \tfrac{1}{2}e^{2(t-2)} - 1 - \tfrac{1}{2}e^{-2} + e^{-2(t-1)}. \end{cases}$$

Case V $(2 \le t < 3)$: $$y_4(t) = \int_{t-2}^{1} (-1).e^{-2\tau} d\tau = \tfrac{1}{2}\left(e^{-2} - e^{-2(t-2)}\right).$$

Case VI ($t > 4$): Since there is no overlap, $y_4(t) = 0$.

Combining all the cases, the result of the convolution $y_4(t) = x(t) * v(t)$ is given by

$$y_4(t) = \begin{cases} \tfrac{1}{2}e^{2t} - \tfrac{1}{2}e^{-2} & (-1 \le t < 0) \\ -e^{2(t-1)} + \tfrac{1}{2}e^{-2} + 1 - \tfrac{1}{2}e^{-2t} & (0 \le t < 1) \\ \tfrac{1}{2}e^{2(t-2)} - 1 - \tfrac{1}{2}e^{-2} + e^{-2(t-1)} & (1 \le t < 2) \\ \tfrac{1}{2}e^{-2} - \tfrac{1}{2}e^{-2(t-2)} & (2 \le t < 3) \\ 0 & \text{elsewhere.} \end{cases}$$

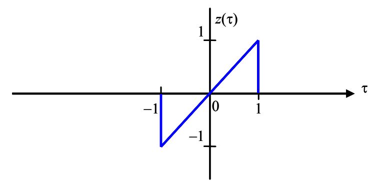
The output is $y_4(t)$ plotted in Fig. S3.6.4(j).

(v) Using the graphical approach, the convolution of $z(t)$ with $z(t)$ is shown in Fig. 3.6.5, where we consider four different cases for different values of t.

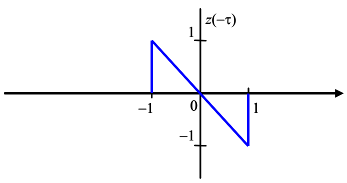
Case I ($t < -2$): Since there is no overlap, $y_5(t) = 0$.

Case II $(-2 \le t < 0)$: $$\begin{cases} y_5(t) = \int_{-1}^{t+1} \tau(t-\tau) d\tau = \left[\tfrac{1}{2}t\tau^2 - \tfrac{1}{3}\tau^3\right]_{-1}^{t+1} \\ \quad = \left[\tfrac{1}{2}t(t+1)^2 - \tfrac{1}{2}t\right] - \left[\tfrac{1}{3}(t+1)^3 + \tfrac{1}{3}\right] = \tfrac{1}{6}\left(t^3 - 6t - 4\right). \end{cases}$$

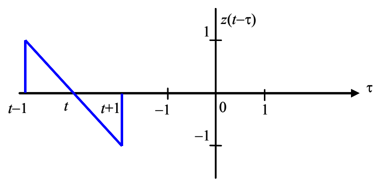
Case III ($0 < t < 2$): $\begin{cases} y_5(t) = \displaystyle\int_{t-1}^{1} \tau(t-\tau)d\tau = \left[\tfrac{1}{2}t\tau^2 - \tfrac{1}{3}\tau^3\right]_{t-1}^{1} \\ \quad = \left[\tfrac{1}{2}t - \tfrac{1}{2}t(t-1)^2\right] - \left[\tfrac{1}{3} - \tfrac{1}{3}(t-1)^3\right] = -\tfrac{1}{6}\left(t^3 + 6t + 4\right). \end{cases}$.



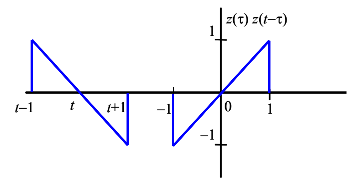
(a) Waveform for $z(\tau)$



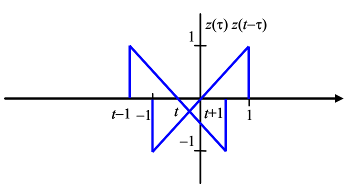
(b) Waveform for $z(-\tau)$



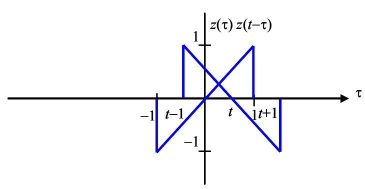
(c) Waveform for $z(t-\tau)$



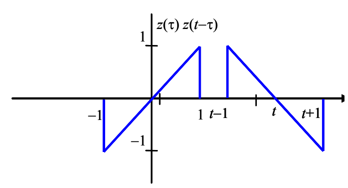
(d) Overlap btw $z(\tau)$ and $z(t-\tau)$ for ($t<-2$)



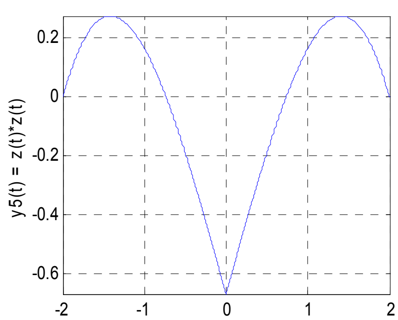
(e) Overlap btw $z(\tau)$ and $z(t-\tau)$ for ($-2\le t<0$)



(f) Overlap btw $z(\tau)$ and $z(t-\tau)$ for ($0\le t<2$)



(g) Overlap btw $z(\tau)$ and $z(t-\tau)$ for ($t>2$)



(h) Convolution output $y_5(t)$

Fig. S3.6.5: Convolution of $z(t)$ with $z(t)$ in Problem 3.6(v).

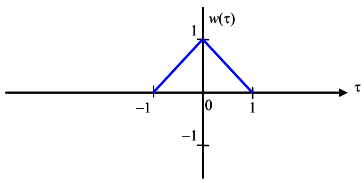
Case IV ($t > 2$): Since there is no overlap, $y_5(t) = 0$.

Combining all the cases, the result of the convolution $y_5(t) = z(t) * z(t)$ is given by

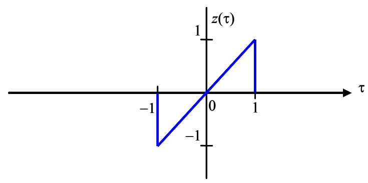
$$y_5(t) = \begin{cases} \frac{1}{6}t^3 - t - \frac{2}{3} & (-2 \le t < 0) \\ -\frac{1}{6}t^3 + t + \frac{2}{3} & (0 \le t < 2) \\ 0 & \text{elsewhere} \end{cases}$$

The output is $y_5(t)$ shown in Fig. S3.6.5(h).

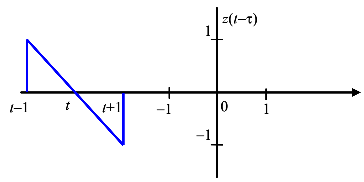
(vi) Using the graphical approach, the convolution of $w(t)$ with $z(t)$ is shown in Fig. 3.6.6, where we consider six different cases for different values of t.



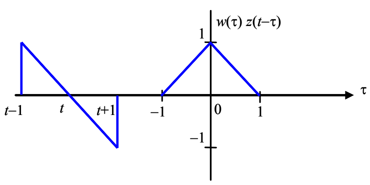
(a) Waveform for $w(\tau)$



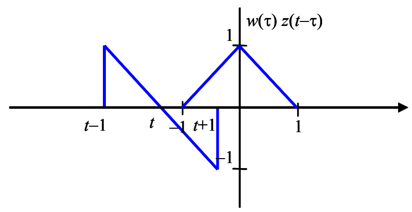
(b) Waveform for $z(\tau)$



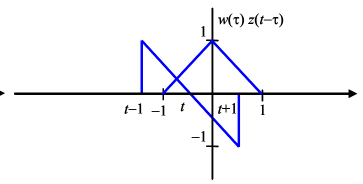
(c) Waveform for $z(t-\tau)$



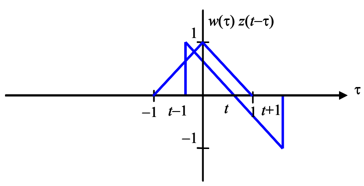
(d) Overlap btw $w(\tau)$ and $z(t-\tau)$ for ($t<-2$)



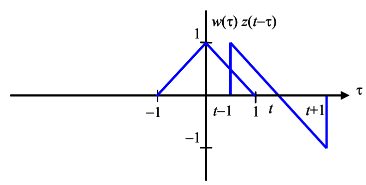
(e) Overlap btw $w(\tau)$ and $z(t-\tau)$ for ($-2 \le t < -1$)



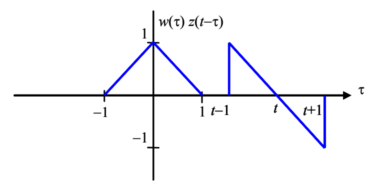
(f) Overlap btw $w(\tau)$ and $z(t-\tau)$ for ($-1 \le t < 0$)



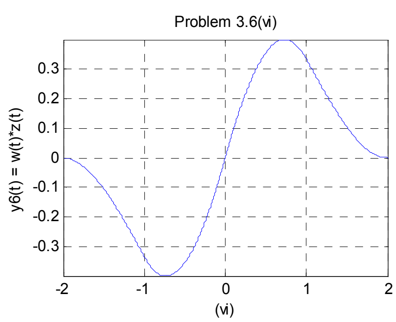
(g) Overlap btw $w(\tau)$ and $z(t-\tau)$ for ($0 \le t < 1$)



(h) Overlap btw $w(\tau)$ and $z(t-\tau)$ for ($1 \le t < 2$)



(i) Overlap btw $w(\tau)$ and $z(t-\tau)$ for ($t>2$)



(j) Convolution output $y_6(t)$

Fig. S3.6.6: Convolution of $w(t)$ with $z(t)$ in Problem 3.6(vi).

Case I ($t < -2$): Since there is no overlap, $y_6(t) = 0$.

