Chapter 5: Continuous-time Fourier Transform

Problem 5.1

(a) The CTFT for $x_1(t)$ is given by

$$X_1(\omega)=\int_{-\infty}^{\infty} x_1(t)e^{-j\omega t}dt=\int_{-\tau}^{0}\left[1+\tfrac{t}{\tau}\right]e^{-j\omega t}dt+\int_{0}^{\tau}\left[1-\tfrac{t}{\tau}\right]e^{-j\omega t}dt$$

$$=\left[\left[1+\tfrac{t}{\tau}\right]\times\frac{e^{-j\omega t}}{(-j\omega)}-\left[\tfrac{1}{\tau}\right]\times\frac{e^{-j\omega t}}{(-j\omega)^2}\right]_{-\tau}^{0}+\left[\left[1-\tfrac{t}{\tau}\right]\times\frac{e^{-j\omega t}}{(-j\omega)}-\left[-\tfrac{1}{\tau}\right]\times\frac{e^{-j\omega t}}{(-j\omega)^2}\right]_{0}^{\tau}$$

$$=\left[\frac{1}{(-j\omega)}-\left[\tfrac{1}{\tau}\right]\times\frac{1}{(-j\omega)^2}+\left[\tfrac{1}{\tau}\right]\times\frac{e^{j\omega\tau}}{(-j\omega)^2}\right]+\left[\left[\tfrac{1}{\tau}\right]\times\frac{e^{-j\omega\tau}}{(-j\omega)^2}-\frac{1}{(-j\omega)}-\left[\tfrac{1}{\tau}\right]\times\frac{1}{(-j\omega)^2}\right]$$

$$=\frac{2}{\omega^2\tau}-\frac{2\cos(\omega\tau)}{\omega^2\tau}=2\times\frac{1-\cos(\omega\tau)}{\omega^2\tau}=2\times\frac{2\sin^2(\omega\tau/2)}{\omega^2\tau}$$

$$=\tau\frac{\sin^2(\omega\tau/2)}{(\omega\tau/2)^2}=\tau\,\mathrm{sinc}^2\left(\frac{\omega\tau}{2\pi}\right).$$

(b) The CTFT for $x_2(t)$ is given by

$$X_2(\omega)=\int_{-\infty}^{\infty} x_2(t)e^{-j\omega t}dt=\int_{-\infty}^{\infty} t^4e^{-at}u(t)e^{-j\omega t}dt\ =\int_{0}^{\infty} t^4e^{-(a+j\omega)t}dt$$

$$=\left[t^4\tfrac{e^{-(a+j\omega)t}}{(-(a+j\omega))}+4t^3\tfrac{e^{-(a+j\omega)t}}{((-(a+j\omega))^2}+12t^2\tfrac{e^{-(a+j\omega)t}}{(-(a+j\omega))^3}+24t\tfrac{e^{-(a+j\omega)t}}{((-(a+j\omega))^4}+24\tfrac{e^{-(a+j\omega)t}}{(-(a+j\omega))^5}\right]_0^{\infty}$$

$$=\left[0+0+0+0+0\right]-\left[0+0+0+0+24\tfrac{1}{(-(a+j\omega))^5}\right]=\tfrac{24}{(a+j\omega)^5}.$$

(c) The CTFT for $x_3(t)$ is given by

$$X_3(\omega)=\int_{-\infty}^{\infty} x_3(t)e^{-j\omega t}dt=\int_{-\infty}^{\infty} e^{-at}\cos(\omega_0 t)u(t)e^{-j\omega t}dt\ =\tfrac{1}{2}\int_{0}^{\infty} e^{-(a+j\omega)t}\left[e^{j\omega_0 t}+e^{-j\omega_0 t}\right]dt$$

$$=\tfrac{1}{2}\int_{0}^{\infty} e^{-(a+j\omega-j\omega_0)t}dt+\tfrac{1}{2}\int_{0}^{\infty} e^{-(a+j\omega+j\omega_0)t}dt\ =\tfrac{1}{2}\left[\tfrac{e^{-(a+j\omega-j\omega_0)t}}{(-(a+j\omega-j\omega_0))}\right]_0^{\infty}+\tfrac{1}{2}\left[\tfrac{e^{-(a+j\omega+j\omega_0)t}}{(-(a+j\omega+j\omega_0))}\right]_0^{\infty}$$

$$=\tfrac{1}{2}\left[0-\tfrac{1}{(-(a+j\omega-j\omega_0))}\right]+\tfrac{1}{2}\left[0-\tfrac{1}{(-(a+j\omega+j\omega_0))}\right]=\tfrac{1}{2}\left[\tfrac{1}{(a+j\omega-j\omega_0)}+\tfrac{1}{(a+j\omega+j\omega_0)}\right]=\tfrac{a+j\omega}{(a+j\omega)^2+\omega_0^2}.$$

(d) The CTFT for $x_4(t)$ is given by

$$X_4(\omega)=\int_{-\infty}^{\infty} x_4(t)e^{-j\omega t}dt=\int_{-\infty}^{\infty}\exp(-\tfrac{t^2}{2\sigma^2})e^{-j\omega t}dt=\int_{-\infty}^{\infty}\exp[-\tfrac{t^2+2j\omega\sigma^2 t}{2\sigma^2}]dt=\int_{-\infty}^{\infty}\exp\left[-\tfrac{t^2+2j\omega\sigma^2 t+(j\omega\sigma^2)^2}{2\sigma^2}\right]\exp\left[\tfrac{(j\omega\sigma^2)^2}{2\sigma^2}\right]dt$$

$$=\exp\left[\tfrac{(j\omega\sigma^2)^2}{2\sigma^2}\right]\cdot\int_{-\infty}^{\infty}\exp\left[-\tfrac{(t+j\omega\sigma^2)^2}{2\sigma^2}\right]dt=\exp\left[\tfrac{(j\omega\sigma^2)^2}{2\sigma^2}\right]\sqrt{2\pi}\sigma=\sqrt{2\pi}\sigma\exp\left[-\tfrac{\omega^2\sigma^2}{2}\right].$$

▌

Problem 5.2

(a) By definition,

$$X_1(\omega)=\int_0^{\pi}3e^{-j\omega t}dt=3\left[\tfrac{e^{-j\omega t}}{(-j\omega)}\right]_0^{\pi}=-\tfrac{3}{j\omega}\left[e^{-j\omega\pi}-1\right]=-\tfrac{3}{j\omega}e^{-j\omega\pi/2}\left[e^{-j\omega\pi/2}-e^{j\omega\pi/2}\right]$$

$$=-\tfrac{3}{j\omega}e^{-j\omega\pi/2}\left[-2j\sin(\omega\pi/2)\right]=6e^{-j\omega\pi/2}\left[\frac{\sin(\omega\pi/2)}{\omega}\right]=6e^{-j\omega\pi/2}\left[\frac{1}{2/\pi}\times\frac{\sin(\omega\pi/2)}{\omega\pi/2}\right]$$

$$=3\pi e^{-j\omega\pi/2}\text{sinc}(\omega/2).$$

(b) By definition,

$$X_2(\omega)=\int_{-0.5T}^{0.5T}0.5e^{-j\omega t}dt+\int_{0.5T}^{1.5T}e^{-j\omega t}dt=0.5\left[\tfrac{e^{-j\omega t}}{(-j\omega)}\right]_{-0.5T}^{0.5T}+\left[\tfrac{e^{-j\omega t}}{(-j\omega)}\right]_{0.5T}^{1.5T}$$

$$=-\tfrac{0.5}{j\omega}\left[e^{-j0.5\omega T}-e^{j0.5\omega T}\right]-\tfrac{1}{j\omega}\left[e^{-j1.5\omega T}-e^{-j0.5\omega T}\right]$$

$$=-\tfrac{0.5}{j\omega}\left[-2j\sin(0.5\omega T)\right]-\tfrac{1}{j\omega}e^{-j\omega T}\left[-2j\sin(0.5\omega T)\right]$$

$$=\left[\tfrac{0.5T}{0.5T}\times\tfrac{\sin(0.5\omega T)}{\omega}\right]+2e^{-j\omega T}\left[\tfrac{0.5T}{0.5T}\times\tfrac{\sin(0.5\omega T)}{\omega}\right]$$

$$=0.5T\text{sinc}\left(\tfrac{0.5\omega \mathrm{T}}{\pi}\right)+Te^{-j\omega T}\text{sinc}\left(\tfrac{0.5\omega \mathrm{T}}{\pi}\right).$$

(c) By definition,

$$X_3(\omega)=\int_0^T\left(1-\tfrac{t}{T}\right)e^{-j\omega t}dt=\left[\left(1-\frac{t}{T}\right)\frac{e^{-j\omega t}}{(-j\omega)}-\left(-\frac{1}{T}\right)\frac{e^{-j\omega t}}{(-j\omega)^2}\right]_0^T$$

$$=\left[0-\left(-\frac{1}{T}\right)\frac{e^{-j\omega T}}{(-j\omega)^2}\right]-\left[\frac{1}{(-j\omega)}-\left(-\frac{1}{T}\right)\frac{1}{(-j\omega)^2}\right]$$

$$=-\frac{1}{\omega^2 T}e^{-j\omega T}+\frac{1}{j\omega}+\frac{1}{\omega^2 T}=\frac{1}{j\omega}+\frac{1}{\omega^2 T}\left(1-e^{-j\omega T}\right).$$

For $\omega = 0$, $\quad X_3(\omega)=\int_0^T\left(1-\frac{t}{T}\right)dt=\left[-\frac{T}{2}\left(1-\frac{t}{T}\right)^2\right]_0^T=0+\frac{T}{2}=\frac{T}{2}.$

(d) By definition,

$$X_4(\omega) = \int_{-T}^{0} \left(1 + \tfrac{t}{T}\right) e^{-j\omega t}\, dt + \int_{0}^{T} \left(1 - \tfrac{t}{T}\right) e^{-j\omega t}\, dt$$

$$= \left[\left(1 + \tfrac{t}{T}\right)\frac{e^{-j\omega t}}{(-j\omega)} - \left(\tfrac{1}{T}\right)\frac{e^{-j\omega t}}{(-j\omega)^2}\right]_{-T}^{0} + \left[\left(1 - \tfrac{t}{T}\right)\frac{e^{-j\omega t}}{(-j\omega)} - \left(-\tfrac{1}{T}\right)\frac{e^{-j\omega t}}{(-j\omega)^2}\right]_{0}^{T}$$

$$= \left[\frac{1}{(-j\omega)} - \left(\tfrac{1}{T}\right)\frac{1}{(-j\omega)^2} - 0 + \left(\tfrac{1}{T}\right)\frac{e^{j\omega T}}{(-j\omega)^2}\right] + \left[0 - \left(-\tfrac{1}{T}\right)\frac{e^{-j\omega T}}{(-j\omega)^2} - \frac{1}{(-j\omega)} + \left(-\tfrac{1}{T}\right)\frac{1}{(-j\omega)^2}\right]$$

$$= \frac{2}{\omega^2 T}\left[1 - \cos(\omega T)\right] = \frac{2\times 2\sin^2(0.5\omega T)}{\omega^2 T} = \frac{4}{1/(0.5^2 T)} \times \frac{\sin^2(0.5\omega T)}{(0.5\omega T)^2} = T\mathrm{sinc}^2\left(\tfrac{0.5\omega T}{\pi}\right).$$

(e) By definition,

$$X_5(\omega) = \int_0^T \left[1 - 0.5\sin\left(\tfrac{\pi t}{T}\right)\right] e^{-j\omega t}\, dt = \underbrace{\int_0^T e^{-j\omega t}\, dt}_{=A} - \underbrace{0.5\int_0^T \sin\left(\tfrac{\pi t}{T}\right) e^{-j\omega t}\, dt}_{=B}$$

We consider different cases for the above integral.

Case I: ($\omega = 0$)

$$X_5(0) = \int_{\infty}^{-\infty} x(t)dt = \int_{-\infty}^{\infty} \left[1 - 0.5\sin\left(\tfrac{\pi t}{T}\right)\right] dt = \int_0^T dt - 0.5\int_0^T \sin\left(\tfrac{\pi t}{T}\right) dt$$

$$= T + \frac{0.5}{\pi/T}\left[\cos\left(\tfrac{\pi t}{T}\right)\right]_0^T = T + \frac{T}{2\pi}\left[\cos(\pi) - \cos(0)\right] = T - \frac{T}{\pi} = T\left(1 - \tfrac{1}{\pi}\right)$$

Case II: ($\omega \neq 0$, $\omega \neq \pi/T$):

$$A = \int_0^T e^{-j\omega t}\, dt = \frac{1}{-j\omega}\left[e^{-j\omega t}\right]_0^T = \frac{1}{-j\omega}\left[e^{-j\omega T} - 1\right] = \frac{1}{j\omega}\left[1 - e^{-j\omega T}\right] \qquad [\omega \neq 0]$$

$$B = 0.5\left[\frac{e^{-j\omega t}}{\frac{\pi^2}{T^2} - \omega^2}\left\{-j\omega\sin\left(\tfrac{\pi t}{T}\right) - \tfrac{\pi}{T}\cos\left(\tfrac{\pi t}{T}\right)\right\}\right]_0^T \qquad \text{for } \omega \neq 0, \pm\tfrac{\pi}{T}$$

$$= \frac{0.5T^2}{\pi^2 - \omega^2 T^2}\left[e^{-j\omega t}\left\{\underbrace{j\omega\sin\left(\tfrac{\pi t}{T}\right)}_{=0 \text{ at } t=0,T} + \tfrac{\pi}{T}\cos\left(\tfrac{\pi t}{T}\right)\right\}\right]_0^T$$

$$= \frac{0.5T^2}{\pi^2 - \omega^2 T^2}\left[-\tfrac{\pi}{T} e^{-j\omega T} - \tfrac{\pi}{T}\right] = \frac{0.5\pi T}{\pi^2 - \omega^2 T^2}\left[1 + e^{-j\omega T}\right]$$

Case III: ($\omega = \pi/T$):

$$B = 0.5\int_0^T \sin\left(\tfrac{\pi t}{T}\right) e^{-j\omega t}\,dt = \tfrac{0.5}{2j}\int_0^T \left[e^{j\frac{\pi t}{T}} - e^{-j\frac{\pi t}{T}}\right] e^{-j\omega t}\,dt = \tfrac{0.5}{2j}\int_0^T \left[e^{-j(\omega-\frac{\pi}{T})t} - e^{-j(\omega+\frac{\pi}{T})t}\right] dt$$

$$= \begin{cases} \tfrac{0.5}{2j}\int_0^T \left[1 - e^{-j\frac{2\pi t}{T}}\right] dt & \omega = \frac{\pi}{T} \\ -\tfrac{0.5}{2j}\int_0^T \left[1 - e^{j\frac{2\pi t}{T}}\right] dt & \omega = -\frac{\pi}{T} \end{cases} \qquad \left[As\ e^{\pm\frac{j2\pi t}{T}}\ is\ periodic\ with\ period\ T,\ \int_0^T e^{\pm\frac{j2\pi t}{T}}\,dt = 0 \right]$$

$$= \pm\tfrac{0.5}{2j}\left[t\right]_0^T = \pm\tfrac{0.5T}{2j}$$

Combining, the above results, the CTFT can be expressed as

$$X_5(\omega) = \begin{cases} T(1-\frac{1}{\pi}) & \omega = 0 \\ \frac{1}{j\omega}\left[1-e^{-j\omega T}\right] \mp \frac{0.5T}{2j} & \omega = \pm\frac{\pi}{T} \\ \frac{1}{j\omega}\left[1-e^{-j\omega T}\right] - \frac{0.5\pi T}{\pi^2-\omega^2T^2}\left[1+e^{-j\omega T}\right] & otherwise \end{cases}$$

$$= \begin{cases} T(1-\frac{1}{\pi}) & \omega = 0 \\ \pm\frac{2T}{j\pi} \mp \frac{T}{4j} & \omega = \pm\frac{\pi}{T} \\ \frac{1}{j\omega}\left[1-e^{-j\omega T}\right] - \frac{0.5\pi T}{\pi^2-\omega^2T^2}\left[1+e^{-j\omega T}\right] & otherwise \end{cases}$$

▌

Problem P5.3

From magnitude and phase spectra shown in Fig. P5.3, the individual CTFT's can be expressed as follows

Fig. P5.3(b): $$X_1(\omega) = \begin{cases} 1\times e^{j0.5\omega} & -W \le \omega \le W \\ 0 & \text{otherwise} \end{cases}$$

Fig. P5.3(c): $$X_2(\omega) = \begin{cases} 1\times e^{-j0.5\omega} & -W \le \omega \le 0 \\ 1\times e^{j0.5\omega} & 0 \le \omega \le W \\ 0 & \text{otherwise} \end{cases}$$

Fig. P5.3(d): $$X_3(\omega) = \begin{cases} 1\times e^{-j\pi/3} & -W \le \omega \le 0 \\ 1\times e^{j\pi/3} & 0 \le \omega \le W \\ 0 & \text{otherwise} \end{cases}$$

Using the CTFT synthesis Eq. (5.9), the function $x_1(t)$ is calculated as follows.

$$x_1(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(\omega)e^{j\omega t}d\omega=\frac{1}{2\pi}\int_{-W}^{W}e^{j0.5\omega}e^{j\omega t}d\omega=\frac{1}{2\pi}\int_{-W}^{W}e^{j(0.5+t)\omega}d\omega$$

$$=\frac{1}{2\pi}\left[\frac{e^{j(0.5+t)\omega}}{j(0.5+t)}\right]_{-W}^{W}=\frac{1}{2\pi}\left[\frac{e^{j(0.5+t)W}-e^{-j(0.5+t)W}}{j(0.5+t)}\right]=\frac{1}{2\pi}\frac{2j\sin\left[(0.5+t)W\right]}{j(0.5+t)}$$

$$=\frac{W}{\pi}\times\frac{\sin\left[(0.5+t)W\right]}{(0.5+t)W}=\frac{W}{\pi}\mathrm{sinc}\left[\tfrac{W}{\pi}(t+0.5)\right].$$

Using the CTFT synthesis Eq. (5.9), the function $x_2(t)$ is calculated as follows.

$$x_2(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(\omega)e^{j\omega t}d\omega=\frac{1}{2\pi}\int_{-W}^{0}e^{j(-0.5+t)\omega}d\omega+\frac{1}{2\pi}\int_{0}^{W}e^{j(0.5+t)\omega}d\omega$$

$$=\frac{1}{2\pi}\left[\frac{e^{j(-0.5+t)\omega}}{j(-0.5+t)}\right]_{-W}^{0}+\frac{1}{2\pi}\left[\frac{e^{j(0.5+t)\omega}}{j(0.5+t)}\right]_{0}^{W}=\frac{1}{2\pi}\left[\frac{1-e^{-j(-0.5+t)W}}{j(-0.5+t)}+\frac{e^{j(0.5+t)W}-1}{j(0.5+t)}\right]$$

$$=\frac{1}{2\pi}\left[\frac{1}{j(t^2-0.25)}+e^{j0.5W}\frac{2jt\sin(tW)-\cos(tW)}{j(t^2-0.25)}\right]$$

$$=\frac{1}{j2\pi(t^2-0.25)}\left[1+e^{j0.5W}\left(2jt\sin(tW)-\cos(tW)\right)\right].$$

Clearly, at ($t=\pm0.5$), $x_2(t)$ is undefined in the above expression. Computing directly, we obtain

At $t = 0.5$: $$x_2(0.5)=\tfrac{1}{2\pi}\int_{0}^{W}e^{j0.5\omega}e^{j0.5\omega}d\omega=\tfrac{1}{2\pi}\int_{0}^{W}e^{j\omega}d\omega=\tfrac{1}{2j\pi}\left[e^{j\omega}\right]_0^W=\tfrac{1}{2j\pi}\left[e^{jW}-1\right].$$

At $t = -0.5$: $$x_2(-0.5)=\tfrac{1}{2\pi}\int_{-W}^{0}e^{-j0.5\omega}e^{-j0.5\omega}d\omega=\tfrac{1}{2\pi}\int_{-W}^{0}e^{-j\omega}d\omega=\tfrac{1}{-2j\pi}\left[e^{-j\omega}\right]_{-W}^{0}=\tfrac{1}{2j\pi}\left[e^{jW}-1\right].$$

Using the CTFT synthesis Eq. (5.9), the function $x_3(t)$ is calculated as follows.

$$x_3(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(\omega)e^{j\omega t}d\omega=\frac{1}{2\pi}\int_{-W}^{0}e^{-j\pi/3}e^{j\omega t}d\omega+\frac{1}{2\pi}\int_{-W}^{0}e^{j\pi/3}e^{j\omega t}d\omega$$

$$=\frac{1}{2\pi}e^{-j\pi/3}\left[\frac{e^{j\omega t}}{jt}\right]_{-W}^{0}+\frac{1}{2\pi}e^{j\pi/3}\left[\frac{e^{j\omega t}}{jt}\right]_{0}^{W}=\frac{1}{2\pi}\left[e^{-j\pi/3}\frac{1-e^{-jWt}}{jt}+e^{j\pi/3}\frac{e^{jWt}-1}{jt}\right]$$

$$=\frac{1}{j2\pi t}\left[2j\sin(Wt+\pi/3)-2j\sin(\pi/3)\right]=\left[\frac{\sin(Wt+\pi/3)-\sin(\pi/3)}{\pi t}\right]$$

Clearly, at ($t = 0$), $x_3(t)$ is undefined in the above expression. Computing directly, we get

$$x_3(0) = \frac{1}{4\pi}\left[(1-j\sqrt{3})\int_{-W}^{0} d\omega + (1+j\sqrt{3})\int_{0}^{W} d\omega\right] = \frac{W}{4\pi}\left[1-j\sqrt{3}+1+j\sqrt{3}\right] = \frac{W}{2\pi}.$$

Although the functions $x_1(t)$, $x_2(t)$, and $x_3(t)$ have the same magnitude spectra, their phase spectra are different. As a result, the time domain representations of these functions are different.

