Chapter 11: Discrete-Time Fourier Series and Transform

Problem 11.1:

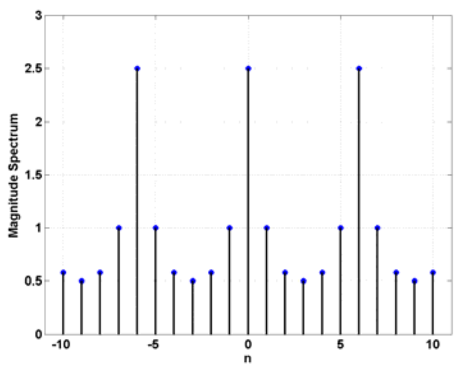
(i) $x[k]=k, \quad \text{for } 0\le k\le 5 \quad \text{and} \quad x[k+6]=x[k]$. $\Omega_0=\dfrac{2\pi}{K_0}=\dfrac{2\pi}{6}=\dfrac{\pi}{3}$.

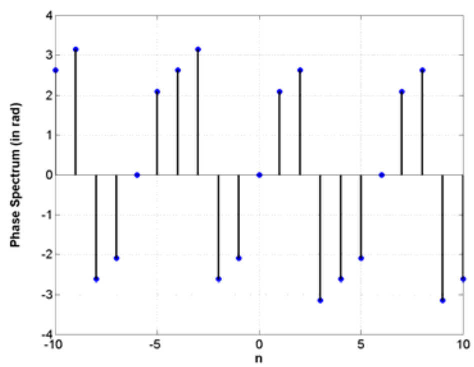
$$\begin{aligned}
D_n &= \frac{1}{K_0}\sum_{k=0}^{5} x[k]e^{-jn\Omega_0 k} = \frac{1}{6}\sum_{k=0}^{5} ke^{-jn\Omega_0 k}\\
&= \frac{1}{6}\left(e^{-jn\Omega_0}+2e^{-j2n\Omega_0}+3e^{-j3n\Omega_0}+4e^{-j4n\Omega_0}+5e^{-j5n\Omega_0}\right)\\
&= \frac{1}{6}\left[\cos(n\Omega_0)+2\cos(2n\Omega_0)+3\cos(3n\Omega_0)+4\cos(4n\Omega_0)+5\cos(5n\Omega_0)\right]\\
&\quad -\frac{1}{6}j\left[\sin(n\Omega_0)+2\sin(2n\Omega_0)+3\sin(3n\Omega_0)+4\sin(4n\Omega_0)+5\sin(5n\Omega_0)\right]
\end{aligned}$$

Expressing $D_n=\frac{1}{6}\sum_{k=0}^{5}k\left(e^{-jn\Omega_0}\right)^k$ and using the series sum formula $\sum_{k=0}^{M}kr^k=\dfrac{r-(M+1)r^{M+1}+Mr^{M+2}}{(1-r)^2}$, we can also represent D_n in a compact form as follows:

$$D_n=\frac{1}{6}\sum_{k=0}^{5}k\left(e^{-jn\Omega_0}\right)^k=\frac{1}{6}\,\frac{e^{-jn\Omega_0}-6e^{-j6n\Omega_0}+5e^{-j7n\Omega_0}}{(1-e^{-jn\Omega_0})^2}, \text{ with } \Omega_0=\frac{\pi}{3}.$$

The magnitude and phase spectra for $-10\le n\le 10$ are shown below.





(ii) $x[k]=\begin{cases}1 & (0\le k\le 2)\\ 0.5 & (3\le k\le 5)\\ 0 & (6\le k\le 8)\end{cases}$ and $x[k+9]=x[k]$

$$\Omega_0=\frac{2\pi}{K_0}=\frac{2\pi}{9}.$$

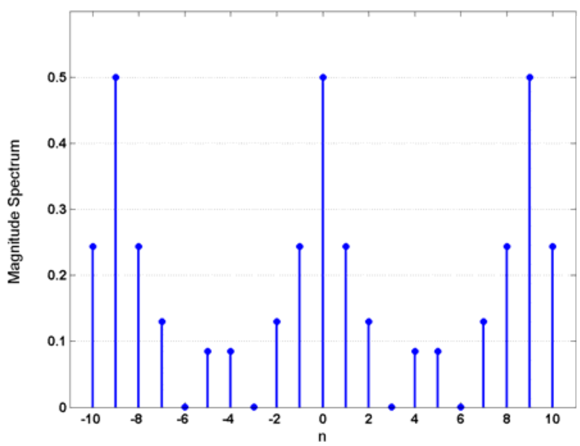
$$D_n = \frac{1}{K_0}\sum_{k=0}^{8} x[k]e^{-jn\Omega_0 k} = \frac{1}{9}\sum_{k=0}^{8} ke^{-jn\Omega_0 k}$$

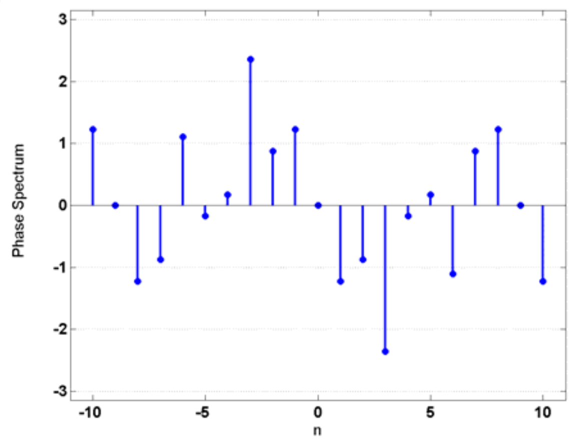
$$= \frac{1}{9}\left(1 + e^{-jn\Omega_0} + e^{-j2n\Omega_0} + 0.5e^{-j3n\Omega_0} + 0.5e^{-j4n\Omega_0} + 0.5e^{-j5n\Omega_0}\right)$$

$$= \frac{1}{9}\left[1 + e^{-jn\Omega_0} + e^{-j2n\Omega_0} + 0.5e^{-j3n\Omega_0}\left(1 + e^{-jn\Omega_0} + e^{-j2n\Omega_0}\right)\right]$$

$$= \frac{1}{9}\left(1 + 0.5e^{-j3n\Omega_0}\right)\left(1 + e^{-jn\Omega_0} + e^{-j2n\Omega_0}\right)$$

The magnitude and phase spectra for $-10 \le n \le 10$ are shown below.

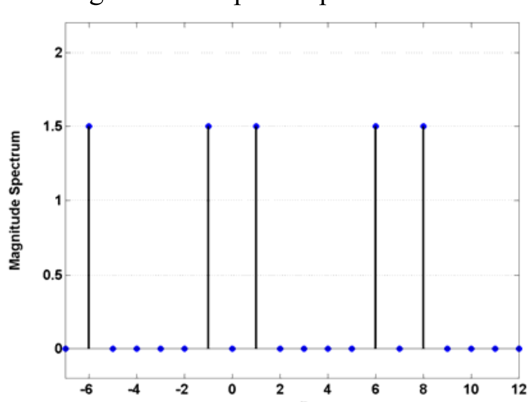


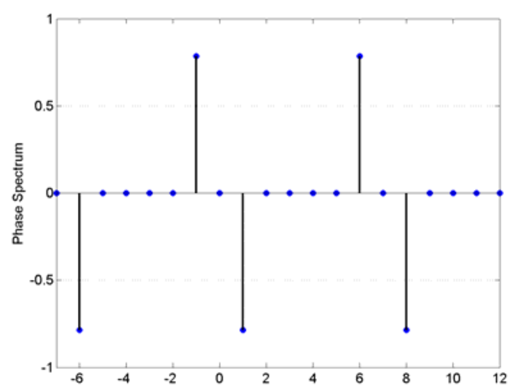


(iii) As solved in Example 11.5, the DTFS coefficients are obtained as (see Eq. 11.19):

$$D_n = \begin{cases} -j\frac{3}{2}e^{j\frac{\pi}{4}} & \text{for } n = 1 \\ j\frac{3}{2}e^{-j\frac{\pi}{4}} & \text{for } n = -1 \\ 0 & \text{elsewhere.} \end{cases}, \quad D_n = D_{n+7}$$

The magnitude and phase spectra for $-7 \le n \le 12$ are shown below.





(iv) Because $\frac{2\pi}{\Omega}=\frac{2\pi}{5\pi/3}=\frac{6}{5}=rational$, $x[k]$ is a periodic function with a fundamental period $K_o=\dfrac{2\pi}{5\pi/3}\cdot m=\dfrac{6}{5}\cdot m=6$ (for m=5). Therefore, $\Omega_0=\dfrac{2\pi}{K_0}=\dfrac{2\pi}{6}=\dfrac{\pi}{3}$.

The function $x[k]$ can be expressed in terms of complex exponential functions as follows.

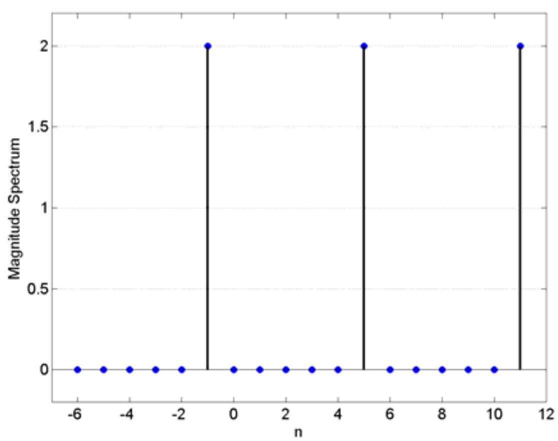
$$x[k]=2e^{j\left(\frac{5\pi}{3}k+\frac{\pi}{4}\right)}=2e^{\frac{j\pi}{4}}\cdot e^{j\frac{5\pi}{3}k}=\sum_{n=0}^{K_0-1}D_n e^{jn\Omega_0 k}$$

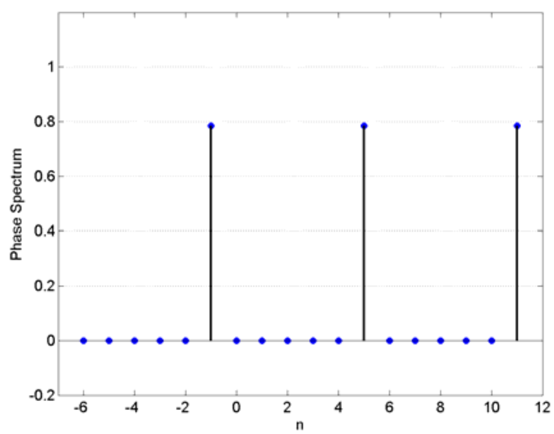
From the above equation, we can state that

$$D_n=\begin{cases}2e^{\frac{j\pi}{4}} & n=5\\ 0 & 0\le n\le 4\end{cases}\quad \text{and } D_n=D_{n+6} \text{ (since } D_n \text{ is periodic with period 6).}$$

Note that $|D_n|=\begin{cases}2 & n=5\\ 0 & 0\le n\le 4\end{cases}$ and $\measuredangle D_n=\begin{cases}\pi/4 & n=5\\ 0 & 0\le n\le 4\end{cases}$.

The amplitude and phase spectrums for $-6\le n\le 11$ are shown below.



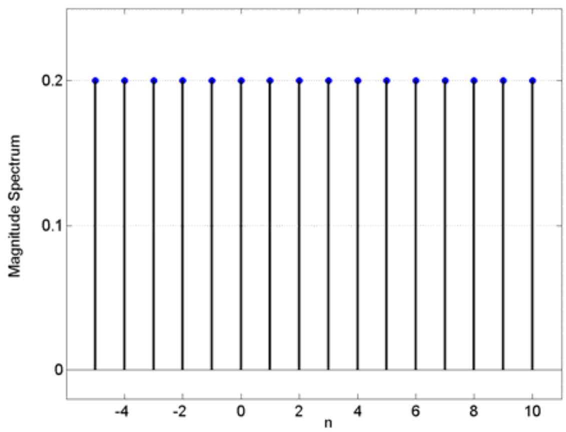


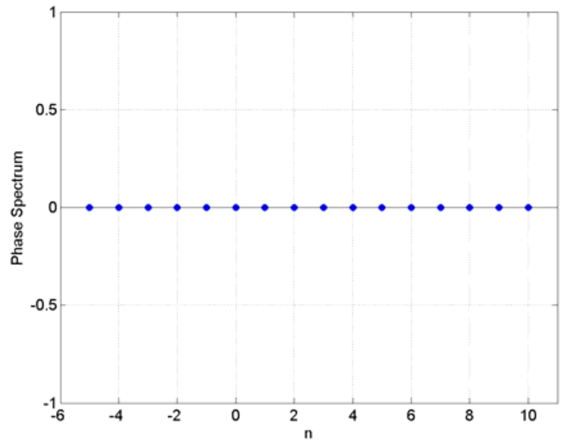
(v) $x[k]=\sum_{m=-\infty}^{\infty}\delta[k-5m]$

$$\Omega_0=\frac{2\pi}{K_0}=\frac{2\pi}{5}.$$

$$D_n=\frac{1}{K_0}\sum_{k=0}^{4}x[k]e^{-jn\Omega_0 k}=\frac{1}{5}\sum_{k=0}^{4}\delta[k]e^{-jn\Omega_0 k}=\frac{1}{5}$$

The magnitude and phase spectra for $-5\le n\le 10$ are shown below.





(vi) $$x[k]=\cos\left(\frac{10\pi k}{3}\right)\cos\left(\frac{2\pi k}{5}\right)=\cos\left(4\pi k-\frac{2\pi k}{3}\right)\cos\left(\frac{2\pi k}{5}\right)=\cos\left(\frac{2\pi k}{3}\right)\cos\left(\frac{2\pi k}{5}\right)$$

$$=\underbrace{\underbrace{\frac{1}{2}\cos\left(\frac{16\pi k}{15}\right)}_{period=15}+\underbrace{\frac{1}{2}\cos\left(\frac{4\pi k}{15}\right)}_{period=15}}_{period=15}$$

$$\Omega_0=\frac{2\pi}{K_0}=\frac{2\pi}{15}.$$

$$x[k]=\frac{1}{2}\cos\left(\frac{16\pi k}{15}\right)+\frac{1}{2}\cos\left(\frac{4\pi k}{15}\right)=\frac{1}{2}\cos\left(2\pi k-\frac{14\pi k}{15}\right)+\frac{1}{2}\cos\left(\frac{4\pi k}{15}\right)$$

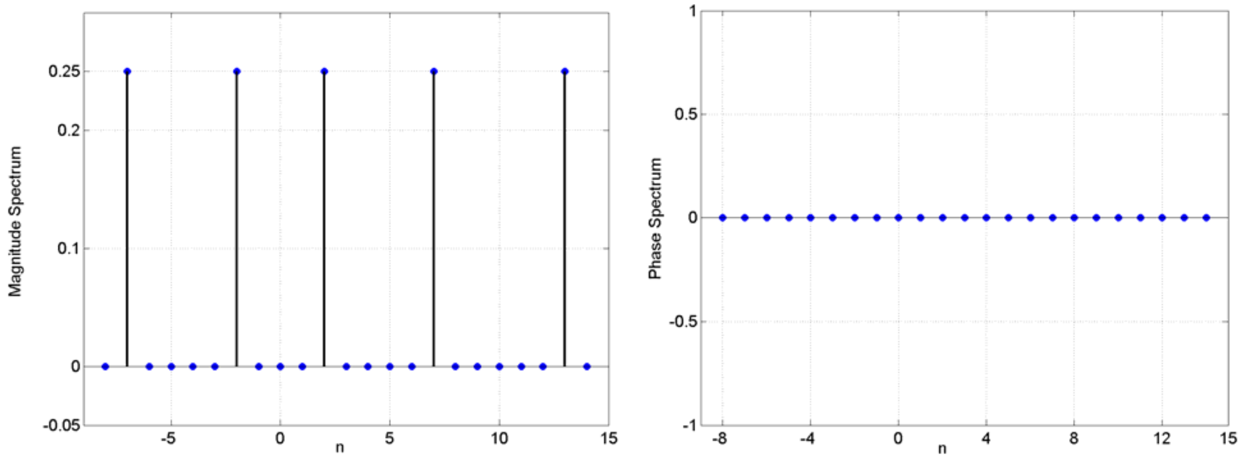
$$=\frac{1}{2}\cos\left(\frac{14\pi k}{15}\right)+\frac{1}{2}\cos\left(\frac{4\pi k}{15}\right)=\frac{1}{2}\cos\left(7\Omega_0 k\right)+\frac{1}{2}\cos\left(2\Omega_0 k\right)$$

$$=\frac{1}{4}\left[e^{j7\Omega_0 k}+e^{-j7\Omega_0 k}+e^{j2\Omega_0 k}+e^{-j2\Omega_0 k}\right]$$

$$=\sum_{n=-7}^{7}D_n e^{jn\Omega_0 k}$$

where $D_n=\begin{cases}1/4 & n=\pm 2,\pm 7\\ 0 & otherwise\end{cases}$

The magnitude and phase spectra for $-8\le n\le 14$ are shown below.