Case II ($-2 \le t < -1$): $$\begin{cases} y_6(t) = \int_{-1}^{t+1} (1+\tau)(t-\tau)d\tau = \left[t\tau + \frac{1}{2}t\tau^2 - \frac{1}{2}\tau^2 - \frac{1}{3}\tau^3\right]_{-1}^{t+1} \\ = \left[t(t+1) + \frac{1}{2}t(t+1)^2 - \frac{1}{2}(t+1)^2 - \frac{1}{3}(t+1)^3\right] - \left[-t + \frac{1}{2}t - \frac{1}{2} + \frac{1}{3}\right] \\ = \frac{1}{6}t^3 + \frac{1}{2}t^2 - \frac{2}{3}. \end{cases}$$

Case III ($-1 < t < 0$): $$\begin{cases} y_6(t) = \int_{-1}^{0} (1+\tau)(t-\tau)d\tau + \int_{0}^{t+1} (1-\tau)(t-\tau)d\tau \\ = \left[t\tau + \frac{1}{2}t\tau^2 - \frac{1}{2}\tau^2 - \frac{1}{3}\tau^3\right]_{-1}^{0} + \left[t\tau - \frac{1}{2}t\tau^2 - \frac{1}{2}\tau^2 + \frac{1}{3}\tau^3\right]_{0}^{t+1} \\ = \left[\frac{1}{2}t + \frac{1}{6}\right] + \left[-\frac{1}{6}t^3 + \frac{1}{2}t^2 + \frac{1}{2}t - \frac{1}{6}\right] = -\frac{1}{6}t^3 + \frac{1}{2}t^2 + t. \end{cases}$$

Case IV ($0 < t < 1$): $$\begin{cases} y_6(t) = \int_{t-1}^{0} (1+\tau)(t-\tau)d\tau + \int_{0}^{1} (1-\tau)(t-\tau)d\tau \\ = \left[t\tau + \frac{1}{2}t\tau^2 - \frac{1}{2}\tau^2 - \frac{1}{3}\tau^3\right]_{t-1}^{0} + \left[t\tau - \frac{1}{2}t\tau^2 - \frac{1}{2}\tau^2 + \frac{1}{3}\tau^3\right]_{0}^{1} \\ = -\left[\frac{1}{6}t^3 + \frac{1}{2}t^2 - \frac{1}{2}t - \frac{1}{6}\right] + \left[\frac{1}{2}t - \frac{1}{6}\right] = -\frac{1}{6}t^3 - \frac{1}{2}t^2 + t. \end{cases}$$

Case V ($1 \le t < 2$): $$\begin{cases} y_6(t) = \int_{t-1}^{1} (1-\tau)(t-\tau)d\tau = \left[t\tau - \frac{1}{2}t\tau^2 - \frac{1}{2}\tau^2 + \frac{1}{3}\tau^3\right]_{t-1}^{1} \\ = \left[t - \frac{1}{2}t - \frac{1}{2} + \frac{1}{3}\right] - \left[t(t-1) - \frac{1}{2}t(t-1)^2 - \frac{1}{2}(t-1)^2 + \frac{1}{3}(t-1)^3\right] \\ = \frac{1}{6}t^3 - \frac{1}{2}t^2 + \frac{2}{3}. \end{cases}$$

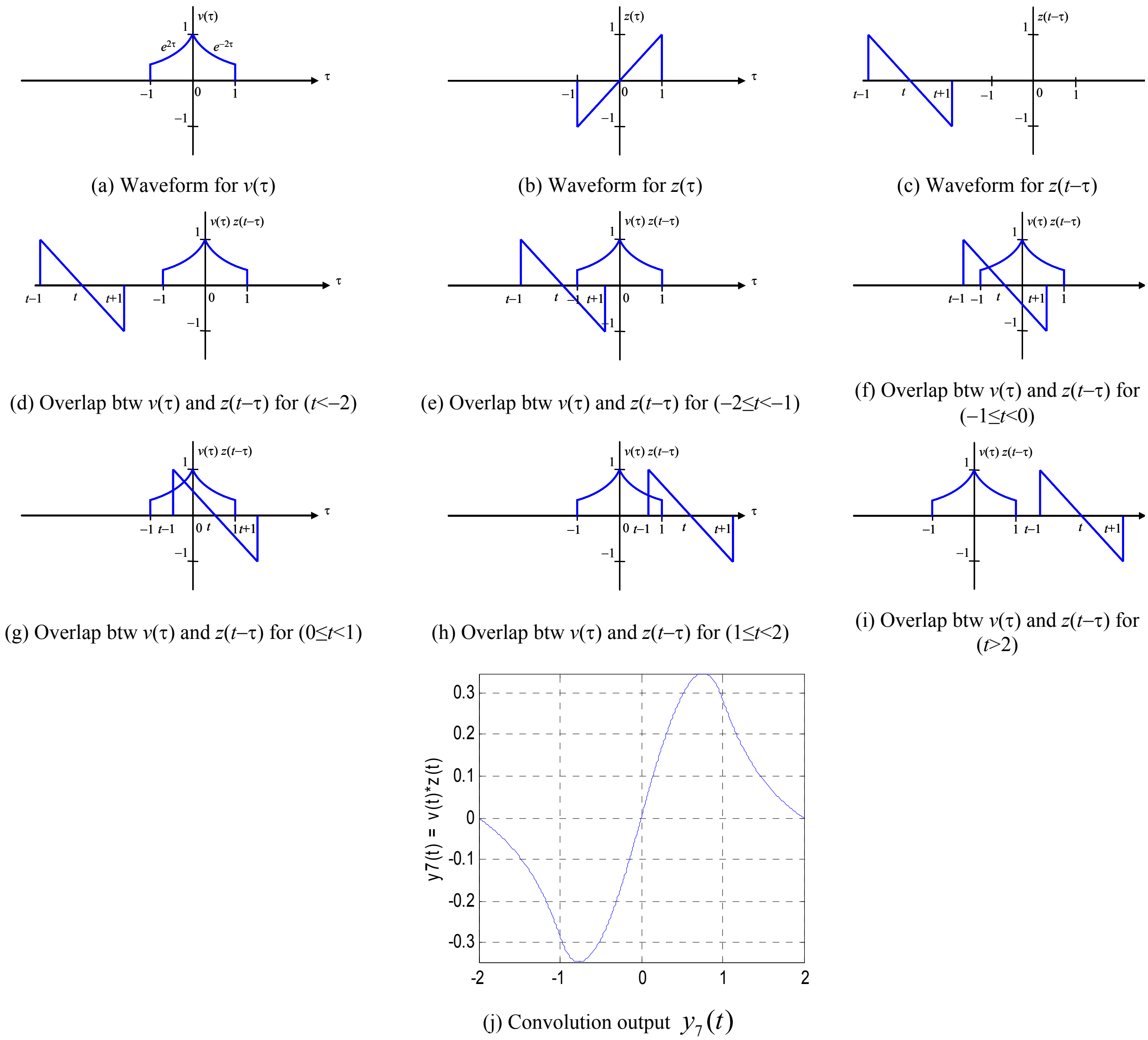
Case VI ($t > 2$): Since there is no overlap, $y_6(t) = 0$.

Combining all the cases, the result of the convolution $y_6(t) = z(t) * w(t)$ is given by

$$y_6(t) = \begin{cases} \frac{1}{6}t^3 + \frac{1}{2}t^2 - \frac{2}{3} & (-2 \le t < -1) \\ -\frac{1}{6}t^3 + \frac{1}{2}t^2 + t & (-1 \le t < 0) \\ -\frac{1}{6}t^3 - \frac{1}{2}t^2 + t & (0 \le t < 1) \\ \frac{1}{6}t^3 - \frac{1}{2}t^2 + \frac{2}{3} & (1 \le t < 2) \\ 0 & \text{elsewhere.} \end{cases}$$

The output is $y_6(t)$ shown in Fig. S3.6.6(j) at the end of the solution of this problem.

(vii) Using the graphical approach, the convolution of $v(t)$ with $z(t)$ is shown in Fig. 3.6.7, where we consider six different cases for different values of t.



(a) Waveform for $v(\tau)$ (b) Waveform for $z(\tau)$ (c) Waveform for $z(t-\tau)$

(d) Overlap btw $v(\tau)$ and $z(t-\tau)$ for ($t<-2$) (e) Overlap btw $v(\tau)$ and $z(t-\tau)$ for ($-2\le t<-1$) (f) Overlap btw $v(\tau)$ and $z(t-\tau)$ for ($-1\le t<0$)

(g) Overlap btw $v(\tau)$ and $z(t-\tau)$ for ($0\le t<1$) (h) Overlap btw $v(\tau)$ and $z(t-\tau)$ for ($1\le t<2$) (i) Overlap btw $v(\tau)$ and $z(t-\tau)$ for ($t>2$)

(j) Convolution output $y_7(t)$

Fig. S3.6.7: Convolution of $v(t)$ with $z(t)$ in Problem 3.6(vii).

Case I ($t<-2$): Since there is no overlap, $y_7(t)=0$.

Case II ($-2\le t<-1$):
$$\begin{cases} y_7(t)=\displaystyle\int_{-1}^{t+1} e^{2\tau}(t-\tau)d\tau=\left[\tfrac{1}{2}(t-\tau)e^{2\tau}+\tfrac{1}{4}e^{2\tau}\right]_{-1}^{t+1} \\ \quad =\left[-\tfrac{1}{2}e^{2(t+1)}+\tfrac{1}{4}e^{2(t+1)}\right]-\left[\tfrac{1}{2}(t+1)e^{-2}+\tfrac{1}{4}e^{-2}\right] \\ \quad =-\tfrac{1}{4}e^{2(t+1)}-\tfrac{1}{2}te^{-2}-\tfrac{3}{4}e^{-2}. \end{cases}$$

Case III ($-1 < t < 0$):

$$y_7(t) = \int_{-1}^{0} e^{2\tau}(t-\tau)d\tau + \int_{0}^{t+1} e^{-2\tau}(t-\tau)d\tau$$
$$= \left[\tfrac{1}{2}(t-\tau)e^{2\tau} + \tfrac{1}{4}e^{2\tau}\right]_{-1}^{0} + \left[-\tfrac{1}{2}(t-\tau)e^{-2\tau} + \tfrac{1}{4}e^{-2\tau}\right]_{0}^{t+1}$$
$$= \left[+\tfrac{1}{2}t + \tfrac{1}{4} - \tfrac{1}{2}te^{-2} - \tfrac{3}{4}e^{-2}\right] + \left[-\tfrac{1}{4}e^{-2(t+1)} + \tfrac{1}{2}t - \tfrac{1}{4}\right]$$
$$= \tfrac{3}{4}e^{-2(t+1)} - \tfrac{1}{2}te^{-2} - \tfrac{3}{4}e^{-2} + t.$$

Case IV ($0 < t < 1$):

$$y_7(t) = \int_{t-1}^{0} e^{2\tau}(t-\tau)d\tau + \int_{0}^{1} e^{-2\tau}(t-\tau)d\tau$$
$$= \left[\tfrac{1}{2}(t-\tau)e^{2\tau} + \tfrac{1}{4}e^{2\tau}\right]_{t-1}^{0} + \left[-\tfrac{1}{2}(t-\tau)e^{-2\tau} + \tfrac{1}{4}e^{-2\tau}\right]_{0}^{1}$$
$$= \left[\tfrac{1}{2}t + \tfrac{1}{4} - \tfrac{3}{4}e^{2(t-1)}\right] + \left[-\tfrac{1}{2}te^{-2} + \tfrac{3}{4}e^{-2} + \tfrac{1}{2}t - \tfrac{1}{4}\right]$$
$$= -\tfrac{3}{4}e^{2(t-1)} - \tfrac{1}{2}te^{-2} + \tfrac{3}{4}e^{-2} + t.$$

Case V ($1 \le t < 2$):

$$y_7(t) = \int_{t-1}^{1} e^{-2\tau}(t-\tau)d\tau = \left[-\tfrac{1}{2}(t-\tau)e^{-2\tau} + \tfrac{1}{4}e^{-2\tau}\right]_{t-1}^{1}$$
$$= \left[-\tfrac{1}{2}e^{2(t+1)} + \tfrac{1}{4}e^{2(t+1)}\right] - \left[\tfrac{1}{2}(t+1)e^{-2} + \tfrac{1}{4}e^{-2}\right]$$
$$= \tfrac{1}{4}e^{-2(t-1)} - \tfrac{1}{2}te^{-2} + \tfrac{3}{4}e^{-2}.$$

Case VI ($t > 2$): Since there is no overlap, $y_7(t) = 0$.