For the special case $W = \pi$, the three functions are plotted in Fig. S5.3. Since $x_2(t)$ is a complex function, its magnitude is plotted in Fig. S5.3. The Matlab code is also included below. ∎

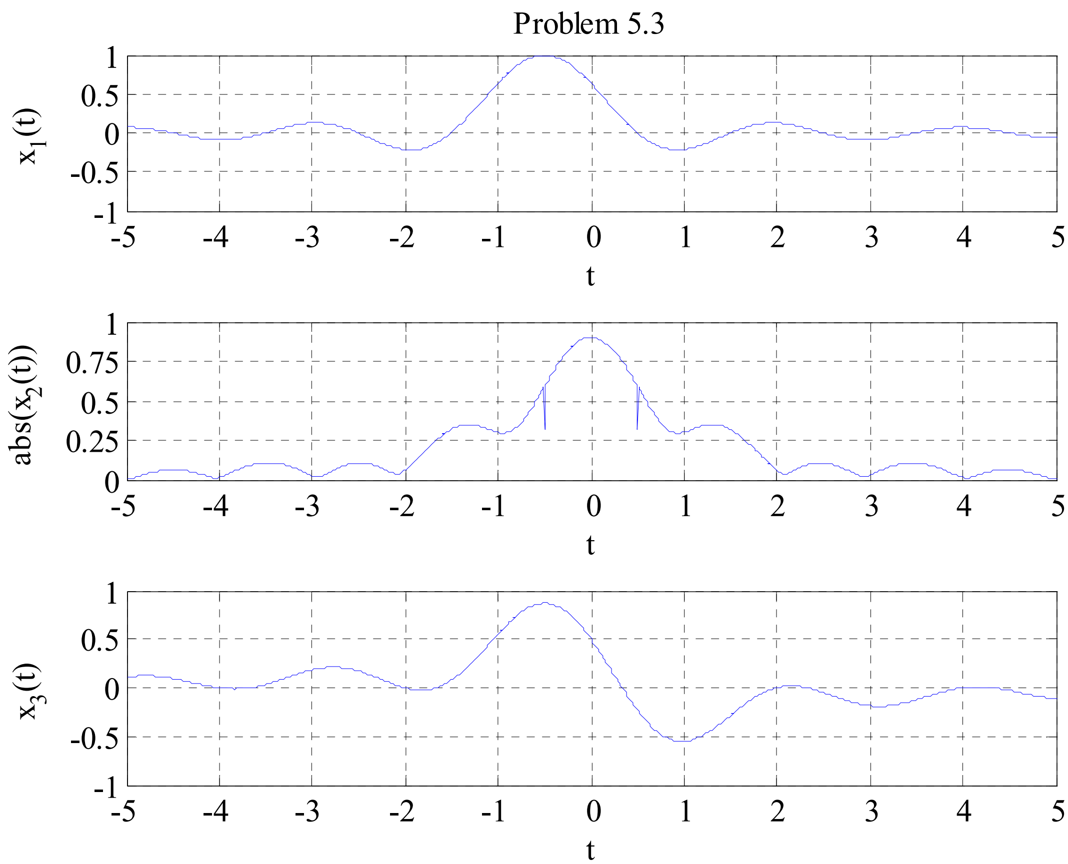


Fig. S5.3. Plots of functions in Problem P5.3.

```
% MATLAB code to plot the functions in Problem 5.3
del = 0.01;
t = -5:del:5;
W = pi ;

x1 = (W/pi)*sinc((W/pi)*(t+0.5)) ;
x2 = 1./(j*2*pi*(t.^2-0.25)).*(1+exp(j*0.5*W)*(2*j*t.*sin(t*W)-cos(t*W)));
x2(t==0.5) = 1./(j*2*pi)*(exp(j*W)-1);
x2(t==-0.5) = 1./(j*2*pi)*(exp(j*W)-1);
x3 = (sin(W*t+pi/3)-sin(pi/3))./(pi*t);
x3(t==0) = W/(2*pi) ;

subplot(3,1,1), plot(t, x1), grid on
title('Problem 5.3');
```

```
xlabel('t')                    % Label of X-axis
ylabel('x_1(t)')               % Label of Y-axis
%
subplot(3,1,2), plot(t, abs(x2)), grid on
xlabel('t')                    % Label of X-axis
ylabel('abs(x_2(t))')          % Label of Y-axis

subplot(3,1,3), plot(t, x3), grid
xlabel('t')                    % Label of X-axis
ylabel('x_3(t)')               % Label of Y-axis
```

Problem 5.4

(a) The partial fraction expansion is given by

$$X_1(\omega) = \frac{(1+j\omega)}{(2+j\omega)(3+j\omega)} \equiv \frac{-1}{(2+j\omega)} + \frac{2}{(3+j\omega)}$$

Calculating the inverse CTFT, we obtain

$$x_1(t) = -e^{-2t}u(t) + 2e^{-3t}u(t).$$

(b) The partial fraction expansion is given by

$$X_2(\omega) = \frac{1}{(1+j\omega)(2+j\omega)(3+j\omega)} \equiv \frac{0.5}{(1+j\omega)} + \frac{-1}{(2+j\omega)} + \frac{0.5}{(3+j\omega)}$$

Calculating the inverse CTFT, we obtain

$$x_2(t) = 0.5e^{-t}u(t) - e^{-2t}u(t) + 0.5e^{-3t}u(t).$$

(c) The partial fraction expansion is given by

$$X_3(\omega) = \frac{1}{(1+j\omega)(2+j\omega)^2(3+j\omega)} \equiv \frac{0.5}{(1+j\omega)} + \frac{0}{(2+j\omega)} + \frac{-1}{(2+j\omega)^2} + \frac{-0.5}{(3+j\omega)}$$

Calculating the inverse CTFT, we obtain

$$x_3(t) = 0.5e^{-t}u(t) - te^{-2t}u(t) + 0.5e^{-3t}u(t).$$

(d) The partial fraction expansion is given by

$$X_4(\omega) = \frac{1}{(1+j\omega)(2+2j\omega+(j\omega)^2)} \equiv \frac{1}{(1+j\omega)} - \frac{1+j\omega}{(2+2j\omega+(j\omega)^2)}$$

or,

$$X_4(\omega) = \frac{1}{(1+j\omega)} - \frac{1+j\omega}{1+(1+j\omega)^2}$$

Calculating the inverse CTFT, we obtain

$$x_4(t) = e^{-t}u(t) - e^{-t}\cos t\, u(t).$$

(e) The partial fraction expansion is given by

$$X_5(\omega) = \frac{1}{(1+j\omega)^2(2+2j\omega+(j\omega)^2)^2} \equiv \frac{1}{(1+j\omega)^2} - \frac{1.50}{(2+2j\omega+(j\omega)^2)} + \frac{0.25(4j\omega+(j\omega)^2)}{(2+2j\omega+(j\omega)^2)^2}$$

or, $$X_5(\omega) = \frac{1}{(1+j\omega)^2} - \frac{1.50}{1+(1+j\omega)^2} + \frac{0.25(4j\omega + (j\omega)^2)}{(1+(1+j\omega)^2)^2}$$

Calculating the inverse CTFT, we obtain

$$x_5(t) = te^{-t}u(t) - 1.50e^{-t}\sin t\, u(t) + \Im^{-1}\left\{\frac{0.25(4j\omega + (j\omega)^2)}{(1+(1+j\omega)^2)^2}\right\}.$$ ▮

Problem 5.5

Consider an arbitrary function $\phi(t)$, and assume that

$$p(t) = \int_{-\infty}^{\infty} e^{j\omega t}\,dt\,.$$

Now, consider the integral

$$\int_{-\infty}^{\infty} \phi(t)p(t-T)dt = \int_{-\infty}^{\infty} \phi(t)\left[\int_{-\infty}^{\infty} e^{j\omega(t-T)}d\omega\right]dt = \underbrace{\int_{-\infty}^{\infty} e^{-j\omega T}\left[\int_{-\infty}^{\infty} \phi(t)e^{j\omega t}dt\right]d\omega}_{\text{Changing the order of integration}} = \underbrace{\int_{-\infty}^{\infty} e^{-j\omega T}\Phi(-\omega)d\omega}_{\Phi(\omega)=\Im\{\phi(t)\}}$$

$$= \underbrace{\int_{\infty}^{-\infty} e^{j\omega' T}\Phi(\omega')(-d\omega')}_{\omega=-\omega',\ d\omega=-d\omega'} = \int_{-\infty}^{\infty} e^{j\omega' T}\Phi(\omega')d\omega'$$

$$= \int_{-\infty}^{\infty} \Phi(\omega)e^{j\omega T}d\omega \qquad \text{..(1)}$$

Note that the right hand side of Eq. (1) is the inverse CTFT of $\Phi(\omega)$ computed at $t = T$, i.e., $\phi(T)$. Hence,

$$\int_{-\infty}^{\infty} \phi(t)p(t-T)dt = \int_{-\infty}^{\infty} \Phi(\omega)e^{j\omega T}d\omega = 2\pi\phi(T)\,.$$

The above equation is valid for any arbitrary $\phi(t)$ if and only if $p(t) = 2\pi\delta(t)$ as can be seen from the following property of the impulse response

$$2\pi\int_{-\infty}^{\infty} \phi(t)\delta(t-T)dt = 2\pi\phi(T)\,.$$

In other words,

$$\int_{-\infty}^{\infty} e^{j\omega t}dt = 2\pi\delta(t)\,.$$

Interchanging the variables, t and ω, we obtain the required identity

$$\int_{-\infty}^{\infty} e^{j\omega t}d\omega = 2\pi\delta(\omega)\,.$$ ▮

Alternate Proof:

Note that the above result can be proved directly from the CTFT pair

$$1 \xleftrightarrow{\text{CTFT}} 2\pi\delta(\omega) .$$

By definition, $$2\pi\delta(\omega) = \int_{-\infty}^{\infty} 1 \times e^{-j\omega t} dt$$

Since $\delta(-\omega) = \delta(\omega)$, $$2\pi\delta(\omega) = 2\pi\delta(-\omega) = \int_{-\infty}^{\infty} e^{j\omega t} dt .$$

▮

Problem 5.6

Using Eq. (5.40), the CTFT for a real-valued even function $x(t)$ can be expressed as

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = 2\int_{0}^{\infty} x(t)\cos(\omega t) dt .$$

Since there is no complex value in the above equation, $X(\omega)$ is real valued, i.e., $\mathrm{Im}\{X(\omega)\} = 0$.

Also, $X(-\omega) = 2\int_{0}^{\infty} x(t)\cos(-\omega t) dt = 2\int_{0}^{\infty} x(t)\cos(\omega t) dt = X(\omega)$.

Therefore, $X(\omega)$ is also an even function with respect to ω. Since, $X(\omega)$ is real valued, $\mathrm{Re}\{X(\omega)\} = \mathrm{Re}\{X(-\omega)\}$. ▮

Problem 5.7

Using Eq. (5.40), the CTFT for a real-valued odd function $x(t)$ can be expressed as

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = -j2\int_{0}^{\infty} x(t)\sin(\omega t) dt .$$

Since $x(t)$ is real, the product $x(t)\sin(\omega t)$ is also real and so is the integral. Therefore, $X(\omega)$ is pure imaginary, i.e., $\mathrm{Re}\{X(\omega)\} = 0$.

Also, $X(-\omega) = 2\int_{0}^{\infty} x(t)\sin(-\omega t) dt = -2\int_{0}^{\infty} x(t)\sin(\omega t) dt = -X(\omega)$.

Therefore, $X(\omega)$ is also an odd function with respect to ω. Since, $X(\omega)$ is imaginary-valued, $\mathrm{Im}\{X(\omega)\} = -\mathrm{Im}\{X(-\omega)\}$. ▮

Problem 5.8

(a) Since $$X_1(-\omega) = \tfrac{5}{2+j(-\omega-5)} = \tfrac{5}{2-j(\omega+5)}$$

is not equal to $$X_1^*(\omega) = \tfrac{5}{2-j(\omega-5)} ,$$

$X_1(\omega)$ does not satisfy the Hermitian property. Its inverse CTFT $x_1(t)$ is not real valued and is complex. Nothing can be stated about the odd and even property of $x_1(t)$ from the Hermitian property.

(b) Since $$X_2(-\omega) = \cos\left(-2\omega + \tfrac{\pi}{6}\right) = \tfrac{\sqrt{3}}{2}\cos(2\omega) + \tfrac{1}{2}\sin(2\omega)$$

is not equal to $$X_2^*(\omega) = \cos\left(2\omega + \tfrac{\pi}{6}\right) = \tfrac{\sqrt{3}}{2}\cos(2\omega) - \tfrac{1}{2}\sin(2\omega),$$

$X_2(\omega)$ does not satisfy the Hermitian property. Its inverse CTFT $x_2(t)$ is not real valued and is complex. Nothing can be stated about the odd and even property of $x_2(t)$ from the Hermitian property.

(c) Since $$X_3(-\omega) = \tfrac{5\sin[4(-\omega-\pi)]}{(-\omega-\pi)} = \tfrac{5\sin[4(\omega+\pi)]}{(\omega+\pi)} = \tfrac{5\sin[4\omega]}{(\omega+\pi)}$$

is not equal to $$X_3^*(\omega) = \tfrac{5\sin[4(\omega-\pi)]}{(\omega-\pi)} = \tfrac{5\sin[4\omega]}{(\omega-\pi)},$$

$X_3(\omega)$ does not satisfy the Hermitian property. Its inverse CTFT $x_3(t)$ is not real valued and is complex. Nothing can be stated about the odd and even property of $x_3(t)$ from the Hermitian property.

(d) Since $X_4(-\omega) = (3+2j)\delta(-\omega-10) + (1-2j)\delta(-\omega+10) = (3+2j)\delta(\omega+10) + (1-2j)\delta(\omega-10)$

is not equal to $$X_4^*(\omega) = (3+2j)\delta(\omega-10) + (1-2j)\delta(\omega+10),$$

$X_4(\omega)$ does not satisfy the Hermitian property. Its inverse CTFT $x_4(t)$ is not real valued and is complex. Nothing can be stated about the odd and even property of $x_4(t)$ from the Hermitian property.

(e) Since $$X_5(-\omega) = \frac{1}{(1-j\omega)(3-j\omega)^2(5+\omega^2)}$$

is equal to $$X_5^*(\omega) = \frac{1}{(1-j\omega)(3-j\omega)^2(5+\omega^2)},$$

$X_4(\omega)$ satisfies the Hermitian property. Its inverse CTFT $x_4(t)$ is real valued.

Since $X_4(\omega)$ is complex (neither pure real-valued or pure imaginary), $x_4(t)$ is neither even nor odd with respect to t. ▌

Problem 5.9

(a) Applying the linearity property,

$$X_1(\omega) = \Im\left\{5 + 3\cos(10t) - 7e^{-2t}\sin(3t)u(t)\right\} = 5\Im\{1\} + 3\Im\{\cos(10t)\} - 7\Im\left\{e^{-2t}\sin(3t)u(t)\right\}.$$

By selecting the appropriate CTFT pairs from Table 5.2, we get

$$X_1(\omega) = 10\delta(\omega)\Im\{1\} + 3\pi\delta(\omega-10) + 3\pi\delta(\omega-10) - \frac{21}{(2+j\omega)^2 + 3^2}.$$

(b) Entry (8) of Table 5.2 provides the CTFT pair

$$\mathrm{sgn}(t) \xleftrightarrow{\text{CTFT}} \tfrac{2}{j\omega}.$$

Using the duality property, $$\tfrac{2}{jt} \xleftrightarrow{\text{CTFT}} 2\pi\,\mathrm{sgn}(-\omega),$$

or, $$\frac{1}{\pi t} \xleftrightarrow{\text{CTFT}} -j\,\text{sgn}(\omega).$$

(c) Entry (7) of Table 5.2 provides the CTFT pair

$$e^{-4|t|} \xleftrightarrow{\text{CTFT}} \frac{8}{4+j\omega}.$$

Using the time shifting property, $e^{-4|t-5|} \xleftrightarrow{\text{CTFT}} \frac{8}{4+j\omega} e^{-j5\omega}$.

Using the frequency differentiation property,

$$t^2\, e^{-4|t-5|} \xleftrightarrow{\text{CTFT}} (j)^2 \frac{d^2}{d\omega^2} \left\{ e^{-j5\omega} \frac{8}{4+j\omega} \right\}$$

or, $$t^2\, e^{-4|t-5|} \xleftrightarrow{\text{CTFT}} 200 e^{-j5\omega} \frac{1}{4+j\omega} + 16 e^{-j5\omega} \frac{1}{(4+j\omega)^3}.$$

(d) Entry (17) of Table 5.2 provides the CTFT pair

$$3\,\text{sinc}(3t) = 3\frac{\sin(3\pi t)}{3\pi t} \xleftrightarrow{\text{CTFT}} \text{rect}\left(\tfrac{\omega}{6\pi}\right)$$

and $$5\,\text{sinc}(5t) = 5\frac{\sin(5\pi t)}{5\pi t} \xleftrightarrow{\text{CTFT}} \text{rect}\left(\tfrac{\omega}{10\pi}\right)$$

Using the multiplication property

$$\pi^2 \times \frac{\sin(3\pi t)}{\pi t} \times \frac{\sin(5\pi t)}{\pi t} \xleftrightarrow{\text{CTFT}} \frac{\pi^2}{2\pi}\left[\text{rect}\left(\tfrac{\omega}{6\pi}\right) * \text{rect}\left(\tfrac{\omega}{10\pi}\right)\right]$$

or, $$\frac{\sin(3\pi t)\sin(5\pi t)}{t^2} \xleftrightarrow{\text{CTFT}} \frac{\pi}{2}\left[\text{rect}\left(\tfrac{\omega}{6\pi}\right) * \text{rect}\left(\tfrac{\omega}{10\pi}\right)\right],$$

or, $$5\frac{\sin(3\pi t)\sin(5\pi t)}{t^2} \xleftrightarrow{\text{CTFT}} \frac{5\pi}{2}\left[\text{rect}\left(\tfrac{\omega}{6\pi}\right) * \text{rect}\left(\tfrac{\omega}{10\pi}\right)\right],$$

where * is the convolution operation.