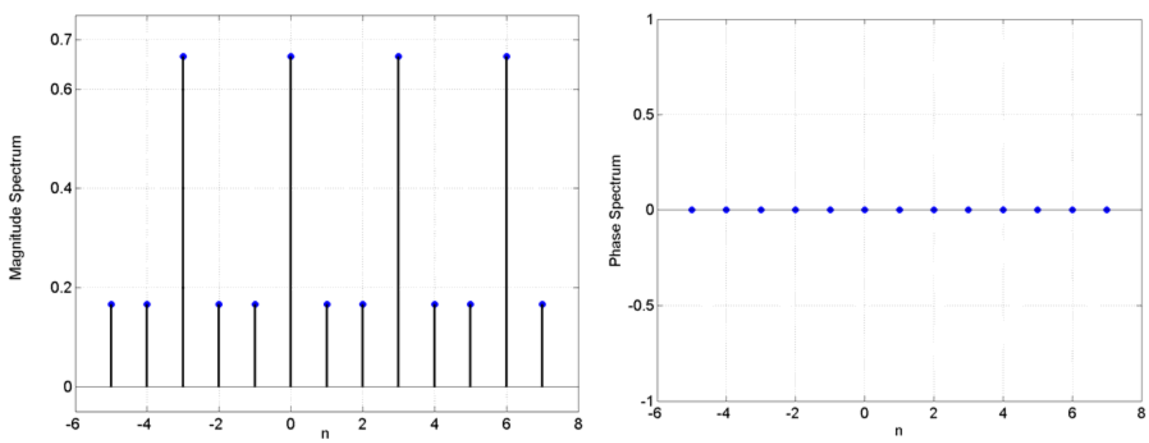


(vii) $x[k] = \left|\cos\left(2\pi k/3\right)\right| = \begin{cases} 1 & k=0 \\ 0.5 & k=1,2 \end{cases}$ and $x[k+3]=x[k]$.

$x[k]$ is a periodic function with fundamental period of 3.

$$\Omega_0 = \frac{2\pi}{K_0} = \frac{2\pi}{3}.$$

$$\begin{aligned} D_n &= \frac{1}{K_0}\sum_{k=0}^{2} x[k]e^{-jn\Omega_0 k} \\ &= \frac{1}{3}\left(1+0.5e^{-jn\Omega_0}+0.5e^{-j2n\Omega_0}\right) \\ &= \begin{cases} 2/3 & n=0 \\ 1/6 & n=1,2 \end{cases} \end{aligned}$$



The magnitude and phase spectra for $-5 \le n \le 7$ are shown below. ▌

```
% MATLAB program for plotting Fourier spectra in problem 11.1
% part(i)
omega = pi/3 ;
%n = [-10:10] ;
n = [-4:4] ;


Dn = (1/6)*(exp(-j*n*omega)+2* exp(-j*2*n*omega)+3* exp(-j*3*n*omega)+4*
exp(-j*4*n*omega)+5* exp(-j*5*n*omega));

mag_Dn   = abs(Dn);
phase_Dn = angle(Dn)
%pha_Dn = unwrap(phase_Dn)

stem(n, mag_Dn, 'filled'),grid
xlabel('n');
ylabel('Magnitude Spectrum')
axis([-11  11  0  3])
print -dtiff plot.tiff

stem(n, phase_Dn, 'filled'),grid
xlabel('n');
ylabel('Phase Spectrum')
%axis([-11  11  -pi  pi])
print -dtiff plot.tiff

%
% part(ii)
omega = 2*pi/9 ;
%n = [-10:10] ;
n = [-4:4] ;


Dn = (1/9)*(1+0.5*exp(-j*3*n*omega)).*(1+exp(-j*n*omega)+exp(-
j*2*n*omega));

mag_Dn   = abs(Dn);
phase_Dn = angle(Dn)

stem(n, mag_Dn, 'filled'),grid
xlabel('n');
ylabel('Magnitude Spectrum')
axis([-11  11  0  0.6])
print -dtiff plot.tiff

stem(n, phase_Dn, 'filled'),grid
xlabel('n');
ylabel('Phase Spectrum')
axis([-11  11  -pi  pi])
print -dtiff plot.tiff

% part(iii)
%omega = 2*pi/7 ;
n = [-7:14] ;

g=-j*1.5*exp(j*pi/4);
h=j*1.5*exp(-j*pi/4);
Dn=[0 g  0  0  0  0 h  0  g  0  0  0  0  h 0  g  0  0 0 0 h 0]

mag_Dn = abs(Dn);
phase_Dn = angle(Dn) ;
```

```
stem(n, mag_Dn, 'filled'),grid
xlabel('n');
ylabel('Magnitude Spectrum')
axis([-7  12  -0.2  2.2])
print -dtiff plot.tiff

stem(n, phase_Dn, 'filled'),grid
xlabel('n');
ylabel('Phase Spectrum')
axis([-7  12  -pi  pi])
print -dtiff plot.tiff


% part(iv)
omega = pi/3 ;
n = [-6:11] ;

mag_Dn = [0  0  0  0  0  2  0  0  0  0  0  2  0  0  0  0  0  2] ;
phase_Dn = [0  0  0  0  0  pi/4  0  0  0  0  0  pi/4  0  0  0  0  0
pi/4] ;

stem(n, mag_Dn, 'filled'),grid
xlabel('n');
ylabel('Magnitude Spectrum')
axis([-7  12  -0.2  2.2])
print -dtiff plot.tiff

stem(n, phase_Dn, 'filled'),grid
xlabel('n');
ylabel('Phase Spectrum')
axis([-7  12  -pi  pi])
print -dtiff plot.tiff

% part(v)
n = [-5:10] ;

mag_Dn = (1/5)*ones(1,16) ;
phase_Dn = zeros(1,16) ;

stem(n, mag_Dn, 'filled'),grid
xlabel('n');
ylabel('Magnitude Spectrum')
axis([-6  11  -0.2  0.3])
print -dtiff plot.tiff

stem(n, phase_Dn, 'filled'),grid
xlabel('n');
ylabel('Phase Spectrum')
axis([-6  11  -pi  pi])
print -dtiff plot.tiff

% part(vi)
n = [-8:14] ;

mag_Dn = [0 0.25 0 0 0 0 0.25  0  0  0  0.25  0  0  0  0  0.25  0  0  0
0  0  0.25  0] ;
phase_Dn = zeros(1,23) ;

stem(n, mag_Dn, 'filled'),grid
xlabel('n');
ylabel('Magnitude Spectrum')
axis([-9  15  -0.05  0.3])
print -dtiff plot.tiff
```

```
stem(n, phase_Dn, 'filled'),grid
xlabel('n');
ylabel('Phase Spectrum')
axis([-9  15  -1  1])
print -dtiff plot.tiff


% part(vii)
n = [-5:7] ;

a=2/3; b=1/6;
mag_Dn = [b b a b b a b b a b b a b] ;
phase_Dn = zeros(1,13) ;

stem(n, mag_Dn, 'filled'),grid
xlabel('n');
ylabel('Magnitude Spectrum')
axis([-6  8  -0.05  0.75])
print -dtiff plot.tiff

stem(n, phase_Dn, 'filled'),grid
xlabel('n');
ylabel('Phase Spectrum')
axis([-6  8  -1  1])
print -dtiff plot.tiff
```

Problem 11.2

(i) $D_n = \begin{cases} 1 & (0 \le k \le 2) \\ 0.5 & (3 \le k \le 5) \\ 0 & (6 \le k \le 8) \end{cases}$ and $\mathrm{D}_{n+9} = D_n$

$x[k]$ is periodic with period 9. Therefore, $\Omega_0 = \dfrac{2\pi}{9}$.

$$x[k] = \sum_{n=0}^{8} D_n e^{jn\Omega_0 k} = \sum_{n=0}^{2} e^{jn(\frac{2\pi}{9})k} + 0.5\sum_{n=3}^{5} e^{jn(\frac{2\pi}{9})k}$$
$$= 1 + e^{j(\frac{2\pi}{9})k} + e^{j(\frac{4\pi}{9})k} + 0.5\left[e^{j(\frac{6\pi}{9})k} + e^{j(\frac{8\pi}{9})k} + e^{j(\frac{10\pi}{9})k}\right]$$
$$= 1 + e^{j(\frac{2\pi}{9})k} + e^{j(\frac{4\pi}{9})k} + 0.5e^{j(\frac{6\pi}{9})k}\left[1 + e^{j(\frac{2\pi}{9})k} + e^{j(\frac{4\pi}{9})k}\right]$$
$$= \left(1 + 0.5e^{j(\frac{6\pi}{9})k}\right)\left(1 + e^{j(\frac{2\pi}{9})k} + e^{j(\frac{4\pi}{9})k}\right)$$

(ii) $D_n = \begin{cases} 1 - j0.5 & (n = -1) \\ 1 & (n = 0) \\ 1 + j0.5 & (n = 1) \\ 0 & (2 \le n \le 5) \end{cases}$ and $\mathrm{D}_{n+7} = D_n$

$x[k]$ is periodic with period 7. Therefore, $\Omega_0 = \dfrac{2\pi}{7}$.

$$x[k] = \sum_{n=-1}^{5} D_n e^{jn\Omega_0 k} = (1 - j0.5)e^{-j\Omega_0 k} + (1 + j0.5)e^{j\Omega_0 k}$$
$$= \left(e^{-j\Omega_0 k} + e^{j\Omega_0 k}\right) + j0.5\left(e^{j\Omega_0 k} - e^{-j\Omega_0 k}\right)$$
$$= 2\cos(\Omega_0 k) - \sin(\Omega_0 k) = \sqrt{5}\cos(\Omega_0 k - \tan^{-1}(0.5))$$
$$\approx \sqrt{5}\cos(\tfrac{2\pi}{7}k - 0.4636)$$

Substituting, different values of *k*, we obtain one period of $x[k]$ ($0 \le k \le 6$) as follows:

```
x[k] = [2.0000, 2.0288, 0.5298, -1.3681, -2.2358, -1.4199, 0.4653]
```

(iii) $D_n = 1 + \frac{3}{4}\sin\left(\frac{\pi n}{8}\right), \quad (0 \le n \le 6)$ and $D_{n+7} = D_n$

$x[k]$ is periodic with period 7. Therefore, $\Omega_0 = \dfrac{2\pi}{7}$.

$$x[k] = \sum_{n=0}^{6} D_n e^{jn\Omega_0 k} = \sum_{n=0}^{6}\left(1 + \tfrac{3}{4}\sin\left(\tfrac{\pi n}{8}\right)\right)e^{jn(\frac{2\pi}{7})k}$$

For $k=0$, $x[k]=\sum_{n=0}^{6}\left(1+\frac{3}{4}\sin\left(\frac{\pi n}{8}\right)\right)=7+\frac{3}{4}\sum_{n=0}^{6}\sin\left(\frac{\pi n}{8}\right)\approx 7+\frac{3}{4}*4.6447=10.4835$.

For $k\neq 0$, $x[k]$ is obtained as follows:

$$x[k]=\sum_{n=0}^{6}\left(1+\frac{3}{4}\sin\left(\frac{\pi n}{8}\right)\right)e^{jn(\frac{2\pi}{7})k}=\sum_{n=0}^{6}e^{jn(\frac{2\pi}{7})k}+\frac{3}{4}\sum_{n=0}^{6}\sin\left(\frac{\pi n}{8}\right)e^{jn(\frac{2\pi}{7})k}$$

$$=\frac{1-e^{j(\frac{2\pi}{7})k.7}}{1-e^{j(\frac{2\pi}{7})k}}+\frac{3}{4}\cdot\frac{1}{2j}\sum_{n=0}^{6}\left[e^{jn(\frac{\pi}{8})}-e^{-jn(\frac{\pi}{8})}\right]e^{jn(\frac{2\pi}{7})k}=\frac{\overbrace{1-e^{j2\pi k}}^{=0}}{1-e^{j(\frac{2\pi}{7})k}}+\frac{3}{j8}\sum_{n=0}^{6}\left[e^{jn\pi(\frac{2k}{7}+\frac{1}{8})}-e^{jn\pi(\frac{2k}{7}-\frac{1}{8})}\right]$$

$$=\frac{3}{j8}\left[\frac{1-e^{j\pi(\frac{2k}{7}+\frac{1}{8}).7}}{1-e^{j\pi(\frac{2k}{7}+\frac{1}{8})}}-\frac{1-e^{j\pi(\frac{2k}{7}-\frac{1}{8}).7}}{1-e^{j\pi(\frac{2k}{7}-\frac{1}{8})}}\right]=\frac{3}{j8}\left[\frac{1-e^{j7\pi/8}}{1-e^{j\pi(\frac{2k}{7}+\frac{1}{8})}}-\frac{1-e^{-j7\pi/8}}{1-e^{j\pi(\frac{2k}{7}-\frac{1}{8})}}\right]$$

$$\approx\frac{3}{j8}\left[\frac{1.9239-j0.3827}{1-e^{j\pi(\frac{2k}{7}+\frac{1}{8})}}-\frac{1.9239+j0.3827}{1-e^{j\pi(\frac{2k}{7}-\frac{1}{8})}}\right]\approx-\frac{0.1435+j0.7215}{1-e^{j\pi(\frac{2k}{7}+\frac{1}{8})}}-\frac{0.1435-j0.7215}{1-e^{j\pi(\frac{2k}{7}-\frac{1}{8})}}$$

$$\approx-\frac{\left(0.1435+j0.7215\right)\left[1-e^{j\pi(\frac{2k}{7}-\frac{1}{8})}\right]+\left(0.1435-j0.7215\right)\left[1-e^{j\pi(\frac{2k}{7}+\frac{1}{8})}\right]}{\left[1-e^{j\pi(\frac{2k}{7}+\frac{1}{8})}\right]\left[1-e^{j\pi(\frac{2k}{7}-\frac{1}{8})}\right]}$$

$$\approx-\frac{0.2870-\left(0.4087+j0.6117\right)e^{j2\pi k/7}-\left(0.4087+j0.6117\right)e^{j2\pi k/7}}{1-1.8478e^{j2\pi k/7}+e^{j4\pi k/7}}$$

$$=-\frac{0.2870-0.8174e^{j2\pi k/7}}{1-1.8478e^{j2\pi k/7}+e^{j4\pi k/7}}$$

Substituting, different values of *k*, we obtain one period of $x[k]$ ($0\le k\le 6$) as follows:

```
x[k] = [10.4835, -1.0626-j0.3735, -0.3844-j0.1220, -0.2948-j0.0341,
        -0.2948+j0.0341, -0.3844+j0.1220, -1.0626+j0.3735].
```

(iv) $D_n=\left(-1\right)^n$, $\left(0\le n\le 7\right)$ and $D_{n+8}=D_n$

$x[k]$ is periodic with period 8. Therefore, $\Omega_0=\dfrac{2\pi}{8}=\dfrac{\pi}{4}$. .

$$x[k]=\sum_{n=0}^{7}D_n e^{jn\Omega_0 k}=\sum_{n=0}^{7}\left(-e^{j\pi k/4}\right)^n=\frac{1-\left(-e^{j\pi k/4}\right)^8}{1+e^{j\pi k/4}}\qquad\left[k\neq 4\right]$$

$$=\frac{1-e^{j2\pi k}}{1+e^{j\pi k/4}}=\frac{0}{1+e^{j\frac{2\pi}{8}k}}=0.$$

For $k=4$, $x[4]=\sum_{n=0}^{7}\left(-e^{j\pi 4/4}\right)^n=\sum_{n=0}^{7}\left(-e^{j\pi}\right)^n=\sum_{n=0}^{7}(1)^n=8.$

Combining, the above result, we obtain,

$$x[k]=\begin{cases}0 & k=0-3,5-7\\ 8 & k=4\end{cases},\quad\text{and}\quad x[k+8]=x[k].$$

(v) $D_n = e^{jn\pi/4}$, $\left(0 \le n \le 7\right)$ and $D_{n+8} = D_n$

$x[k]$ is periodic with period 8. Therefore, $\Omega_0 = \dfrac{2\pi}{8} = \dfrac{\pi}{4}$.

$$x[k] = \sum_{n=0}^{7} D_n e^{jn\Omega_0 k} = \sum_{n=0}^{7} e^{jn\pi/4} e^{jn(\pi/4)k} = \sum_{n=0}^{7} e^{jn(\pi/4)(k+1)} = \frac{1 - e^{j8(\pi/4)(k+1)}}{1 - e^{j(\pi/4)(k+1)}} \qquad [k \ne 7]$$

$$= \frac{1 - e^{j2\pi(k+1)}}{1 - e^{j(\pi/4)(k+1)}} = \frac{0}{1 - e^{j(\pi/4)(k+1)}} = 0.$$

For $k = 7$, $x[7] = \sum_{n=0}^{7} e^{jn(\pi/4)(7+1)} = \sum_{n=0}^{7} e^{j2n\pi} = \sum_{n=0}^{7} 1 = 8.$

Combining, the above result, we obtain,

$$x[k] = \begin{cases} 0 & 0 \le k \le 6 \\ 8 & k = 7 \end{cases}, \text{ and } \quad x[k+8] = x[k].$$

```
% Problem 11.2, MATLAB Calculation and Verification

% Part (ii)
k = 0:6
x2 = sqrt(5)*cos(2*pi*k/7-0.4636);
x2 = [2.0000, 2.0288, 0.5298, -1.3681, -2.2358, -1.4199, 0.4653]


% Part (iii)
x3_1= zeros(1,7);
x3_2= zeros(1,7);
x3_2(1)= 10.4835;
for k = 0:6
   n = 0:6;
   a =1+(3/4)*sin(n*pi/8);
   b = exp(j*n*2*pi*k/7);
   x3_1(k+1) = sum(a.*b);
end
x3_1
%
for k = 1:6
   a = exp(j*2*pi*k/7);
   b = 0.2870 - 0.8174*a;
   c = 1-1.8478*a + a.*a;
   x3_2(k+1) = -b./c ;
end
x3_2
x3 = 10.4835, -1.0626-j0.3735, -0.3844-j0.1220, -0.2948-j0.0341,
-0.2948+j0.0341, -0.3844+j0.1220, -1.0626+j0.3735

% Part (iv)
x4= zeros(1,8);
for k = 0:7
   n = 0:7;
   b = (-exp(j*pi*k/4)).^n ;
   x4(k+1) = sum(b);
end
% x4 = [0 0 0 0 8 0 0 0] ;
```

```
% Part (v)
x5= zeros(1,8);
for k = 0:7
   n = 0:7;
   b = exp(j*n*pi*(k+1)/4) ;
   x5(k+1) = sum(b);
end
x5
% x5 = [0 0 0 0 0 0 0 8] ;
```

Problem 11.3

(i) $\sum_{k=-\infty}^{\infty} |x[k]| = \sum_{k=-\infty}^{\infty} |2| = \sum_{k=-\infty}^{\infty} 2 = 2 + \lim_{M\to\infty} 2\sum_{k=1}^{M} 2 = 2 + \lim_{M\to\infty} 4M = \infty$ ➔ The DTFT does not exist.