Combining all the cases, the result of the convolution $y_7(t) = z(t) * v(t)$ is given by

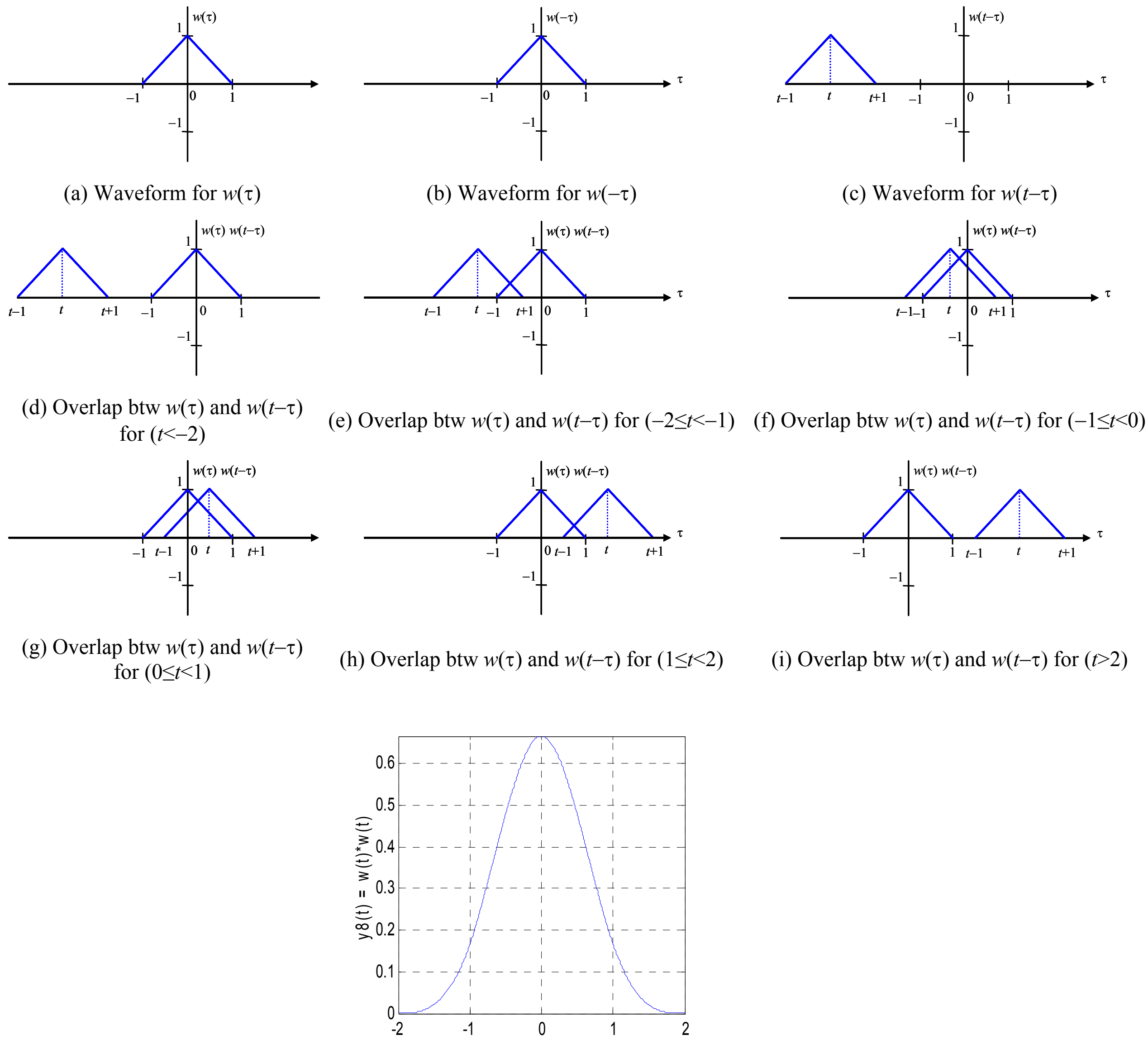
$$y_7(t) = \begin{cases} -\tfrac{1}{4}e^{2(t+1)} - \tfrac{1}{2}te^{-2} - \tfrac{3}{4}e^{-2} & (-2 \le t < -1) \\ \tfrac{3}{4}e^{-2(t+1)} - \tfrac{1}{2}te^{-2} - \tfrac{3}{4}e^{-2} + t & (-1 \le t < 0) \\ -\tfrac{3}{4}e^{2(t-1)} - \tfrac{1}{2}te^{-2} + \tfrac{3}{4}e^{-2} + t & (0 \le t < 1) \\ \tfrac{1}{4}e^{-2(t-1)} - \tfrac{1}{2}te^{-2} + \tfrac{3}{4}e^{-2} & (1 \le t < 2) \\ 0 & \text{elsewhere.} \end{cases}$$

The output is $y_7(t)$ shown in Fig. S3.6.7(j) at the end of the solution of this problem.

(viii) Since $w(t) = 1 - |t|$, therefore, the expression for $w(t-\tau)$ is

$$w(t-\tau) = 1 - |t-\tau| = \begin{cases} 1-(t-\tau) & \text{if } \tau < t \\ 1-(\tau-t) & \text{if } \tau > t. \end{cases}$$

Using the graphical approach, the convolution of $w(t)$ with $w(t)$ is shown in Fig. 3.6.8, where we consider six different cases for different values of t.



(a) Waveform for $w(\tau)$

(b) Waveform for $w(-\tau)$

(c) Waveform for $w(t-\tau)$

(d) Overlap btw $w(\tau)$ and $w(t-\tau)$ for $(t<-2)$

(e) Overlap btw $w(\tau)$ and $w(t-\tau)$ for $(-2\le t<-1)$

(f) Overlap btw $w(\tau)$ and $w(t-\tau)$ for $(-1\le t<0)$

(g) Overlap btw $w(\tau)$ and $w(t-\tau)$ for $(0\le t<1)$

(h) Overlap btw $w(\tau)$ and $w(t-\tau)$ for $(1\le t<2)$

(i) Overlap btw $w(\tau)$ and $w(t-\tau)$ for $(t>2)$

(j) Convolution output $y_8(t)$

Fig. S3.6.8: Convolution of $w(t)$ with $w(t)$ in Problem 3.6(viii).

Case I ($t<-2$): Since there is no overlap, $y_8(t)=0$.

Case II ($-2\le t<-1$):

$$
\begin{cases}
y_8(t)=\displaystyle\int_{-1}^{t+1}(1+\tau)(1+t-\tau)\,d\tau \\
\quad =\displaystyle\int_{-1}^{t+1}(1-\tau^2)\,d\tau+t\int_{-1}^{t+1}(1+\tau)\,d\tau=\left[\tau-\tfrac{1}{3}\tau^3+\tfrac{1}{2}t(1+\tau)^2\right]_{-1}^{t+1} \\
\quad =\tfrac{1}{6}t^3+t^2+2t+\tfrac{4}{3}.
\end{cases}
$$

Case III ($-1 < t < 0$):

$$y_8(t) = \int_{-1}^{t}(1+\tau)(1-t+\tau)d\tau + \int_{t}^{0}(1+\tau)(1+t-\tau)d\tau + \int_{0}^{t+1}(1-\tau)(1+t-\tau)d\tau$$

$$= \int_{-1}^{t}(1+\tau)^2 d\tau - t\int_{-1}^{t}(1+\tau)d\tau + \int_{t}^{0}(1-\tau^2)d\tau + t\int_{t}^{0}(1+\tau)d\tau + \int_{0}^{t+1}(1-\tau)^2 d\tau + t\int_{0}^{t+1}(1-\tau)d\tau$$

$$= \left[\tfrac{1}{3}(1+t)^3\right] - t\left[\tfrac{1}{2}(1+t)^2\right] - \left[t - \tfrac{1}{3}t^3\right] - t\left[t + \tfrac{1}{2}t^2\right] + \left[-\tfrac{1}{3}t^3 - \tfrac{1}{3}\right] + t\left[(t+1) - \tfrac{1}{2}(t+1)^2\right]$$

$$= \tfrac{2}{3} - t^2 - \tfrac{1}{2}t^3.$$

Case IV ($0 < t < 1$):

$$y_8(t) = \int_{t-1}^{0}(1+\tau)(1-t+\tau)d\tau + \int_{0}^{t}(1-\tau)(1+t-\tau)d\tau + \int_{t}^{1}(1-\tau)(1+t-\tau)d\tau$$

$$= \int_{t-1}^{0}(1+\tau)^2 d\tau - t\int_{t-1}^{0}(1+\tau)d\tau + \int_{0}^{t}(1-\tau)^2 d\tau + t\int_{0}^{t}(1-\tau)d\tau + \int_{t}^{1}(1-\tau)^2 d\tau + t\int_{t}^{1}(1-\tau)d\tau$$

$$= \left[\tfrac{1}{3} - \tfrac{1}{3}t^3\right] - t\left[\tfrac{1}{2} - \tfrac{1}{2}t^2\right] + \left[-\tfrac{1}{3}(1-t)^3 + \tfrac{1}{3}\right] + t\left[t - \tfrac{1}{2}t^2\right] + \left[\tfrac{1}{3}(1-t)^3\right] + t\left[\tfrac{1}{2}(1-t)^2\right]$$

$$= \tfrac{2}{3} - t^2 + \tfrac{1}{2}t^3.$$

Case V ($1 \le t < 2$):

$$\begin{cases} y_8(t) = \displaystyle\int_{t-1}^{1}(1-\tau)(1-t+\tau)d\tau \\ \quad = \displaystyle\int_{t-1}^{1}(1-\tau^2)d\tau - t\int_{t-1}^{1}(1-\tau)d\tau = \left[\tau - \tfrac{1}{3}\tau^3 - t(\tau - \tfrac{1}{2}\tau^2)\right]_{t-1}^{1}. \\ \quad = -\tfrac{1}{6}t^3 + t^2 - 2t + \tfrac{4}{3} \end{cases}$$

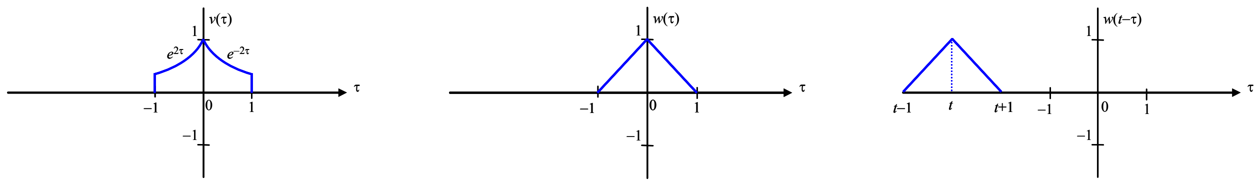
Case VI ($t > 2$): Since there is no overlap, $y_8(t) = 0$.