(e) Entry (17) of Table 5.2 provides the CTFT pair

$$3\,\text{sinc}(3t) = 3\frac{\sin(3\pi t)}{3\pi t} \xleftrightarrow{\text{CTFT}} \text{rect}\left(\tfrac{\omega}{6\pi}\right)$$

and $$4\,\text{sinc}(4t) = 4\frac{\sin(4\pi t)}{4\pi t} \xleftrightarrow{\text{CTFT}} \text{rect}\left(\tfrac{\omega}{8\pi}\right).$$

Using the time differentiation property,

$$\frac{1}{\pi}\frac{d}{dt}\frac{\sin(4\pi t)}{t} \xleftrightarrow{\text{CTFT}} (j\omega)\text{rect}\left(\tfrac{\omega}{8\pi}\right).$$

Using the convolution property

$$\pi^2 \times \frac{\sin(3\pi t)}{\pi t} * \frac{1}{\pi}\frac{d}{dt}\frac{\sin(4\pi t)}{t} \xleftrightarrow{\text{CTFT}} \frac{\pi^2}{2\pi}\left[\text{rect}\left(\tfrac{\omega}{6\pi}\right) \times j\omega\text{rect}\left(\tfrac{\omega}{8\pi}\right)\right]$$

or, $$\frac{\sin(3\pi t)}{t} * \frac{d}{dt}\frac{\sin(4\pi t)}{t} \xleftrightarrow{\text{CTFT}} \frac{\pi}{2}\left[\text{rect}\left(\tfrac{\omega}{6\pi}\right) \times j\omega\text{rect}\left(\tfrac{\omega}{8\pi}\right)\right],$$

or, $$4\frac{\sin(3\pi t)}{t} * \frac{d}{dt}\frac{\sin(4\pi t)}{t} \xleftrightarrow{\text{CTFT}} j2\pi\,\text{rect}\left(\tfrac{\omega}{6\pi}\right).$$ ▌

Problem 5.10

Using the linearity property,

$$
\begin{aligned}
X(\omega) &= \Im\left\{\left[\tfrac{6}{13}e^{-2t} - \tfrac{6}{13}\cos(3t) + \tfrac{4}{13}\sin(3t)\right]u(t)\right\} \\
&= \tfrac{6}{13}\Im\left\{e^{-2t}u(t)\right\} - \tfrac{6}{13}\Im\{\cos(3t)u(t)\} + \tfrac{4}{13}\Im\{\sin(3t)u(t)\} \\
&= \tfrac{6}{13}\tfrac{1}{2+j\omega} - \tfrac{6}{13}\left[\tfrac{\pi}{2}\delta(\omega-3) + \tfrac{\pi}{2}\delta(\omega+3)\right] - \tfrac{6}{13}\tfrac{j\omega}{9-\omega^2} + \tfrac{4}{13}\left[\tfrac{-j\pi}{2}\delta(\omega-3) + \tfrac{j\pi}{2}\delta(\omega+3)\right] + \tfrac{4}{13}\tfrac{3}{9-\omega^2} \\
&= \tfrac{6}{13}\left[\tfrac{1}{2+j\omega} - \tfrac{j\omega}{9-\omega^2} + \tfrac{2}{9-\omega^2}\right] - \tfrac{\pi}{26}\left[6\delta(\omega-3) + 6\delta(\omega+3) + 4j\delta(\omega-3) - 4j\delta(\omega+3)\right] \\
&= \tfrac{6}{13}\left[\tfrac{(9-\omega^2)+(2-j\omega)(2+j\omega)}{(2+j\omega)(9-\omega^2)}\right] - \tfrac{\pi}{26}\left[(6+j4)\delta(\omega-3) + (6-j4)\delta(\omega+3)\right] \\
&= \tfrac{6}{(2+j\omega)(9-\omega^2)} - \tfrac{\pi}{13}\left[(3+j2)\delta(\omega-3) + (3-j2)\delta(\omega+3)\right]
\end{aligned}
$$

which is the required result. ▮

Problem 5.11

From the definition of CTFT, $F\{x(at)\} = \int_{-\infty}^{\infty} x(at)e^{-j\omega t}\,dt$.

We consider two different cases ($a > 0$) and ($a < 0$)

Case 1: Assume $a > 0$. Substitute $r = at$ in the above expression. The upper and lower limits of integration stay the same and $dr = a\,dt$. The final result is

$$F\{x(at)\} = \int_{-\infty}^{\infty} x(r)e^{-j\left(\frac{\omega}{a}\right)r}\,\frac{dr}{a} = \frac{1}{a}\int_{-\infty}^{\infty} x(r)e^{-j\left(\frac{\omega}{a}\right)r}\,dr = \frac{1}{a}X\left(\frac{\omega}{a}\right).$$

Case 2: Assume $a < 0$. Substitute $r = at$ in the above expression. The upper limit of integration is $r \to -\infty$ and the lower limit of integration is $r \to \infty$, and $dr = a\,dt$. The final result is

$$F\{x(at)\} = \int_{-\infty}^{\infty} x(r)e^{-j\left(\frac{\omega}{a}\right)r}\,\frac{dr}{a} = \frac{1}{a}\int_{\infty}^{-\infty} x(r)e^{-j\left(\frac{\omega}{a}\right)r}\,dr = -\frac{1}{a}\int_{-\infty}^{\infty} x(r)e^{-j\left(\frac{\omega}{a}\right)r}\,dr = -\frac{1}{a}X\left(\frac{\omega}{a}\right).$$

Combining the two cases, yields $F\{x(at)\} = \frac{1}{|a|}X\left(\frac{\omega}{a}\right)$. ▮

Problem 5.12

Comparing with Fig. 5.9(a), we observe that

$$h(t) = x_1\left(\tfrac{t}{2}\right).$$

Using the scaling property, $H(\omega) = 2X_1(2\omega)$

or, $H(\omega) = \frac{2}{\omega^2}\left[2\omega\sin(4\omega) + \cos(2\omega) - 1\right]$,

which simplifies to $H(\omega) = 16\,\mathrm{sinc}\left(\frac{4\omega}{\pi}\right) - 4\,\mathrm{sinc}^2\left(\frac{\omega}{\pi}\right)$. ▮

Problem 5.13

Using the definition of CTFT, we obtain

$$F\{e^{j\omega_0 t}x(t)\} = \int_{-\infty}^{\infty} e^{j\omega_0 t}x(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} x(t)e^{-j(\omega-\omega_0)t}dt = X(\omega-\omega_0)\,.$$

Problem 5.14

Using the convolution property,

$$x(t) * u(t) \xleftrightarrow{\text{CTFT}} X(\omega)\left[2\pi\delta(\omega) + \tfrac{1}{j\omega}\right]\,,$$

or, $$\int_{-\infty}^{\infty} x(\tau)u(t-\tau)d\tau \xleftrightarrow{\text{CTFT}} 2\pi X(0)\delta(\omega) + \tfrac{X(\omega)}{j\omega}\,,$$

or, $$\int_{-\infty}^{\infty} x(\tau)u(-(\tau-t))d\tau \xleftrightarrow{\text{CTFT}} 2\pi X(0)\delta(\omega) + \tfrac{X(\omega)}{j\omega}\,,$$

or, $$\int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{\text{CTFT}} 2\pi X(0)\delta(\omega) + \tfrac{X(\omega)}{j\omega}\,.$$

Problem 5.15

(a) Using the time scaling property, $x(2t) \xleftrightarrow{\text{CTFT}} \frac{1}{2}X\left(\frac{\omega}{2}\right)$.

Using the frequency shifting property, $e^{-j5t}x(2t) \xleftrightarrow{\text{CTFT}} \frac{1}{2}X\left(\frac{\omega+5}{2}\right)$.

Substituting the value of $X(\omega)$, we obtain

$$\Im\left\{e^{-j5t}x(2t)\right\} = \tfrac{1}{2}\begin{cases} 1-\frac{|\omega+5|}{3} & |\omega+5| \le 3 \\ 0 & \text{elsewhere} \end{cases}$$

$$= \begin{cases} \frac{\omega+11}{12} & -11 \le \omega \le -5 \\ \frac{1-\omega}{12} & -5 \le \omega \le -1 \\ 0 & \text{elsewhere.} \end{cases}$$

(b) Using the frequency differentiation property,

$$(jt)^2 x(t) \xleftrightarrow{\text{CTFT}} \tfrac{d^2X}{d\omega^2}\,,$$

or, $$t^2 x(t) \xleftrightarrow{\text{CTFT}} -\tfrac{d^2X}{d\omega^2}\,.$$

The CTFT of $t^2 x(t)$ is given by

$$F\{t^2x(t)\} = -\tfrac{d^2}{d\omega^2}\left[\Delta\left(\tfrac{\omega}{3}\right)\right] = -\tfrac{d}{d\omega}\left[\text{rect}\left(\tfrac{\omega}{3}\right)\right] = -[\delta(\omega+3)-\delta(\omega-3)] = [\delta(\omega-3)-\delta(\omega+3)]\,.$$

(c) Express $(t+5)\frac{dx}{dt} = t\frac{dx}{dt} + 5\frac{dx}{dt}$.

Using the time differentiation property, the CTFT of $\frac{dx}{dt}$ is given by

$$\frac{dx}{dt} \xleftrightarrow{\text{CTFT}} j\omega X(\omega).$$

Applying the frequency differentiation property to the above CTFT pair, gives

$$t\frac{dx}{dt} \xleftrightarrow{\text{CTFT}} j\frac{d}{d\omega}\left[j\omega X(\omega)\right] = -X(\omega) - \omega\frac{dX}{d\omega}.$$

The CTFT of $(t+5)\frac{dx}{dt}$ is given by

$$\Im\left\{(t+5)\frac{dx}{dt}\right\} = -X(\omega) - \omega\frac{dX}{d\omega} + 5j\omega X(\omega).$$

Substituting the value of $X(\omega)$, we obtain

$$\Im\left\{(t+5)\frac{dx}{dt}\right\} = \begin{cases} j5\omega\left(1-\frac{\omega}{3}\right) - \left(1-\frac{2\omega}{3}\right) & 0 \le \omega \le 3 \\ j5\omega\left(1+\frac{\omega}{3}\right) - \left(1+\frac{2\omega}{3}\right) & -3 \le \omega \le 0 \\ 0 & \text{elsewhere.} \end{cases}$$

(d) Using the time multiplication property,

$$x(t)\cdot x(t) \xleftrightarrow{\text{CTFT}} \frac{1}{2\pi}\left[X(\omega) * X(\omega)\right],$$

which implies that

$$F\{x(t)\cdot x(t)\} = \frac{1}{2\pi}\left[\Delta\left(\frac{\omega}{3}\right) * \Delta\left(\frac{\omega}{3}\right)\right].$$

(e) Using the time convolution property,

$$x(t) * x(t) \xleftrightarrow{\text{CTFT}} X(\omega)\cdot X(\omega),$$

which reduces to

$$F\{x(t) * x(t)\} = \begin{cases} \left[1-\frac{|\omega|}{3}\right]^2 & |\omega| \le 3 \\ 0 & \text{elsewhere} \end{cases} = \begin{cases} 1+\frac{\omega^2}{9}-\frac{2|\omega|}{3} & |\omega| \le 3 \\ 0 & \text{elsewhere.} \end{cases}$$

(f) Using the time multiplication property,

$$x(t)\cdot\cos\omega_0 t \xleftrightarrow{\text{CTFT}} \frac{1}{2\pi}X(\omega) * \pi\delta(\omega-\omega_0) + \frac{1}{2\pi}X(\omega) * \pi\delta(\omega+\omega_0),$$

or, $$x(t)\cdot\cos\omega_0 t \xleftrightarrow{\text{CTFT}} \frac{1}{2}X(\omega-\omega_0) + \frac{1}{2}X(\omega+\omega_0)$$

Case I: For $\omega_0 = 3/2$, we obtain

$$x(t)\cdot\cos(3t/2) \xleftrightarrow{\text{CTFT}} \frac{1}{2}X\left(\omega-\frac{3}{2}\right) + \frac{1}{2}X\left(\omega+\frac{3}{2}\right).$$

The two replicas overlap over $(-3/2 \le \omega < 3/2)$, therefore,

$$F\{x(t)\cos(3t/2)\} = \begin{cases} \frac{1}{2} + \frac{\omega+3/2}{6} & -\frac{9}{2} \le \omega \le -\frac{3}{2} \\ 1 & -\frac{3}{2} \le \omega \le \frac{3}{2} \\ \frac{1}{2} + \frac{\omega-3/2}{6} & \frac{3}{2} \le \omega \le \frac{9}{2} \\ 0 & \text{elsewhere.} \end{cases}$$

Case II: For $\omega_0 = 3$, we obtain

$$x(t)\cdot\cos 3t \xleftrightarrow{\text{CTFT}} \tfrac{1}{2}X(\omega-3) + \tfrac{1}{2}X(\omega+3).$$

Since there is no overlap between the two shifted replicas,

$$F\{x(t)\cos 3t\} = \frac{1}{2}\begin{cases} 1 - \frac{|\omega+3|}{3} & |\omega+3| \le 3 \\ 1 - \frac{|\omega-3|}{3} & |\omega-3| \le 3 \\ 0 & \text{elsewhere.} \end{cases}$$

or,

$$F\{x(t)\cos 3t\} = \begin{cases} \frac{1}{2} - \frac{|\omega+3|}{6} & -6 \le \omega < 0 \\ \frac{1}{2} - \frac{|\omega-3|}{6} & 0 \le \omega < 6 \\ 0 & \text{elsewhere.} \end{cases}$$

Case III: For $\omega_0 = 6$, we obtain

$$x(t)\cdot\cos 6t \xleftrightarrow{\text{CTFT}} \tfrac{1}{2}X(\omega-6) + \tfrac{1}{2}X(\omega+6).$$

Since there is no overlap between the two shifted replicas,

$$F\{x(t)\cos 3t\} = \frac{1}{2}\begin{cases} 1 - \frac{|\omega+6|}{3} & |\omega+6| \le 3 \\ 1 - \frac{|\omega-6|}{3} & |\omega-6| \le 3 \\ 0 & \text{elsewhere.} \end{cases}$$

or,

$$F\{x(t)\cos 3t\} = \begin{cases} \frac{1}{2} - \frac{|\omega+6|}{6} & -9 \le \omega < -3 \\ \frac{1}{2} - \frac{|\omega-6|}{6} & 3 \le \omega < 9 \\ 0 & \text{elsewhere.} \end{cases}$$ ▌

Problem 5.16

(a) From Table 5.2,
$$\frac{5}{2+j\omega} \xleftrightarrow{\text{inverse CTFT}} 5e^{-2t}u(t).$$

Using the frequency shifting property,

$$\frac{5}{2+j(\omega-5)} \xleftrightarrow{\text{inverse CTFT}} \left[5e^{-2t}u(t)\right] \times e^{j5t}$$

implying that
$$x_1(t) = 5e^{(-2+j5)t}u(t).$$

(b) From Table 5.2,
$$\cos(2t) \xleftrightarrow{\text{CTFT}} \pi[\delta(\omega-2) + \delta(\omega+2)].$$

Using the duality property,

$$\pi[\delta(t-2)+\delta(t+2)] \xleftrightarrow{\text{CTFT}} 2\pi\cos(-2\omega) = 2\pi\cos(2\omega).$$

Using the frequency shifting property,

$$\pi[\delta(t-2)+\delta(t+2)]\, e^{-j\frac{\pi}{12}t} \xleftrightarrow{\text{CTFT}} 2\pi\cos\left(2(\omega+\tfrac{\pi}{12})\right),$$

implying that

$$x_2(t) = \tfrac{1}{2}[\delta(t-2)+\delta(t+2)]\, e^{-j\frac{\pi}{12}t} = \tfrac{1}{2}\left[\delta(t-2)e^{-j\frac{\pi}{12}t} + \delta(t+2)e^{-j\frac{\pi}{12}t}\right]$$

or, $\quad x_2(t) = \tfrac{1}{2}\left[\delta(t-2)e^{-j\frac{\pi}{6}} + \delta(t+2)e^{j\frac{\pi}{6}}\right] = \tfrac{1}{4}\left[(\sqrt{3}-j)\delta(t-2) + (\sqrt{3}+j)\delta(t+2)e^{j\frac{2\pi}{3}}\right].$

(c) From Table 5.2, $\quad \mathrm{rect}\left(\tfrac{t}{4}\right) \xleftrightarrow{\text{CTFT}} 4\mathrm{sinc}\left(\tfrac{4\omega}{2\pi}\right) = 4\,\frac{\sin(4\pi\omega/2\pi)}{(4\pi\omega/2\pi)} = 2\,\frac{\sin(2\omega)}{\omega}.$

Using the time scaling property,

$$\mathrm{rect}\left(\tfrac{t}{4\cdot 2}\right) \xleftrightarrow{\text{CTFT}} 2\cdot 2\,\frac{\sin(4\omega)}{2\omega} = 2\,\frac{\sin(4\omega)}{\omega}.$$

Using the frequency shifting property,

$$\mathrm{rect}\left(\tfrac{t}{8}\right) e^{j\pi t} \xleftrightarrow{\text{CTFT}} 2\,\frac{\sin(4(\omega-\pi))}{(\omega-\pi)},$$

implying that

$$x_3(t) = \tfrac{5}{2}\,\mathrm{rect}\left(\tfrac{t}{8}\right) e^{j\pi t}.$$

(d) Using the linearity property, we obtain

$$x_4(t) = \Im^{-1}\{(3+2j)\,\delta(\omega-10) + (1-2j)\delta(\omega+10)\}$$

$$= (3+2j)\,\Im^{-1}\{\delta(\omega-10)\} + (1-2j)\Im^{-1}\{\delta(\omega+10)\}$$

$$= \tfrac{(3+2j)}{2\pi}e^{j10t} + \tfrac{(1-2j)}{2\pi}e^{-j10t}.$$

Expanding the exponential terms using the Euler's formula, we obtain

$$x_4(t) = \tfrac{(3+2j)}{2\pi}(\cos 10t + j\sin 10t) + \tfrac{(1-2j)}{2\pi}(\cos 10t - j\sin 10t)$$

or, $\quad x_4(t) = \tfrac{2}{\pi}\cos 10t - \tfrac{(2-j)}{\pi}\sin 10t.$

(e) Taking the partial fraction expansion

$$X_5(\omega) = \frac{1}{(1+j\omega)(3+j\omega)^2(5+\omega^2)} \equiv \frac{A}{(1+j\omega)} + \frac{B}{(3+j\omega)} + \frac{C}{(3+j\omega)^2} + \frac{jD\omega+E}{(5+\omega^2)}$$

where $\quad A = 0.0625, \quad B = 0.25, \quad C = 0.125, \quad D = -0.3125, \quad$ and $\quad E = 0.6876.$

Calculating the inverse CTFT transform yields

$$x_5(t) \cong Ae^{-t}u(t) + Be^{-3t}u(t) + Cte^{-3t}u(t) + D\cos(\sqrt{5}t)u(t) + \left(E/\sqrt{5}\right)\sin(\sqrt{5}t)u(t).$$ ▌

Problem 5.17

(i) The functions are plotted in Fig. S5.17. The MATLAB code used to generate the plots is given below.

```
% MATLAB code to plot the functions in Problem 5.17
t = -10:0.01:10 ;
t4 = 0:0.001:10000 ;                    % for plotting x4(t)
t = t + eps;
t4 = t4 + eps;                          %for plotting x4(t)
%
x1 = exp(-2*abs(t));                    % a=2
x2 = exp(-2*t).*(cos(5*t)).*(t>=0);     % a=2, w=5
x3 = (t.^4).*exp(-2*t).*(t>=0);         % a=2
x4 = sin(log(t4));
x5 = 1./t ;
x6 = cos(pi./(2*t)) ;
x7 = exp(-(t.^2)./(2*3*3)) ;            % sigma=3
%
subplot(4,2,1), plot(t, x1), grid
xlabel('t')                             % Label of X-axis
ylabel('x1(t)')                         % Label of Y-axis
axis([-5  5  0 1.3])
%
subplot(4,2,3), plot(t, x2), grid
xlabel('t')                             % Label of X-axis
ylabel('x2(t)')                         % Label of Y-axis %
axis([-5  5  -0.5 1.3])
%
subplot(4,2,4), plot(t, x3), grid
xlabel('t')                             % Label of X-axis
ylabel('x3(t)')                         % Label of Y-axis %
axis([-4  4  0 0.4])
%
subplot(4,2,5), plot(t4, x4), grid
xlabel('t')                             % Label of X-axis
ylabel('x4(t)')                         % Label of Y-axis %
axis([0.001  10000  -1.3 1.3])
%
subplot(4,2,6), plot(t, x5), grid
xlabel('t')                             % Label of X-axis
ylabel('x5(t)')                         % Label of Y-axis
axis([-1  1  -100 100])
%
subplot(4,2,7), plot(t, x6), grid
xlabel('t')                             % Label of X-axis
ylabel('x6(t)')                         % Label of Y-axis %
axis([-5  5  -1.3 1.3])
%
subplot(4,2,8), plot(t, x7), grid
xlabel('t')                             % Label of X-axis
ylabel('x7(t)')                         % Label of Y-axis %
axis([-5  5  0 1.3])
```