(ii) $\sum_{k=-\infty}^{\infty} |x[k]| = \sum_{k=-2}^{2} \left(3-|k|\right) = 1+2+3+2+1 = 9 < \infty$ ➔ The DTFT exists

(iii) $\sum_{k=-\infty}^{\infty} |x[k]| = \sum_{k=-\infty}^{\infty} \underbrace{\left|k3^{-|k|}\right|}_{=even} = 2\sum_{k=1}^{\infty}\left|k3^{-|k|}\right| = 2\sum_{k=1}^{\infty} k3^{-k} = 2\times\sum_{k=1}^{\infty} k(1/3)^k = 2\times\frac{1/3}{(1-1/3)^2} = \frac{3}{2} < \infty$

Therefore, the DTFT exists.

(iv) $\sum_{k=-\infty}^{\infty} |x[k]| = \sum_{k=-\infty}^{\infty} \left|\alpha^k \cos\left(\omega_0 k\right) u[k]\right| = \sum_{k=0}^{\infty}\left|\alpha^k \cos\left(\omega_0 k\right)\right|$

$$< \sum_{k=0}^{\infty}\left|\alpha^k\right|\left|\cos\left(\omega_0 k\right)\right| \qquad \left[\text{Using Schwartz's inequality}\right]$$

$$\le \sum_{k=0}^{\infty}|\alpha|^k \qquad \left[\because\ \left|\cos\left(\omega_0 k\right)\right| \le 1\right]$$

$$= \frac{1}{1-|\alpha|} < \infty \qquad \left[\because\ |\alpha| < 1\right]$$

Therefore, the DTFT exists.

(v) $\sum_{k=-\infty}^{\infty} |x[k]| = \sum_{k=-\infty}^{\infty} \left|\alpha^k \sin\left(\omega_0 k+\phi\right) u[k]\right| = \sum_{k=0}^{\infty}\left|\alpha^k \sin\left(\omega_0 k+\phi\right)\right| < \sum_{k=0}^{\infty}\left|\alpha^k\right|\underbrace{\left|\sin\left(\omega_0 k+\phi\right)\right|}_{\le 1}$

$$\le \sum_{k=0}^{\infty}|\alpha|^k$$

$$= \frac{1}{1-|\alpha|} < \infty \qquad \left[\because\ |\alpha| < 1\right]$$

Therefore, the DTFT exists.

(vi) $\sum_{k=-\infty}^{\infty} |x[k]| = \sum_{k=-\infty}^{\infty} \underbrace{\left| \frac{\sin\left(\frac{\pi k}{5}\right)\sin\left(\frac{\pi k}{7}\right)}{\pi^2 k^2} \right|}_{=even} = |x[0]| + 2\sum_{k=1}^{\infty} |x[k]|$

Note that $x[0] = \left.\frac{\sin\left(\frac{\pi k}{5}\right)\sin\left(\frac{\pi k}{7}\right)}{\pi^2 k^2}\right|_{k=0} = \left.\frac{\sin\left(\frac{\pi k}{5}\right)\sin\left(\frac{\pi k}{7}\right)}{35.\frac{\pi k}{5}\cdot\frac{\pi k}{7}}\right|_{k=0} = \frac{1}{35}$. Therefore,

$$\begin{aligned}
\sum_{k=-\infty}^{\infty} |x[k]| &= \tfrac{1}{35} + 2\sum_{k=1}^{\infty} |x[k]| = \tfrac{1}{35} + 2\sum_{k=1}^{\infty} \left| \frac{\sin\left(\frac{\pi k}{5}\right)\sin\left(\frac{\pi k}{7}\right)}{\pi^2 k^2} \right| \le \tfrac{1}{35} + 2\sum_{k=1}^{\infty} \frac{\left|\sin\left(\frac{\pi k}{5}\right)\right|\left|\sin\left(\frac{\pi k}{7}\right)\right|}{\pi^2 k^2} \\
&\le \tfrac{1}{35} + 2\sum_{k=1}^{\infty} \frac{1}{\pi^2 k^2} \qquad \left[\because \left|\sin\left(\tfrac{\pi k}{5}\right)\right|, \left|\sin\left(\tfrac{\pi k}{7}\right)\right| \le 1\right] \\
&= \tfrac{1}{35} + \tfrac{2}{\pi^2}\sum_{k=1}^{\infty} \frac{1}{k^2} \quad < \infty
\end{aligned}$$

(vii) $x[k] = \sum_{m=-\infty}^{\infty} \delta[k - 5m - 3]$ is a periodic function with period 5. The sum in any one period, $\sum_{<5>} |x[k]| = 1$. Therefore $\sum_{k=-\infty}^{\infty} |x[k]|$ is infinite, and hence the DTFT does not exist.

(viii) $x[k]$ is a periodic function with period 7. The sum in any one period, $\sum_{<7>} |x[k]| = 9$. Therefore $\sum_{k=-\infty}^{\infty} |x[k]|$ is infinite, and hence the DTFT does not exist.

(ix) $\sum_{k=-\infty}^{\infty} |x[k]| = \sum_{k=-\infty}^{\infty} \underbrace{\left| e^{j(0.2\pi k + 45^0)} \right|}_{=1} = \sum_{k=-\infty}^{\infty} 1 = \infty$ ➔ The DTFT does not exist.

(x) $x[k] = \underbrace{k3^{-k}u[k]}_{=aperiodic} + \underbrace{e^{j(0.2\pi k + 45^0)}}_{=periodic}$. Although, the aperiodic component decays very quickly, the periodic component oscillates forever. Therefore $\sum_{k=-\infty}^{\infty} |x[k]|$ is infinite, and hence the DTFT does not exist. ▮

Problem 11.4

(a)

(i) $x[k] = 2$ is a periodic signal with period $K_0 = 1$. Therefore, we first need to calculate the DTFS coefficients, and then use Eq. (11.36a) to obtain the DTFT.

From Table 11.2, we obtain $D_n = 2$. Using, Eq. (11.36a), the DTFT is obtained as

$$X\left(\Omega\right)=2\pi\sum_{n=-\infty}^{\infty}D_n\delta\left(\Omega-\tfrac{2\pi n}{K_0}\right)=4\pi\sum_{n=-\infty}^{\infty}\delta\left(\Omega-2\pi n\right)$$

(ii) $$X(\Omega)=\sum_{k=-\infty}^{\infty}\underbrace{x[k]}_{=even}\cdot e^{-j\Omega k}=\sum_{k=-2}^{2}\underbrace{x[k]\cdot\cos(\Omega k)}_{=even\ function}-j\underbrace{\sum_{k=-2}^{2}\underbrace{x[k]\cdot\sin(\Omega k)}_{=odd\ function}}_{=0}$$

$$=x[0]+2\sum_{k=1}^{2}x[k]\cdot\cos(\Omega k)=3+2\sum_{k=1}^{2}(3-k)\cdot\cos(\Omega k)$$

$$=3+4\cos(\Omega)+2\cos(2\Omega)$$

(iii) $$X(\Omega)=\sum_{k=-\infty}^{\infty}x[k]\cdot e^{-j\Omega k}=\sum_{k=-\infty}^{\infty}k3^{-|k|}e^{-j\Omega k}=\sum_{k=-\infty}^{-1}k3^{k}e^{-j\Omega k}+\sum_{k=1}^{\infty}k3^{-k}e^{-j\Omega k}$$

$$=-\sum_{k=1}^{\infty}k3^{-k}e^{j\Omega k}+\sum_{k=1}^{\infty}k3^{-k}e^{-j\Omega k}=\sum_{k=1}^{\infty}k\left(\tfrac{1}{3}e^{-j\Omega}\right)^k-\sum_{k=1}^{\infty}k\left(\tfrac{1}{3}e^{j\Omega}\right)^k$$

$$=\frac{\tfrac{1}{3}e^{-j\Omega}}{\left(1-\tfrac{1}{3}e^{-j\Omega}\right)^2}-\frac{\tfrac{1}{3}e^{j\Omega}}{\left(1-\tfrac{1}{3}e^{j\Omega}\right)^2}\qquad\left[\because\sum_{k=1}^{\infty}kr^k=\frac{r}{(1-r)^2}\right]$$

$$=\frac{\tfrac{1}{3}e^{-j\Omega}\left(1-\tfrac{1}{3}e^{j\Omega}\right)^2-\tfrac{1}{3}e^{j\Omega}\left(1-\tfrac{1}{3}e^{-j\Omega}\right)^2}{\left(1-\tfrac{1}{3}e^{-j\Omega}\right)^2\left(1-\tfrac{1}{3}e^{j\Omega}\right)^2}$$

$$=\frac{\tfrac{1}{3}e^{-j\Omega}\left(1-\tfrac{2}{3}e^{j\Omega}+\tfrac{1}{9}e^{j2\Omega}\right)-\tfrac{1}{3}e^{j\Omega}\left(1-\tfrac{2}{3}e^{-j\Omega}+\tfrac{1}{9}e^{-j2\Omega}\right)}{\left(1-\tfrac{1}{3}e^{-j\Omega}-\tfrac{1}{3}e^{j\Omega}+\tfrac{1}{9}\right)^2}$$

$$=\frac{\tfrac{8}{27}\left(e^{-j\Omega}-e^{j\Omega}\right)}{\left(\tfrac{10}{9}-\tfrac{2}{3}\cos(\Omega)\right)^2}=\frac{-\tfrac{16}{27}j\sin(\Omega)}{\left(\tfrac{10}{9}-\tfrac{2}{3}\cos(\Omega)\right)^2}=\frac{-\tfrac{16}{27}j\sin(\Omega)}{\tfrac{100}{81}-\tfrac{40}{27}\cos(\Omega)+\tfrac{4}{9}\cos^2(\Omega)}$$

$$=\frac{-\tfrac{16}{27}j\sin(\Omega)}{\tfrac{100}{81}-\tfrac{40}{27}\cos(\Omega)+\tfrac{2}{9}\left[1+\cos(2\Omega)\right]}=\frac{-\tfrac{16}{27}j\sin(\Omega)}{\tfrac{118}{81}-\tfrac{40}{27}\cos(\Omega)+\tfrac{2}{9}\cos(2\Omega)}$$

(iv) $$X(\Omega)=\sum_{k=-\infty}^{\infty}x[k]\cdot e^{-j\Omega k}=\sum_{k=-\infty}^{\infty}\alpha^k\cos\left(\omega_0k\right)u[k]e^{-j\Omega k}=\sum_{k=0}^{\infty}\alpha^k\cos(\omega_0k)e^{-j\Omega k}$$

$$=\tfrac{1}{2}\sum_{k=0}^{\infty}\alpha^k[e^{j\omega_0k}+e^{-j\omega_0k}]e^{-j\Omega k}=\tfrac{1}{2}\sum_{k=0}^{\infty}\alpha^k\left[e^{-j(\Omega+\omega_0)k}+e^{-j(\Omega-\omega_0)k}\right]$$

$$=\tfrac{1}{2}\sum_{k=0}^{\infty}\left(\alpha e^{-j(\Omega+\omega_0)}\right)^k+\tfrac{1}{2}\sum_{k=0}^{\infty}\left(\alpha e^{-j(\Omega-\omega_0)}\right)^k=\frac{1/2}{1-\alpha e^{-j(\Omega+\omega_0)}}+\frac{1/2}{1-\alpha e^{-j(\Omega-\omega_0)}}$$

$$=\frac{1-\tfrac{\alpha}{2}\left(e^{-j(\Omega+\omega_0)}+e^{-j(\Omega-\omega_0)}\right)}{\left(1-\alpha e^{-j(\Omega+\omega_0)}\right)\left(1-\alpha e^{-j(\Omega-\omega_0)}\right)}=\frac{1-\alpha e^{-j\Omega}\cos(\omega_0)}{\left(1-\alpha e^{-j(\Omega+\omega_0)}\right)\left(1-\alpha e^{-j(\Omega-\omega_0)}\right)}$$

(v) $X(\Omega)=\sum_{k=-\infty}^{\infty}x[k]\cdot e^{-j\Omega k}=\sum_{k=-\infty}^{\infty}\alpha^k\sin\left(\omega_0 k++\phi\right)u[k]e^{-j\Omega k}=\sum_{k=0}^{\infty}\alpha^k\sin\left(\omega_0 k+\phi\right)e^{-j\Omega k}$

$$=\tfrac{1}{2j}\sum_{k=0}^{\infty}\alpha^k[e^{j(\omega_0 k+\phi)}-e^{-(j\omega_0 k+\phi)}]e^{-j\Omega k}=\tfrac{1}{2j}\sum_{k=0}^{\infty}\alpha^k\left[e^{-j(\Omega+\omega_0)k}e^{j\phi}-e^{-j(\Omega-\omega_0)k}e^{-j\phi}\right]$$

$$=\tfrac{1}{2j}e^{j\phi}\sum_{k=0}^{\infty}\left(\alpha e^{-j(\Omega+\omega_0)}\right)^k-\tfrac{1}{2j}e^{-j\phi}\sum_{k=0}^{\infty}\left(\alpha e^{-j(\Omega-\omega_0)}\right)^k=\frac{\tfrac{1}{2j}e^{j\phi}}{1-\alpha e^{-j(\Omega+\omega_0)}}-\frac{\tfrac{1}{2j}e^{-j\phi}}{1-\alpha e^{-j(\Omega-\omega_0)}}$$

$$=\frac{\sin(\phi)+\tfrac{\alpha}{2j}\left[e^{-j(\Omega+\omega_0+\phi)}-e^{-j(\Omega-\omega_0-\phi)}\right]}{\left(1-\alpha e^{-j(\Omega+\omega_0)}\right)\left(1-\alpha e^{-j(\Omega-\omega_0)}\right)}=\frac{\sin(\phi)-\tfrac{\alpha}{2j}e^{-j\Omega}\left[e^{j(\omega_0+\phi)}-e^{-j(\omega_0+\phi)}\right]}{\left(1-\alpha e^{-j(\Omega+\omega_0)}\right)\left(1-\alpha e^{-j(\Omega-\omega_0)}\right)}$$

$$=\frac{\sin(\phi)-\alpha e^{-j\Omega}\sin(\omega_0+\phi)}{\left(1-\alpha e^{-j(\Omega+\omega_0)}\right)\left(1-\alpha e^{-j(\Omega-\omega_0)}\right)}$$

(vi) Using the DTFT pair $\frac{\sin(Wk)}{\pi k}\xleftrightarrow{\text{DTFT}}\begin{cases}1 & |\Omega|\le W\\ 0 & W<|\Omega|\le\pi\end{cases}$, we obtain the following:

$$\frac{\sin(\frac{\pi}{5}k)}{\pi k}\xleftrightarrow{\text{DTFT}}X_1(\Omega)=\begin{cases}1 & |\Omega|\le\frac{\pi}{5}\\ 0 & \frac{\pi}{5}<|\Omega|\le\pi\end{cases};\quad \frac{\sin(\frac{\pi}{7}k)}{\pi k}\xleftrightarrow{\text{DTFT}}X_2(\Omega)=\begin{cases}1 & |\Omega|\le\frac{\pi}{7}\\ 0 & \frac{\pi}{7}<|\Omega|\le\pi\end{cases}$$

Using the frequency convolution property of the DTFT, we obtain

$$X(\Omega)=\Im\left[\left\{\frac{\sin(\frac{\pi}{5}k)}{\pi k}\right\}\times\left\{\frac{\sin(\frac{\pi}{7}k)}{\pi k}\right\}\right]=\frac{1}{2\pi}\int_{-\pi}^{\pi}X_1(\theta)X_2(\Omega-\theta)d\theta$$

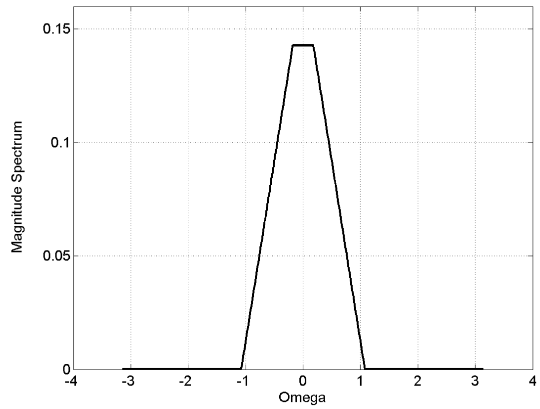
Using the graphical convolution method, it can easily be shown that

$$\int_{-\pi}^{\pi}X_1(\theta)X_2(\Omega-\theta)d\theta=\begin{cases}\Omega+\frac{12\pi}{35} & -\frac{12\pi}{35}\le\Omega<-\frac{2\pi}{35}\\ \frac{12\pi}{35} & -\frac{2\pi}{35}\le\Omega<\frac{2\pi}{35}\\ \Omega-\frac{12\pi}{35} & \frac{2\pi}{35}\le\Omega<\frac{12\pi}{35}\\ 0 & otherwise\end{cases}$$

Therefore, the DTFT $X(\Omega)$ is obtained as:

$$X(\Omega)\Big|_{|\Omega|\le\pi}=\begin{cases}\frac{1}{2\pi}\left(\Omega+\frac{12\pi}{35}\right) & -\frac{12\pi}{35}\le\Omega<-\frac{2\pi}{35}\\ \frac{1}{7} & -\frac{2\pi}{35}\le\Omega<\frac{2\pi}{35}\\ \frac{1}{2\pi}\left(\Omega-\frac{12\pi}{35}\right) & \frac{2\pi}{35}\le\Omega<\frac{12\pi}{35}\\ 0 & otherwise\end{cases}\qquad\text{with}\quad X(\Omega)=X(\Omega+2\pi)$$