Combining all the cases, the result of the convolution $y_8(t) = w(t) * w(t)$ is given by

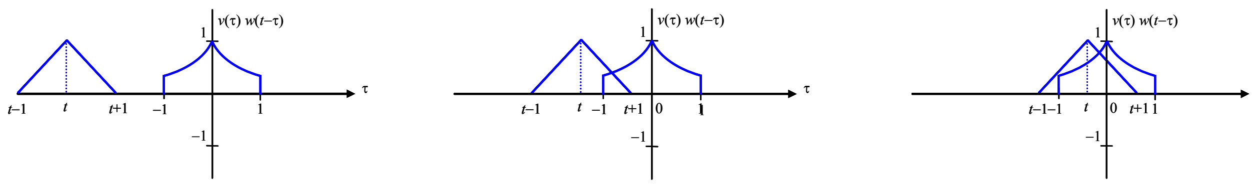
$$y_8(t) = \begin{cases} \tfrac{1}{6}t^3 + t^2 + 2t + \tfrac{4}{3} & (-2 \le t < -1) \\ \tfrac{2}{3} - t^2 - \tfrac{1}{2}t^3 & (-1 \le t < 0) \\ \tfrac{2}{3} - t^2 + \tfrac{1}{2}t^3 & (0 \le t < 1) \\ -\tfrac{1}{6}t^3 + t^2 - 2t + \tfrac{4}{3} & (1 \le t < 2) \\ 0 & \text{elsewhere.} \end{cases}$$

The output is $y_8(t)$ shown in Fig. S3.6.8(j) at the end of the solution of this problem.

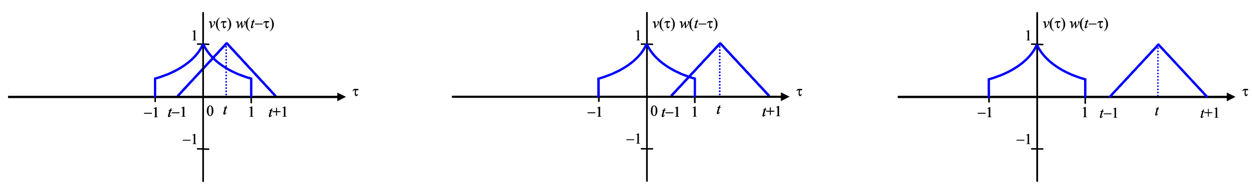
(ix) Using the graphical approach, the convolution of $v(t)$ with $w(t)$ is shown in Fig. 3.6.9, where we consider six different cases for different values of t.



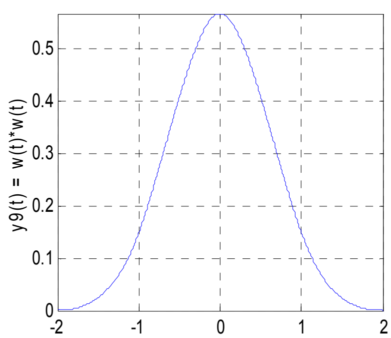
(a) Waveform for $v(\tau)$ (b) Waveform for $w(\tau)$ (c) Waveform for $w(t-\tau)$



(d) Overlap btw $v(\tau)$ and $w(t-\tau)$ for ($t<-2$) (e) Overlap btw $v(\tau)$ and $w(t-\tau)$ for ($-2\le t<-1$) (f) Overlap btw $v(\tau)$ and $w(t-\tau)$ for ($-1\le t<0$)



(g) Overlap btw $v(\tau)$ and $w(t-\tau)$ for ($0\le t<1$) (h) Overlap btw $v(\tau)$ and $w(t-\tau)$ for ($1\le t<2$) (i) Overlap btw $v(\tau)$ and $w(t-\tau)$ for ($t>2$)



(j) Convolution output $y_9(t)$

Fig. S3.6.9: Convolution of $v(t)$ with $w(t)$ in Problem 3.6(ix).

Since $w(t) = 1 - |t|$, therefore, the expression for $w(t - \tau)$ is

$$w(t-\tau) = 1 - |t-\tau| = \begin{cases} 1-(t-\tau) & \text{if } \tau < t \\ 1-(\tau-t) & \text{if } \tau > t. \end{cases}$$

Case I ($t < -2$): Since there is no overlap, $y_9(t) = 0$.

$$\text{Case II } (-2 \le t < -1): \begin{cases} y_9(t) = \displaystyle\int_{-1}^{t+1} e^{2\tau}(1-\tau+t)d\tau = (1+t)\int_{-1}^{t+1} e^{2\tau}d\tau - \int_{-1}^{t+1} \tau e^{2\tau}d\tau \\ \quad = \tfrac{1}{2}(1+t)\left(e^{2(t+1)} - e^{-2}\right) - \left(\tfrac{1}{2}(t+1)e^{2(t+1)} - \tfrac{1}{4}e^{2(t+1)} + \tfrac{1}{2}e^{-2} + \tfrac{1}{4}e^{-2}\right) \\ \quad = \tfrac{1}{4}e^{2(t+1)} - \tfrac{1}{2}te^{-2} - \tfrac{5}{4}e^{-2}. \end{cases}$$

$$\text{Case III } (-1 < t < 0): \begin{cases} y_9(t) = \displaystyle\int_{-1}^{t} e^{2\tau}(1-t+\tau)d\tau + \int_{t}^{0} e^{2\tau}(1+t-\tau)d\tau + \int_{0}^{t+1} e^{-2\tau}(1+t-\tau)d\tau \\ \quad = (1-t)\left[\tfrac{1}{2}e^{2\tau}\right]_{-1}^{t} + \left[\tfrac{1}{2}\tau e^{2\tau} - \tfrac{1}{4}e^{2\tau}\right]_{-1}^{t} + (1+t)\left[\tfrac{1}{2}e^{2\tau}\right]_{t}^{0} - \left[\tfrac{1}{2}\tau e^{2\tau} - \tfrac{1}{4}e^{2\tau}\right]_{t}^{0} \\ \quad + (1+t)\left[-\tfrac{1}{2}e^{-2\tau}\right]_{0}^{t+1} - \left[-\tfrac{1}{2}\tau e^{-2\tau} - \tfrac{1}{4}e^{-2\tau}\right]_{0}^{t+1} \\ \quad = \tfrac{1}{4}e^{-2(t+1)} - \tfrac{1}{2}e^{2t} + \tfrac{1}{2}te^{-2} + \tfrac{1}{4}e^{-2} + t + 1. \end{cases}.$$

$$\text{Case IV } (0 < t < 1): \begin{cases} y_9(t) = \displaystyle\int_{t-1}^{0} e^{2\tau}(1-t+\tau)d\tau + \int_{0}^{t} e^{2\tau}(1-t+\tau)d\tau + \int_{t}^{1} e^{-2\tau}(1+t-\tau)d\tau \\ \quad = (1-t)\left[\tfrac{1}{2}e^{2\tau}\right]_{t-1}^{0} + \left[\tfrac{1}{2}\tau e^{2\tau} - \tfrac{1}{4}e^{2\tau}\right]_{t-1}^{0} + (1-t)\left[\tfrac{1}{2}e^{2\tau}\right]_{0}^{0} + \left[\tfrac{1}{2}\tau e^{2\tau} - \tfrac{1}{4}e^{2\tau}\right]_{0}^{0} \\ \quad + (1+t)\left[-\tfrac{1}{2}e^{-2\tau}\right]_{t}^{1} - \left[-\tfrac{1}{2}\tau e^{-2\tau} - \tfrac{1}{4}e^{-2\tau}\right]_{t}^{1} \\ \quad = \tfrac{1}{4}e^{2(t-1)} - \tfrac{1}{2}e^{-2t} - \tfrac{1}{2}te^{-2} + \tfrac{1}{4}e^{-2} - t + 1. \end{cases}$$

$$\text{Case V } (1 \le t < 2): \begin{cases} y_9(t) = \displaystyle\int_{t-1}^{1} e^{-2\tau}(1-t+\tau)d\tau = (1-t)\int_{t-1}^{1} e^{-2\tau}d\tau + \int_{t-1}^{1} \tau e^{-2\tau}d\tau \\ \quad = -\tfrac{1}{2}(1-t)\left(e^{-2} - e^{-2(t-1)}\right) + \left(-\tfrac{1}{2}e^{-2} - \tfrac{1}{4}e^{-2} + \tfrac{1}{2}(t-1)e^{-2(t-1)} + \tfrac{1}{4}e^{-2(t-1)}\right). \\ \quad = \tfrac{1}{4}e^{-2(t-1)} + \tfrac{1}{2}te^{-2} - \tfrac{5}{4}e^{-2}. \end{cases}$$

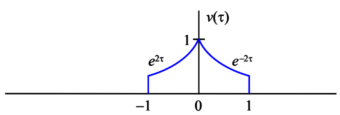
Case VI ($t > 2$): Since there is no overlap, $y_9(t) = 0$.

Combining all the cases, the result of the convolution $y_9(t) = v(t) * w(t)$ is given by

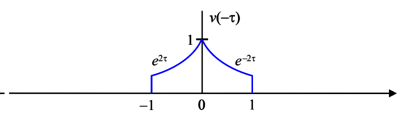
$$y_9(t)=\begin{cases}\frac{1}{4}e^{2(t+1)}-\frac{1}{2}te^{-2}-\frac{5}{4}e^{-2} & (-2\le t<-1)\\ \frac{1}{4}e^{-2(t+1)}-\frac{1}{2}e^{2t}+\frac{1}{2}te^{-2}+\frac{1}{4}e^{-2}+t+1 & (-1\le t<0)\\ \frac{1}{4}e^{2(t-1)}-\frac{1}{2}e^{-2t}-\frac{1}{2}te^{-2}+\frac{1}{4}e^{-2}-t+1 & (0\le t<1)\\ \frac{1}{4}e^{-2(t-1)}+\frac{1}{2}te^{-2}-\frac{5}{4}e^{-2} & (1\le t<2)\\ 0 & \text{elsewhere.}\end{cases}$$

The output is $y_9(t)$ shown in Fig. S3.6.9(j) at the end of the solution of this problem.