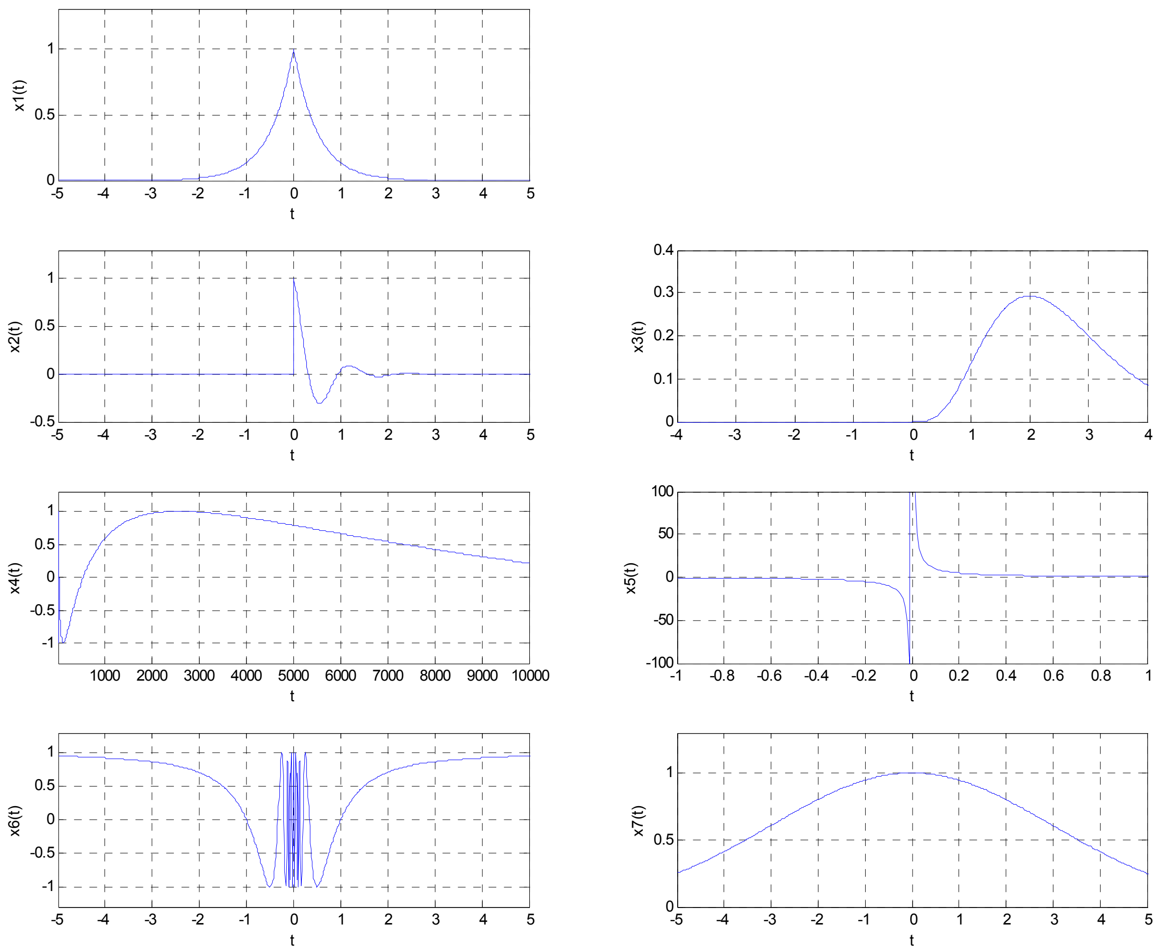


Fig. S5.17: Time-domain Waveforms for Problem 5.17

(ii)

(a) $$\int_{-\infty}^{\infty}|x1(t)|\,dt = \int_{-\infty}^{\infty}\left|e^{-a|t|}\right|dt = \int_{-\infty}^{0} e^{at}\,dt + \int_{0}^{\infty} e^{-at}\,dt = \tfrac{1}{a}e^{at}\Big|_{-\infty}^{0} + \tfrac{1}{(-a)}e^{-at}\Big|_{0}^{\infty} = \tfrac{1}{a}\left[1-0\right] - \tfrac{1}{a}\left[0-1\right] = \tfrac{2}{a} < \infty.$$

Since the condition in Eq. (5.59) is satisfied, the CTFT for $x1(t)$ exists.

(b) $$\int_{-\infty}^{\infty}|x2(t)|\,dt = \int_{-\infty}^{\infty}\left|e^{-at}\cos(\omega_0 t)u(t)\right|dt = \tfrac{1}{2}\int_{0}^{\infty}\left|e^{-at}\left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right]\right|dt \le \tfrac{1}{2}\left|\underbrace{\int_{0}^{\infty} e^{-(a-j\omega_0)t}\,dt}_{I}\right| + \tfrac{1}{2}\left|\underbrace{\int_{0}^{\infty} e^{-(a+j\omega_0)t}\,dt}_{II}\right|$$

Integral I is given by

$$I = \tfrac{1}{2}\int_{0}^{\infty} e^{-(a-j\omega_0)t}\,dt = \tfrac{1}{2}\left[\frac{e^{-(a-j\omega_0)t}}{-(a-j\omega_0)}\right]_{0}^{\infty} = \tfrac{1}{2}\left[0 - \frac{1}{-(a-j\omega_0)}\right] = \tfrac{1}{2}\left[\frac{1}{(a-j\omega_0)}\right],$$

while Integral II is given by

$$II = \frac{1}{2}\int_0^\infty e^{-(a+j\omega_0)t}dt = \frac{1}{2}\left[\frac{e^{-(a+j\omega_0)t}}{-(a+j\omega_0)}\right]_0^\infty = \frac{1}{2}\left[0 - \frac{1}{-(a+j\omega_0)}\right] = \frac{1}{2}\left[\frac{1}{(a+j\omega_0)}\right].$$

Therefore,

$$\int_{-\infty}^{\infty}|x_4(t)|dt = |I| + |II| = \frac{1}{2}\frac{1}{|(a-j\omega_0)|} + \frac{1}{2}\frac{1}{|(a+j\omega_0)|} = \frac{1}{\sqrt{a^2+\omega_0^2}} < \infty.$$

Since the condition in Eq. (5.59) is satisfied, the CTFT for $x2(t)$ exists.

(c) $$\int_{-\infty}^{\infty}|x3(t)|dt = \int_{-\infty}^{\infty}\left|t^4e^{-at}u(t)\right|dt = \int_0^\infty t^4e^{-at}dt = \left[t^4\frac{e^{-at}}{(-a)} + 4t^3\frac{e^{-at}}{(-a)^2} + 12t^2\frac{e^{-at}}{(-a)^3} + 24t\frac{e^{-at}}{(-a)^4} + 24\frac{e^{-at}}{(-a)^5}\right]_0^\infty$$

$$= \left[0+0+0+0+0\right] - \left[0+0+0+0+24\frac{1}{(-a)^5}\right] = \frac{24}{a^5} < \infty.$$

Since the condition in Eq. (5.59) is satisfied, the CTFT for $x_3(t)$ exists.

(d) The function

$$x4(t) = \sin(\ln(t))u(t)$$

is plotted in Fig. S5.17. Note that the horizontal axis uses a logarithmic scale. It is observed that the function oscillates like a sine wave (although not with a constant period). Therefore, the function has an infinite number of maximas and minimas. In addition,

$$\int_{-\infty}^{\infty}|x4(t)|dt = \int_0^\infty|\sin(\ln(t))|dt \longrightarrow \infty$$

Therefore, the CTFT for $x4(t)$ does not exist.

(e) $$\int_{-\infty}^{\infty}|x5(t)|dt = \int_{-\infty}^{\infty}\left|\tfrac{1}{t}\right|dt = 2\int_0^\infty \tfrac{1}{t}dt = 2\left[\ln(t)\right]_0^\infty \longrightarrow \infty$$

Since the condition in Eq. (5.59) is not satisfied, the CTFT for $x5(t)$ does not exist.

(f) $$\int_{-\infty}^{\infty}|x6(t)|dt = \int_{-\infty}^{\infty}\left|\cos\left(\tfrac{\pi/2}{t}\right)\right|dt \longrightarrow \infty.$$

Clearly the area enclosed by the cosine term would be infinite. Since the condition in Eq. (5.59) is not satisfied, the CTFT for $x6(t)$ does not exist. Also, it can be checked that $x6(t)$ has an infinite number of maximas and minimas, which is a second violation of the existence of the CTFT.

(g) $$\int_{-\infty}^{\infty}|x7(t)|dt = \int_{-\infty}^{\infty}\left|\exp(-\tfrac{t^2}{2\sigma^2})\right|dt = \int_{-\infty}^{\infty}\exp(-\tfrac{t^2}{2\sigma^2})dt = \sqrt{2\pi}\,\sigma < \infty.$$

In evaluating the above result, we used the fact that the area enclosed by a bell curve is 1. Mathematically, this implies that

$$\int_{\infty}^{\infty}\frac{1}{\sqrt{2\pi}\sigma}\exp\left[\frac{t^2}{2\sigma^2}\right]dt = \int_{\infty}^{\infty}\frac{1}{\sqrt{2\pi}\sigma}\exp\left[\frac{(t-m)^2}{2\sigma^2}\right]dt = 1,$$

where m is a constant. Since the condition in Eq. (5.59) is satisfied, the CTFT for $x_7(t)$ exists. ▌

Problem 5.18

(a) From the solution of Problem P4.11(a), the CTFS coefficients D_n of the rectangular pulse train are obtained as

$$D_n = \begin{cases} \frac{3}{2} & n = 0 \\ 0 & \text{even } n, n \neq 0 \\ \frac{3}{jn\pi} & \text{odd } n, \end{cases}$$

with fundamental frequency $\omega_0 = 1$ radians/s. Therefore, the CTFT is given by

$$X1(\omega) = 2\pi \sum_{n=-\infty}^{\infty} D_n \delta(\omega - n\omega_0) = 3\pi\delta(\omega) - j6 \sum_{\substack{n=-\infty \\ \text{odd } n}}^{\infty} \delta(\omega - n) .$$

(b) From the solution of Problem P4.11(b), the CTFS coefficients D_n of the rectangular pulse train are obtained as

$$D_n = \begin{cases} \frac{3}{4} & n = 0 \\ -\frac{0.5}{n\pi}\sin(0.5n\pi) & n \neq 0 \end{cases}$$

with fundamental frequency $\omega_0 = \pi/T$ radians/s. Therefore, the CTFT is given by

$$X2(\omega) = 2\pi \sum_{n=-\infty}^{\infty} D_n \delta(\omega - n\omega_0) = 1.5\pi\delta(\omega) - \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{n}\sin(0.5n\pi)\delta(\omega - \tfrac{n\pi}{T}) .$$

(c) From the solution of Problem P4.11(c), the CTFS coefficients D_n of the rectangular pulse train are obtained as

$$D_n = \begin{cases} \frac{1}{2}, & n = 0 \\ \frac{1}{j2n\pi}, & n \neq 0. \end{cases}$$

with fundamental frequency $\omega_0 = 2\pi/T$ radians/s. Therefore, the CTFT is given by

$$X3(\omega) = 2\pi \sum_{n=-\infty}^{\infty} D_n \delta(\omega - n\omega_0) = \pi\delta(\omega) - j \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{n}\delta(\omega - \tfrac{2n\pi}{T}) .$$

(d) From the solution of Problem P4.11(d), the CTFS coefficients D_n of the rectangular pulse train are obtained as

$$D_n = \begin{cases} \frac{1}{2}, & n = 0 \\ 0, & \text{even } n, n \neq 0 \\ \frac{2}{(n\pi)^2} & \text{odd } n, n \neq 0. \end{cases}$$

with fundamental frequency $\omega_0 = \pi/T$ radians/s. Therefore, the CTFT is given by

$$X4(\omega)=2\pi\sum_{n=-\infty}^{\infty}D_n\delta(\omega-n\omega_0)=\pi\delta(\omega)+\frac{4}{\pi}\sum_{\substack{n=-\infty\\ n\neq 0\\ \text{odd } n}}^{\infty}\frac{1}{n^2}\delta(\omega-\tfrac{n\pi}{T}).$$

(e) From the solution of Problem P4.11(e), the CTFS coefficients D_n of the rectangular pulse train are obtained as

$$D_n=\begin{cases}\frac{1}{2}\left(1-\frac{1}{\pi}\right) & n=0\\ \mp j\left(\frac{1}{\pi}-\frac{1}{8}\right) & n=\pm 1\\ \frac{1}{2\pi(n^2-1)} & 0\neq n=even\\ \frac{1}{jn\pi} & \pm 1\neq n=odd\end{cases}=\begin{cases}0.3408 & n=0\\ \mp j0.1933 & n=\pm 1\\ \frac{0.1592}{n^2-1} & 0\neq n=even\\ \frac{-j0.3183}{n} & \pm 1\neq n=odd\end{cases}$$

with fundamental frequency $\omega_0=\pi/T$ radians/s. Therefore, the CTFT is given by

$$X5(\omega)=2\pi\sum_{n=-\infty}^{\infty}D_n\delta(\omega-n\omega_0)=0.6816$$

$$=0.6816\pi\delta(\omega)+j0.3866\pi\delta(\omega+\tfrac{\pi}{T})-j0.3866\pi\delta(\omega-\tfrac{\pi}{T})+\sum_{\substack{n=-\infty\\ n\neq 0,1\\ \text{even } n}}^{\infty}\frac{1}{n^2-1}\delta(\omega-\tfrac{n\pi}{T})-j2\sum_{\substack{n=-\infty\\ n\neq 0,1\\ \text{odd } n}}^{\infty}\frac{1}{n}\delta(\omega-\tfrac{n\pi}{T}).$$

▮

Problem 5.19

(a) From the solution of Problem P5.2(a), the CTFT of the aperiodic signal is given by

$$X_1(\omega)=3\pi e^{-j\omega\pi/2}\mathrm{sinc}(\omega/2)=\begin{cases}3\pi & \omega=0\\ \frac{3}{j\omega}(1-e^{-j\omega\pi}) & \omega\neq 0.\end{cases}$$

The signal shown in Fig. P4.6(a) is a periodic signal with a fundamental period $T_0=2\pi$ with one period matching the function shown in Fig. P5.2(a). The fundamental frequency $\omega_0=1$ and the exponential CTFS coefficients are given by

$$D_n=\frac{1}{T_0}X_1(\omega)\Big|_{\omega=n\omega_0}=\frac{1}{2\pi}\begin{cases}3\pi & n=0\\ \frac{3}{jn\omega_0}(1-e^{-jn\omega_0\pi}) & n\neq 0\end{cases}=\begin{cases}\frac{3}{2} & n=0\\ \frac{3}{j2\pi n}(1-e^{-jn\pi}) & n\neq 0\end{cases}$$

which simplifies to

$$D_n==\begin{cases}\frac{3}{2} & n=0\\ \frac{3}{jn\pi} & \text{odd } n\\ 0 & \text{even } n, n\neq 0.\end{cases}$$

(b) From the solution of Problem P5.2(b), the CTFT of the aperiodic signal is given by

$$X_2(\omega)=0.5T\mathrm{sinc}\left(\frac{0.5\omega T}{\pi}\right)+Te^{-j\omega T}\mathrm{sinc}\left(\frac{0.5\omega T}{\pi}\right)=\begin{cases}1.5T & \omega=0\\ \frac{\sin(\omega T/2)}{\omega}\left(1+2e^{-j\omega T}\right) & \omega\neq 0.\end{cases}$$

The signal shown in Fig. P4.6(b) is a periodic signal with a fundamental period $T_0 = 2T$ with one period matching the function shown in Fig. P5.2(b). The fundamental frequency $\omega_0 = \pi/T$ and the exponential CTFS coefficients are given by

$$D_n = \frac{1}{T_0} X_2(\omega)\Big|_{\omega=n\omega_0} = \frac{1}{2T}\begin{cases} 1.5T & \omega = 0 \\ \frac{\sin(n\omega_0 T/2)}{n\omega_0}\left(1+2e^{-jn\omega_0 T}\right) & \omega \neq 0 \end{cases} = \begin{cases} \frac{3}{4} & n = 0 \\ \frac{\sin(n\pi/2)}{2n\pi}(1-e^{-jn\pi}) & n \neq 0 \end{cases}$$

which simplifies to

$$D_n = = \begin{cases} \frac{3}{4} & n = 0 \\ -\frac{1}{2n\pi} & n = 4k+1 \\ \frac{1}{2n\pi} & n = 4k+3 \\ 0 & \text{even } n, n \neq 0. \end{cases}$$

(c) From the solution of Problem P5.2(c), the CTFT of the aperiodic signal is given by

$$X_3(\omega) = \begin{cases} 0.5T & \omega = 0 \\ \frac{1}{j\omega} + \frac{1}{\omega^2 T}\left(1-e^{-j\omega T}\right) & \omega \neq 0. \end{cases}$$

The signal shown in Fig. P4.6(c) is a periodic signal with a fundamental period $T_0 = T$ with one period matching the function shown in Fig. P5.2(c). The fundamental frequency $\omega_0 = 2\pi/T$ and the exponential CTFS coefficients are given by

$$D_n = \frac{1}{T_0} X_3(\omega)\Big|_{\omega=n\omega_0} = \frac{1}{T}\begin{cases} 0.5T & \omega = 0 \\ \frac{1}{jn\omega_0} - \frac{1}{n^2\omega_0^2 T}\left(1-e^{-jn\omega_0 T}\right) & \omega \neq 0 \end{cases} = \begin{cases} 0.5 & n = 0 \\ \frac{1}{j2n\pi} - \frac{1}{4n^2\pi^2}\left(1-e^{-j2n\pi}\right) & n \neq 0 \end{cases}$$

which simplifies to

$$D_n = \begin{cases} 0.5 & n = 0 \\ \frac{1}{j2n\pi} & n \neq 0. \end{cases}$$

(d) From the solution of Problem P5.2(d), the CTFT of the aperiodic signal is given by

$$X_3(\omega) = T\text{sinc}^2\left(\tfrac{0.5\omega T}{\pi}\right) = \begin{cases} T & \omega = 0 \\ T\text{sinc}^2\left(\tfrac{0.5\omega T}{\pi}\right) & \omega \neq 0. \end{cases}$$

The signal shown in Fig. P4.6(d) is a periodic signal with a fundamental period $T_0 = 2T$ with one period matching the function shown in Fig. P5.2(d). The fundamental frequency $\omega_0 = \pi/T$ and the exponential CTFS coefficients are given by

$$D_n = \frac{1}{T_0} X_3(\omega)\Big|_{\omega=n\omega_0} = \frac{1}{2T}\begin{cases} T & \omega = 0 \\ T\text{sinc}^2\left(\tfrac{0.5n\omega_0 T}{\pi}\right) & \omega \neq 0. \end{cases} = \begin{cases} 0.5 & n = 0 \\ 0.5\text{sinc}^2(0.5n) & n \neq 0 \end{cases}$$

which simplifies to

$$D_n = \begin{cases} 0.5 & n = 0 \\ 0 & \text{even } n, n \neq 0 \\ \frac{2}{(n\pi)^2} & \text{odd } n, n \neq 0. \end{cases}$$

(e) From the solution of Problem P5.2(e), the CTFT of the aperiodic signal is obtained as

$$X_5(\omega)=\begin{cases} T(1-\frac{1}{\pi}) & \omega=0\\ \pm\frac{2T}{j\pi}\mp\frac{T}{4j} & \omega=\pm\frac{\pi}{T}\\ \frac{1}{j\omega}\left[1-e^{-j\omega T}\right]-\frac{0.5\pi T}{\pi^2-\omega^2T^2}\left[1+e^{-j\omega T}\right] & otherwise\end{cases}$$