The DTFT $X(\Omega)$ is plotted below.




```
% MATLAB program for plotting Fourier spectra in problem 11.4(vi)
X=zeros(1,length([-pi:0.01:pi]));
k = 1:10000 ;

ind=1;
c=2*pi*pi;
for w = -pi:0.01:pi
   X(ind)=1/35;
   w1=w+2*pi/35; w2=w-2*pi/35; w3=w+12*pi/35; w4=w-12*pi/35;
   for k = 1:10000
      X(ind)=X(ind)+(cos(w1*k)+cos(w2*k)-cos(w3*k)-cos(w4*k))/(c*k*k);
   end
   ind = ind+1;
end
%
mag_X   = abs(X);
phase_X = angle(X)
%pha_X = unwrap(phase_X)

Omega = -pi:0.01:pi
plot(Omega, mag_X),grid
xlabel('Omega');
ylabel('Magnitude Spectrum')
%axis([-pi  pi  0  3])
print -dtiff plot.tiff

plot(Omega, phase_X),grid
xlabel('Omega');
ylabel('Phase Spectrum')
%axis([-pi  pi  -pi  pi])
print -dtiff plot.tiff
```

(vii) $x[k]=\sum_{m=-\infty}^{\infty}\delta[k-5m-3]$ is a periodic function with period 5. Therefore, we first need to calculate the DTFS coefficients, and then use Eq. (11.36a) to obtain the DTFT.

$$\Omega_0=\frac{2\pi}{K_0}=\frac{2\pi}{5}.$$

$$D_n = \frac{1}{K_0}\sum_{k=0}^{4} x[k]e^{-jn\Omega_0 k} = \frac{1}{5}\sum_{k=0}^{4}\delta[k-3]e^{-jn\Omega_0 k} = \frac{1}{5}e^{-j3n\Omega_0} = \frac{1}{5}e^{\frac{-j6n\pi}{5}}$$

Using, Eq. (11.36a), the DTFT is obtained as

$$X(\Omega) = 2\pi\sum_{n=-\infty}^{\infty} D_n\delta\left(\Omega - \tfrac{2\pi n}{K_0}\right) = 2\pi\sum_{n=-\infty}^{\infty}\frac{1}{5}e^{\frac{-j6n\pi}{5}}\delta\left(\Omega - 2\pi n\right) = \frac{2\pi}{5}\sum_{n=-\infty}^{\infty} e^{\frac{-j6n\pi}{5}}\delta\left(\Omega - 2\pi n\right)$$

(viii) $x[k] = \begin{cases} 3-|k| & |k| \le 3 \\ 0 & \text{otherwise} \end{cases}$ and $x[k+7] = x[k]$

$x[k]$ is a periodic (and even) function. Fundamental period = 7. Therefore, $\Omega_0 = \frac{2\pi}{7}$. The DTFS coefficients are calculated as follows.

$$D_r = \frac{1}{7}\sum_{k=-3}^{3} x[k]\cdot e^{-jr\Omega_0 k} = \frac{1}{7}\sum_{k=-3}^{3}\underbrace{x[k]\cdot\cos(r\Omega_0 k)}_{=even\ function} - \underbrace{\frac{j}{7}\sum_{k=-4}^{4}\underbrace{x[k]\cdot\sin(r\Omega_0 k)}_{=odd\ function}}_{=0} = \frac{1}{7}\sum_{k=-3}^{3}\left(3-|k|\right)\cdot\cos(r\Omega_0 k)$$

$$= \frac{1}{7}\left[3 + 2\sum_{k=1}^{3}\left(3-k\right)\cdot\cos(r\Omega_0 k)\right] = \frac{3}{7} + \frac{2}{7}\left[2\cos(r\Omega_0) + \cos(2r\Omega_0)\right]$$

$$= \frac{3}{7} + \frac{4}{7}\cos\left(\frac{2\pi r}{7}\right) + \frac{2}{7}\cos\left(\frac{4\pi r}{7}\right)$$

$$= \frac{1}{7}\left[3 + 4\cos\left(\frac{2\pi r}{7}\right) + 2\cos\left(\frac{4\pi r}{7}\right)\right]$$

with $\Omega_0 = \frac{2\pi}{7}$.

The DTFT is then given by the following expression.

$$X(\Omega) = 2\pi\sum_{r=-\infty}^{\infty} D_r\delta\left(\Omega - \tfrac{2\pi r}{K_0}\right) = 2\pi\sum_{r=-\infty}^{\infty}\left\{\frac{3}{7} + \frac{4}{7}\cos\left(\frac{2\pi r}{7}\right) + \frac{2}{7}\cos\left(\frac{4\pi r}{7}\right)\right\}\delta\left(\Omega - \tfrac{2\pi r}{7}\right)$$

(ix) $x[k] = e^{j(0.2\pi k + 45^0)}$

$\Omega = 0.2\pi$, and $K_0 = \frac{2\pi}{\Omega} = \frac{2\pi}{0.2\pi} = 10$. In other words, $x[k]$ is a periodic signal with periodicity 10, and $\Omega_0 = \frac{2\pi}{K_0} = \frac{2\pi}{10} = 0.2\pi$.

$$x[k] = e^{j(0.2\pi k + 45^0)} = e^{j\pi/4}e^{j0.2\pi k} = \frac{1+j}{\sqrt{2}}e^{j0.2\pi k} = \sum_{r=0}^{9} D_r e^{jr\Omega_0 k}$$

Comparing the two left-most expressions we obtain $D_{r,0\le r\le 9} = \begin{cases} \frac{1+j}{\sqrt{2}} & r = 1 \\ 0 & otherwise \end{cases}$.

As $D_n = D_{n+10}$, $D_n = \begin{cases} \dfrac{1+j}{\sqrt{2}} & n = 1 \pm 10m \\ 0 & otherwise \end{cases} = \dfrac{1+j}{\sqrt{2}} \sum_{m=-\infty}^{\infty} \delta[n-10m]$

The DTFT is therefore given by the following expression.

$$X(\Omega) = 2\pi \sum_{n=-\infty}^{\infty} D_n \delta\left(\Omega - \tfrac{2n\pi}{K_0}\right) \qquad \text{where } D_n = \begin{cases} \dfrac{1+j}{\sqrt{2}} & n = 1+10m \\ 0 & otherwise \end{cases}$$

$$= \sqrt{2}\pi(1+j) \sum_{m=-\infty}^{\infty} \delta\left(\Omega - \tfrac{2\pi}{10} - 2\pi m\right)$$

(x) $x[k] = k3^{-k} u[k] + e^{j(0.2\pi k + 45^0)}$

$$X(\Omega) = \Im\left\{k3^{-k} u[k]\right\} + \Im\left\{e^{j(0.2\pi k + 45^0)}\right\} \qquad \left[\text{applying the linearity property}\right]$$

$$= \frac{\frac{1}{3} e^{-j\Omega}}{\left(1 - \frac{1}{3} e^{-j\Omega}\right)^2} + \sqrt{2}\pi(1+j) \sum_{m=-\infty}^{\infty} \delta(\Omega - \tfrac{2\pi}{10} - 2\pi m)$$

(b)

(i) $x[k] = k, \quad \text{for } 0 \le k \le 5 \quad \text{and} \quad x[k+6] = x[k]$

$$D_n = \frac{1}{6} \frac{e^{-jn\pi/3} - 6e^{-j6n\pi/3} + 5e^{-j7n\pi/3}}{(1 - e^{-jn\pi/3})^2}$$

Using, Eq. (11.36a), the DTFT is obtained as, $X(\Omega) = 2\pi \sum_{n=-\infty}^{\infty} D_n \delta\left(\Omega - \tfrac{2\pi n}{6}\right)$.

(ii) $x[k] = \begin{cases} 1 & (0 \le k \le 2) \\ 0.5 & (3 \le k \le 5) \\ 0 & (6 \le k \le 8) \end{cases}$ and $x[k+9] = x[k]$

$$\Omega_0 = \frac{2\pi}{K_0} = \frac{2\pi}{9}.$$

$$D_n = \frac{1}{9}\left(1 + 0.5e^{-j6n\pi/9}\right)\left(1 + e^{-j2n\pi/9} + e^{-j4n\pi/9}\right)$$

Using, Eq. (11.36a), the DTFT is obtained as, $X(\Omega) = 2\pi \sum_{n=-\infty}^{\infty} D_n \delta\left(\Omega - \tfrac{2\pi n}{9}\right)$.

(iii) $x[k] = 3\sin\left(\tfrac{2\pi}{7} k + \tfrac{\pi}{4}\right)$

$$D_n = \begin{cases} -j\tfrac{3}{2}e^{j\frac{\pi}{4}} & \text{for } n=1 \\ j\tfrac{3}{2}e^{-j\frac{\pi}{4}} & \text{for } n=-1 \\ 0 & \text{elsewhere.} \end{cases}, \; D_n = D_{n+7}$$

Using, Eq. (11.36a), the DTFT is obtained as, $X(\Omega) = 2\pi \sum_{n=-\infty}^{\infty} D_n \delta\left(\Omega - \tfrac{2\pi n}{7}\right)$.

(iv) $x[k] = 2e^{j\left(\frac{5\pi}{3}k + \frac{\pi}{4}\right)}$

$$D_n = \begin{cases} 2e^{\frac{j\pi}{4}} & n=5 \\ 0 & 0 \le n \le 4 \end{cases} \text{ and } D_n = D_{n+6}$$

Using, Eq. (11.36a), the DTFT is obtained as, $X(\Omega) = 2\pi \sum_{n=-\infty}^{\infty} D_n \delta\left(\Omega - \tfrac{2\pi n}{6}\right)$.

(v) $x[k] = \sum_{m=-\infty}^{\infty} \delta(k - 5m)$

$$D_n = \frac{1}{5}$$

Using, Eq. (11.36a), the DTFT is obtained as, $X(\Omega) = 2\pi \sum_{n=-\infty}^{\infty} D_n \delta\left(\Omega - \tfrac{2\pi n}{6}\right) = \dfrac{2\pi}{5} \sum_{n=-\infty}^{\infty} \delta\left(\Omega - \tfrac{2\pi n}{5}\right)$.

(vi) $x[k] = \cos(10\pi k/3)\cos(2\pi k/5)$

$$D_n = \begin{cases} 1/4 & n = \pm 2, \pm 7 \\ 0 & otherwise \end{cases}, \; D_n = D_{n+15}$$

Using, Eq. (11.36a), the DTFT is obtained as, $X(\Omega) = 2\pi \sum_{n=-\infty}^{\infty} D_n \delta\left(\Omega - \tfrac{2\pi n}{15}\right)$.

(vii) $x[k] = \left|\cos(2\pi k/3)\right|$.

$$D_n = \begin{cases} 2/3 & n=0 \\ 1/6 & n=1,2 \end{cases}, \; D_n = D_{n+3}$$

Using, Eq. (11.36a), the DTFT is obtained as,

$$\begin{aligned} X(\Omega) &= 2\pi \sum_{n=-\infty}^{\infty} D_n \delta\left(\Omega - \tfrac{2\pi n}{3}\right) \\ &= \frac{4\pi}{3} \sum_{m=-\infty}^{\infty} \delta(\Omega - 2\pi m) + \frac{\pi}{3} \sum_{m=-\infty}^{\infty} \left\{ \delta\left(\Omega - 2\pi m - \frac{2\pi}{3}\right) + \delta\left(\Omega - 2\pi m - \frac{4\pi}{3}\right) \right\} \end{aligned}$$

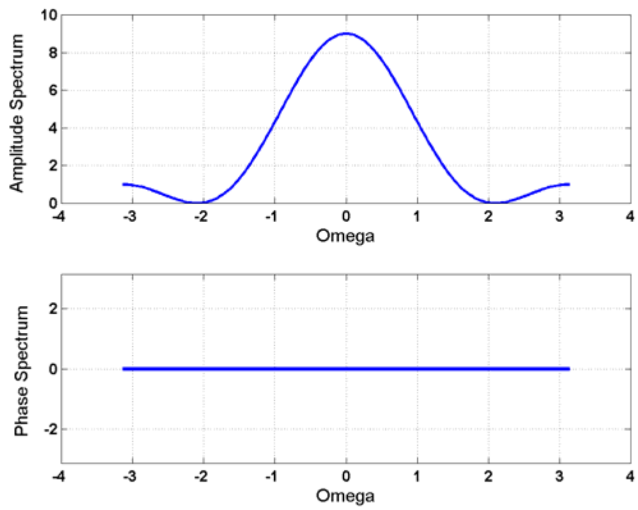
▌

Extras – Spectrum Plots for DTFTs in Problem 11.4

11.4(a)(ii) $X(\Omega) = 3 + 4\cos(\Omega) + 2\cos(2\Omega)$

$$|X(\Omega)| = |3 + 4\cos(\Omega) + 2\cos(2\Omega)|$$

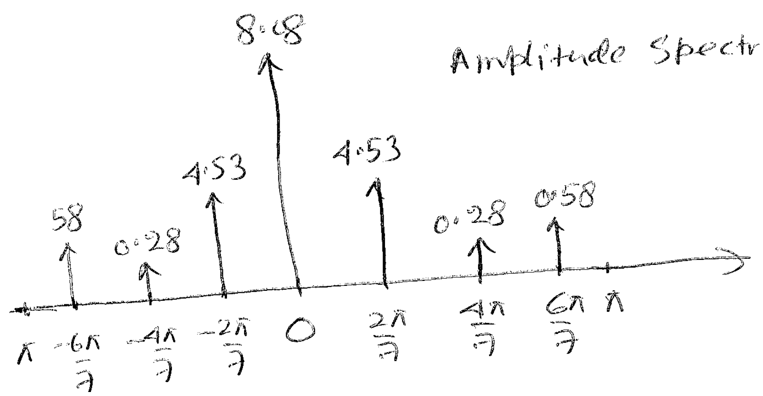
$$\measuredangle X(\Omega) = \begin{cases} 0 & 3 + 4\cos(\Omega) + 2\cos(2\Omega) \geq 0 \\ \pi & 3 + 4\cos(\Omega) + 2\cos(2\Omega) < 0 \end{cases}$$

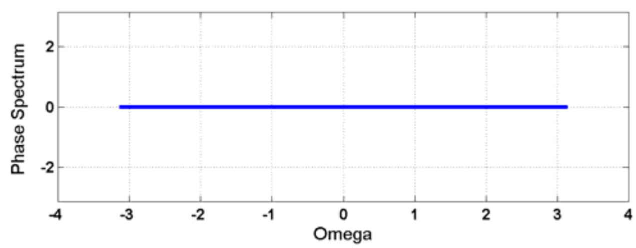


11.4(a)(viii) $X(\Omega) = 2\pi \sum_{r=-\infty}^{\infty} \left\{ \frac{3}{7} + \frac{4}{7}\cos\left(\frac{2\pi r}{7}\right) + \frac{2}{7}\cos\left(\frac{4\pi r}{7}\right) \right\} \delta\left(\Omega - \frac{2\pi r}{7}\right)$

$$|X(\Omega)| = \frac{2\pi}{7} \sum_{r=-\infty}^{\infty} \left| 3 + 4\cos\left(\frac{2\pi r}{7}\right) + 2\cos\left(\frac{4\pi r}{7}\right) \right| \delta\left(\Omega - \frac{2\pi r}{7}\right)$$

$$\measuredangle X(\Omega) = \begin{cases} 0 & \Omega \neq \frac{2\pi r}{7} \\ 0 & 3 + 4\cos(\Omega) + 2\cos(2\Omega) \geq 0, \Omega = \frac{2\pi r}{7} \\ \pi & 3 + 4\cos(\Omega) + 2\cos(2\Omega) < 0, \Omega = \frac{2\pi r}{7} \end{cases}$$



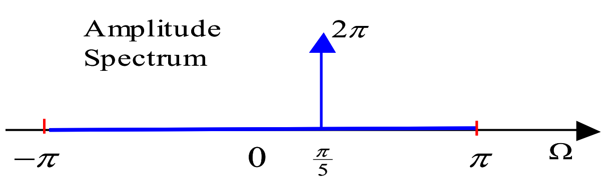


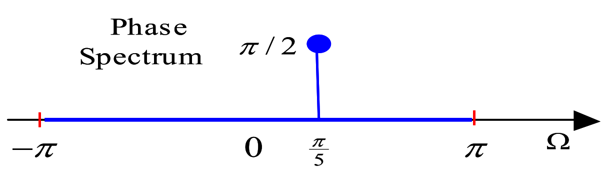
(ix) $X(\Omega) = \sqrt{2}\pi(1+j)\sum_{m=-\infty}^{\infty}\delta\left(\Omega - \tfrac{2\pi}{10} - 2\pi m\right)$

$$\left|X(\Omega)\right| = 2\pi\sum_{m=-\infty}^{\infty}\delta\left(\Omega - \tfrac{\pi}{5} - 2\pi m\right)$$

$$\measuredangle X(\Omega) = \begin{cases} \dfrac{\pi}{2} & \Omega = \tfrac{\pi}{5} - 2\pi m, m \in \mathbb{Z} \\ 0 & otherwise \end{cases}$$

The DTFT spectrum for one period is shown below.