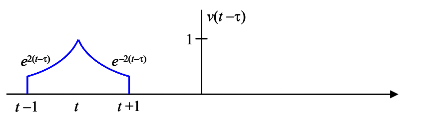
(x) Using the graphical approach, the convolution of $v(t)$ with $v(t)$ is shown in Fig. 3.6.10, where we consider six different cases for different values of t.



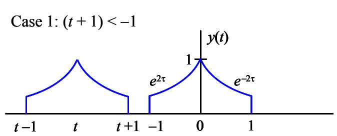
(a) Waveform for $v(\tau)$



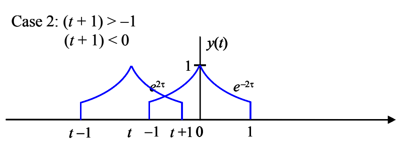
(b) Waveform for $v(-\tau)$



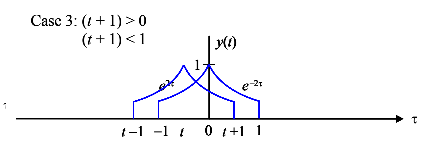
(c) Waveform for $v(t-\tau)$



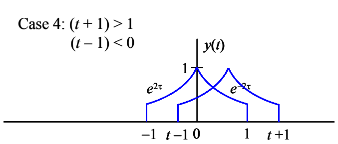
(d) Overlap btw $v(\tau)$ and $v(t-\tau)$ for ($t<-2$)



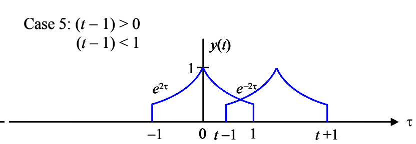
(e) Overlap btw $v(\tau)$ and $v(t-\tau)$ for ($-2\le t<-1$)



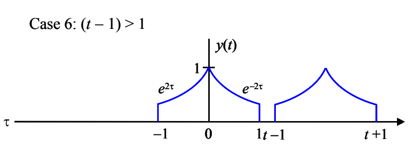
(f) Overlap btw $v(\tau)$ and $v(t-\tau)$ for ($-1\le t<0$)



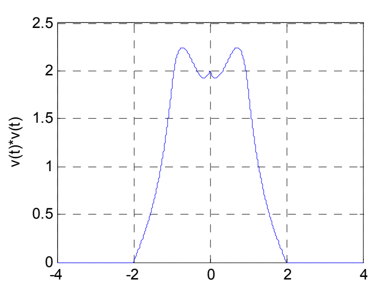
(g) Overlap btw $v(\tau)$ and $v(t-\tau)$ for ($0\le t<1$)



(h) Overlap btw $v(\tau)$ and $v(t-\tau)$ for ($1\le t<2$)



(i) Overlap btw $v(\tau)$ and $v(t-\tau)$ for ($t>2$)



(j) Convolution output $y_{10}(t)$

Fig. S3.4.10: Convolution of $v(t)$ with $v(t)$ in Problem 3.6(x).

Case I ($t<-2$): Since there is no overlap, $y_{10}(t)=0$.

Case II ($-2\le t<-1$):

$$y_{10}(t) = \int_{-1}^{t+1} e^{2\tau}e^{-2(t-\tau)}d\tau = e^{-2t}\int_{-1}^{t+1} e^{4\tau}d\tau = e^{-2t}\left[\frac{e^{4\tau}}{4}\right]_{-1}^{t+1} = e^{-2t}\left[\frac{e^{4(t+1)} - e^{-4}}{4}\right] = \frac{1}{4}\left[e^{2t+4} - e^{-(2t-4)}\right]$$

Case III ($-1 < t < 0$):

$$\begin{cases} y_{10}(t) = \int_{-1}^{t} e^{2\tau}e^{2(t-\tau)}d\tau + \int_{t}^{0} e^{2\tau}e^{-2(t-\tau)}d\tau + \int_{0}^{t+1} e^{-2\tau}e^{-2(t-\tau)}d\tau \\ \quad = (t+1)e^{2t} + \tfrac{1}{4}e^{-2t}\left(1 - e^{4t}\right) + (t+1)e^{-2t} \\ \quad = \left(t + \tfrac{3}{4}\right)e^{2t} + \left(t + \tfrac{5}{4}\right)e^{2t}. \end{cases}.$$

Case IV ($0 < t < 1$):

$$\begin{cases} y_{10}(t) = \int_{t-1}^{0} e^{2\tau}e^{2(t-\tau)}d\tau + \int_{0}^{t} e^{-2\tau}e^{2(t-\tau)}d\tau + \int_{t}^{1} e^{-2\tau}e^{-2(t-\tau)}d\tau \\ \quad = (1-t)e^{2t} + \tfrac{1}{4}e^{2t}\left(1 - e^{-4t}\right) + (1-t)e^{-2t} \\ \quad = \left(\tfrac{5}{4} - t\right)e^{2t} + \left(\tfrac{3}{4} - t\right)e^{-2t}. \end{cases}$$

Case V ($1 \le t < 2$): $$y_{10}(t) = \int_{t-1}^{1} e^{-2\tau}e^{2(t-\tau)}d\tau = \tfrac{1}{4}e^{2t}\left(e^{-4(t-1)} - e^{-4}\right).$$

Case VI ($t > 2$): Since there is no overlap, $y_{10}(t) = 0$.

Combining all the cases, the result of the convolution $y_{10}(t) = v(t) * v(t)$ is given by

$$y_{10}(t) = \begin{cases} \frac{1}{4}\left(e^{2t+4} - e^{-(2t-4)}\right) & (-2 \le t < -1) \\ \left(t + \tfrac{3}{4}\right)e^{2t} + \left(t + \tfrac{5}{4}\right)e^{2t} & (-1 \le t < 0) \\ \left(\tfrac{5}{4} - t\right)e^{2t} + \left(\tfrac{3}{4} - t\right)e^{-2t} & (0 \le t < 1) \\ \tfrac{1}{4}e^{2t}\left(e^{-4(t-1)} - e^{-4}\right) & (1 \le t < 2) \\ 0 & \text{elsewhere.} \end{cases}$$

The output $y_{10}(t)$ is shown in Fig. S3.6.10(j). ▮

Problem 3.7

(a) Distributive property: By definition, $x_1(t) * z(t) = \int_{-\infty}^{\infty} x_1(\tau)z(t-\tau)d\tau$.

Substituting $z(t) = x_2(t) + x_3(t)$, we obtain

$$x_1(t)*\big(x_2(t)+x_3(t)\big)=\int_{-\infty}^{\infty}x_1(\tau)\big(x_2(t-\tau)+x_3(t-\tau)\big)d\tau$$

or, $$x_1(t)*\big(x_2(t)+x_3(t)\big)=\underbrace{\int_{-\infty}^{\infty}x_1(\tau)x_2(t-\tau)d\tau}_{x_1(t)*x_2(t)}+\underbrace{\int_{-\infty}^{\infty}x_1(\tau)x_3(t-\tau)d\tau}_{x_1(t)*x_3(t)}$$

or, $$x_1(t)*\big(x_2(t)+x_3(t)\big)=x_1(t)*x_2(t)+x_1(t)*x_3(t).$$

(b) Associative property: By definition, $x_1(t)*w(t)=\int_{-\infty}^{\infty}x_1(\tau)w(t-\tau)d\tau$.

Substituting $$w(t)=x_2(t)*x_3(t)=\int_{-\infty}^{\infty}x_3(\alpha)x_2(t-\alpha)d\alpha,$$

we obtain $$x_1(t)*\big(x_2(t)*x_3(t)\big)=\int_{-\infty}^{\infty}x_1(\tau)\left(\int_{-\infty}^{\infty}x_3(\alpha)x_2(t-\tau-\alpha)d\alpha\right)d\tau.$$

Changing the order of integrations, the above equation simplifies to

$$x_1(t)*\big(x_2(t)*x_3(t)\big)=\int_{-\infty}^{\infty}x_3(\alpha)\left(\int_{-\infty}^{\infty}x_1(\tau)x_2(t-\tau-\alpha)d\tau\right)d\alpha.$$

Similarly, expanding

$$\big(x_1(t)*x_2(t)\big)*x_3(t)=\int_{-\infty}^{\infty}x_3(\tau)\left(\int_{-\infty}^{\infty}x_1(\alpha)x_2(t-\tau-\alpha)d\alpha\right)d\tau.$$

Since the right hand side of the two expressions are equal, we obtain the following.

$$x_1(t)*\big(x_2(t)*x_3(t)\big)=\big(x_1(t)*x_2(t)\big)*x_3(t).$$

(c) Scaling Property: We consider the two cases $\alpha > 0$ and $\alpha < 0$ separately.

Case I: ($\alpha = k$) where k is a positive constant.

By definition, $$x_1(kt)*z(kt)=\int_{-\infty}^{\infty}x_1(k\tau)z(kt-k\tau)d\tau$$

Substituting $p = k\tau$, we get $x_1(kt)*z(kt)=\int_{-\infty}^{\infty}x_1(p)z(kt-p)\dfrac{dp}{k}=\dfrac{1}{k}y(kt)$.

Case II: ($\alpha = -k$) where k is a positive constant.

By definition, $$x_1(-kt)*z(-kt)=\int_{-\infty}^{\infty}x_1(-k\tau)z(-kt+k\tau)d\tau$$

Substituting $p = -k\tau$, we get $x_1(-kt) * z(-kt) = \int_{\infty}^{-\infty} x_1(p) z(-kt - p) \frac{dp}{(-k)}$.