The signal shown in Fig. P4.6(e) is a periodic signal with a fundamental period $T_0 = 2T$ and whose one period is identical to the function shown in Fig. P5.2(e). Therefore, the fundamental frequency $\omega_0 = \pi/T$ and the exponential CTFS coefficients are given by

$$D_n=\frac{1}{T_0}X(n\omega_0)=\frac{1}{2T}X(n\omega_0)$$

$$=\frac{1}{2T}\times\begin{cases} T(1-\frac{1}{\pi}) & n=0\\ \pm\frac{1}{j\omega_0}\left[1-e^{\mp j\omega_0 T}\right]\mp\frac{0.5T}{2j} & n=\pm1\\ \frac{1}{jn\omega_0}\left[1-e^{-jn\omega_0 T}\right]-\frac{0.5\pi T}{\pi^2-(n\omega_0)^2T^2}\left[1+e^{-jn\omega_0 T}\right] & otherwise\end{cases}$$

$$=\frac{1}{2T}\times\begin{cases} T(1-\frac{1}{\pi}) & n=0\\ \pm\frac{T}{j\pi}\left[1-e^{\mp j\pi}\right]\mp\frac{0.5T}{2j} & n=\pm1\\ \frac{T}{jn\pi}\left[1-e^{-jn\pi}\right]-\frac{0.5\pi T}{\pi^2-n^2\pi^2}\left[1+e^{-jn\pi}\right] & otherwise\end{cases}$$

$$=\begin{cases} \frac{1}{2}-\frac{1}{2\pi} & n=0\\ \pm\frac{1}{j\pi}\mp\frac{0.5T}{2j} & n=\pm1\\ \frac{1}{j2n\pi}\left[1-(-1)^n\right]+\frac{1}{4\pi(n^2-1)}\left[1+(-1)^n\right] & otherwise\end{cases}$$

$$=\begin{cases} \frac{1}{2}\left(1-\frac{1}{\pi}\right) & n=0\\ \mp j\left(\frac{1}{\pi}-\frac{1}{8}\right) & n=\pm1\\ \frac{1}{2\pi(n^2-1)} & 0\neq n=even\\ \frac{1}{jn\pi} & \pm1\neq n=odd\end{cases}$$

▌

Problem 5.20

(a) Calculating the CTFT of both sides and applying the time differentiation property, yields

$$(j\omega)^3Y(\omega)+6(j\omega)^2Y(\omega)+11(j\omega)Y(\omega)+6Y(\omega)=X(\omega),$$

or,
$$\left((j\omega)^3+6(j\omega)^2+11(j\omega)+6\right)Y(\omega)=X(\omega),$$

or,
$$H(\omega)=\frac{Y(\omega)}{X(\omega)}=\frac{1}{(j\omega)^3+6(j\omega)^2+11(j\omega)+6}.$$

The impulse response $h(t)$ can be obtained by calculating the inverse CTFT of $H(\omega)$, which can be expressed as

$$H(\omega)=\frac{1}{(1+j\omega)(2+j\omega)(3+j\omega)}\equiv\frac{0.5}{(1+j\omega)}+\frac{-1}{(2+j\omega)}+\frac{0.5}{(3+j\omega)}$$

Calculating the inverse CTFT, we obtain

$$h(t) = 0.5e^{-t}u(t) - e^{-2t}u(t) + 0.5e^{-3t}u(t)\,.$$

(b) Calculating the CTFT of both sides and applying the time differential property, yields

$$(j\omega)^2 Y(\omega) + 3(j\omega)Y(\omega) + 2Y(\omega) = X(\omega),$$

or,

$$\left((j\omega)^2 + 3(j\omega) + 2\right)Y(\omega) = X(\omega),$$

or,

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1}{(j\omega)^2 + 3(j\omega) + 2}\,.$$

The impulse response $h(t)$ can be obtained by calculating the inverse CTFT of $H(\omega)$, which can be expressed as

$$H(\omega) = \frac{1}{(j\omega)^2 + 3(j\omega) + 2} \equiv \frac{1}{(1 + j\omega)} - \frac{1}{(2 + j\omega)}$$

Calculating the inverse CTFT, we obtain

$$h(t) = e^{-t}u(t) - e^{-2t}u(t)\,.$$

(c) Calculating the CTFT of both sides and applying the time differentiation property, yields

$$(j\omega)^2 Y(\omega) + 2(j\omega)Y(\omega) + Y(\omega) = X(\omega),$$

or,

$$\left((j\omega)^2 + 1(j\omega) + 1\right)Y(\omega) = X(\omega),$$

or,

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1}{(j\omega)^2 + 2(j\omega) + 1}\,.$$

The impulse response $h(t)$ can be obtained by calculating the inverse CTFT of $H(\omega)$, which can be expressed as

$$H(\omega) = \frac{1}{(1 + j\omega)^2}$$

Calculating the inverse CTFT, we obtain

$$h(t) = te^{-t}u(t)\,.$$

(d) Calculating the CTFT of both sides and applying the time differentiation property, yields

$$(j\omega)^2 Y(\omega) + 6(j\omega)Y(\omega) + 8Y(\omega) = (j\omega)X(\omega) + 4X(\omega),$$

or,

$$\left((j\omega)^2 + 6(j\omega) + 8\right)Y(\omega) = \left((j\omega) + 4\right)X(\omega),$$

or,

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{(j\omega) + 4}{(j\omega)^2 + 6(j\omega) + 8} = \frac{1}{2 + j\omega}\,.$$

The impulse response $h(t)$ can be obtained by calculating the inverse CTFT of $H(\omega)$, which is given by

$$h(t)=e^{-2t}u(t).$$

(e) Calculating the CTFT of both sides and applying the time differential property, yields

$$(j\omega)^3 Y(\omega)+8(j\omega)^2 Y(\omega)+19(j\omega)Y(\omega)+12Y(\omega)=X(\omega),$$

or,
$$\left((j\omega)^3+8(j\omega)^2+19(j\omega)+12\right)Y(\omega)=X(\omega),$$

or,
$$H(\omega)=\frac{Y(\omega)}{X(\omega)}=\frac{1}{(j\omega)^3+8(j\omega)^2+19(j\omega)+12}.$$

The impulse response $h(t)$ can be obtained by calculating the inverse CTFT of $H(\omega)$, which can be expressed as

$$H(\omega)=\frac{1}{(j\omega)^3+8(j\omega)^2+19(j\omega)+12}\equiv\frac{1/6}{(1+j\omega)}+\frac{-1/2}{(3+j\omega)}+\frac{1/3}{(4+j\omega)}$$

Calculating the inverse CTFT, we obtain

$$h(t)=\tfrac{1}{6}e^{-t}u(t)-\tfrac{1}{2}e^{-3t}u(t)+\tfrac{1}{3}e^{-4t}u(t).$$

▌

Problem 5.21

(a) Calculating the CTFT of the input and output signals, we obtain

$$X(\omega)=\frac{1}{2+j\omega} \quad \text{and} \quad Y(\omega)=\frac{5}{2+j\omega}.$$

The transfer function is given by

$$H(\omega)=\frac{Y(\omega)}{X(\omega)}=5.$$

Calculating the inverse CTFT, the impulse response is given by

$$h(t)=5\,\delta(t).$$

In the frequency domain, the input-output relationship is given by

$$Y(\omega)=5X(\omega)$$

or, in the time domain,
$$y(t)=5x(t).$$

(b) Calculating the CTFT of the input and output signals, we obtain

$$X(\omega)=\frac{1}{2+j\omega} \quad \text{and} \quad Y(\omega)=\frac{3}{2+j\omega}e^{-j4\omega}.$$

The transfer function is given by

$$H(\omega)=\frac{Y(\omega)}{X(\omega)}=3e^{-j4\omega}.$$

Calculating the inverse CTFT, the impulse response is given by

$$h(t)=3\,\delta(t-4).$$

In the frequency domain, the input-output relationship is given by

$$Y(\omega) = 3e^{-j4\omega} X(\omega)$$

or, in the time domain, $$y(t) = 3x(t-4).$$

(c) Calculating the CTFT of the input and output signals, we obtain

$$X(\omega) = \frac{1}{2+j\omega} \quad \text{and} \quad Y(\omega) = \frac{6}{(2+j\omega)^4}.$$

The transfer function is given by

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{6}{(2+j\omega)^3}.$$

Taking the inverse CTFT, the impulse response is given by

$$h(t) = 3t^2 e^{-2t} u(t).$$

In the frequency domain, the input-output relationship is given by

$$(2+j\omega)^3 Y(\omega) = 6X(\omega),$$

or, $$\left[(8 + 12(j\omega) + 6(j\omega)^2 + (j\omega)^3\right] Y(\omega) = 6X(\omega).$$

Calculating the inverse CTFT, the resulting differential equation is obtained as

$$\frac{d^3y}{dt^3} + 6\frac{d^2y}{dt^2} + 12\frac{dy}{dt} + 8y(t) = 6x(t).$$

(d) Calculating the CTFT of the input and output signals, we get

$$X(\omega) = \frac{1}{2+j\omega} \quad \text{and} \quad Y(\omega) = \frac{1}{(1+j\omega)} + \frac{1}{(3+j\omega)}.$$

The transfer function is given by

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{(4+2j\omega)(2+j\omega)}{(1+j\omega)(3+j\omega)} = \frac{2(2+j\omega)^2}{(1+j\omega)(3+j\omega)}.$$

Using partial fraction expansion, the transfer function is given by

$$H(\omega) = \frac{2(2+j\omega)^2}{(1+j\omega)(3+j\omega)} \equiv 2 + \frac{1}{(1+j\omega)} - \frac{1}{(3+j\omega)}$$

Taking the inverse CTFT, the impulse response is given by

$$h(t) = 2\delta(t) + e^{-t}u(t) + e^{-3t}u(t).$$

In the frequency domain, the input-output relationship is given by

$$(1+j\omega)(3+j\omega)Y(\omega) = 2(2+j\omega)^2 X(\omega),$$

or, $$\left[(3 + 4(j\omega) + (j\omega)^2\right] Y(\omega) = 2\left[(4 + 4(j\omega) + (j\omega)^2\right] X(\omega).$$

Calculating the inverse CTFT, the resulting differential equation is obtained as

$$\frac{d^2y}{dt^2}+4\frac{dy}{dt}+3y(t)=2\frac{d^2x}{dt^2}+8\frac{dx}{dt}+8x(t).$$

▌

Problem 5.22

The transfer function of the RC series circuit is given by

$$H(\omega)=\frac{1/(j\omega C)}{R+1/(j\omega C)}=\frac{1}{(1+j\omega CR)}=\frac{1}{CR}\times\frac{1}{1/(CR)+j\omega}.$$

Calculating the inverse CTFT, the impulse response of the system is obtained as

$$h(t)=\frac{1}{CR}\times e^{-t/(CR)}.$$

The output response is calculated by convolving the input signal with the impulse response $h(t)$ in the time domain. Figure S5.22 shows the convolution using graphical approach.

We consider the three cases separately:

Case I ($t <= -T/2$): Since there is no overlap between $h(\tau)$ and $v(t-\tau)$, the output $y(t)$ is 0.

Case II ($-T/2 < t <= T/2$): The output $y(t)$ is given by

$$y(t)=\frac{1}{CR}\int_0^{t+T/2} e^{-\tau/(CR)}dt=\frac{1}{CR}(-CR)\,e^{-\tau/(CR)}\Big|_0^{t+T/2}=1-e^{-T/(2CR)}e^{-t/(CR)}.$$

Case III ($t > T/2$): The output $y(t)$ is given by

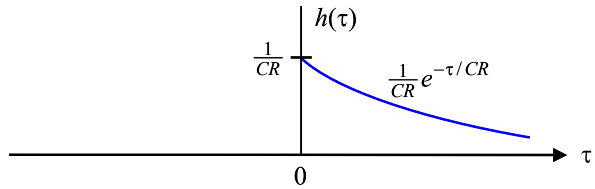
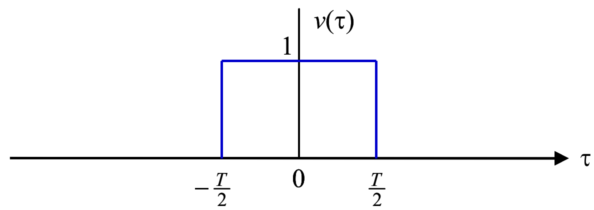
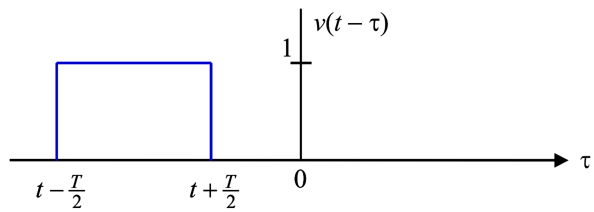
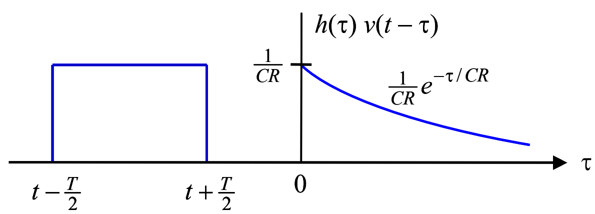
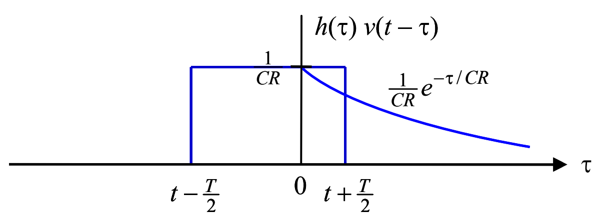
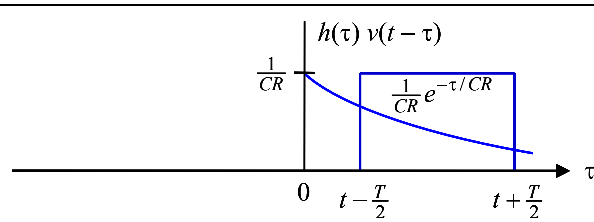
$$y(t)=\frac{1}{CR}\int_{t-T/2}^{t+T/2} e^{-\tau/(CR)}dt=\frac{1}{CR}(-CR)e^{-\tau/(CR)}\Big|_{t-T/2}^{t+T/2}=\left[e^{T/(2CR)}-e^{-T/(2CR)}\right]e^{-t/(CR)}.$$

Combining the three cases, we obtain

$$y(t)=\begin{cases}0 & t\le -T/2\\ 1-e^{-T/(2CR)}e^{-t/(CR)} & -T/2<t\le T/2\\ \left[e^{T/(2CR)}-e^{-T/(2CR)}\right]e^{-t/(CR)} & t>T/2\end{cases}$$

For $T=2,\ R=1M\Omega,\ C=1\mu F,$ $\quad y(t)=\begin{cases}0 & t\le -1\\ 1-e^{-(t+1)} & -1<t\le 1\\ (\underbrace{e-1/e}_{\approx 2.3504})\,e^{-t} & t>1\end{cases}$

The above output $y(t)$ is plotted in the last row of Fig. S5.22. The output response matches our expectation from our circuit theory knowledge. At $t=-T/2$, the input voltage becomes 1 volt, and the capacitor starts charging resulting in an increase in the output voltage. The increase continues until $t=T/2$ at which the input becomes zero. After $t=T/2$, the capacitor starts discharging resulting in an exponential decrease of the output voltage. The output voltage becomes zero at $t=\infty$. ▌

<table>
<tr><td>$h(\tau)$</td><td>
</td></tr>
<tr><td>$v(\tau)$</td><td>
</td></tr>
<tr><td>$v(t-\tau)$</td><td>
</td></tr>
<tr><td>Case I:
$(t < -T/2)$</td><td>
</td></tr>
<tr><td>Case II:
$(-T/2 < t < T/2)$</td><td>
</td></tr>
<tr><td>Case III:
$(t > T/2)$</td><td>
</td></tr>
<tr><td>$y(t)$

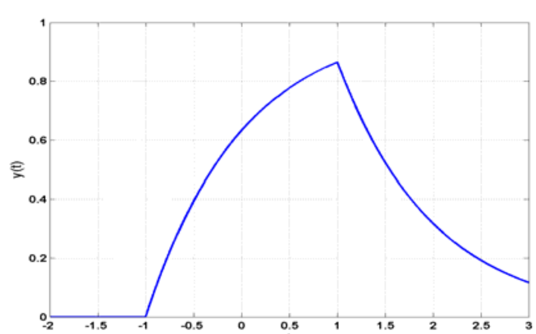
$(for\ T = 2,$
$R = 1\ \mathrm{M}\Omega,\ C = 1\ \mu\mathrm{F})$</td><td>
</td></tr>
</table>

Figure S5.22: Convolution of the input signal $v(t)$ with the impulse response $h(t)$ in Problem 5.22.

Program: The MATLAB code to plot y(t) in Problem 5.22

```
t = -2:0.001:3;
%  P5.20(a)
y = 0*(t<=-1)+(1-exp(-t-1)).*(t>-1).*(t<=1)+(exp(1)-exp(-1))*exp(-t).*(t>1);
plot(t,y); grid on;
xlabel('t');
ylabel('y(t)');
```

Problem 5.23

(i) As determined in Problem 5.22, the transfer function of the RC series circuit is given by

$$H(\omega) = \frac{1}{CR} \times \frac{1}{1/(CR) + j\omega}.$$

Calculating the CTFT of the input, we obtain

$$X_1(\omega) = \pi\left[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right].$$

Using the modulation property, the CTFT of the output signal is obtained as

$$Y(\omega) = H(\omega)X_1(\omega) = \frac{\pi}{CR} \times \frac{1}{1/(CR) + j\omega}\delta(\omega - \omega_0) + \frac{\pi}{CR} \times \frac{1}{1/(CR) + j\omega}\delta(\omega + \omega_0)$$

which reduces to

$$Y(\omega) = \frac{\pi}{CR} \times \frac{1}{1/(CR) + j\omega_0}\delta(\omega - \omega_0) + \frac{\pi}{CR} \times \frac{1}{1/(CR) - j\omega_0}\delta(\omega + \omega_0),$$

or,
$$Y(\omega) = \frac{\pi}{CR} \times \frac{1/(CR) - j\omega_0}{1/(CR)^2 + \omega_0^2}\delta(\omega - \omega_0) + \frac{\pi}{CR} \times \frac{1/(CR) + j\omega_0}{1/(CR)^2 + \omega_0^2}\delta(\omega + \omega_0),$$

or,
$$\begin{aligned} Y(\omega) = {} & \frac{\pi}{CR} \times \frac{1/(CR)}{1/(CR)^2 + \omega_0^2}\left[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right] \\ & + \frac{\pi}{CR} \times \frac{-j\omega_0}{1/(CR)^2 + \omega_0^2}\left[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)\right]. \end{aligned}$$

Calculating the inverse CTFT, we obtain

$$Y(\omega) = \frac{1}{CR} \times \frac{1/(CR)}{1/(CR)^2 + \omega_0^2}\cos(\omega_0 t) + \frac{1}{CR} \times \frac{-j\omega_0}{1/(CR)^2 + \omega_0^2}\, j\sin(\omega_0 t),$$

or,

$$y(t) = \frac{1}{1 + C^2R^2\omega_0^2}\left[\cos(\omega_0 t) + \omega_0 CR\sin(\omega_0 t)\right],$$

which can be expressed as

$$y(t)=\frac{1}{\sqrt{1+C^2R^2\omega_0^2}}\cos\left[\omega_0 t-\tan^{-1}(\omega_0 CR)\right].$$

(b) As determined in Problem 5.22, the transfer function of the RC series circuit is given by

$$H(\omega)=\frac{1}{CR}\times\frac{1}{1/(CR)+j\omega}.$$

Calculating the CTFT of the input, we obtain

$$X_1(\omega)=j\pi\left[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)\right].$$

Using the modulation property, the CTFT of the output signal is given by

$$Y(\omega)=H(\omega)X_1(\omega)=\frac{\pi}{jCR}\times\frac{1}{1/(CR)+j\omega}\delta(\omega-\omega_0)-\frac{\pi}{jCR}\times\frac{1}{1/(CR)+j\omega}\delta(\omega+\omega_0)$$

which reduces to

$$Y(\omega)=\frac{\pi}{jCR}\times\frac{1}{1/(CR)+j\omega_0}\delta(\omega-\omega_0)-\frac{\pi}{jCR}\times\frac{1}{1/(CR)-j\omega_0}\delta(\omega+\omega_0),$$

or,
$$Y(\omega)=\frac{\pi}{jCR}\times\frac{1/(CR)-j\omega_0}{1/(CR)^2+\omega_0^2}\delta(\omega-\omega_0)-\frac{\pi}{jCR}\times\frac{1/(CR)+j\omega_0}{1/(CR)^2+\omega_0^2}\delta(\omega+\omega_0),$$

or,
$$\begin{aligned}Y(\omega)=&\frac{\pi}{jCR}\times\frac{1/(CR)}{1/(CR)^2+\omega_0^2}\left[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)\right]\\&+\frac{\pi}{jCR}\times\frac{-j\omega_0}{1/(CR)^2+\omega_0^2}\left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right].\end{aligned}$$

Calculating the inverse CTFT, we obtain

$$Y(\omega)=\frac{1}{CR}\times\frac{1/(CR)}{1/(CR)^2+\omega_0^2}\sin(\omega_0 t)-\frac{1}{CR}\times\frac{\omega_0}{1/(CR)^2+\omega_0^2}\cos(\omega_0 t),$$

or,

$$y(t)=\frac{1}{1+C^2R^2\omega_0^2}\left[\sin(\omega_0 t)-\omega_0 CR\cos(\omega_0 t)\right],$$

which can be expressed as

$$y(t)=\frac{1}{\sqrt{1+C^2R^2\omega_0^2}}\sin\left[\omega_0 t-\tan^{-1}(\omega_0 CR)\right].$$

Problem 5.24

The transfer functions obtained in Problem 5.20 are as follows:

(a) $H(\omega)=\dfrac{1}{(j\omega)^3+6(j\omega)^2+11(j\omega)+6}$

(b) $H(\omega)=\dfrac{1}{(j\omega)^2+3(j\omega)+2}$

(c) $H(\omega)=\dfrac{1}{(j\omega)^2+2(j\omega)+1}$

(d) $H(\omega)=\dfrac{(j\omega)+4}{(j\omega)^2+6(j\omega)+8}=\dfrac{1}{2+j\omega}$.