(x) $X(\Omega) = \dfrac{\frac{1}{3}e^{-j\Omega}}{\left(1-\frac{1}{3}e^{-j\Omega}\right)^2} + \sqrt{2}\pi(1+j)\sum_{m=-\infty}^{\infty}\delta(\Omega - \frac{2\pi}{10} - 2\pi m)$

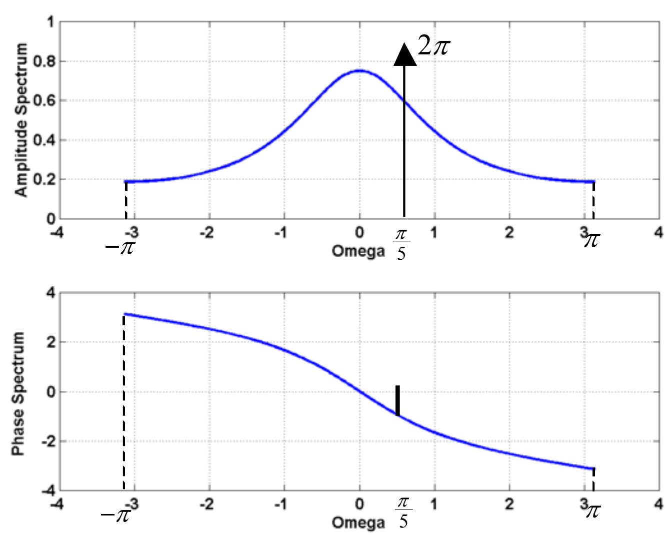
Note that there are two components of $X(\Omega)$. At $\Omega = \frac{2\pi}{10} - 2\pi m$, the Diract delta function will have infinite value, and the first component can be ignored. At other values of Ω, the second component will have zero magnitude, and value of $X(\Omega)$ will be equal to the first component.

Note that $\left|\dfrac{\frac{1}{3}e^{-j\Omega}}{\left(1-\frac{1}{3}e^{-j\Omega}\right)^2}\right| = \left|\dfrac{\frac{1}{3}e^{-j\Omega}}{\left(1-\frac{1}{3}\cos(\Omega)+j\frac{1}{3}\sin(\Omega)\right)^2}\right| = \dfrac{\frac{1}{3}}{\left(1-\frac{1}{3}\cos(\Omega)\right)^2+\frac{1}{9}\sin^2(\Omega)} = \dfrac{3}{10-6\cos(\Omega)}$.

Therefore, $|X(\Omega)| \approx \begin{cases} 2\pi\delta\left(\Omega - \frac{2\pi}{10} - 2\pi m\right) & \Omega = \frac{2\pi}{10} - 2\pi m \\ \dfrac{3}{10-6\cos(\Omega)} & otherwise \end{cases}$

$$\measuredangle X(\Omega) = \measuredangle\left[\frac{\frac{1}{3}e^{-j\Omega}}{\left(1-\frac{1}{3}e^{-j\Omega}\right)^2}\right] + \measuredangle\left[\sqrt{2}\pi(1+j)\delta\left(\Omega - \frac{2\pi}{10} - 2\pi m\right)\right]$$
$$= \begin{cases} \dfrac{\pi}{2} - \Omega - 2\measuredangle\tan^{-1}\left(\dfrac{\sin(\Omega)}{3-\cos(\Omega)}\right) & \Omega = \frac{2\pi}{10} - 2\pi m, m \in \mathbb{Z} \\ -\Omega - 2\measuredangle\tan^{-1}\left(\dfrac{\sin(\Omega)}{3-\cos(\Omega)}\right) & otherwise \end{cases}$$

The DTFT spectrum for one period is shown below.



<u>Problem 11.5:</u>

$$X_1(\Omega)=\sum_{k=-\infty}^{\infty} x_1[k]\cdot e^{-j\Omega k} \quad \text{and} \quad X_2(\Omega)=\sum_{k=-\infty}^{\infty} x_2[k]\cdot e^{-j\Omega k}$$

(i) $$X(\Omega)=\sum_{k=-\infty}^{\infty} x[k]\cdot e^{-j\Omega k}=\sum_{k=-\infty}^{\infty}(-1)^k x_1[k]e^{-j\Omega k}=\sum_{k=-\infty}^{\infty} x_1[k]\left(-e^{-j\Omega}\right)^k$$
$$=\sum_{k=-\infty}^{\infty} x_1[k]\left(e^{-j(\Omega-\pi)}\right)^k=X_1(\Omega-\pi)$$

(ii) $x_2[k]\xleftrightarrow{\text{DTFT}}X_2(\Omega)$

Applying the time shifting property, we obtain, $h[k]=x_2[k-4]\xleftrightarrow{\text{DTFT}}X_2(\Omega)e^{-j4\Omega}=H(\Omega)$.

Using the DTFT pair $kx[k]\xleftrightarrow{\text{DTFT}} j\frac{dX(\Omega)}{d\Omega}$, we obtain the following two DTFT pairs:

$$\Im\left\{\underbrace{kh[k]}_{=g[k]}\right\}=j\frac{dH(\Omega)}{d\Omega}=j\frac{d\left\{X_2(\Omega)e^{-j4\Omega}\right\}}{d\Omega}=j\left[e^{-j4\Omega}\frac{dX_2(\Omega)}{d\Omega}-j4X_2(\Omega)e^{-j4\Omega}\right]$$
$$=\left[j4X_2(\Omega)+\frac{dX_2(\Omega)}{d\Omega}\right]e^{-j4\Omega}=G(\Omega)$$

$$\Im\left\{kg[k]\right\}=\Im\left\{k^2h[k]\right\}=j\frac{dG(\Omega)}{d\Omega}=j\frac{d\left\{\left[j4X_2(\Omega)+\frac{dX_2(\Omega)}{d\Omega}\right]e^{-j4\Omega}\right\}}{d\Omega}$$
$$=j\left[j4e^{-j4\Omega}\frac{dX_2(\Omega)}{d\Omega}+16X_2(\Omega)e^{-j4\Omega}+e^{-j4\Omega}\frac{d^2X_2(\Omega)}{d^2\Omega}-j4e^{-j4\Omega}\frac{dX_2(\Omega)}{d\Omega}\right]$$
$$=\left[16jX_2(\Omega)+j\frac{d^2X_2(\Omega)}{d^2\Omega}\right]e^{-j4\Omega}=j\left[\frac{d^2X_2(\Omega)}{d^2\Omega}+16X_2(\Omega)\right]e^{-j4\Omega}$$

Using the above results, the DTFT of the given signal is now obtained as follows:

$$\Im\left\{(k-5)^2x_2[k-4]\right\}=\Im\left\{(k-5)^2h[k]\right\}=\Im\left\{(k^2-10k+25)h[k]\right\}$$
$$=\Im\left\{k^2h[k]\right\}-10\Im\left\{kh[k]\right\}+25\Im\left\{h[k]\right\}$$
$$=j\left[\frac{d^2X_2(\Omega)}{d^2\Omega}+16X_2(\Omega)\right]e^{-j4\Omega}-10\left[j4X_2(\Omega)+\frac{dX_2(\Omega)}{d\Omega}\right]e^{-j4\Omega}+25X_2(\Omega)e^{-j4\Omega}$$
$$=\left[j\frac{d^2X_2(\Omega)}{d^2\Omega}+j16X_2(\Omega)-j40X_2(\Omega)-10\frac{dX_2(\Omega)}{d\Omega}+25X_2(\Omega)\right]e^{-j4\Omega}$$
$$=\left[j\frac{d^2X_2(\Omega)}{d^2\Omega}-10\frac{dX_2(\Omega)}{d\Omega}+\left(25-j24\right)X_2(\Omega)\right]e^{-j4\Omega}$$

(iii) $x_1[k]\xleftrightarrow{\text{DTFT}}X_1(\Omega)$

Applying the time shifting property, we obtain, $h[k]=x_1[k-3]\xleftrightarrow{\text{DTFT}}X_1(\Omega)e^{-j3\Omega}=H(\Omega)$.

Applying the time inversion property, we obtain, $g[k]=x_1[3-k]\xleftrightarrow{\text{DTFT}}H(-\Omega)=X_1(-\Omega)e^{j3\Omega}=G(\Omega)$.

Applying the time multiplication property, we obtain,

$$p[k]=kg[k]=kx_1[3-k]\xleftrightarrow{\text{DTFT}} j\frac{dG(\Omega)}{d\Omega}=j\frac{d\left\{X_1(-\Omega)e^{j3\Omega}\right\}}{d\Omega}=\left[j\frac{dX_1(-\Omega)}{d\Omega}-3X_1(-\Omega)\right]e^{j3\Omega}=P(\Omega).$$

Now applying the frequency shifting property, we obtain

$$\Im\left\{ke^{-j4k}\,x_1[3-k]\right\} = \Im\left\{e^{-j4k}\,p[k]\right\} = P(\Omega+4) = \left[j\frac{dX_1(-(\Omega+4))}{d\Omega} - 3X_1(-(\Omega+4))\right]e^{j3(\Omega+4)}$$

(iv)
$$\Im\left\{\sum_{m=-\infty}^{\infty}\left[x_1[k-4m]+x_2[k-6m]\right]\right\} = \left\{\sum_{m=-\infty}^{\infty}x_1[k-4m]\right\} + \Im\left\{\sum_{m=-\infty}^{\infty}x_2[k-6m]\right\}$$

$$= \sum_{m=-\infty}^{\infty}\Im\left\{x_1[k-4m]\right\} + \sum_{m=-\infty}^{\infty}\Im\left\{x_2[k-6m]\right\}$$

$$= \sum_{m=-\infty}^{\infty}X_1(\Omega)e^{-j4m\Omega} + \sum_{m=-\infty}^{\infty}X_2(\Omega)e^{-j6m\Omega}$$

$$= X_1(\Omega)\sum_{m=-\infty}^{\infty}e^{-j4m\Omega} + X_2(\Omega)\sum_{m=-\infty}^{\infty}e^{-j6m\Omega}$$

In order to calculate the infinite series sum, we will take help of the DTFT pair

$$\Im\{1\} = \sum_{k=-\infty}^{\infty}e^{-j\Omega k} = \sum_{m=-\infty}^{\infty}e^{-j\Omega m} = 2\pi\sum_{m=-\infty}^{\infty}\delta\left(\Omega-2m\pi\right)$$

Substituting the $\Omega=4\Omega'$ in the above expression, we obtain the following identity:

$$\sum_{m=-\infty}^{\infty}e^{-j4\Omega'\mathrm{m}} = 2\pi\sum_{m=-\infty}^{\infty}\delta\left(4\Omega'-2m\pi\right) = \frac{2\pi}{4}\sum_{m=-\infty}^{\infty}\delta\left(\Omega'-\frac{2m\pi}{4}\right) = \frac{\pi}{2}\sum_{m=-\infty}^{\infty}\delta\left(\Omega'-\frac{m\pi}{2}\right)$$

Similarly, we obtain, $\displaystyle\sum_{m=-\infty}^{\infty}e^{-j6\Omega'\mathrm{m}} = 2\pi\sum_{m=-\infty}^{\infty}\delta\left(6\Omega'-2m\pi\right) = \frac{\pi}{3}\sum_{m=-\infty}^{\infty}\delta\left(\Omega'-\frac{m\pi}{3}\right)$.

Using the above results, we obtain:

$$\Im\left\{\sum_{m=-\infty}^{\infty}\left[x_1[k-4m]+x_2[k-6m]\right]\right\} = X_1(\Omega)\sum_{m=-\infty}^{\infty}e^{-j4m\Omega} + X_2(\Omega)\sum_{m=-\infty}^{\infty}e^{-j6m\Omega}$$

$$= \tfrac{\pi}{2}X_1(\Omega)\sum_{m=-\infty}^{\infty}\delta\left(\Omega-\tfrac{m\pi}{2}\right) + \tfrac{\pi}{3}X_2(\Omega)\sum_{m=-\infty}^{\infty}\delta\left(\Omega-\tfrac{m\pi}{3}\right)$$

$$= \tfrac{\pi}{2}\sum_{m=-\infty}^{\infty}X_1(\Omega-\tfrac{m\pi}{2})\delta\left(\Omega-\tfrac{m\pi}{2}\right) + \tfrac{\pi}{3}\sum_{m=-\infty}^{\infty}X_2(\Omega-\tfrac{m\pi}{3})\delta\left(\Omega-\tfrac{m\pi}{3}\right).$$

(v) Using the time-shifting and time-inversion properties, we

Applying the time shifting property, we obtain, $h[k] = x_1[k-5] \xleftrightarrow{\text{DTFT}} X_2(\Omega)e^{-j5\Omega} = H(\Omega)$.

Applying the time inversion property, we obtain,

$$g[k] = x_1[5-k] = h[-k] \xleftrightarrow{\text{DTFT}} H(-\Omega) = X_1(-\Omega)e^{j5\Omega} = G(\Omega).$$

Similarly, we obtain the pair, $p[k] = x_2[7-k] \xleftrightarrow{\text{DTFT}} X_2(-\Omega)e^{j7\Omega} = P(\Omega)$

Applying the frequency convolution property, the DTFT of the given signal is obtained as:

$$\Im\{x_1[5-k]x_2[7-k]\} = \Im\{g[k]p[k]\} = \tfrac{1}{2\pi}\int_{(2\pi)} G(\theta)P(\Omega-\theta)d\theta$$

$$= \tfrac{1}{2\pi}\int_{(2\pi)} X_1(-\theta)e^{j5\theta}X_2(\theta-\Omega)e^{j7(\Omega-\theta)}d\theta$$

$$= \tfrac{1}{2\pi}e^{j7\Omega}\int_{(2\pi)} X_1(-\theta)X_2(\theta-\Omega)e^{-j2\theta}d\theta$$

▌

Problem 11.6

(i) $$X(\Omega) = \frac{4e^{-j\Omega}}{1-5e^{-j\Omega}+6e^{-j2\Omega}} = \frac{4e^{-j\Omega}}{\left(1-2e^{-j\Omega}\right)\left(1-3e^{-j\Omega}\right)} = \frac{A}{1-2e^{-j\Omega}} + \frac{B}{1-3e^{-j\Omega}}$$

where $A = \left.\dfrac{4e^{-j\Omega}}{1-3e^{-j\Omega}}\right|_{2e^{-j\Omega}=1} = \dfrac{4/2}{1-1.5} = -4$, $B = \left.\dfrac{4e^{-j\Omega}}{1-2e^{-j\Omega}}\right|_{3e^{-j\Omega}=1} = \dfrac{4/3}{1-\frac{2}{3}} = 4$

Substituting the values of *A* and *B*, we get

$$X(\Omega) = -\frac{4}{1-2e^{-j\Omega}} + \frac{4}{1-3e^{-j\Omega}}$$

Using the DTFT pair $-\alpha^k u[-k-1] \xleftrightarrow{\text{DTFT}} \frac{1}{1-\alpha e^{-j\Omega}}$ $\quad |\alpha|>1$, the sequence $x[k]$ is obtained as follows.

$$x[k] = 4\times 2^k u[-k-1] - 4\times 3^k u[-k-1] = 4(2^k - 3^k)u[-k-1]$$

(ii) $$X(\Omega) = \frac{2e^{-j2\Omega}}{\left(1-4e^{-j\Omega}\right)^2\left(1-2e^{-j\Omega}\right)} = \frac{A}{1-4e^{-j\Omega}} + \frac{B}{\left(1-4e^{-j\Omega}\right)^2} + \frac{C}{1-2e^{-j\Omega}}$$

where $B = \left.\dfrac{2e^{-j2\Omega}}{1-2e^{-j\Omega}}\right|_{e^{-j\Omega}=1/4} = \dfrac{1/8}{1-1/2} = \dfrac{1}{4}$, $C = \left.\dfrac{2e^{-j2\Omega}}{\left(1-4e^{-j\Omega}\right)^2}\right|_{e^{-j\Omega}=1/2} = \dfrac{1/2}{(1-2)^2} = \dfrac{1}{2}$.

Substituting $B=\dfrac{1}{4}$ and $C=\dfrac{1}{2}$ in $X(\Omega)$ expression and comparing the numerators, we get $A=-\dfrac{3}{4}$.

Substituting the values of *A* and *B* and *C*, we get

$$X(\Omega) = \frac{-3/4}{1-4e^{-j\Omega}} + \frac{1/4}{\left(1-4e^{-j\Omega}\right)^2} + \frac{1/2}{1-2e^{-j\Omega}}$$

Using the DTFT pairs

$$-\alpha^k u[-k-1] \xleftrightarrow{\text{DTFT}} \frac{1}{1-\alpha e^{-j\Omega}} \quad |\alpha|>1, \text{ and } -(k+1)\alpha^k u[-k-2] \xleftrightarrow{\text{DTFT}} \frac{1}{\left(1-\alpha e^{-j\Omega}\right)^2} \quad |\alpha|>1,$$

the sequence $x[k]$ is obtained as follows:

$$x[k]=\left(\tfrac{3}{4}4^k-\tfrac{1}{2}2^k\right)u[-k-1]-\tfrac{1}{4}(k+1)4^k u[-k-2]$$
$$=-\tfrac{1}{16}\delta[k+1]+\left\{\tfrac{3}{4}4^k-\tfrac{1}{2}2^k-\tfrac{1}{4}(k+1)4^k\right\}u[-k-2]$$
$$=-\tfrac{1}{16}\delta[k+1]-\tfrac{1}{2}\left\{2^k-(1-k/2)4^k\right\}u[-k-2].$$

(iii) $X(\Omega)=8\sin(7\Omega)\cos(9\Omega)=4\sin(16\Omega)-4\sin(2\Omega)=-j2\left[e^{j16\Omega}+e^{j2\Omega}-e^{-j2\Omega}-e^{-j16\Omega}\right]$

Using the DTFT pair $\delta[k-k_0]\xleftrightarrow{\text{DTFT}}e^{-j\Omega k_0}$, the sequence $x[k]$ is obtained as:

$$x[k]=-j2\left\{\delta[k+16]+\delta[k+2]-\delta[k-2]-\delta[k-16]\right\}$$

(iv) $X(\Omega)=\dfrac{4e^{-j4\Omega}}{10-6\cos\Omega}=\dfrac{4e^{-j4\Omega}}{10-3(e^{j\Omega}+e^{-j\Omega})}=\dfrac{-4e^{-j5\Omega}}{3-10e^{-j\Omega}+3e^{-j2\Omega}}=\dfrac{-4e^{-j5\Omega}}{(3-e^{-j\Omega})(1-3e^{-j\Omega})}$

$$=-4e^{-j5\Omega}\left[\frac{-1/8}{3-e^{-j\Omega}}+\frac{3/8}{1-3e^{-j\Omega}}\right]=\frac{1}{6}\cdot\frac{e^{-j5\Omega}}{1-(1/3)e^{-j\Omega}}-\frac{3}{2}\cdot\frac{e^{-j5\Omega}}{1-3e^{-j\Omega}}$$

Using the DTFT pairs $p^k u[k]\xleftrightarrow{\text{DTFT}}\frac{1}{1-pe^{-j\Omega}}\quad |p|<1$, $-\alpha^k u[-k-1]\xleftrightarrow{\text{DTFT}}\frac{1}{1-\alpha e^{-j\Omega}}\quad |\alpha|>1$ and the time shifting property of the Fourier transform, the sequence $x[k]$ is obtained as follows.

$$x[k]=\left\{\tfrac{1}{6}(\tfrac{1}{3})^{k'}u[k']+\tfrac{3}{2}(3)^{k'}u[-k'-1]\right\}_{k'=k-5}=\tfrac{1}{6}(\tfrac{1}{3})^{k-5}u[k-5]+\tfrac{3}{2}(3)^{k-5}u[-k+4]$$

(v) $x[k]=\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi}\underbrace{X(\Omega)}_{=even}e^{jk\Omega}d\Omega=\dfrac{1}{2\pi}\int_{-\pi}^{\pi}\underbrace{X(\Omega)\cos(\Omega k)}_{=even}d\Omega+\underbrace{\dfrac{j}{2\pi}\int_{-\pi}^{\pi}\underbrace{X(\Omega)\sin(\Omega k)}_{=odd}d\Omega}_{=0}$

$$=\frac{2}{2\pi}\int_{0}^{\pi}X(\Omega)\cos(\Omega k)d\Omega=\frac{1}{\pi}\int_{\pi/4}^{3\pi/4}\cos(\Omega k)d\Omega=\frac{1}{\pi}\left[\frac{\sin(\Omega k)}{k}\right]_{0.25\pi}^{3\pi/4}$$
$$=\frac{1}{\pi k}\left[\sin(3\pi k/4)-\sin(\pi k/4)\right]$$

Alternative Solution:

$X(\Omega)=X_1(\Omega)-X_2(\Omega)$

where $X_1(\Omega)=\begin{cases}1 & |\Omega|<0.75\pi\\ 0 & 0.75\pi\le|\Omega|<\pi\end{cases}$ and $X_2(\Omega)=\begin{cases}1 & |\Omega|<0.25\pi\\ 0 & 0.25\pi\le|\Omega|<\pi\end{cases}$.