By changing the order of the upper and lower limits

$$x_1(-kt) * z(-kt) = -\frac{1}{(-k)} \int_{-\infty}^{\infty} x_1(p) z((-k)t - p) dp = \frac{1}{k} y(-kt)$$

Collectively, Cases I and II prove the scaling property. ▮

Problem 3.8

We know that $$u(t) \to (1 - e^{-t}) u(t).$$

Then $$\underbrace{\frac{u(t)}{dt}}_{\delta(t)} \to \frac{d}{dt}\left[(1 - e^{-t}) u(t)\right],$$

which simplifies to $$\delta(t) \to \delta(t) + e^{-t} u(t) - e^{-t}\delta(t)$$

or, $$\delta(t) \to e^{-t} u(t).$$

Hence, the impulse response is given by $h(t) = e^{-t} u(t)$. ▮

Problem 3.9

Convolution of two signals that are, respectively, nonzero over the range $[t_{\ell 1}, t_{u1}]$ and $[t_{\ell 2}, t_{u2}]$ is nonzero over the range $[t_{\ell 1} + t_{\ell 2}, t_{u1} + t_{u2}]$. Therefore, $t_{\ell 1} + t_{\ell 2} = -5$ and $t_{u1} + t_{u2} = 6$. By substituting, $t_{\ell 1} = -2$ and $t_{u1} = 3$, the values of $t_{\ell 2} = -5 + 2 = -3$ and $t_{u2} = 6 - 3 = 3$. The possible nonzero range of the impulse response $h(t)$ is therefore $[-3, 3]$. ▮

Problem 3.10

In Example 3.8, it was shown that

$$\left[x(t) = e^{-t}u(t)\right] * \left[h(t) = \begin{cases} 1 - t & 0 \le t \le 1 \\ 0 & \text{otherwise} \end{cases}\right] = \begin{cases} 0 & t < 0 \\ 2 - t - 2e^{-t} & 0 \le t \le 1 \\ e^{-(t-1)} - 2e^{-t} & t > 1. \end{cases}$$

Using the commutative property, Eq. (3.3), we interchange the impulse response and the input to obtain

$$y(t) = \left[x(t) = \begin{cases} 1 - t & 0 \le t \le 1 \\ 0 & \text{otherwise} \end{cases}\right] * \left[h(t) = e^{-t}u(t)\right] = \begin{cases} 0 & t < 0 \\ 2 - t - 2e^{-t} & 0 \le t \le 1 \\ e^{-(t-1)} - 2e^{-t} & t > 1. \end{cases}$$

▮

Problem 3.11

By inspection, we note that $x'(t) = x(t - 2)$ and $h'(t) = h(t - 4)$. Using the shifting property, Eq. (3.40), the output

$$y'(t) = x'(t) * h'(t) = x(t-2) * h(t-4) = y(t-6)\,.$$

Therefore, the convolution output for the shifted input $x'(t) = x(t-2)$ and shifted impulse response $h'(t) = h(t-4)$ is given by

$$y'(t) = y(t-6) = \begin{cases} 0 & (t-6) < 0 \\ 2-(t-6)-2e^{-(t-6)} & 0 \le (t-6) \le 1 \\ e^{-((t-6)-1)} - 2e^{-(t-6)} & (t-6) > 1, \end{cases}$$

or,

$$y'(t) = \begin{cases} 0 & t < 6 \\ 8-t-2e^{-(t-6)} & 6 \le t \le 7 \\ e^{-(t-7)} - 2e^{-(t-6)} & t > 7. \end{cases}$$

<u>Problem 3.12</u>

(i) System $h1(t)$ is NOT memoryless since $h1(t) \neq 0$ for $t \neq 0$.
System $h1(t)$ is causal since $h1(t) = 0$ for $t < 0$.
System $h1(t)$ is BIBO stable since

$$\int_{-\infty}^{\infty} |h1(t)|\,dt = \int_{-\infty}^{\infty} \delta(t)dt + \int_{-\infty}^{\infty} e^{-5t}u(t)dt = 1 + \left[-\tfrac{1}{5}e^{-5t}\right]_0^{\infty} = \frac{6}{5} < \infty\,.$$

(ii) System $h2(t)$ is NOT memoryless since $h3(t) \neq 0$ for $t \neq 0$.
System $h2(t)$ is causal since $h2(t) = 0$ for $t < 0$.
System $h2(t)$ is BIBO stable since

$$\int_{-\infty}^{\infty} |h2(t)|\,dt = \int_{-\infty}^{\infty} e^{-2t}u(t)dt = \int_{0}^{\infty} e^{-2t}dt = \left[-\tfrac{1}{2}e^{-2t}\right]_0^{\infty} = \frac{1}{2} < \infty\,.$$

(iii) System $h3(t)$ is NOT memoryless since $h3(t) \neq 0$ for $t \neq 0$.
System $h3(t)$ is causal since $h3(t) = 0$ for $t < 0$.
System $h3(t)$ is BIBO stable since

$$\int_{-\infty}^{\infty} |h3(t)|\,dt = \int_{-\infty}^{\infty} e^{-5t}\sin(2\pi t)u(t)dt = \int_{0}^{\infty} e^{-5t}\sin(2\pi t)dt < \infty\,.$$

(iv) System $h4(t)$ is NOT memoryless since $h4(t) \neq 0$ for $t \neq 0$.
System $h4(t)$ is NOT causal since $h4(t) \neq 0$ for $t < 0$.
System $h4(t)$ is BIBO stable since

$$\int_{-\infty}^{\infty} |h4(t)|\,dt = \int_{-\infty}^{0} e^{2t}dt + \int_{0}^{\infty} e^{-2t}dt + \int_{-1}^{1} 1\,dt = 3 < \infty\,.$$

(v) System $h5(t)$ is NOT memoryless since $h5(t) \neq 0$ for $t \neq 0$.
System $h5(t)$ is NOT causal since $h5(t) \neq 0$ for $t < 0$.
System $h5(t)$ is BIBO stable since

$$\int_{-\infty}^{\infty} |h5(t)|\,dt = \int_{-4}^{4} t\,dt = \left.\frac{t^2}{2}\right|_{-4}^{4} = 16 < \infty .$$

(vi) System $h6(t)$ is NOT memoryless since $h6(t) \neq 0$ for $t \neq 0$.
System $h6(t)$ is NOT causal since $h6(t) \neq 0$ for $t < 0$.
System $h6(t)$ is NOT BIBO stable since

$$\int_{-\infty}^{\infty} |h6(t)|\,dt = \int_{-\infty}^{\infty} |\sin(10t)|\,dt = \infty .$$

Consider the bounded input signal $\sin(10t)$. If this signal is applied to the system, the output can be calculated as:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} \sin(10\tau)\sin(10t-10\tau)d\tau$$

The output at t=0 is given by,

$$y(0) = \int_{-\infty}^{\infty} \sin(10\tau)\sin(-10\tau)d\tau = -\int_{-\infty}^{\infty} \sin^2(10\tau)d\tau = -\tfrac{1}{2}\int_{-\infty}^{\infty} \big(1-\cos(20\tau)\big)d\tau$$

$$= -\tfrac{1}{2}\underbrace{\int_{-\infty}^{\infty} d\tau}_{=\infty} + \tfrac{1}{2}\underbrace{\int_{-\infty}^{\infty} \cos(20\tau)d\tau}_{=\textit{finite value}} = -\infty$$

It is observed that the output becomes unbounded even if the input is always bounded. This is because the system is not BIBO stable.

(vii) System $h7(t)$ is NOT memoryless since $h7(t) \neq 0$ for $t \neq 0$.
System $h7(t)$ is causal since $h7(t) = 0$ for $t < 0$.
System $h7(t)$ is NOT BIBO stable since

$$\int_{-\infty}^{\infty} |h7(t)|\,dt = \int_{0}^{\infty} \cos(5t)dt = \infty .$$

Consider the bounded input signal $\cos(5t)$. If this signal is applied to the system, the output can be calculated as:

$$y(t) = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau = \int_{-\infty}^{\infty} \cos(5t-5\tau)\cos(5\tau)u(\tau)d\tau = \int_{0}^{\infty} \cos(5t-5\tau)\cos(5\tau)d\tau .$$

The output at t=0 is given by,

$$y(0) = \int_{0}^{\infty} \cos(-5\tau)\cos(5\tau)d\tau = \int_{0}^{\infty} \cos^2(5\tau)d\tau = \tfrac{1}{2}\int_{0}^{\infty} \big(1+\cos(10\tau)\big)d\tau$$

$$= \tfrac{1}{2}\underbrace{\int_{0}^{\infty} d\tau}_{=\infty} + \tfrac{1}{2}\underbrace{\int_{0}^{\infty} \cos(10\tau)d\tau}_{=\textit{finite value}} = \infty$$

It is observed that the output becomes unbounded at t=0 even if the input is always bounded. This proves that the system is not BIBO stable.

(viii) System $h8(t)$ is NOT memoryless since $h8(t) \neq 0$ for $t \neq 0$.
System $h8(t)$ is NOT causal since $h8(t) \neq 0$ for $t < 0$.
System $h8(t)$ is BIBO stable since

$$\int_{-\infty}^{\infty} |h8(t)|\,dt = \int_{-\infty}^{\infty} 0.95^{|t|}dt = 2\int_{0}^{\infty} 0.95^{t}\,dt = 2\int_{0}^{\infty} e^{t\ln(0.95)}dt = \frac{2}{\ln(0.95)}\left[e^{t\ln(0.95)}\right]_0^{\infty}$$

$$= \frac{2}{\ln(0.95)}\left[0-1\right] = -\frac{2}{\ln(0.95)} = 39 < \infty$$

(ix) System $h9(t)$ is NOT memoryless since $h8(t) \neq 0$ for $t \neq 0$.
System $h9(t)$ is NOT causal since $h8(t) = 0$ for $t < 0$.
System $h8(t)$ is BIBO stable since

$$\int_{-\infty}^{\infty} |h9(t)|\,dt = \int_{-1}^{1} 1dt = 2 < \infty .$$

∎

Problem 3.13

(i) $y(t) = x(t) * h(t) = x(t) * \left[\delta(t) - \delta(t-2)\right] = x(t) - x(t-2)$.

From the input-output relationship, we observe that the output at time t depends on the values of the input at time (t – 2). Therefore, the system is NOT memoryless. However, it is causal.

(ii) $y(t) = x(t) * h(t) = x(t) * rect(t/2) = \int_{-\infty}^{\infty} rect(\tau/2)x(t-\tau)d\tau = \int_{-1}^{1} x(t-\tau)d\tau$.

From the input-output relationship, we observe that the output at time t depends on the values of the input from time (t – 1) to (t + 1). Therefore, the system is NOT memoryless and NOT causal.

(iii) $y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = 2\int_{-\infty}^{\infty} e^{-4(t-\tau)}u(t-\tau)x(\tau)d\tau = 2\int_{-\infty}^{t} e^{-4(t-\tau)}x(\tau)d\tau = 2e^{-4t}\int_{-\infty}^{t} e^{4\tau}x(\tau)d\tau$.

From the input-output relationship, we observe that the output at time t depends on the values of the input from time (–∞, t). Therefore, the system is NOT memoryless. However, it is causal.

(iv) $y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} \left[1-e^{-4(t-\tau)}\right]u(t-\tau)x(\tau)d\tau = \int_{-\infty}^{t} \left[1-e^{-4(t-\tau)}\right]x(\tau)d\tau$.

From the input-output relationship, we observe that the output at time t depends on the values of the input from time (–∞, t). Therefore, the system is NOT memoryless. However, it is causal. ∎

Problem 3.14

(i) System (i) is invertible with the impulse response $h1_i(t)$ of its inverse system given by

$$h1_i(t) = \tfrac{1}{5}\delta(t+2) .$$

(ii) System (ii) will be invertible if there exists an impulse response $h2_i(t)$ such that

$$h2(t) * h2_i(t) = \delta(t).$$

Substituting the value of $h2(t)$, we get

$$h2_i(t) + h2_i(t+2) = \delta(t)$$

which simplifies to $$h2_i(t) = \delta(t-2) - h2_i(t-2).$$

Substituting the value of $h2_i(t-2) = \delta(t-4) - h2_i(t-4)$ in the earlier expression gives

$$h2_i(t) = \delta(t-2) - \delta(t-4) + h2_i(t-4).$$

Iterating the above procedure yields,

$$h2_i(t) = \sum_{m=1}^{\infty} (-1)^{m+1} \delta(t-2m).$$

Therefore, the system is invertible with the impulse response of the inverse system given above.