(e) $H(\omega)=\dfrac{1}{(j\omega)^3+8(j\omega)^2+19(j\omega)+12}$

The MATLAB code to plot the magnitude and phase spectra is given below:

```
w = -5:0.001:5;
%
%  P5.20(a)
H = 1./((j*w).^3+6*(j*w).^2+11*(j*w)+6);
subplot(5,2,1)
plot(w,abs(H)); grid on;
xlabel('\omega (radians/s)');
ylabel('P5.20(a): |H_1(\omega)|'); axis tight
subplot(5,2,2)
plot(w,angle(H)); grid on;
xlabel('\omega (radians/s)');
ylabel('P5.20(a): <H_1(\omega)'); axis tight
%
%  P5.20(b)
H = 1./((j*w).^2+3*(j*w)+2);
subplot(5,2,3)
plot(w,abs(H)); grid on;
xlabel('\omega (radians/s)');
ylabel('P5.20(b): |H_2(\omega)|'); axis tight
subplot(5,2,4)
plot(w,angle(H)); grid on;
xlabel('\omega (radians/s)');
ylabel('P5.20(b): <H_2(\omega)');
axis tight
%
% P5.20(c)
H = 1./((j*w).^2+2*(j*w)+1);
subplot(5,2,5)
plot(w,abs(H)); grid on;
xlabel('\omega (radians/s)');
ylabel('P5.20(c): |H_3(\omega)|'); axis tight
subplot(5,2,6)
plot(w,angle(H)); grid on;
xlabel('\omega (radians/s)');
ylabel('P5.20(c): <H_3(\omega)'); axis tight
%
% P5.20(d)
H = (4+j*w)./((j*w).^2+6*(j*w)+8);
subplot(5,2,7)
plot(w,abs(H)); grid on;
xlabel('\omega (radians/s)');
ylabel('P5.20(d): |H_4(\omega)|'); axis tight
```

```
subplot(5,2,8)
plot(w,angle(H)); grid on;
xlabel('\omega (radians/s)');
ylabel('P5.20(d): <H_4(\omega)'); axis tight
%
% P5.20(e)
H = 1./((j*w).^3+8*(j*w).^2+19*(j*w)+12);
subplot(5,2,9)
plot(w,abs(H)); grid on;
xlabel('\omega (radians/s)');
ylabel('P5.20(e): |H_5(\omega)|');
axis tight
subplot(5,2,10)
plot(w,angle(H)); grid on;
xlabel('\omega (radians/s)');
ylabel('P5.20(e): <H_5(\omega)'); axis tight
```

The spectra are plotted in Fig. S5.24. ▌

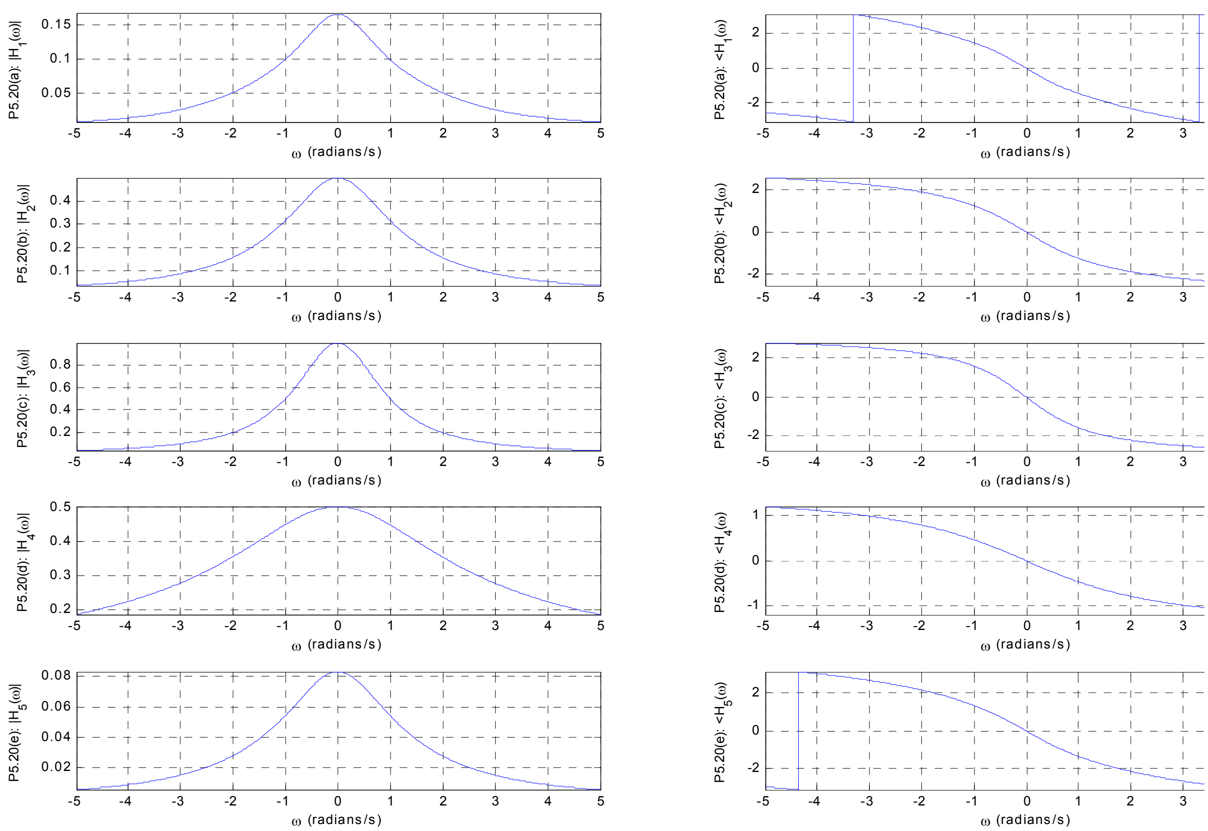


Fig. S5.24: Gain and Phase responses for Problem 5.24.

Problem 5.25

The transfer functions obtained in Problem 5.21 are as follows:

(a) $H(\omega) = \dfrac{Y(\omega)}{X(\omega)} = 5$.

(b) $H(\omega) = \dfrac{Y(\omega)}{X(\omega)} = 3e^{-j4\omega}$.

(c) $H(\omega) = \dfrac{Y(\omega)}{X(\omega)} = \dfrac{6}{(2+j\omega)^3}$.

(d) $H(\omega) = \dfrac{Y(\omega)}{X(\omega)} = \dfrac{(4+2j\omega)(2+j\omega)}{(1+j\omega)(3+j\omega)} = \dfrac{2(2+j\omega)^2}{(1+j\omega)(3+j\omega)}$.

The MATLAB code to plot the magnitude and phase spectra is given below:

```
w = -5:0.001:5;
%
%  P5.21(a)
H = 5*ones(size(w));
subplot(4,2,1)
plot(w,abs(H)); grid on;
xlabel('\omega (radians/s)');
ylabel('P5.21(a): |H_1(\omega)|'); axis([-5 5 0 5.25]);
subplot(4,2,2)
plot(w,angle(H)); grid on;
xlabel('\omega (radians/s)');
ylabel('P5.21(a): <H_1(\omega)'); axis tight
%
%  P5.21(b)
H = 3*exp(-j*4*w);
subplot(4,2,3)
plot(w,abs(H)); grid on;
xlabel('\omega (radians/s)');
ylabel('P5.21(b): |H_2(\omega)|'); axis([-5 5 0 3.25]);
subplot(4,2,4)
plot(w,angle(H)); grid on;
xlabel('\omega (radians/s)');
ylabel('P5.21(b): <H_2(\omega)'); axis tight
%
%  P5.21(c)
H = 6./((2+j*w).^3);
subplot(4,2,5)
plot(w,abs(H)); grid on;
xlabel('\omega (radians/s)');
ylabel('P5.21(c): |H_3(\omega)|'); axis tight
subplot(4,2,6)
plot(w,angle(H)); grid on;
xlabel('\omega (radians/s)');
ylabel('P5.21(c): <H_3(\omega)'); axis tight
%
%  P5.21(d)
H = 2*(2+j*w).^2./((1+j*w).*(3+j*w));
subplot(4,2,7)
plot(w,abs(H)); grid on;
```

```
xlabel('\omega (radians/s)');
ylabel('P5.21(d): |H_4(\omega)|'); axis tight
subplot(4,2,8)
plot(w,angle(H)); grid on;
xlabel('\omega (radians/s)');
ylabel('P5.21(d): <H_4(\omega)'); axis tight
```

The gain and phase spectra are plotted in Fig. S5.25.

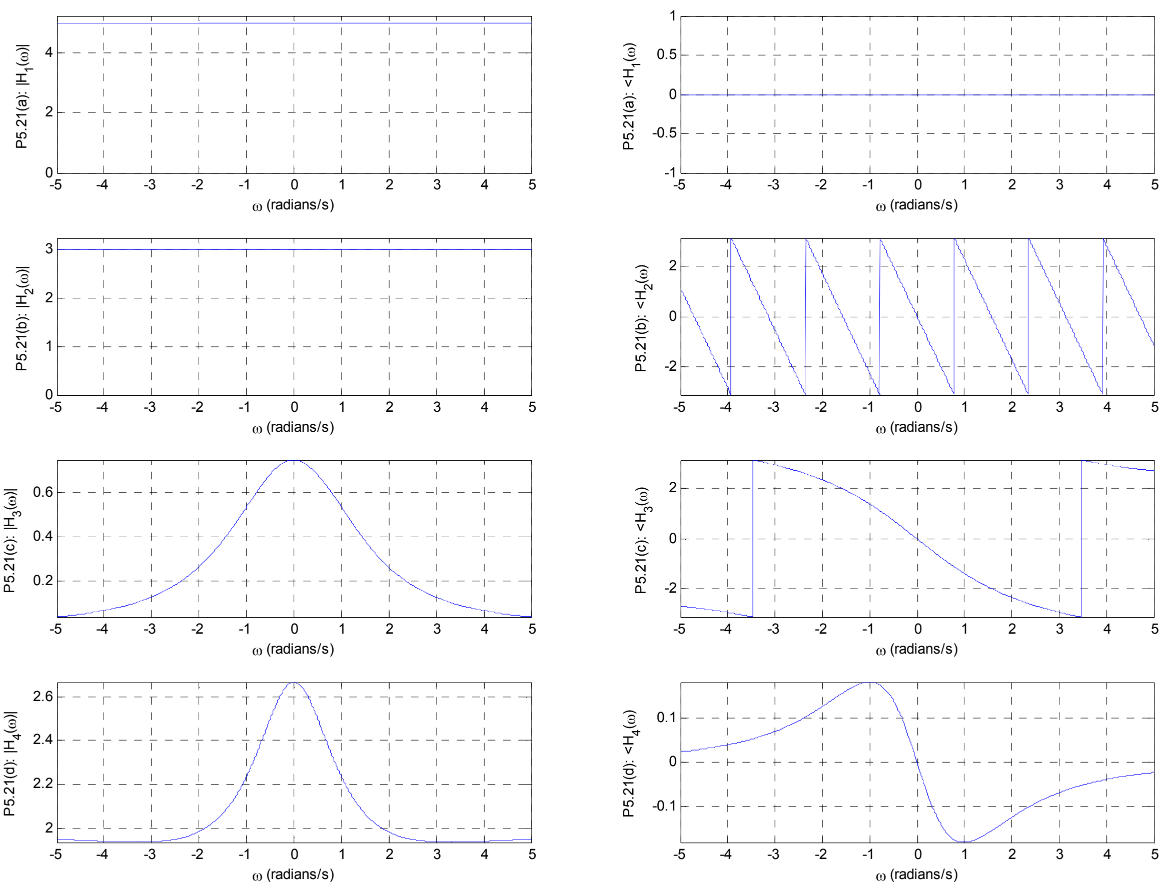


Fig. S5.25: Gain and phase responses for Problem 5.25.

Problem 5.26

Case II with input signal given by $\sin(\omega_0 t)$:

Let us assume that the transfer function $H(\omega_0) = A(\omega_0) + jB(\omega_0)$ at the fundamental frequency ω_0 of the sine wave. From the Hermitian property, we note that the real component of $A(\omega_0)$ of $H(\omega_0)$ is even, while the imaginary component $B(\omega_0)$ of $H(\omega_0)$ is odd. Therefore,

$$H(-\omega_0) = A(\omega_0) - jB(\omega_0).$$

Using the modulation property, the CTFT of the output of the system is given by

$$Y(\omega) = H(\omega) \times j\pi\left[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)\right],$$

or, $$Y(\omega) = j\pi\left[H(-\omega_0)\delta(\omega+\omega_0) - H(\omega_0)\delta(\omega-\omega_0)\right].$$

Expressing $H(\omega_0)$ and $H(-\omega_0)$ in terms of their real and imaginary components, we obtain

$$Y(\omega) = j\pi A(\omega_0)\left[\delta(\omega+\omega_0) - \delta(\omega-\omega_0)\right] + \pi B(\omega_0)\left[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)\right].$$

Calculating the inverse CTFT, the output $y(t)$ is obtained as

$$\begin{aligned} y(t) &= A(\omega_0)\sin(\omega_0 t) + B(\omega_0)\cos(\omega_0 t) \\ &= \sqrt{(A(\omega_0))^2 + (B(\omega_0))^2}\,\sin\left\{\omega_0 t + \tan^{-1}(B(\omega_0)/A(\omega_0))\right\}. \\ &= \left|H(\omega_0)\right|\sin\left\{\omega_0 t + < H(\omega_0)\right\} \end{aligned}$$

where

$$\left|H(\omega_0)\right| = \sqrt{(A(\omega_0))^2 + (B(\omega_0))^2} \quad \text{and} \quad < H(\omega_0) = \tan^{-1}(B(\omega_0)/A(\omega_0)).$$

Case I with input signal given by cos($\omega_0 t$):

Using the modulation property, the CTFT of the output of the system is given by

$$Y(\omega) = H(\omega)\times\pi\left[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)\right],$$

or, $$Y(\omega) = \pi\left[H(-\omega_0)\delta(\omega+\omega_0) + H(\omega_0)\delta(\omega-\omega_0)\right].$$

Expressing $H(\omega_0)$ and $H(-\omega_0)$ in terms of their real and imaginary components, we get

$$Y(\omega) = \pi A(\omega_0)\left[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)\right] - j\pi B(\omega_0)\left[\delta(\omega+\omega_0) - \delta(\omega-\omega_0)\right].$$

Taking the inverse CTFT, the output $y(t)$ is given by

$$\begin{aligned} y(t) &= A(\omega_0)\cos(\omega_0 t) - B(\omega_0)\sin(\omega_0 t) \\ &= \sqrt{(A(\omega_0))^2 + (B(\omega_0))^2}\,\cos\left\{\omega_0 t + \tan^{-1}(B(\omega_0)/A(\omega_0))\right\}. \\ &= \left|H(\omega_0)\right|\cos\left\{\omega_0 t + < H(\omega_0)\right\} \end{aligned}$$

where

$$\left|H(\omega_0)\right| = \sqrt{(A(\omega_0))^2 + (B(\omega_0))^2} \quad \text{and} \quad < H(\omega_0) = \tan^{-1}(B(\omega_0)/A(\omega_0)).$$

The above result states that the output of an LTI system with real-valued impulse response and a sinusoidal signal at the input is another sinusoidal signal of the same fundamental frequency as the input. Only the magnitude and phase of the sinusoidal signal are modified. ▌

Problem 5.27

(i) With $x_1(t) = \cos(\omega_0 t)$, the output of the RC circuit shown in Fig. P5.22 is given by

$$y(t) = \left|H(\omega_0)\right|\cos\left(\omega_0 t + < H(\omega_0)\right)$$

where $$H(\omega) = \frac{1}{1 + j\omega CR}.$$

Substituting the value of the magnitude and phase of $H(\omega_0)$ at the fundamental frequency $\omega = \omega_0$, the output is given by

$$y(t) = \frac{1}{\sqrt{1+(\omega_0 CR)^2}} \cos\left(\omega_0 t - \tan^{-1}(\omega CR)\right).$$

(ii) As in part (i), the output of the RC circuit shown in Fig. P5.22 for $x_2(t) = \sin(\omega_0 t)$, is given by

$$y(t) = |H(\omega_0)| \sin\left(\omega_0 t + < H(\omega_0)\right)$$

where
$$H(\omega) = \frac{1}{1 + j\omega CR}.$$

Substituting the value of the magnitude and phase of $H(\omega_0)$ at the fundamental frequency $\omega = \omega_0$, the output is given by

$$y(t) = \frac{1}{\sqrt{1+(\omega_0 CR)^2}} \sin\left(\omega_0 t - \tan^{-1}(\omega CR)\right).$$

The answers obtained above match with those obtained in Problem 5.23. ▮

Problem 5.28

(i) Based on the solution of Problem 5.26,

$$\sin(3t) \longrightarrow |H(3)| \sin(3t + < H(3)).$$

For $R = 1\text{M}\Omega$, and $C = 0.1\mu\text{F}$, the transfer function is given by

$$H(\omega) = \frac{1}{1 + j0.1\omega}.$$

At $\omega = 3$ radians/s, the magnitude and phase of the RC circuit is given by

$$|H(3)| = \frac{1}{\sqrt{1+0.3^2}} = 0.9578 \quad \text{and} \quad < H(3) = -16.70^0 .$$

The output is given by

$$y_1(t) = 0.9578 \sin\left(3t - 16.70^0\right).$$

(ii) Based on the solution of Problem 5.26,

$$\cos(3t) \longrightarrow |H(3)| \cos(3t + < H(3))$$

and

$$\sin\left(6t + 30^0\right) \longrightarrow |H(6)| \sin\left(6t + 30^0 + < H(6)\right)$$

At $\omega = 3$ radians/s, the magnitude and phase of the RC circuit is given by

$$|H(3)| = \frac{1}{\sqrt{1+0.3^2}} = 0.9578 \quad \text{and} \quad < H(3) = -16.70^0 .$$

Similarly, at $\omega = 6$ radians/s, the magnitude and phase of the RC circuit is given by

$$|H(6)| = \frac{1}{\sqrt{1+0.6^2}} = 0.8575 \quad \text{and} \quad < H(6) = -30.96^0 .$$

Using the linearity property, the output is given by

$$y_1(t) = 0.9578\cos\left(3t - 16.70^0\right) - 4.2875\sin\left(6t - 0.96^0\right).$$

(iii) Based on the solution of Problem 5.26,

$$\cos(2t) \longrightarrow |H(2)|\cos(2t + < H(2))$$

and

$$\sin(2000t) \longrightarrow |H(2000)|\sin(2000t + < H(2000))$$

At ω = 2 radians/s, the magnitude and phase of the RC circuit is given by

$$|H(2)| = \frac{1}{\sqrt{1+0.3^2}} = 0.9806 \quad \text{and} \quad < H(3) = -11.31^0 .$$

Similarly, at ω = 2000 radians/s, the magnitude and phase of the RC circuit is given by

$$|H(2000)| = \frac{1}{\sqrt{1+2000^2}} = 0.0050 \quad \text{and} \quad < H(2000) = -89.71^0 .$$

Using the linearity property, the output is given by

$$y_1(t) = 0.9806\cos\left(2t - 11.3^{'0}\right) + 0.0050\sin\left(2000t - 89.71^0\right).$$

(iv) Based on Eq. (5.75)

$$\exp(j3t) \longrightarrow |H(3)|\exp(j3t + j < H(3))$$

and

$$\exp(j2000t) \longrightarrow |H(2000)|\exp(j3t + j < H(2000))$$

At ω = 3 radians/s, the magnitude and phase of the RC circuit is given by

$$|H(3)| = \frac{1}{\sqrt{1+0.3^2}} = 0.9578 \quad \text{and} \quad < H(3) = -16.70^0 .$$

Similarly, at ω = 2000 radians/s, the magnitude and phase of the RC circuit is given by

$$|H(2000)| = \frac{1}{\sqrt{1+2000^2}} = 0.0050 \quad \text{and} \quad < H(2000) = -89.71^0 .$$

Using the linearity property, the output is given by

$$y_1(t) = 0.9578\exp\left(j3t - j16.70^0\right) + 0.0050\exp\left(j2000t - 89.71^0\right). \qquad \blacksquare$$

Problem 5.29

(a) In Example 3.6, it was shown that

$$y(t) = e^{-t}u(t) * e^{-2t}u(t) = \left[e^{-t} - e^{-2t}\right] u(t).$$

(b) From Table 5.2, the CTFT of $x(t)$ and $h(t)$ are obtained as

$$X(\omega) = \tfrac{1}{1+j\omega}, \quad \text{and} \quad H(\omega) = \tfrac{1}{2+j\omega}.$$

The CTFT of the output is then given by

$$Y(\omega) = H(\omega)X(\omega) = \tfrac{1}{(1+j\omega)}\tfrac{1}{(2+j\omega)} = \tfrac{1}{1+j\omega} - \tfrac{1}{2+j\omega}.$$

Calculating the inverse CTFT results in the output signal

$$y(t) = \left[e^{-t} - e^{-2t}\right] u(t).$$

(c) As $H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1}{2+j\omega}$, the Fourier-domain input-output relationship can be expressed as

$$j\omega Y(\omega) + 2Y(\omega) = X(\omega).$$

Calculating the inverse CTFT of both sides results in the following differential equation

$$\frac{dy}{dt} + 2y(t) = x(t).$$

The output can be obtained by solving the differential equation with input $x(t) = e^{-t}u(t)$

and zero initial conditions $y(0^-) = 0$.