Therefore,

$$x[k]=\Im^{-1}\left\{X_1(\Omega)-X_2(\Omega)\right\}=\Im^{-1}\left\{X_1(\Omega)\right\}-\Im^{-1}\left\{X_2(\Omega)\right\}=\frac{1}{\pi k}\left[\sin(3\pi k/4)-\sin(\pi k/4)\right]\quad\blacksquare$$

Problem 11.7:

(a) Assuming $x[k]$ is a real-valued function, the Hermitian symmetry property can be proved as follows.

$$X^*(\Omega) = \left[X(\Omega)\right]^* = \left[\sum_{k=-\infty}^{\infty} x[k]e^{-j\Omega k}\right]^* = \sum_{k=-\infty}^{\infty}\left[x[k]e^{-j\Omega k}\right]^*$$

$$= \sum_{k=-\infty}^{\infty} x[k]\left[e^{-j\Omega k}\right]^* \qquad \left[\because x[k] \text{ is real-valued function}\right]$$

$$= \sum_{k=-\infty}^{\infty} x[k]e^{j\Omega k}$$

$$= X(-\Omega)$$

(b)

(i) $X(-\Omega) = \dfrac{4e^{j\Omega}}{1-5e^{j\Omega}+6e^{j2\Omega}}$..

$$X^*(\Omega) = \frac{4e^{j\Omega}}{1-5e^{j\Omega}+6e^{j2\Omega}}.$$

Since $X^*(\Omega) = X(-\Omega)$, the DT sequence is real-valued.

(ii) $X(-\Omega) = \dfrac{2e^{j2\Omega}}{\left(1-4e^{j\Omega}\right)^2\left(1-2e^{j\Omega}\right)}$; $\quad X^*(\Omega) = \dfrac{2e^{j2\Omega}}{\left(1-4e^{j\Omega}\right)^2\left(1-2e^{j\Omega}\right)}$

Since $X^*(\Omega) = X(-\Omega)$, the DT sequence is real-valued.

(iii) $X(-\Omega) = 8\sin(-7\Omega)\cos(-9\Omega) = -8\sin(7\Omega)\cos(9\Omega)$; $X^*(\Omega) = 8\sin(7\Omega)\cos(9\Omega)$.

Since $X^*(\Omega) \neq X(-\Omega)$, the DT sequence is not real-valued.

(iv) $X(-\Omega) = \dfrac{4e^{j4\Omega}}{10-6\cos(-\Omega)} = \dfrac{4e^{j4\Omega}}{10-6\cos(\Omega)}$; $X^*(\Omega) = \dfrac{4e^{j4\Omega}}{10-6\cos(\Omega)}$.

Since $X^*(\Omega) = X(-\Omega)$, the DT sequence is real-valued.

(v) $X(-\Omega) = X(\Omega) = X^*(\Omega)$, and therefore the DT sequence is real valued. ▌

Problem 11.8:

$$\frac{dX(\Omega)}{d\Omega} = \frac{d}{d\Omega}\left[\sum_{k=-\infty}^{\infty} x[k]e^{-j\Omega k}\right] = \sum_{k=-\infty}^{\infty} \frac{d}{d\Omega}\left\{x[k]e^{-j\Omega k}\right\} = \sum_{k=-\infty}^{\infty} x[k]\frac{d}{d\Omega}\left\{e^{-j\Omega k}\right\}$$

$$= \sum_{k=-\infty}^{\infty} x[k](-jk)e^{-j\Omega k} = \sum_{k=-\infty}^{\infty} (-jkx[k])e^{-j\Omega k}$$

$$= \Im\left\{-jkx[k]\right\}$$

In other words, $-jkx[k] \xleftrightarrow{\text{DTFT}} \frac{dX}{d\Omega}$.

Problem 11.9

$$\Im\left\{x[k]*h[k]\right\} = \sum_{k=-\infty}^{\infty} \left\{x[k]*h[k]\right\}e^{-j\Omega k} = \sum_{k=-\infty}^{\infty}\left(\sum_{m=-\infty}^{\infty} x[m]h[k-m]\right)e^{-j\Omega k} = \sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} x[m]h[k-m]e^{-j\Omega k}$$

$$= \sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} x[m]h[k-m]e^{-j\Omega k} = \sum_{m=-\infty}^{\infty} x[m]\underbrace{\sum_{k=-\infty}^{\infty} h[k-m]e^{-j\Omega k}}_{=\Im\{h[k-m])}$$

$$= \sum_{m=-\infty}^{\infty} x[m]H(\Omega)e^{-j\Omega m} = H(\Omega)\underbrace{\sum_{m=-\infty}^{\infty} x[m]e^{-j\Omega m}}_{=X(\Omega)}$$

$$= X(\Omega)H(\Omega)$$

Problem 11.10

$$\Im\left\{x[k-k_0]\right\} = \sum_{k=-\infty}^{\infty} x[k-k_0]e^{-j\Omega k} = \sum_{k=-\infty}^{\infty} x[k-k_0]e^{-j\Omega(k-k_0)}e^{-j\Omega k_0}$$

$$= e^{-j\Omega k_0}\sum_{m=-\infty}^{\infty} x[m]e^{-j\Omega m} \qquad \left[\text{substituting } k-k_0 = m\right]$$

$$= e^{-j\Omega k_0}X(\Omega)$$

Problem 11.11:

(i) $$H(\Omega) = \frac{1}{\left(1-0.3e^{-j\Omega}\right)\left(1-0.5e^{-j\Omega}\right)\left(1-0.7e^{-j\Omega}\right)} = \frac{k_1}{1-0.3e^{-j\Omega}} + \frac{k_2}{1-0.5e^{-j\Omega}} + \frac{k_3}{1-0.7e^{-j\Omega}}$$

where the partial fraction coefficients are calculated as

$$k_1 = \left.\frac{1}{\left(1-0.5e^{-j\Omega}\right)\left(1-0.7e^{-j\Omega}\right)}\right|_{0.3e^{-j\Omega}=1} = \frac{1}{(1-\frac{0.5}{0.3})(1-\frac{0.7}{0.3})} = \frac{9}{8}$$

$$k_2 = \left.\frac{1}{\left(1-0.3e^{-j\Omega}\right)\left(1-0.7e^{-j\Omega}\right)}\right|_{0.5e^{-j\Omega}=1} = \frac{1}{(1-\frac{0.3}{0.5})(1-\frac{0.7}{0.5})} = -\frac{25}{4}$$

$$k_3 = \left.\frac{1}{\left(1-0.3e^{-j\Omega}\right)\left(1-0.5e^{-j\Omega}\right)}\right|_{0.7e^{-j\Omega}=1} = \frac{1}{(1-\frac{0.3}{0.7})(1-\frac{0.5}{0.7})} = \frac{49}{8}$$

In other words, $H(\Omega) = \dfrac{9/8}{1-0.3e^{-j\Omega}} + \dfrac{-25/4}{1-0.5e^{-j\Omega}} + \dfrac{49/8}{1-0.7e^{-j\Omega}}$

Calculating the inverse DTFT of each partial fractions, $h[k]$ is obtained as

$$h[k] = \left[\tfrac{9}{8}(0.3)^k - \tfrac{25}{4}(0.5)^k + \tfrac{49}{8}(0.7)^k\right]u[k]$$

(ii) $$H(\Omega) = \frac{1}{\left(1-0.3e^{-j\Omega}\right)\left(1-0.5e^{-j\Omega}\right)\left(1-0.7e^{-j\Omega}\right)} = \frac{1}{\left(1-0.8e^{-j\Omega}+0.15e^{-j2\Omega}\right)\left(1-0.7e^{-j\Omega}\right)}$$

$$= \frac{1}{1-1.5e^{-j\Omega}+0.71e^{-j2\Omega}-0.105e^{-j3\Omega}} = \frac{Y(\Omega)}{X(\Omega)}$$

Or, $\left(1-1.5e^{-j\Omega}+0.71e^{-j2\Omega}-0.105e^{-j3\Omega}\right)Y(\Omega) = X(\Omega)$

Calculating the inverse DTFT of both sides, the input-output relationship is obtained as

$$y[k]-1.5y[k+1]+0.71y[k+2]-0.105y[k+3] = x[k]$$

(iii) $$X(\Omega)\big|_{|\Omega|\le\pi} = \pi\delta(\Omega) + \frac{1}{1-e^{-j\Omega}}$$

$$H(\Omega) = \frac{1}{\left(1-0.3e^{-j\Omega}\right)\left(1-0.5e^{-j\Omega}\right)\left(1-0.7e^{-j\Omega}\right)}$$

$$Y(\Omega)\big|_{|\Omega|\le\pi} = X(\Omega)H(\Omega) = \left[\pi\delta(\Omega) + \frac{1}{1-e^{-j\Omega}}\right] \times \frac{1}{\left(1-0.3e^{-j\Omega}\right)\left(1-0.5e^{-j\Omega}\right)\left(1-0.7e^{-j\Omega}\right)}$$

$$= \underbrace{\frac{\pi\delta(\Omega)}{\left(1-0.3e^{-j\Omega}\right)\left(1-0.5e^{-j\Omega}\right)\left(1-0.7e^{-j\Omega}\right)}}_{=A(\Omega)} + \underbrace{\frac{1}{\left(1-0.3e^{-j\Omega}\right)\left(1-0.5e^{-j\Omega}\right)\left(1-0.7e^{-j\Omega}\right)\left(1-e^{-j\Omega}\right)}}_{=B(\Omega)}$$

Using the properties of continuous impulse function $\delta(\cdot)$, $A(\Omega)$ can be simplified as:

$$A(\Omega) = \frac{\pi\delta(\Omega)}{(1-0.3)(1-0.5)(1-0.7)} = \tfrac{200}{21}\pi\delta(\Omega)$$

The function $B(\Omega)$ can be decomposed into four partial fractions as follows.

$$B(\Omega) = \frac{1}{\left(1-0.3e^{-j\Omega}\right)\left(1-0.5e^{-j\Omega}\right)\left(1-0.7e^{-j\Omega}\right)\left(1-e^{-j\Omega}\right)}$$

$$= \frac{k_1}{1-0.3e^{-j\Omega}} + \frac{k_2}{1-0.5e^{-j\Omega}} + \frac{k_3}{1-0.7e^{-j\Omega}} + \frac{k_4}{1-e^{-j\Omega}}$$

where the partial fraction coefficients are calculated as

$$k_1 = \left.\frac{1}{\left(1-0.5e^{-j\Omega}\right)\left(1-0.7e^{-j\Omega}\right)\left(1-e^{-j\Omega}\right)}\right|_{0.3e^{-j\Omega}=1} = \frac{1}{(1-\frac{0.5}{0.3})(1-\frac{0.7}{0.3})(1-\frac{1}{0.3})} = -\frac{27}{56}$$

$$k_2 = \frac{1}{\left(1-0.3e^{-j\Omega}\right)\left(1-0.7e^{-j\Omega}\right)\left(1-e^{-j\Omega}\right)}\Bigg|_{0.5e^{-j\Omega}=1} = \frac{1}{(1-\frac{0.3}{0.5})(1-\frac{0.7}{0.5})(1-\frac{1}{0.5})} = \frac{25}{4}$$

$$k_3 = \frac{1}{\left(1-0.3e^{-j\Omega}\right)\left(1-0.5e^{-j\Omega}\right)\left(1-e^{-j\Omega}\right)}\Bigg|_{0.7e^{-j\Omega}=1} = \frac{1}{(1-\frac{0.3}{0.7})(1-\frac{0.5}{0.7})(1-\frac{1}{0.7})} = -\frac{343}{24}$$

$$k_4 = \frac{1}{\left(1-0.3e^{-j\Omega}\right)\left(1-0.5e^{-j\Omega}\right)\left(1-0.7e^{-j\Omega}\right)}\Bigg|_{e^{-j\Omega}=1} = \frac{1}{(1-0.3)(1-0.5)(1-0.7)} = \frac{200}{21}$$

$$Y(\Omega)\Big|_{|\Omega|\le\pi} = \tfrac{200}{21}\pi\delta(\Omega) + \frac{-27/56}{1-0.3e^{-j\Omega}} + \frac{25/4}{1-0.5e^{-j\Omega}} + \frac{-343/24}{1-0.7e^{-j\Omega}} + \frac{200/21}{1-e^{-j\Omega}}$$

$$= \tfrac{200}{21}\left[\pi\delta(\Omega) + \frac{1}{1-e^{-j\Omega}}\right] + \frac{-27/56}{1-0.3e^{-j\Omega}} + \frac{25/4}{1-0.5e^{-j\Omega}} + \frac{-343/24}{1-0.7e^{-j\Omega}}$$

$$y[k] = \left[\tfrac{200}{21} - \tfrac{27}{56}(0.3)^k + \tfrac{25}{4}(0.5)^k - \tfrac{343}{24}(0.7)^k\right]u[k]$$

(iv) Note that $\{u[k]\} * \{b^k u[k]\} = \begin{cases}(k+1)u[k] & b=1\\ \frac{1}{1-b}\left[1-b^{k+1}\right]u[k] & b\ne 1\end{cases}$

Therefore,

$$y[k] = x[k] * h[k]$$

$$= u[k] * \left\{\left[\tfrac{9}{8}(0.3)^k - \tfrac{25}{4}(0.5)^k + \tfrac{49}{8}(0.7)^k\right]u[k]\right\}$$

$$= \tfrac{9}{8}\left\{u[k]*(0.3)^k u[k]\right\} - \tfrac{25}{4}\left\{u[k]*(0.5)^k u[k]\right\} + \tfrac{49}{8}\left\{u[k]*(0.7)^k u[k]\right\}$$

$$= \left\{\frac{9}{8}\cdot\frac{1-(0.3)^{k+1}}{1-0.3} - \frac{25}{4}\cdot\frac{1-(0.5)^{k+1}}{1-0.5} + \frac{49}{8}\cdot\frac{1-(0.7)^{k+1}}{1-0.7}\right\}u[k]$$

$$= \left\{\frac{45}{28}\left(1-(0.3)^{k+1}\right) - \frac{25}{2}\left(1-(0.5)^{k+1}\right) + \frac{245}{12}\left(1-(0.7)^{k+1}\right)\right\}u[k]$$

$$= \left\{\left(\frac{45}{28} - \frac{25}{2} + \frac{245}{12}\right) - \frac{45}{28}(0.3)^{k+1} + \frac{25}{2}(0.5)^{k+1} - \frac{245}{12}(0.7)^{k+1}\right\}u[k]$$

$$= \left\{\frac{200}{21} - \frac{45\times0.3}{28}0.3^k + \frac{25\times0.5}{2}0.5^k - \frac{245\times0.7}{12}0.7^k\right\}u[k]$$

$$= \left[\tfrac{200}{21} - \tfrac{27}{56}(0.3)^k + \tfrac{25}{4}(0.5)^k - \tfrac{343}{24}(0.7)^k\right]u[k]$$

Note that the results obtained in parts (iii) and (iv) are identical. ▌

Problem 11.12:

(i) $y[k]+y[k-1]+\frac{1}{4}y[k-2]=x[k]-x[k-2]$

Calculating the DTFT of both sides and rearranging the terms (see section 11.6), the transfer function can be expressed as

$$H(\Omega)=\frac{1-e^{-j2\Omega}}{1+e^{-j\Omega}+0.5e^{-j2\Omega}}=\frac{1-e^{-j2\Omega}}{\left(1+\frac{1}{2}e^{-j\Omega}\right)^2}=\frac{1}{\left(1+0.5e^{-j\Omega}\right)^2}-\frac{1}{\left(1+0.5e^{-j\Omega}\right)^2}\cdot e^{-j2\Omega}.$$