(iii) System (iii) will be invertible if there exists an impulse response $h3_i(t)$ such that

$$h3(t) * h3_i(t) = \delta(t).$$

Substituting the value of $h3(t)$, we get

$$h3_i(t+1) + h3_i(t-1) = \delta(t)$$

which simplifies to $$h3_i(t) = \delta(t-1) - h3_i(t-2).$$

Substituting the value of $h3_i(t-2) = \delta(t-3) - h3_i(t-4)$ in the earlier expression yields

$$h3_i(t) = \delta(t-1) - \delta(t-3) + h3_i(t-4).$$

Iterating the above procedure yields,

$$h3_i(t) = \sum_{m=1}^{\infty} (-1)^{m+1} \delta(t+1-2m).$$

(iv) System (iv) will be invertible if there exists an impulse response $h4_i(t)$ such that

$$h4(t) * h4_i(t) = \delta(t).$$

Substituting the value of $h4(t)$, we get

$$\int_{-\infty}^{\infty} h4_i(\tau) u(t-\tau) d\tau = \delta(t)$$

which simplifies to $$\int_{-\infty}^{t} h4_i(\tau) d\tau = \delta(t).$$

Differentiating both sides of the above expression with respect to t, we obtain

$$h4_i(t) = \tfrac{d}{dt}\left(\delta(t)\right).$$

In other words, system (iv) is an integrator. As expected, its inverse system is a differentiator.

(v) System (v) will be invertible if there exists an impulse response $h5_i(t)$ such that

$$h5(t) * h5_i(t) = \delta(t).$$

Substituting the value of $h5(t)$, we obtain

$$\int_{-\infty}^{\infty} h5_i(\tau)\mathrm{rect}(\tfrac{t-\tau}{4})d\tau = \delta(t),$$

which simplifies to
$$\int_{t-4}^{t+4} h5_i(\tau)d\tau = \delta(t),$$

which is expressed as
$$\underbrace{\int_{-\infty}^{t+4} h5_i(\tau)d\tau}_{\text{Substitute } \alpha=\tau-4} - \underbrace{\int_{-\infty}^{t-4} h5_i(\tau)d\tau}_{\text{Substitute } \alpha=\tau+4} = \delta(t),$$

or,
$$\int_{-\infty}^{t} h5_i(\alpha+4)d\alpha - \int_{-\infty}^{t} h5_i(\alpha-4)d\alpha = \delta(t)$$

Taking the derivative of both sides of the equation with respect to t, we obtain

$$h5_i(t+4) - h5_i(t-4) = \tfrac{d}{dt}\big(\delta(t)\big).$$

which can be expressed as

$$h5_i(t) = \sum_{m=0}^{\infty} \tfrac{d}{dt}\big(\delta(t-4-8m)\big).$$

(vi) System (vi) will be invertible if there exists an impulse response $h6_i(t)$ such that

$$h6(t) * h6_i(t) = \delta(t).$$

Substituting the value of $h6(t)$, we obtain

$$\int_{-\infty}^{\infty} h6_i(\tau)e^{-2(t-\tau)}u(t-\tau)d\tau = \delta(t),$$

which simplifies to
$$e^{-2t}\int_{-\infty}^{t} h6_i(\tau)e^{2\tau}d\tau = \delta(t)$$

or,
$$\int_{-\infty}^{t} h6_i(\tau)e^{2\tau}d\tau = \delta(t)e^{2t}.$$

Taking the derivative of both sides of the equation with respect to t, we obtain

$$h6_i(t)e^{2t} = \tfrac{d}{dt}\Big(\delta(t)e^{2t}\Big) = e^{2t}\tfrac{d}{dt}\big(\delta(t)\big) + 2\delta(t)e^{2t}$$

or,
$$h6_i(t) = \tfrac{d}{dt}\big(\delta(t)\big) + 2\delta(t).$$ ▌

Problem 3.15

By inspection, $v(t) = x(t) + h_2(t) * y(t)$, and $y(t) = v(t) * h_1(t)$.

Therefore, $y(t) = [x(t) + h_2(t) * y(t)] * h_1(t) = x(t) * h_1(t) + h_2(t) * h_1(t) * y(t)$.

By rearranging the terms on both sides of the equation, we obtain

$$[\delta(t) - h_2(t) * h_1(t)] * y(t) = h_1(t) * x(t).$$

▌

Problem 3.16

The output of the system is given by

$$y(t) = x(t) * h(t) = e^{j\omega_0 t} * h(t) = \int_{-\infty}^{\infty} h(\tau) e^{j\omega_0 (t-\tau)} d\tau = e^{j\omega_0 t} \int_{-\infty}^{\infty} h(\tau) e^{-j\omega_0 \tau} d\tau .$$

Defining $H(\omega) = \int_{-\infty}^{\infty} h(t) e^{j\omega t} dt$, the output is given by

$$y(t) = e^{j\omega_o t} H(\omega)|_{\omega=\omega_o} .$$

▌

Problem 3.17

In P3.16, it is shown that $e^{j\omega_o t} \to e^{j\omega_o t} H(\omega)|_{\omega=\omega_o} = e^{j\omega_o t} H(\omega_0)$.

Applying the linearity property, we obtain

$$Ae^{j(\omega_o t+\theta)} = \left(Ae^{j\theta}\right) e^{j\omega_o t} \to \left(Ae^{j\theta}\right) e^{j\omega_o t} H(\omega_o) = Ae^{j(\omega_o t+\theta)} H(\omega_o).$$

By expressing $H(\omega)$ in polar format, $H(\omega) = |H(\omega)|\, e^{j<H(\omega)}$ with $|H(\omega)|$ being the magnitude and $<H(\omega)$ being the phase of $H(\omega)$, we obtain

$$Ae^{j(\omega_o t+\theta)} \to Ae^{j(\omega_o t+\theta+<H(\omega_0))} |H(\omega_0)|.$$

Decomposing the above expression into its real and imaginary components

$$A\cos(\omega_o t + \theta) + jA\sin(\omega_o t + \theta) \to A\cos(\omega_o t + \theta + <H(\omega_0))|H(\omega_o)| + jA\sin(\omega_o t + \theta + <H(\omega_0))|H(\omega_o)|$$

Since the impulse response of the system is real-valued, the real part of the output arises due to the real part of the input, and the imaginary part of the output arises due to the imaginary part of the input. Therefore, by separating the real and imaginary components, we obtain:

$$A\cos(\omega_o t + \theta) \to A|H(\omega_o)|\cos(\omega_o t + \theta + <H(\omega_0))$$

and

$$A\sin(\omega_o t + \theta) \to A|H(\omega_o)|\sin(\omega_o t + \theta + <H(\omega_0)).$$

The above result implies that an LTIC system only changes the magnitude and phase of the sinusoidal input. The output is still sinusoidal with the same fundamental frequency as that of the input signal. ▌

Problem 3.18

Express $-3\sin(2\pi t + \pi/4) \to 5\cos(2\pi t)$

as $3\sin\left(2\pi t+5\pi/4\right)\rightarrow 5\sin\left(2\pi t+\pi/2\right)=3\times(5/3)\sin\left(2\pi t+5\pi/4-3\pi/4\right)$.

Comparing with $A\sin(\omega_o t+\theta)\rightarrow A|H(\omega_o)|\sin(\omega_o t+\theta+<H(\omega_0))$,

we note that $|H(\omega_o)|=5/3$ and $<H(\omega_0)=-3\pi/4$.

The transfer function of the system at ω = 2π is therefore given by

$$H(\omega)\big|_{\omega=2\pi}=\tfrac{5}{3}\,e^{-j3\pi/4}.$$

Problem 3.19

(i) $\ddot{y}(t)+4\dot{y}(t)+8y(t)=\dot{x}(t)+x(t)$ with $x(t)=e^{-4t}u(t), y(0)=0,$ and $\dot{y}(0)=0.$

The Matlab code is included below with both the analytical and computational plots included in Fig. S3.19.1.

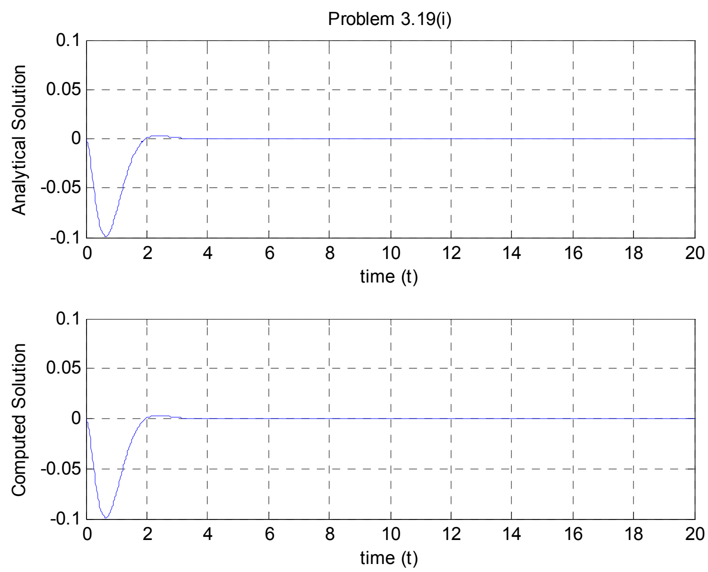


Fig. S3.19.1: Analytical (top) and computational (bottom) plots for Problem 3.2 part (i)

```
% MATLAB Code for Problem 3.19(i)
tspan = [0:0.02:20];
%Analytical Solution from Problem 3.2
t = tspan;
yanalytical = 3/8*(exp(-2*t).*cos(2*t)-exp(-2*t).*sin(2*t)-exp(-4*t));
subplot(211);
plot(t,yanalytical);
title('Problem 3.19(i)');
xlabel('time (t)');
ylabel('Analytical Solution');
grid on
%Computational Solution
y0 = [0; 0]
[t2,y] = ode23('myfunc4problem3_19a',tspan,y0);
subplot(212);
plot(t2,y(:,2));
xlabel('time (t)');
ylabel('Computed Solution');
grid on
% Include the following function in a separate file < myfunc4problem3_19a.m>
```

```
function [ydot] = myfunc4problem3_19a(t,y)
ydot(1,1) = -4*y(1) - 8*y(2) - 3*exp(-4*t)*(t >= 0);
ydot(2,1) = y(1);
%---end of the function----------------------
```

(ii) $\ddot{y}(t)+6\dot{y}(t)+4y(t)=\dot{x}(t)+x(t)$ with $x(t)=\cos(6t)u(t), y(0)=2,$ and $\dot{y}(0)=0.$

The Matlab code is included below with both the analytical and computational plots included in Fig. S3.19.2.