Zero-input Response: Due to zero initial condition, the zero-input response is $y_{zi}(t) = 0$.

Zero-state Response: The characteristics equation is given by $(s + 2) = 0$ resulting in a single pole at $s = -2$. The homogenous component of the zero-state response is given by

$$y_{zs}^{h}(t) = Ae^{-2t}.$$

Since the input $x(t) = \exp(-t)\, u(t)$, the particular solution is of the form

$$y_{zs}^{p}(t) = Ke^{-t} \text{ for } t \geq 0.$$

Inserting the particular solution in the differential equation results in $K = 1$. Therefore,

$$y_{zs}^{p}(t) = e^{-t}u(t).$$

The overall zero-state response is, therefore, given by

$$y_{zs}(t) = Ae^{-2t} + e^{-t}$$

for $t \geq 0$. To determine the value of A, we insert the initial condition $y(0^-) = 0$ giving

$$A + 1 = 0 \Rightarrow A = -1$$

or, $A = -1$. The zero state response is given by

$$y_{zs}(t) = \left(e^{-t} - e^{-2t}\right)u(t).$$

Total Response: By adding the zero-input and zero-state responses, the overall output is given by

$$y(t) = \underbrace{y_{zi}(t)}_{=0} + y_{zs}(t) = \left(e^{-t} - e^{-2t}\right)u(t).$$

It is observed that Methods (a) – (c) yield the same result. ▌

Problem 5.30

(a) For part (a), we assume that $T = 1$ in $H_1(\omega)$ and $H_2(\omega)$. From the solution of Problem P5.19(a), the CTFT of Fig. P4.6(a) is given by

$$X1(\omega) = 3\pi\delta(\omega) - j6\sum_{\substack{n=-\infty \\ \text{odd } n}}^{\infty} \frac{1}{n}\delta(\omega - n).$$

Output for $H1(\omega)$: Since $H1(\omega)$ eliminates all frequency components outside the range $|\omega| \le 4$ (as $T = 1$), the output is given by

$$Y1(\omega) = j2\delta(\omega+3) + j6\delta(\omega+1) + 3\pi\delta(\omega) - j6\delta(\omega-1) - j2\delta(\omega-3).$$

Output for $H2(\omega)$: Since $H2(\omega)$ eliminates all frequency components outside the range $4 \le |\omega| \le 8$, the output is given by

$$Y1(\omega) = j\tfrac{6}{7}\delta(\omega+7) + j\tfrac{6}{5}\delta(\omega+5) - j\tfrac{6}{5}\delta(\omega-1) - j\tfrac{6}{7}\delta(\omega-3).$$

(b) From the solution of Problem P5.19(b), the CTFT of Fig. P4.6(b) is given by

$$X2(\omega) = 1.5\pi\delta(\omega) - \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \tfrac{1}{n}\sin(0.5n\pi)\delta(\omega - \tfrac{n\pi}{T}).$$

Output for $H1(\omega)$: Since $H1(\omega)$ eliminates all frequency components outside the range $|\omega| \le 4/T$, the output is given by

$$Y2(\omega) = -\delta(\omega + \tfrac{\pi}{T}) + 1.5\pi\delta(\omega) - \delta(\omega - \tfrac{\pi}{T}).$$

Output for $H2(\omega)$: Since $H2(\omega)$ eliminates all frequency components outside the range $4/T \le |\omega| \le 8/T$, the output is given by

$$Y2(\omega) = 0.$$

(c) From the solution of Problem P5.19(c), the CTFT of Fig. P4.6(c) is given by

$$X3(\omega) = \pi\delta(\omega) - j\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \tfrac{1}{n}\delta(\omega - \tfrac{2n\pi}{T}).$$

Output for $H1(\omega)$: Since $H1(\omega)$ eliminates all frequency components outside the range $|\omega| \le 4/T$, the output is given by

$$Y3(\omega) = \pi\delta(\omega).$$

Output for $H2(\omega)$: Since $H2(\omega)$ eliminates all frequency components outside the range $4/T \le |\omega| \le 8/T$, the output is given by

$$Y3(\omega) = j\delta(\omega + \tfrac{2\pi}{T}) - j\delta(\omega - \tfrac{2\pi}{T}).$$

(d) From the solution of Problem P5.19(d), the CTFT of Fig. P4.6(d) is given by

$$X4(\omega) = 2\pi \sum_{n=-\infty}^{\infty} D_n \delta(\omega - n\omega_0) = \pi\delta(\omega) + \tfrac{4}{\pi} \sum_{\substack{n=-\infty \\ n\neq 0 \\ \text{odd } n}}^{\infty} \tfrac{1}{n^2} \delta(\omega - \tfrac{n\pi}{T}).$$

Output for $H1(\omega)$: Since $H1(\omega)$ eliminates all frequency components outside the range $|\omega| \leq 4/T$, the output is given by

$$Y4(\omega) = \tfrac{4}{\pi}\delta(\omega + \tfrac{\pi}{T}) + \pi\delta(\omega) + \tfrac{4}{\pi}\delta(\omega - \tfrac{\pi}{T}).$$

Output for $H2(\omega)$: Since $H2(\omega)$ eliminates all frequency components outside the range $4/T \leq |\omega| \leq 8/T$, the output is given by

$$Y4(\omega) = 0.$$

(e) From the solution of Problem P5.19(e), the CTFT of Fig. P4.6(e) is given by

$$X5(\omega) = 2\pi \sum_{n=-\infty}^{\infty} D_n \delta(\omega - n\omega_0)$$

$$= 0.6816\pi\delta(\omega) + j0.3866\pi\delta(\omega + \tfrac{\pi}{T}) - j0.3866\pi\delta(\omega - \tfrac{\pi}{T}) + \sum_{\substack{n=-\infty \\ n\neq 0 \\ \text{even } n}}^{\infty} \tfrac{1}{n^2-1} \delta(\omega - \tfrac{n\pi}{T})$$

$$- j2 \sum_{\substack{n=-\infty \\ n\neq 0,1 \\ \text{odd } n}}^{\infty} \tfrac{1}{n} \delta(\omega - \tfrac{n\pi}{T}).$$

Output for $H1(\omega)$: Since $H1(\omega)$ eliminates all frequency components outside the range $|\omega| \leq 4/T$, the output is given by

$$Y5(\omega) = j0.3866\pi\delta(\omega + \tfrac{\pi}{T}) + 0.6816\pi\delta(\omega) - j0.3866\pi\delta(\omega - \tfrac{\pi}{T}).$$

Output for $H2(\omega)$: Since $H2(\omega)$ eliminates all frequency components outside the range $4/T \leq |\omega| \leq 8/T$, the output is given by

$$Y5(\omega) = \tfrac{1}{3}\delta(\omega + \tfrac{2\pi}{T}) + \tfrac{1}{3}\delta(\omega - \tfrac{2\pi}{T}).$$

Problem 5.31

(a) The magnitude spectra of the two systems are calculated below

$$|H_1(\omega)| = \frac{\sqrt{400+\omega^2}}{\sqrt{400+\omega^2}} = 1$$

$$|H_2(\omega)| = \begin{cases} 1 & |\omega| \geq 20 \\ 0 & \text{elsewhere.} \end{cases}$$

The magnitude spectra are plotted in Fig. S5.31. From Fig. S5.31(a), we observe that the magnitude $|H_1(\omega)|$ is 1 at all frequencies. Therefore, System $H_1(\omega)$ is an all pass filter.

From Fig. S5.31(b), we observe that the magnitude $|H_2(\omega)|$ is zero at frequencies below 20 radians/s. At frequencies above 20 radians/s, the magnitude is 1. Therefore, System $H_2(\omega)$ is a highpass filter.

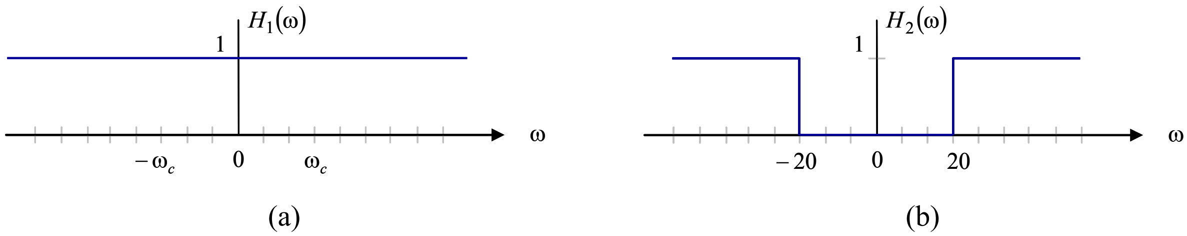


Fig. S5.31: Magnitude Spectra for Problem 5.31.

(b) Calculating the inverse CTFT, the impulse response of the two systems is given by

$$h_1(t) = \Im^{-1}\left\{\frac{40-20-j\omega}{20+j\omega}\right\} = \Im^{-1}\left\{\frac{40}{20+j\omega}\right\} - \Im^{-1}\{1\} = 40e^{-20t}u(t) - \delta(t).$$

$$h_2(t) = \Im^{-1}\left\{1-\text{rect}(\tfrac{\omega}{40})\right\} = \Im^{-1}\{1\} - \Im^{-1}\left\{\text{rect}(\tfrac{\omega}{40})\right\} = \delta(t) - \tfrac{20}{\pi}\text{sinc}(\tfrac{20t}{\pi}).$$

▌

Problem 5.32

The transfer functions for the three LTIC systems are given by

System (a): $$H_1(\omega) = \frac{2}{(1+j\omega)^2}.$$

System (b): $$H_2(\omega) = \pi\delta(\omega) + \frac{1}{j\omega}.$$

System (c): $$H_3(\omega) = -2 + \frac{5}{2+j\omega} = \frac{1-j2\omega}{2+j\omega}$$

The following Matlab code generates the magnitude and phase spectra of the three LTIC systems.

```
%MATLAB Program for Problem P5.32
%System (a)
clear;                              % clear the MATLAB environment
num_coeff = [2];                    % NUM coeffs. in decreasing powers of s
denom_coeff = [1 2 1];              % DEN coeffs. in decreasing powers of s
sys = tf(num_coeff,denom_coeff);    % specify the transfer function
figure(1)
bode(sys,{0.02,100}); grid;         % sketch the Bode plots
title('Bode Plot for System-1')
%System (b)
clear;                              % clear the MATLAB environment
num_coeff = [1];                    % NUM coeffs. in decreasing powers of s
denom_coeff = [1 0];                % DEN coeffs. in decreasing powers of s
sys = tf(num_coeff,denom_coeff);    % specify the transfer function
figure(2)
bode(sys,{0.02,100}); grid;         % sketch the Bode plots
title('Bode Plot for System-2')
%System (vc)
```

```
clear;                                  % clear the MATLAB environment
num_coeff = [-2 1];                     % NUM coeffs. in decreasing powers of s
denom_coeff = [1 2];                    % DEN coeffs. in decreasing powers of s
sys = tf(num_coeff,denom_coeff);        % specify the transfer function
figure(3)
bode(sys,{0.02,100}); grid;             % sketch the Bode plots
title('Bode Plot for System-3')
```

The resulting Bode plots are shown in Fig. S5.32.

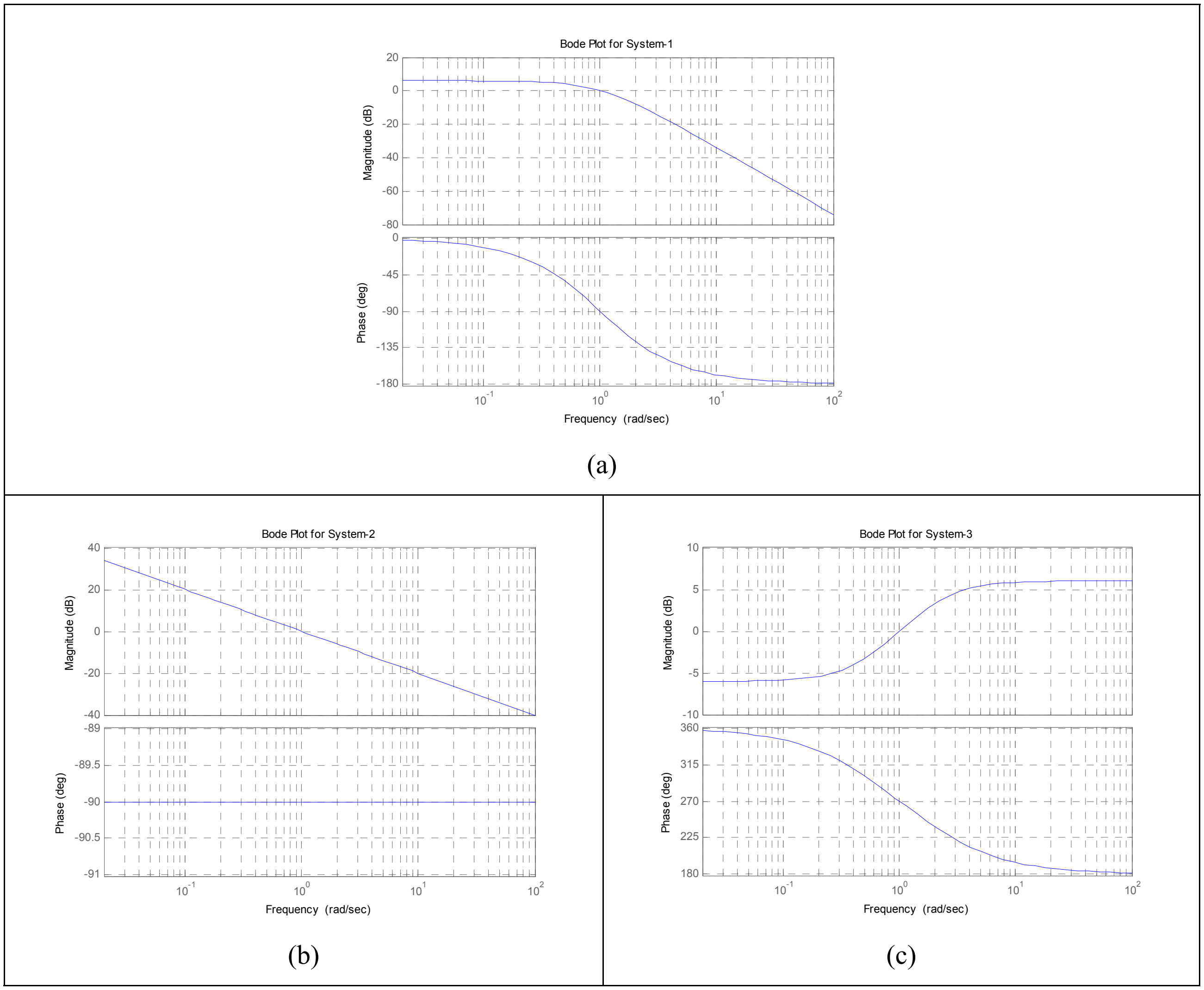


Figure S5.32. Magnitude and phase spectra for systems in Problem 5.32.

Calculating Output:

System (a): Using the convolution property, the output of system (a) is given by

$$Y_1(\omega) = \frac{2}{(1+j\omega)^2} \times \pi\left[\delta(\omega-1)+\delta(\omega+1)\right] = 2\pi\left(\tfrac{1}{(1+j1)^2}\delta(\omega-1)+\tfrac{1}{(1-j1)^2}\delta(\omega+1)\right)$$
$$= -j\pi\left[\delta(\omega-1)-\delta(\omega+1)\right].$$

Calculating the inverse CTFT, we obtain

$$y_1(t) = \sin t\,.$$

System (b): Using the convolution property, the output of system (b) is given by

$$Y_2(\omega) = \left[\pi\delta(\omega)+\frac{1}{j\omega}\right] \times \pi\left[\delta(\omega-1)+\delta(\omega+1)\right] = \pi\left[\tfrac{1}{j}\delta(\omega-1)+\tfrac{1}{-j}\delta(\omega+1)\right]$$
$$= -j\pi\left[\delta(\omega-1)-\delta(\omega+1)\right].$$

Calculating the inverse CTFT, we obtain

$$y_2(t) = \sin t\,.$$

System (c): Using the convolution property, the output of system (c) is given by

$$Y_2(\omega) = \frac{1-j2\omega}{2+j\omega} \times \pi\left[\delta(\omega-1)+\delta(\omega+1)\right] = \pi\left[\tfrac{1-j2}{2+j}\delta(\omega-1)+\tfrac{1+j2}{2-j}\delta(\omega+1)\right]$$
$$= -j\pi\left[\delta(\omega-1)-\delta(\omega+1)\right].$$

Calculating the inverse CTFT, we obtain

$$y_3(t) = \sin t\,.$$

To explain why the three systems produce the same output for input $x(t)$ = cos$t$, consider Eq. (5.77), which for $\omega_0 = 1$ is given by

$$\cos(t) \xrightarrow{\text{Hermitian Symmetric } H(\omega)} |H(1)|\cos(\omega_0 t + < H(1))\,.$$

In other words, the output for $x(t) = \cos(t)$ depends only on the magnitude and phase of the system at ω = 1. For the three systems, we note that

$$|H_1(1)| = |H_2(1)| = |H_3(1)| = 1 \qquad \text{and}$$

$$< H_1(1) = H_2(1) = H_3(1) = -\tfrac{\pi}{2}\,.$$

Since the magnitudes and phases of the three system transfer functions at ω = 1 are identical, the three systems produce the same output $y(t) = \sin t$ for $x(t) = \cos(t)$. ▌