(ii) Rearranging the terms, the transfer function $H(\Omega)$ can be expressed as

$$H(\Omega)=\frac{1-e^{-j2\Omega}}{\left(1+\frac{1}{2}e^{-j\Omega}\right)^2}=\frac{1}{\left(1+0.5e^{-j\Omega}\right)^2}-\frac{1}{\left(1+0.5e^{-j\Omega}\right)^2}\cdot e^{-j2\Omega}.$$

Using Entry 7 of Table 11.2, we get the following DTFT pair

$$\left(k+1\right)(-0.5)^k u[k]\leftrightarrow\frac{1}{\left(1+0.5e^{-j\Omega}\right)^2}$$

Applying the DTFT time shifting property (see Eq. 11.42) on the above DTFT pair, we get

$$\left(k'+1\right)(-0.5)^{k'}u[k']\Big|_{k'=k-2}\leftrightarrow\frac{1}{\left(1+0.5e^{-j\Omega}\right)^2}\cdot e^{-j2\Omega}$$

Or,
$$\left(k-1\right)(-0.5)^{k-2}u[k-2]\leftrightarrow\frac{1}{\left(1+0.5e^{-j\Omega}\right)^2}\cdot e^{-j2\Omega}$$

By calculating the inverse DTFT of $H(\Omega)$, the impulse response $h[k]$ is given by

$$\begin{aligned}h[k]&=\left(k+1\right)(-0.5)^k u[k]-\left(k-1\right)(-0.5)^{k-2}u[k-2]\\&=-4\delta[k]-(3k-5)(-0.5)^k u[k]\end{aligned}$$

(iii) $y[k]=y[k]*h[k]=\left\{-4\delta[k]-(3k-5)(-0.5)^k u[k]\right\}*\left\{0.5^k u[k]\right\}$

$$\begin{aligned}&=-4\times0.5^k u[k]-\left\{(3k-5)(-0.5)^k u[k]\right\}*\left\{0.5^k u[k]\right\}\\&=-4\times0.5^k u[k]-3\left\{k(-0.5)^k u[k]\right\}*\left\{0.5^k u[k]\right\}+5\left\{(-0.5)^k u[k]\right\}*\left\{0.5^k u[k]\right\}\end{aligned}$$

Using the following two known convolution results

$$\left(ka^k u[k]\right)*\left(b^k u[k]\right)=\begin{cases}\frac{k(k+1)}{2}a^k u[k] & a=b\\ \frac{a}{(a-b)^2}\left[ka^{k+1}-(k+1)a^k b+b^{k+1}\right]u[k] & a\neq b.\end{cases}$$

$$\left(a^k u[k]\right)*\left(b^k u[k]\right)=\begin{cases}(k+1)a^k u[k] & a=b\\ \frac{1}{a-b}\left(a^{k+1}-b^{k+1}\right)u[k] & a\neq b,\end{cases}$$

the sequence $y[k]$ is obtained as follows

$$\begin{aligned}
y[k] &= -4\times 0.5^k u[k] - 3\left\{k(-0.5)^k u[k]\right\} * \left\{0.5^k u[k]\right\} + 5\left\{(-0.5)^k u[k]\right\} * \left\{0.5^k u[k]\right\} \\
&= -4\times 0.5^k u[k] + 1.5\left[k(-0.5)^{k+1} - 0.5(k+1)(-0.5)^k + 0.5^{k+1}\right]u[k] + 5\left[0.5^{k+1} - (-0.5)^{k+1}\right]u[k] \\
&= \left\{-4\times 0.5^k + \left[1.5k(-0.5)^{k+1} - 0.75(k+1)(-0.5)^k + 1.5\times 0.5^{k+1}\right] + \left[5\times 0.5^{k+1} - 5\times(-0.5)^{k+1}\right]\right\}u[k] \\
&= \left\{(-4+0.75+2.5)0.5^k + (-0.75k - 0.75k - 0.75 + 2.5)(-0.5)^k\right\}u[k] \\
&= \left\{-0.75\times 0.5^k + (1.75 - 1.5k+)(-0.5)^k\right\}u[k]
\end{aligned}$$

(iv) $X(\Omega) = \dfrac{1}{1-\frac{1}{2}e^{-j\Omega}}$; $H(\Omega) = \dfrac{1-e^{-j2\Omega}}{\left(1+\frac{1}{2}e^{-j\Omega}\right)^2}$

$$Y(\Omega) = X(\Omega)H(\Omega) = \frac{1-e^{-j2\Omega}}{\left(1-\frac{1}{2}e^{-j\Omega}\right)\left(1+\frac{1}{2}e^{-j\Omega}\right)^2} = \frac{A}{1-\frac{1}{2}e^{-j\Omega}} + \frac{B}{1+\frac{1}{2}e^{-j\Omega}} + \frac{C}{\left(1+\frac{1}{2}e^{-j\Omega}\right)^2}$$

$$A = \left.\frac{1-e^{-j2\Omega}}{\left(1+\frac{1}{2}e^{-j\Omega}\right)^2}\right|_{e^{-j\Omega}=2} = -\frac{3}{4},\ C = \left.\frac{1-e^{-j2\Omega}}{\left(1-\frac{1}{2}e^{-j\Omega}\right)}\right|_{e^{-j\Omega}=-2} = -\frac{3}{2}.$$

Substituting $A = -\dfrac{3}{4}$ and $C = -\dfrac{3}{2}$ in $X(\Omega)$ expression and comparing the numerators, we get $B = \dfrac{13}{4}$.

Substituting the values of *A* and *B* and *C*, we get

$$X(\Omega) = \frac{-3/4}{1-\frac{1}{2}e^{-j\Omega}} + \frac{13/4}{1+\frac{1}{2}e^{-j\Omega}} - \frac{3/2}{\left(1+\frac{1}{2}e^{-j\Omega}\right)^2}.$$

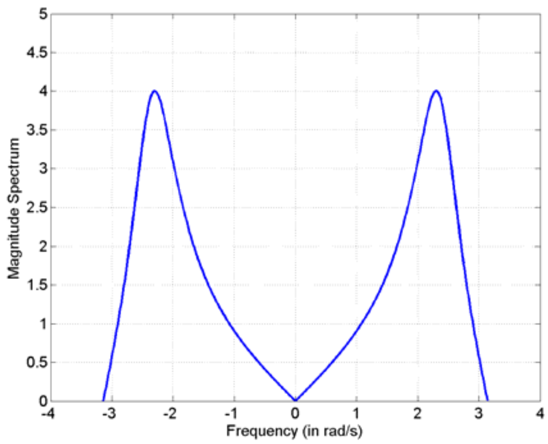
Using the DTFT pairs in Table 11.2, the sequence $y[k]$ is obtained as follows.

$$y[k] = \left\{-\tfrac{3}{4}(\tfrac{1}{2})^k + \tfrac{13}{4}(-\tfrac{1}{2})^k - \tfrac{3}{2}(k+1)(-\tfrac{1}{2})^k\right\}u[k] = \left\{-\tfrac{3}{4}(\tfrac{1}{2})^k + \left[\tfrac{7}{4} - 1.5k)\right](-\tfrac{1}{2})^k\right\}u[k].$$

Note that the results obtained in parts (iii) and (iv) are identical. ▮

```
% Problem 11.12
%The MATLAB code to plot the magnitude frequency response in Problem 11.12.
omega = [-pi:0.01:pi] ;
H = (1-exp(-j*2*omega))./(1+ exp(-j*omega) + 0.5*exp(-j*2*omega));
abs_H = abs(H);

plot(omega, abs_H), grid
xlabel('Frequency (in rad/s)')                  % Label of X-axis
ylabel('Magnitude Spectrum')                    % Label of Y-axis %
axis([-4  4  0  5])
print -dtiff plot.tiff
```



Problem 11.13:

(i) $x[k]=u[k]$ and $h[k]=4^{-|k|}$

To solve this problem, we will use the following DTFT pairs

$$p^k u[k] \leftrightarrow \frac{1}{1-pe^{-j\Omega}} \qquad |p|<1$$

$$p^{-k} u[-k] \leftrightarrow \frac{1}{1-pe^{j\Omega}} \qquad |p|<1$$

$$-p^k u[-k-1] \leftrightarrow \frac{1}{1-pe^{-j\Omega}} \qquad |p|>1$$

The impulse response can be expressed as

$$h[k]=4^{-|k|}=4^{-k}u[k]+4^k u[-k-1]=0.25^k u[k]+0.25^{-k}u[-k]-\delta[k]$$

The transfer function $H(\Omega)$ can be expressed as:

$$H(\Omega)=\frac{1}{1-0.25e^{-j\Omega}}+\frac{1}{1-0.25e^{j\Omega}}-1=\frac{1}{1-0.25e^{-j\Omega}}+\frac{-4e^{-j\Omega}}{1-4e^{-j\Omega}}-1$$
$$=\frac{-3.75e^{-j\Omega}}{\left(1-0.25e^{-j\Omega}\right)\left(1-4e^{-j\Omega}\right)}$$

On the other hand,

$$X(\Omega)\Big|_{|\Omega|\le\pi}=\pi\delta(\Omega)+\frac{1}{1-e^{-j\Omega}}$$

Therefore, $Y(\Omega)$ can be expressed as

$$Y(\Omega)\Big|_{|\Omega|\le\pi}=X(\Omega)H(\Omega)=\left[\pi\delta(\Omega)+\frac{1}{1-e^{-j\Omega}}\right]\times\frac{-3.75e^{-j\Omega}}{\left(1-0.25e^{-j\Omega}\right)\left(1-4e^{-j\Omega}\right)}$$
$$=\pi\delta(\Omega)\times\frac{-3.75e^{-j\Omega}}{\left(1-0.25e^{-j\Omega}\right)\left(1-4e^{-j\Omega}\right)}+\frac{-3.75e^{-j\Omega}}{\left(1-e^{-j\Omega}\right)\left(1-0.25e^{-j\Omega}\right)\left(1-4e^{-j\Omega}\right)}$$
$$=\pi\delta(\Omega)\times\frac{-3.75}{(1-0.25)(1-4)}+\frac{-3.75e^{-j\Omega}}{\left(1-e^{-j\Omega}\right)\left(1-0.25e^{-j\Omega}\right)\left(1-4e^{-j\Omega}\right)}$$
$$=\tfrac{5}{3}\pi\delta(\Omega)+\frac{k_1}{1-e^{-j\Omega}}+\frac{k_2}{1-0.25e^{-j\Omega}}+\frac{k_3}{1-4e^{-j\Omega}}$$

The partial fraction coefficients can be calculated as $k_1=\frac{5}{3}, k_2=-\frac{1}{3}, k_3=-\frac{4}{3}$.

$$Y(\Omega)\Big|_{|\Omega|\le\pi}=\frac{5}{3}\left[\pi\delta(\Omega)+\frac{1}{1-e^{-j\Omega}}\right]-\frac{1/3}{1-0.25e^{-j\Omega}}-\frac{4/3}{1-4e^{-j\Omega}}$$

Using the DTFT pairs mentioned above, $y[k]$ can be expressed as

$$y[k] = \tfrac{5}{3}u[k] - \tfrac{1}{3}\times 0.25^k u[k] + \tfrac{4}{3}\times 4^k u[-k-1] = \left(\tfrac{5}{3} - \tfrac{1}{3}\times 0.25^k\right)u[k] + \tfrac{4}{3}\times 0.25^{-k}u[-k-1]$$

$$= \begin{cases} \tfrac{5}{3} - \tfrac{1}{3}\times 0.25^k & k \ge 0 \\ \tfrac{4}{3}\times 0.25^{-k} & k < 0 \end{cases}$$

(ii) $x[k] = 2^{-k}u[k]$ and $h[k] = 2^k u[-k-1]$

Using the DTFT pairs

$$p^k u[k] \leftrightarrow \frac{1}{1-pe^{-j\Omega}} \qquad |p| < 1$$

$$-p^k u[-k-1] \leftrightarrow \frac{1}{1-pe^{-j\Omega}} \qquad |p| > 1$$

the DTFT $X(\Omega)$ and $H(\Omega)$ are obtained as:

$$X(\Omega) = \frac{1}{1-0.5e^{-j\Omega}};\ H(\Omega) = -\frac{1}{1-2e^{-j\Omega}}.$$

Therefore, the DTFT $Y(\Omega)$ can be expressed as

$$Y(\Omega) = X(\Omega)H(\Omega) = -\frac{1}{\left(1-0.5e^{-j\Omega}\right)\left(1-2e^{-j\Omega}\right)} = \frac{k_1}{1-0.5e^{-j\Omega}} + \frac{k_2}{1-2e^{-j\Omega}}.$$

The partial fraction coefficients can be calculated as $k_1 = \frac{1}{3}, k_2 = -\frac{4}{3}$.

Using the DTFT pairs mentioned above, $y[k]$ can be calculated as

$$y[k] = \Im^{-1}\left[\frac{-1/3}{1-e^{-j\Omega}} + \frac{-4/3}{1-2e^{-j\Omega}}\right] = \tfrac{1}{3}\times 2^{-k}u[k] - \tfrac{4}{3}\times 2^k u[-k-1].$$

(iii) $x[k] = u[k] - u[k-9] = \begin{cases} 1 & 0 \le k \le 8 \\ 0 & otherwise \end{cases}$ and $h[k] = 3^k u[-k+4]$

$$X(\Omega) = \sum_{k=-\infty}^{\infty} x[k]e^{-j\Omega k} = \sum_{k=0}^{8} 1\cdot e^{-j\Omega k} = \frac{1-e^{-j9\Omega}}{1-e^{-j\Omega}}$$

Using the DTFT pair, $-p^k u[-k-1] \leftrightarrow \dfrac{1}{1-pe^{-j\Omega}}$ $\quad |p| > 1$, we get

$$-3^k u[-k-1] \leftrightarrow \frac{1}{1-3e^{-j\Omega}}$$

Or, $-3^{(k-5)}u[-(k-5)-1] \leftrightarrow \dfrac{e^{-j5\Omega}}{1-3e^{-j\Omega}}$

Or, $3^k u[-k+4] \leftrightarrow -3^5 \dfrac{e^{-j5\Omega}}{1-3e^{-j\Omega}}$

In other words, $H(\Omega) = -3^5 \dfrac{e^{-j5\Omega}}{1-3e^{-j\Omega}}$

Using the convolution property of DTFT, $Y(\Omega)$ can be expressed as

$$Y(\Omega) = X(\Omega)H(\Omega) = -3^5 \frac{1-e^{-j9\Omega}}{1-e^{-j\Omega}} \times \frac{e^{-j5\Omega}}{1-3e^{-j\Omega}} = -3^5 e^{-j5\Omega} \frac{1-e^{-j9\Omega}}{\left(1-e^{-j\Omega}\right)\left(1-3e^{-j\Omega}\right)}$$

$$= 3^5 \left[e^{-j14\Omega} - e^{-j5\Omega}\right] \frac{1}{\left(1-e^{-j\Omega}\right)\left(1-3e^{-j\Omega}\right)}$$

Noting that $\dfrac{1}{\left(1-e^{-j\Omega}\right)\left(1-3e^{-j\Omega}\right)} = 0.5e^{j\Omega}\left[\dfrac{1}{1-3e^{-j\Omega}} - \dfrac{1}{1-e^{-j\Omega}}\right]$, $Y(\Omega)$ can be expressed as

$$Y(\Omega) = 3^5 \left[e^{-j14\Omega} - e^{-j5\Omega}\right] \times 0.5e^{j\Omega}\left[\frac{1}{1-3e^{-j\Omega}} - \frac{1}{1-e^{-j\Omega}}\right]$$

$$= 0.5 \times 3^5 \left[e^{-j13\Omega} - e^{-j4\Omega}\right]\left[\frac{1}{1-3e^{-j\Omega}} - \frac{1}{1-e^{-j\Omega}}\right]$$

$$= 0.5 \times 3^5 \left\{\left[e^{-j13\Omega} - e^{-j4\Omega}\right] \underbrace{\frac{1}{1-3e^{-j\Omega}}}_{=Q(\Omega)} + e^{-j4\Omega} \underbrace{\frac{1-e^{-j9\Omega}}{1-e^{-j\Omega}}}_{=X(\Omega)}\right\}$$

Using the DTFT pair, $q[k] = -3^k u[-k-1] \leftrightarrow \dfrac{1}{1-3e^{-j\Omega}} = Q(\Omega)$, and the linearity and time shifting property of the DTFT, we obtain

$$y[k] = 0.5 \times 3^5 \left\{q[k-13] - q[k-4] + x[k-4]\right\}$$

$$= 0.5 \times 3^5 \left\{-3^k \times 3^{-13} u[-k+12] + 3^k \times 3^{-4} u[-k+3] + u[k-4] - u[k-13]\right\}$$

$$= 0.5 \times 3^5 \left\{0.5 \times 3^{k+1} u[-k+3] - 0.5 \times 3^{k-8} u[-k+12] + u[k-4] - u[k-13]\right\}$$

(iv) $x[k] = k5^{-k} u[k]$ and $h[k] = 5^k u[-k]$

$$X(\Omega) = \Im\left\{k5^{-k} u[k]\right\} = \Im\left\{k0.2^k u[k]\right\} = \Im\left\{(k+1)0.2^k u[k]\right\} - \Im\left\{0.2^k u[k]\right\}$$

$$= \frac{1}{\left(1-0.2e^{-j\Omega}\right)^2} - \frac{1}{1-0.2e^{-j\Omega}} = \frac{0.2e^{-j\Omega}}{\left(1-0.2e^{-j\Omega}\right)^2}$$

$$H(\Omega) = \Im\left\{5^k u[-k]\right\} = \Im\left\{\delta[k] + 5^k u[-k-1]\right\} = 1 - \frac{1}{1-5e^{-j\Omega}} = \frac{-5e^{-j\Omega}}{1-5e^{-j\Omega}}$$

Therefore, the DTFT $Y(\Omega)$ can be expressed as

$$Y(\Omega) = X(\Omega)H(\Omega) = \frac{-e^{-j2\Omega}}{\left(1-5e^{-j\Omega}\right)\left(1-0.2e^{-j\Omega}\right)^2} = \frac{k_1}{1-5e^{-j\Omega}} + \frac{k_2}{\left(1-0.2e^{-j\Omega}\right)^2} + \frac{k_3}{1-0.2e^{-j\Omega}}.$$

The first two partial fraction coefficients can be calculated as $k_1 = \dfrac{-25}{576}, k_2 = \dfrac{25}{24}$.