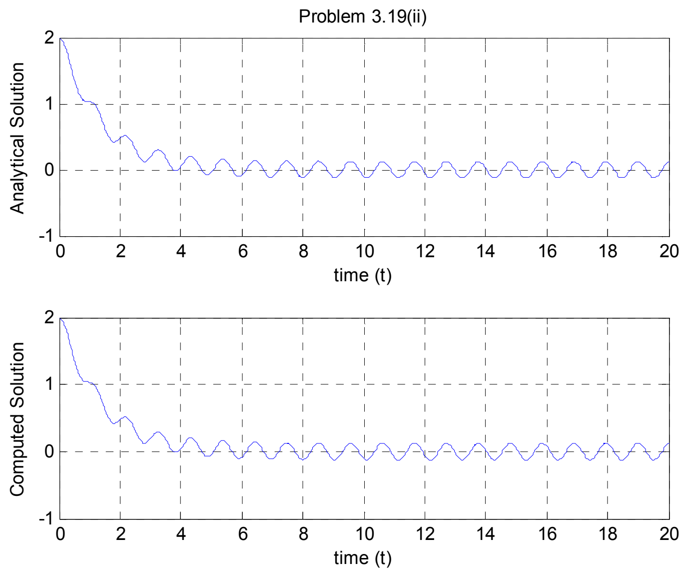


Fig. S3.19.2: Analytical (top) and computational (bottom) plots for Problem 3.2 part (ii)

```
% MATLAB Code for Problem 3.19(ii)
tspan = [0:0.02:20];
%Analytical Solution from Problem 3.2
t = tspan;
yanalytical = -0.1962*exp(-5.2361*t)+2.1169*exp(0.7639*t)+0.0793*cos(6*t);
yanalytical = yanalytical+0.0983*sin(6*t);
subplot(211);
plot(t,yanalytical);
title('Problem 3.19(ii)');
xlabel('time (t)');
ylabel('Analytical Solution');
myaxis = axis;
grid on
%Computational Solution
y0 = [0; 2]
[t2,y] = ode23('myfunc4problem3_19b',tspan,y0);
subplot(212);
plot(t2,y(:,2));
xlabel('time (t)');
ylabel('Computed Solution');
axis(myaxis);
grid on
% Include the following function in a separate file < myfunc4problem3_19b.m>
function [ydot] = myfunc4problem3_19b(t,y)
ydot(1,1) = -6*y(1) - 4*y(2) - 6*sin(6*t) + cos(6*t);
ydot(2,1) = y(1);
```

(iii) $\ddot{y}(t)+2\dot{y}(t)+y(t)=\ddot{x}(t)$ with $x(t)=\left[\cos(t)+\sin(2t)\right]u(t), y(0)=3,$ and $\dot{y}(0)=1.$

The Matlab code is included below with both the analytical and computational plots included in Fig. S3.19.3.

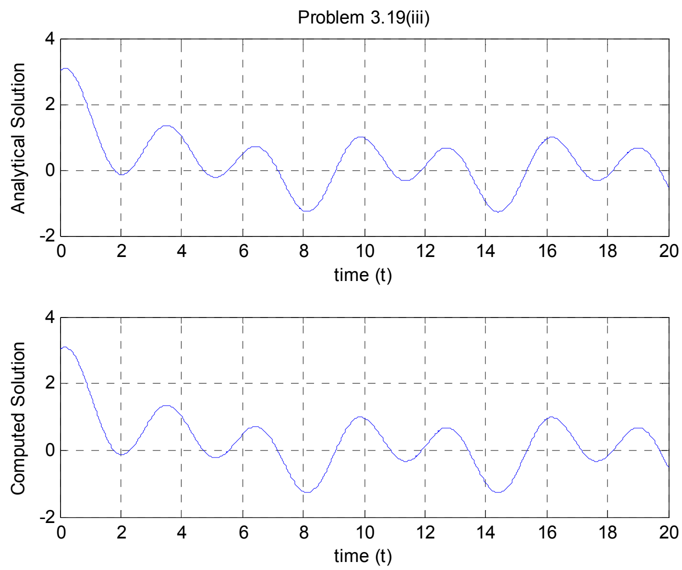


Fig. S3.19.3: Analytical (top) and computational (bottom) plots for Problem 3.2 part (iii)

```
% MATLAB Code for Problem 3.19(iii)
tspan = [0:0.02:20];
%Analytical Solution from Problem 3.2
t = tspan;
yanalytical = 2.36*exp(-t)+2.9*t.*exp(-t);
yanalytical = yanalytical - 0.5*sin(t)+0.64*cos(2*t)+0.48*sin(2*t);
subplot(211);
plot(t,yanalytical);
title('Problem 3.19(iii)');
xlabel('time (t)');
ylabel('Analytical Solution');
myaxis = axis;
grid on
%Computational Solution
y0 = [1; 3]
[t2,y] = ode23('myfunc4problem3_19c',tspan,y0);
subplot(212);
plot(t2,y(:,2));
xlabel('time (t)');
ylabel('Computed Solution');
axis(myaxis);
grid on
% Include the following function in a separate file < myfunc4problem3_19c.m>
function [ydot] = myfunc4problem3_19c(t,y)
ydot(1,1) = -2*y(1) - y(2) - cos(t) - 4*sin(2*t);
ydot(2,1) = y(1);
```

(iv) $\ddot{y}(t)+4y(t)=5x(t)$ with $x(t)=4te^{-t}u(t), y(0)=-2,$ and $\dot{y}(0)=0.$

The Matlab code is included below with both the analytical and computational plots included in Fig. S3.19.4.

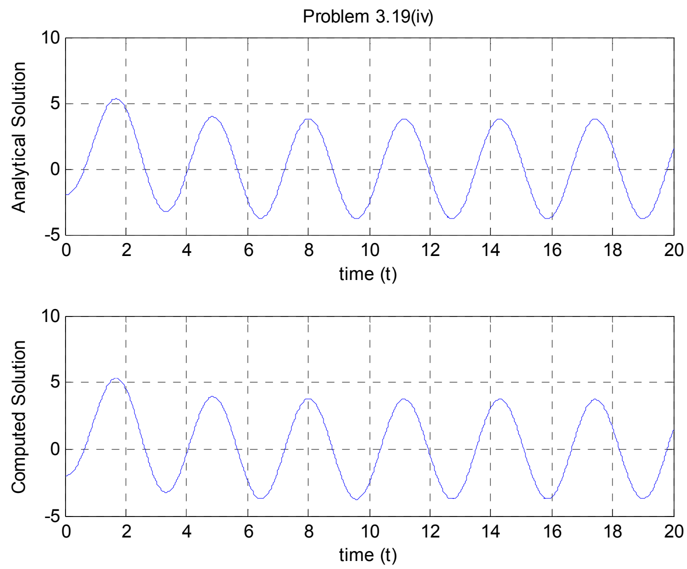


Fig. S3.19.4: Analytical (top) and computational (bottom) plots for Problem 3.2 part (iv)

```
% MATLAB Code for Problem 3.19(iv)
tspan = [0:0.02:20];
%Analytical Solution from Problem 3.2
t = tspan;
yanalytical = -3.6*cos(2*t)-1.2*sin(2*t)+1.6*exp(-t)+4*t.*exp(-t);
subplot(211);
plot(t,yanalytical);
title('Problem 3.19(iv)');
xlabel('time (t)');
ylabel('Analytical Solution');
myaxis = axis;
grid on
%Computational Solution
y0 = [0; -2]
[t2,y] = ode23('myfunc4problem3_19d',tspan,y0);
subplot(212);
plot(t2,y(:,2));
xlabel('time (t)');
ylabel('Computed Solution');
axis(myaxis);
grid on
% Include the following function in a separate file < myfunc4problem3_19d.m>
function [ydot] = myfunc4problem3_19d(t,y)
ydot(1,1) = - 4*y(2) +5*4*t.*exp(-t);
ydot(2,1) = y(1);
```

(v) $\frac{d^4y}{dt^4} + 2\frac{d^2y}{dt^2} + y(t) = x(t) \quad with \quad x(t) = 2u(t), y(0) = \ddot{y}(0) = \dddot{y}(0) = 0, \quad and\ \dot{y}(0) = 1.$

The Matlab code is included below with both the analytical and computational plots included in Fig. S3.19.5.

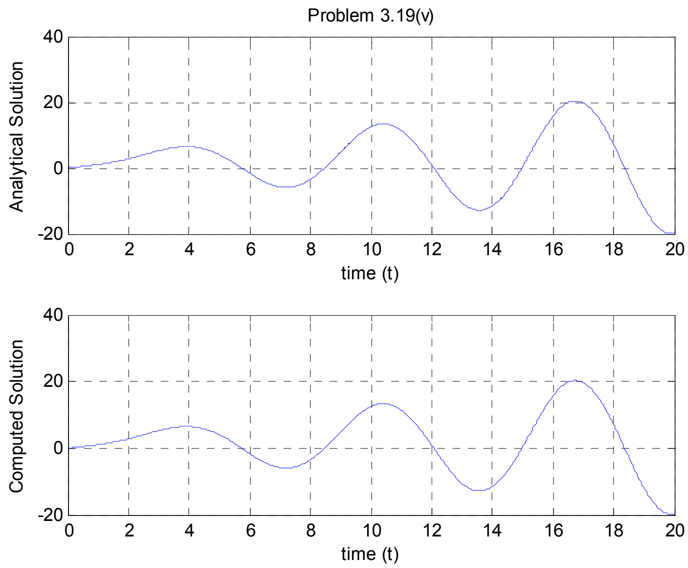


Fig. S3.19.5: Analytical (top) and computational (bottom) plots for Problem 3.2 part (v)

```
% MATLAB Code for Problem 3.19(iii)
tspan = [0:0.02:20];
%Analytical Solution from Problem 3.2
t = tspan;
%yanalytical = -0.25*exp(t)-0.75*exp(-t)-cos(t)+0.5*sin(t)+2;
yanalytical = 1.50*sin(t)-2*cos(t)-t.*sin(t)-0.5*t.*cos(t) +2;
subplot(211);
plot(t,yanalytical);
title('Problem 3.19(v)');
xlabel('time (t)');
ylabel('Analytical Solution');
myaxis = axis;
grid on
%Computational Solution
y0 = [0; 0; 1; 0];
[t2,y] = ode23('myfunc4problem3_19e',tspan,y0);
subplot(212);
plot(t2,y(:,4));
xlabel('time (t)');
ylabel('Computed Solution');
axis(myaxis);
grid on
% Include the following function in a separate file < myfunc4problem3_19e.m>
function [ydot] = myfunc4problem3_19e(t,y)
ydot(1,1) = -2*y(2) - y(4) + 2;
ydot(2,1) = y(1);
ydot(3,1) = y(2);
ydot(4,1) = y(3);
```