Problem 5.33

The MATLAB code for calculating the CTFTs is listed below.

```
% Problem 5_33(i)
ws = 200*pi;          % sampling rate
Ts = 2*pi/ws;         % sampling interval
tmin = -2; tmax = 2;
t = tmin:Ts:tmax;     % define time instants
x = sin(5*pi*t);
y = fft(x);           % fft computes CTFT
z = (2*pi*Ts/(tmax-tmin))*y;% scale the magnitude of y
z = fftshift(z);      % centre CTFT about w = 0
```

```
w = -ws/2:ws/length(z):ws/2-ws/length(z);
subplot(221); plot(w,abs(z));  % CTFT plot of cos(w0*t)
axis([-20*pi 20*pi 0 max(abs(z))]);
xlabel('\omega (radians/s)');
ylabel('|x_1(t)|');
title('x_1(t) = sin(5\pi t): Magnitude spectrum')
grid on
subplot(222); plot(w,angle(z).*abs(z)/max(abs(z)));
axis([-20*pi 20*pi -0.5*pi 0.5*pi]);
xlabel('\omega (radians/s)');
ylabel('<x_1(t)')
title('x_1(t) = sin(5\pi t): Phase spectrum')
grid on
% end
%
% Problem 5_33(ii)
ws = 1000*pi;        % sampling rate
Ts = 2*pi/ws;       % sampling interval
tmin = -1.25; tmax = 1.25;
t = tmin:Ts:tmax;   % define time instants
x = sin(8*pi*t)+sin(20*pi*t);
y = fft(x);         % fft computes CTFT
z = (2*pi*Ts/(tmax-tmin))*y;% scale the magnitude of y
z = fftshift(z);    % centre CTFT about w = 0
w = -ws/2:ws/length(z):ws/2-ws/length(z);
subplot(223); plot(w,abs(z));  % CTFT plot of cos(w0*t)
axis([-40*pi 40*pi 0 max(abs(z))]);
xlabel('\omega (radians/s)');
ylabel('|x_2(t)|');
title('x_2(t) = sin(8\pi t)+sin(20\pi t): Magnitude spectrum')
grid on
subplot(224); plot(w,angle(z).*abs(z)/max(abs(z)));
axis([-40*pi 40*pi -0.5*pi 0.5*pi]);
xlabel('\omega (radians/s)');
ylabel('<x_2(t)')
title('x_2(t) = sin(8\pi t)+sin(20\pi t): Phase spectrum')
grid on
% end
```

The magnitude and phase spectra are shown in Fig. S5.33. ▮

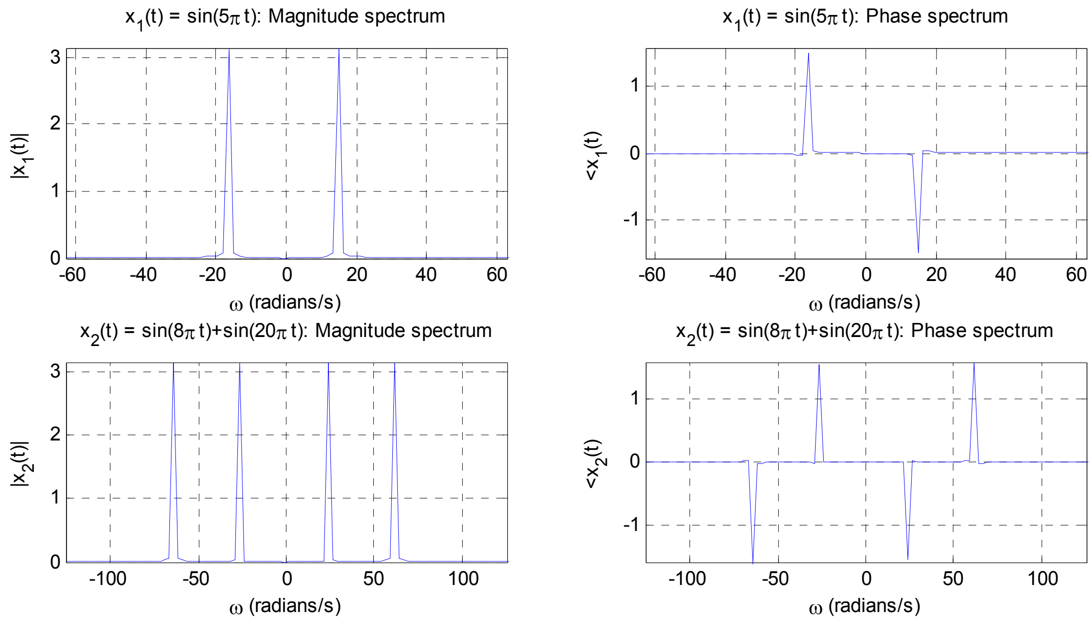


Figure S5.33. Magnitude and phase spectra for the sinusoidal signals in Problem 5.33.

Problem 5.35

The MATLAB code for calculating the output is listed below.

```
% Problem 5.35
t = -5:0.001:5;
% time waveforms with N samples each
x = exp(-t).*(t>=0);
h = exp(-2*t).*(t>=0);
% CTFT calculated for (2N-1) samples
Xfreq = fft(x,length(x)+length(h)-1);
Hfreq = fft(h,length(x)+length(h)-1);
% Scale the ffts
Xfreq = (2*pi*0.001/10) * Xfreq;
Hfreq = (2*pi*0.001/10) * Hfreq;
% Output
Yfreq = Xfreq .* Hfreq;
y = ifft(Yfreq);
y = (10/(2*pi*0.001))*y;
% plot
plot([-10:0.001:10],real(y));
xlabel('time (t)');
ylabel('output y(t)');
title('Problem 5.35');
```

The magnitude and phase spectra are shown in Fig. S5.35. ▌

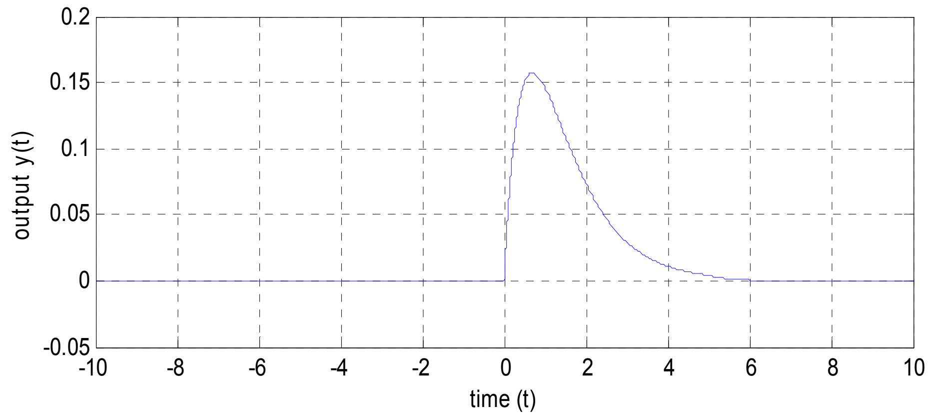


Figure S5.35: Output waveform for Problem 5.35.

Problem 5.36

(i) Bode plots for the LTIC systems specified in Problem 5.20:

The MATLAB code for calculating the Bode plots for the LTIC systems specified in Problem 5.20 is listed below.

```
% Problem 5.36
% Bode plot for Problem 5.20(a)
figure(1)
w = 0.01:0.01:100;
num = [1];
den = [1 6 11 6];
sys = tf(num,den);
bode(sys,{0.01,100});
xlabel('\omega (radians/s)');
title('P5.20(a)');
grid on
%
% Bode plot for Problem 5.20(b)
figure(2)
w = 0.01:0.01:100;
num = [1];
den = [1 3 2];
sys = tf(num,den);
bode(sys,{0.01,100});
xlabel('\omega (radians/s)');
title('P5.20(b)');
grid on
%
% Bode plot for Problem 5.20(c)
figure(3)
w = 0.01:0.01:100;
num = [1];
den = [1 2 1];
sys = tf(num,den);
bode(sys,{0.01,100});
xlabel('\omega (radians/s)');
title('P5.20(c)');
grid on
```

```
%
% Bode plot for Problem 5.20(d)
figure(4)
w = 0.01:0.01:100;
num = [1 4];
den = [1 6 8];
sys = tf(num,den);
bode(sys,{0.01,100});
xlabel('\omega (radians/s)');
title('P5.20(d)');
grid on
%
% Bode plot for Problem 5.20(e)
figure(5)
w = 0.01:0.01:100;
num = [1];
den = [1 8 19 12];
sys = tf(num,den);
bode(sys,{0.01,100});
xlabel('\omega (radians/s)');
title('P5.20(e)');
grid on
```

The Bode plots are shown in Fig. S5.36.1 to Fig. S5.36.5 below.

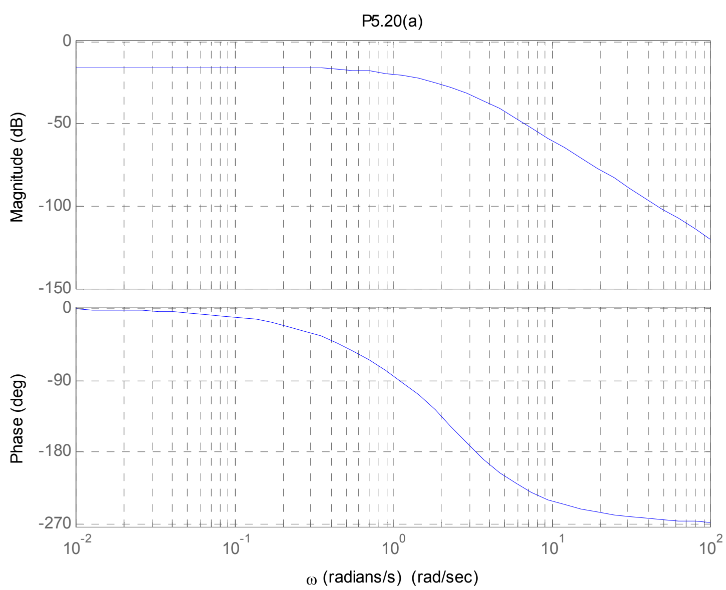


Figure S5.36.1: Bode plot for LTI system specified in Problem 5.20(a) as required in Problem 5.36.

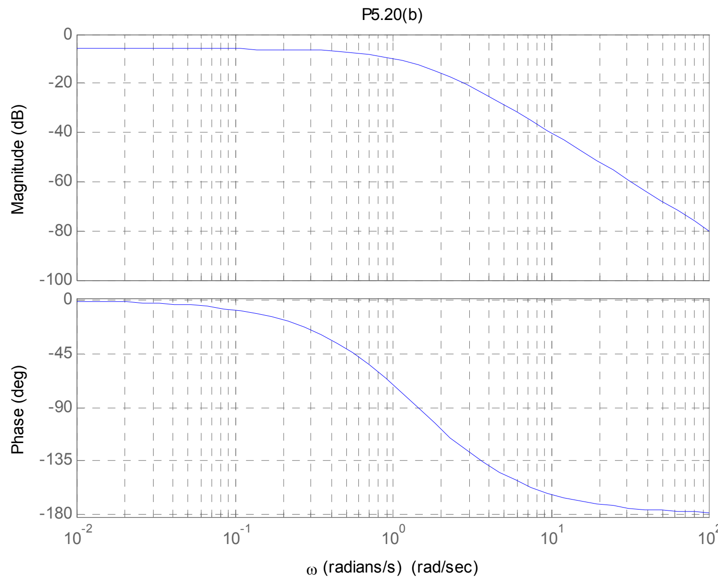


Figure S5.36.2: Bode plot for LTI system specified in Problem 5.20(b) as required in Problem 5.36.

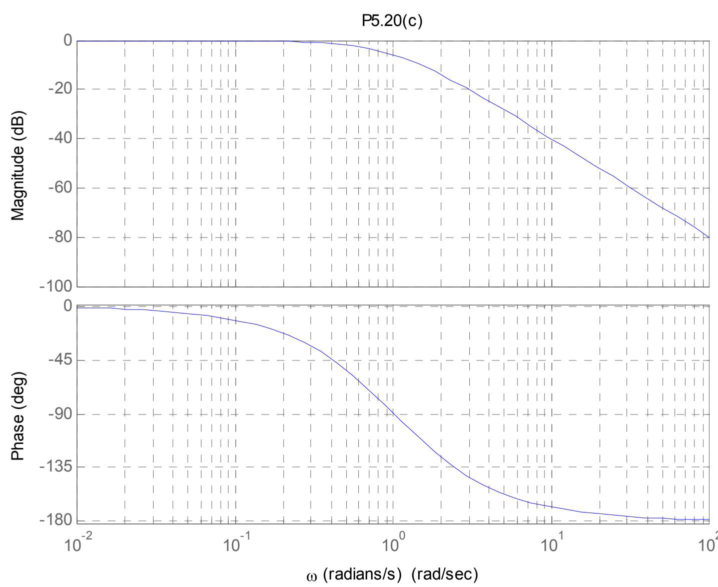


Figure S5.36.3: Bode plot for LTI system specified in Problem 5.20(c) as required in Problem 5.36.

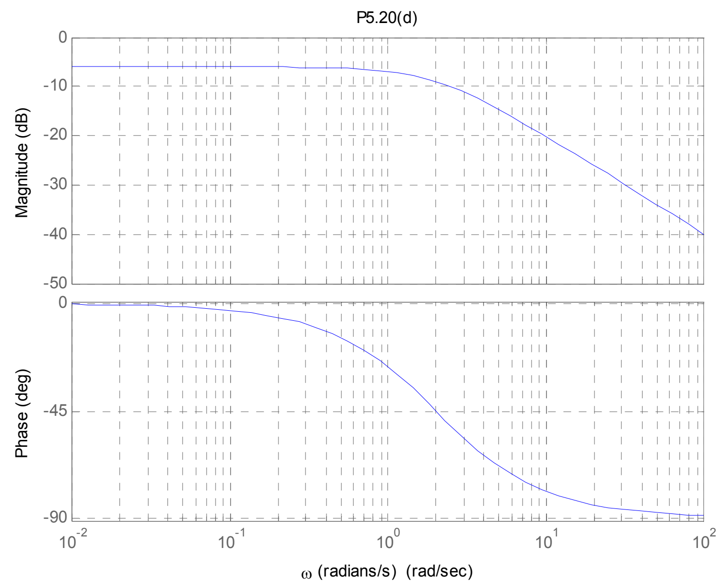


Figure S5.36.4: Bode plot for LTI system specified in Problem 5.20(d) as required in Problem 5.36.

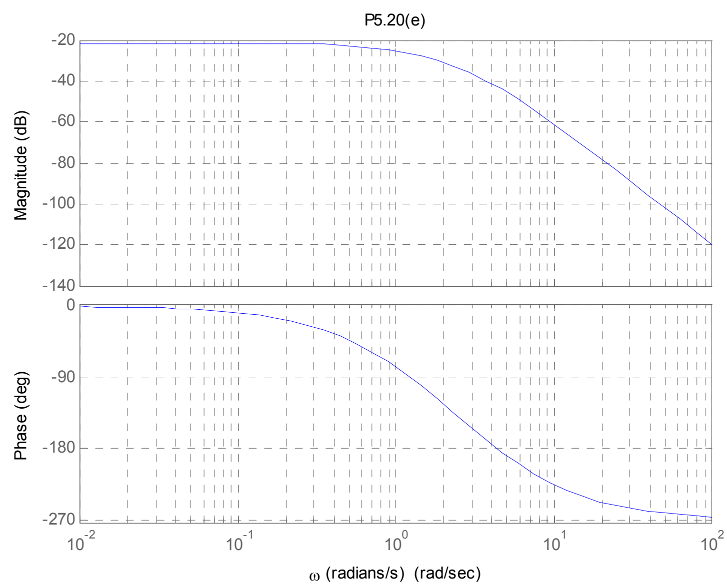


Figure S5.36.5: Bode plot for LTI system specified in Problem 5.20(e) as required in Problem 5.36.

The Bode plots have a one to one correspondence with the magnitude and phase spectra plotted in Problem 5.20.

(ii) Bode plots for the LTIC systems specified in Problem 5.21:

Since the transfer function $H(\omega) = 5$ in part (a), the magnitude plot for part (a) is constant at 5 for all frequencies. The phase is 0.

The transfer function $H(\omega) = 3e^{-j4\omega}$ in part (b). Therefore, the magnitude plot for part (b) is constant at 3 for all frequencies. The phase is -4ω represented by a straight line with a slope of -4. These two plots are not plotted.

The MATLAB code for calculating the Bode plots for the LTIC systems specified in parts (c) and (d) of Problem 5.21 is listed below. The plots are shown in Fig. S5.36.6 and S5.36.7.

```
% Problem 5.36
% Bode plot for Problem 5.21(c)
figure(1)
w = 0.01:0.01:100;
num = [6];
den = [1 6 12 8];
sys = tf(num,den);
bode(sys,{0.01,100});
xlabel('\omega (radians/s)');
title('P5.20(c)');
grid on
% Bode plot for Problem 5.21(c)
figure(2)
w = 0.01:0.01:100;
num = [2 8 8];
den = [1 4 3];
sys = tf(num,den);
bode(sys,{0.01,100});
xlabel('\omega (radians/s)');
title('P5.20(d)');
grid on
```

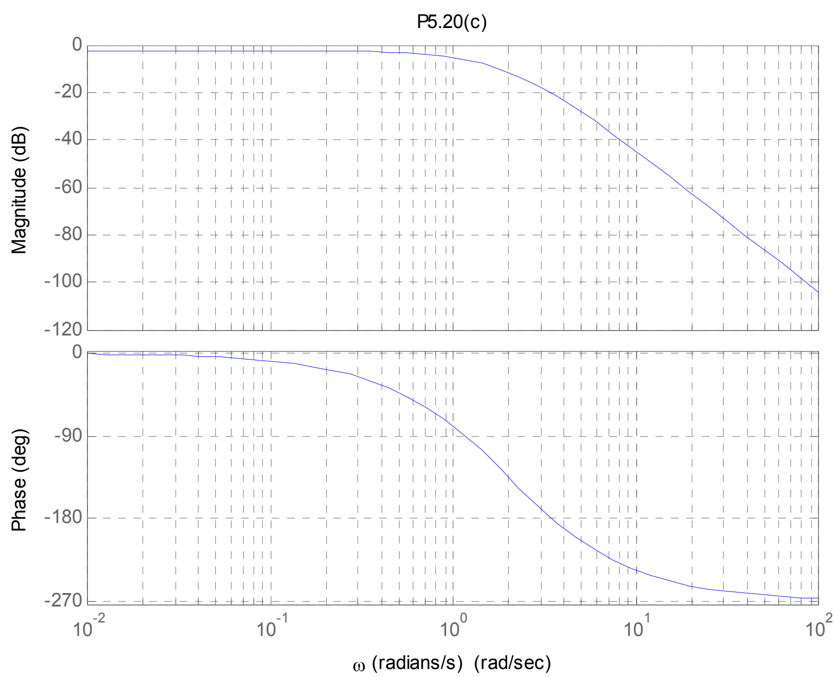


Figure S5.36.6: Bode plot for LTI system specified in Problem 5.21(c) as required in Problem 5.36.

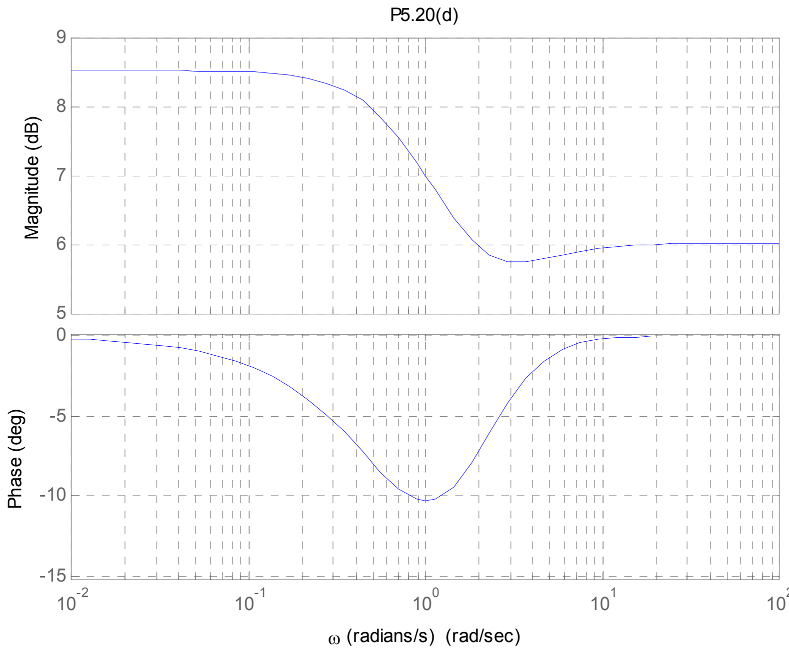


Figure S5.36.7: Bode plot for LTI system specified in Problem 5.21(d) as required in Problem 5.36.