Substituting $k_1 = \dfrac{-25}{576}, k_2 = \dfrac{25}{24}$ in $Y(\Omega)$ expression and comparing the numerators, we get $k_3 = \dfrac{-575}{576}$.

Substituting the values of A and B and C, we get

$$Y(\Omega)=\frac{-25/576}{1-5e^{-j\Omega}}+\frac{25/24}{\left(1-0.2e^{-j\Omega}\right)^2}+\frac{-575/576}{1-0.2e^{-j\Omega}}.$$

Using the DTFT pairs mentioned above, $y[k]$ can be calculated as

$$\begin{aligned}y[k]&=\tfrac{25}{576}\times 5^k u[-k-1]+\tfrac{25}{24}(k+1)\times 0.2^k u[k]-\tfrac{575}{576}\times 0.2^k u[k]\\&=\tfrac{25}{576}\times 5^k u[-k-1]+\tfrac{25}{24}(k+\tfrac{1}{24})\times 0.2^k u[k].\end{aligned}$$

(v) In the textbook, there are typos in the $x[k]$ and $h[k]$ expressions. The correct expressions are as follows:

$$x[k]=u[k+2]-u[k-3]=\begin{cases}1 & -2\le k\le 2\\ 0 & otherwise\end{cases}\quad \text{and}\quad h[k]=u[k-5]-u[k-6]=\delta[k-5]$$

$$X(\Omega)=\sum_{k=-\infty}^{\infty}x[k]e^{-j\Omega k}=\sum_{k=-2}^{2}1\cdot e^{-j\Omega k}=e^{j2\Omega}\frac{1-(e^{-j\Omega})^5}{1-e^{-j\Omega}}=e^{j2\Omega}\frac{1-e^{-j5\Omega}}{1-e^{-j\Omega}}$$

$$H(\Omega)=\Im\{\delta[k-5]\}=e^{-j5\Omega}$$

Therefore, $Y(\Omega)$ can be expressed as

$$Y(\Omega)=X(\Omega)H(\Omega)=e^{-j3\Omega}\frac{1-e^{-j5\Omega}}{1-e^{-j\Omega}}=e^{-j3\Omega}\sum_{k=0}^{4}e^{-jk\Omega}=\sum_{k=3}^{7}1.e^{-jk\Omega},$$

resulting in the following output:

$$y[k]=\begin{cases}1 & 3\le k\le 7\\ 0 & otherwise\end{cases}=u[k-3]-u[k-8].$$

Problem 11.14:

$$H(\Omega)=\frac{1}{1+3e^{-j\Omega}}=\frac{1+3e^{j\Omega}}{(1+3e^{-j\Omega})(1+3e^{j\Omega})}=\frac{1+3\cos(\Omega)+j3\sin(\Omega)}{10+6\cos(\Omega)}$$

(i) $\mathrm{Re}\{H(\Omega)\}=\dfrac{1+3\cos(\Omega)}{10+6\cos(\Omega)}$

(ii) $\mathrm{Im}\{H(\Omega)\}=\dfrac{3\sin(\Omega)}{10+6\cos(\Omega)}$

(iii) $|H(\Omega)|=\left|\dfrac{1}{1+3e^{-j\Omega}}\right|=\dfrac{1}{|1+3\cos(\Omega)-j3\sin(\Omega)|}=\dfrac{1}{\sqrt{(1+3\cos(\Omega))^2+9\sin^2(\Omega)}}=\dfrac{1}{\sqrt{10+6\cos(\Omega)}}.$

(iv) $\sphericalangle H(\Omega)=\tan^{-1}\left(\dfrac{3\sin(\Omega)}{1+3\cos(\Omega)}\right)$

The frequency responses $\mathrm{Re}\{H(\Omega)\}$, $\mathrm{Im}\{H(\Omega)\}$, $|H(\Omega)|$, and $\measuredangle H(\Omega)$, for $-\pi \le \Omega \le \pi$, are shown in Fig. S11.14. ▌

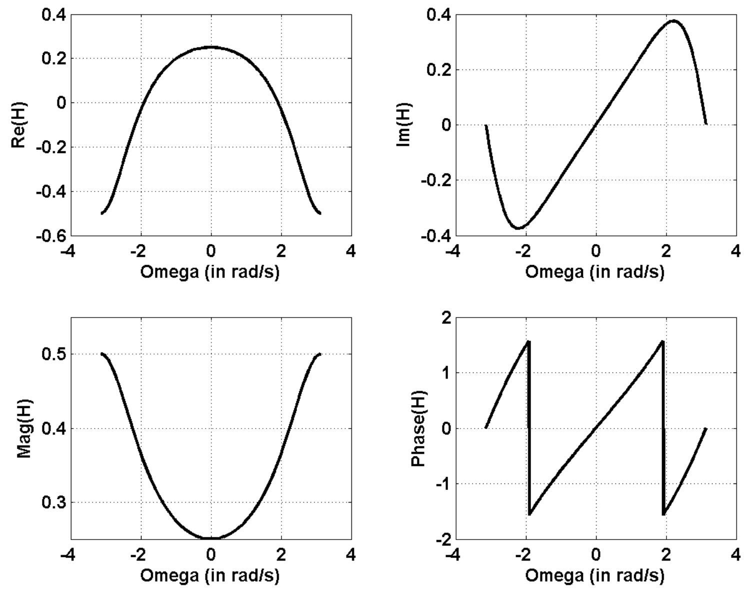


Fig. S11.14: Frequency response plots of $H(\Omega)$ in Problem 11.14.

```
% Problem 11.14
%The MATLAB code to plot the frequency responses in Problem 11.14.
omega = [-pi:0.01:pi] ;
ReH = (1+3*cos(omega))./(10+6*cos(omega));
ImH = 3*sin(omega)./(10+6*cos(omega));
MagH = 1./sqrt(10+6*cos(omega));
PhaseH = atan(3*sin(omega)./(1+3*cos(omega)));

subplot(2,2,1), plot(omega, ReH), grid
xlabel('Omega (in rad/s)')              % Label of X-axis
%axis([-4  4  0  5])
ylabel('Re(H)')        % Label of Y-axis
%
subplot(2,2,2), plot(omega, ImH), grid
xlabel('Omega (in rad/s)')                % Label of X-axis
ylabel('Im(H)')           % Label of Y-axis %
%
subplot(2,2,3), plot(omega, MagH), grid
xlabel('Omega (in rad/s)')                    % Label of X-axis
ylabel('Mag(H)')            % Label of Y-axis %

subplot(2,2,4), plot(omega, PhaseH), grid
xlabel('Omega (in rad/s)')                    % Label of X-axis
ylabel('Phase(H)')            % Label of Y-axis %

print -dtiff plot.tiff        % Save the figure as a TIFF file
```

<u>Problem 11.15:</u>

(i) $$H_1(\Omega)=\frac{-3.75e^{-j\Omega}}{\left(1-0.25e^{-j\Omega}\right)\left(1-4e^{-j\Omega}\right)}=\frac{-3.75e^{-j\Omega}}{1-4.25e^{-j\Omega}+e^{-j2\Omega}}=\frac{-3.75e^{-j\Omega}}{e^{-j\Omega}\left[e^{j\Omega}-4.25+e^{-j\Omega}\right]}$$

$$=\frac{-3.75}{2\cos(\Omega)-4.25}=\frac{3.75}{4.25-2\cos(\Omega)}.$$

Note that $\left[4.25-2\cos(\Omega)\right]$ is a positive real-valued function. Therefore,

$$\left|H_1(\Omega)\right|=\left|\frac{3.75}{4.25-2\cos(\Omega)}\right|=\frac{3.75}{4.25-2\cos(\Omega)}, \text{ and } \sphericalangle H_1(\Omega)=0.$$

The magnitude and phase response plots are shown in Fig. S11.15(i).

(ii) $$H_2(\Omega)=-\frac{1}{1-2e^{-j\Omega}}$$

$$\left|H_2(\Omega)\right|=\frac{1}{\left|1-2\cos(\Omega)+j2\sin(\Omega)\right|}=\frac{1}{\sqrt{5-4\cos(\Omega)}}$$

$$\sphericalangle H_2(\Omega)=\sphericalangle(-1)-\sphericalangle\left(1-2\cos(\Omega)+j2\sin(\Omega)\right)=\pi-\tan^{-1}\left(\frac{2\sin(\Omega)}{1-2\cos(\Omega)}\right).$$

The magnitude and phase response plots are shown in Fig. S11.15(ii).

(iii) $$H_3(\Omega)=-3^5\,\frac{e^{-j5\Omega}}{1-3e^{-j\Omega}}$$

$$\left|H_3(\Omega)\right|=\frac{3^5}{\left|1-3\cos(\Omega)+j3\sin(\Omega)\right|}=\frac{243}{\sqrt{10-6\cos(\Omega)}}$$

$$\sphericalangle H_3(\Omega)=\sphericalangle\left(-3^5\right)+\sphericalangle\left(e^{-j5\Omega}\right)-\sphericalangle\left(1-3\cos(\Omega)+j3\sin(\Omega)\right)=\pi-5\Omega-\tan^{-1}\left(\frac{3\sin(\Omega)}{1-3\cos(\Omega)}\right).$$

The magnitude and phase response plots are shown in Fig. S11.15(iii).

(iv) $$H_4(\Omega)=\frac{-5e^{-j\Omega}}{1-5e^{-j\Omega}}$$

$$\left|H_4(\Omega)\right|=\frac{5}{\left|1-5\cos(\Omega)+j5\sin(\Omega)\right|}=\frac{1}{\sqrt{26-10\cos(\Omega)}}$$

$$\sphericalangle H_4(\Omega)=\sphericalangle(-5)+\sphericalangle\left(e^{-j\Omega}\right)-\sphericalangle\left(1-5\cos(\Omega)+j5\sin(\Omega)\right)=\pi-\Omega-\tan^{-1}\left(\frac{5\sin(\Omega)}{1-5\cos(\Omega)}\right).$$

The magnitude and phase response plots are shown in Fig. S11.15(iv).

(v) $H_5(\Omega)=e^{-j5\Omega}$ [Assuming $h[k]=u[k-5]-u[k-6]=\delta[k-5]$]

$$\left|H_5(\Omega)\right|=1$$

$$\angle H_5(\Omega) = \angle\left(e^{-j5\Omega}\right) = -5\Omega$$

The magnitude and phase response plots are shown in Fig. S11.15(v).

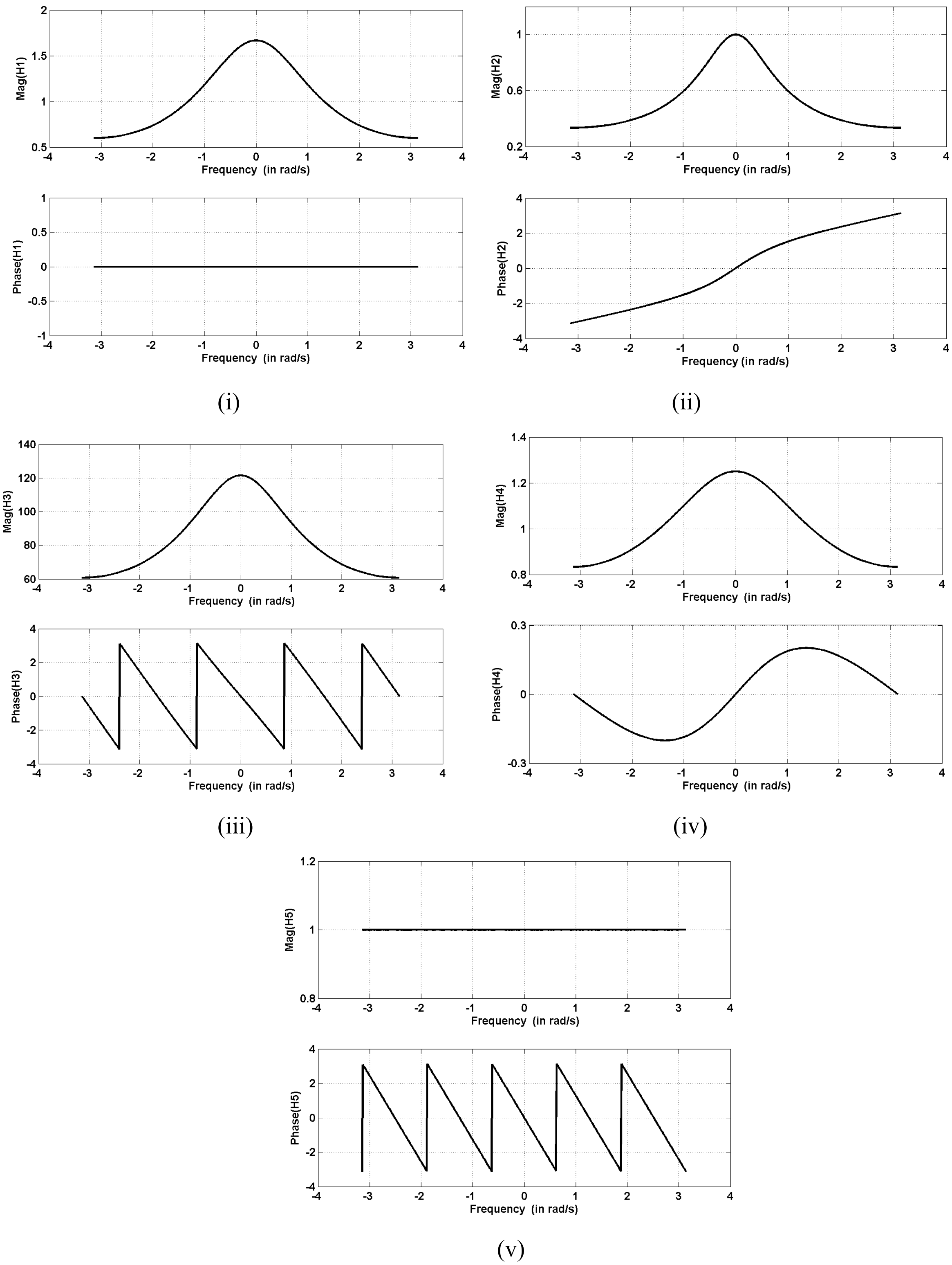


Fig. S11.15: Magnitude and Phase response plots for Problem 11.15.

```
% Problem 11.15
%The MATLAB code to plot the amplitude and phase responses in Problem 11.15.
omega = [-pi:0.01:pi] ;

%part(i)
NumH=3.75;
DenH=(4.25-2*cos(omega));
H1 = NumH./(DenH+eps);
MagH1 = H1;
PhaseH1 = 0*omega;
subplot(2,1,1), plot(omega, MagH1), grid
xlabel(' Frequency (in rad/s)')           % Label of X-axis
ylabel('Mag(H1)')             % Label of Y-axis %
subplot(2,1,2), plot(omega, PhaseH1), grid
xlabel(' Frequency (in rad/s)')           % Label of X-axis
ylabel('Phase(H1)')             % Label of Y-axis %
print -dtiff plot.tiff         % Save the figure as a TIFF file
%
%part(ii)
NumH=-1;
DenH=1-2*exp(-j*omega);
H2 = NumH./(DenH+eps);
MagH2 = abs(H2);
PhaseH2 = phase(H2);
subplot(2,1,1), plot(omega, MagH2), grid
xlabel(' Frequency (in rad/s)')           % Label of X-axis
ylabel('Mag(H2)')             % Label of Y-axis %
axis([-4  4  0.2  1.2])
subplot(2,1,2), plot(omega, PhaseH2), grid
xlabel('Frequency (in rad/s)')                % Label of X-axis
ylabel('Phase(H2)')             % Label of Y-axis %
print -dtiff plot.tiff         % Save the figure as a TIFF file

%part(iii)
omega = [-pi:0.01:pi] ;
NumH=(-3^5)*exp(-j*5*omega);
DenH=1-3*exp(-j*omega);
H3 = NumH./(DenH+eps);
MagH3 = abs(H3);
PhaseH3 = phase(H3);
%Unwrapping phase over [-pi,pi] %P=unwrap(PhaseH3,pi);
bin = round(PhaseH3/(2*pi));
PhaseH3 = PhaseH3-bin*2*pi;    %mod(PhaseH3,pi)
subplot(2,1,1), plot(omega, MagH3), grid
xlabel(' Frequency  (in rad/s)')           % Label of X-axis
ylabel('Mag(H3)')             % Label of Y-axis %
%axis([-4  4  0.2  1.2])
subplot(2,1,2), plot(omega, PhaseH3), grid
xlabel(' Frequency  (in rad/s)')          % Label of X-axis
ylabel('Phase(H3)')             % Label of Y-axis %
print -dtiff plot.tiff         % Save the figure as a TIFF file
%
%part(iv)
omega = [-pi:0.01:pi] ;
NumH=-5*exp(-j*omega);
DenH=1+NumH;
H4 = NumH./(DenH+eps);
MagH4 = abs(H4);
PhaseH4 = phase(H4);
%Unwrapping phase over [-pi,pi]
%bin = round(PhaseH4/(2*pi));
```

```
%PhaseH4 = PhaseH4-bin*2*pi;
subplot(2,1,1), plot(omega, MagH4), grid
xlabel(' Frequency  (in rad/s)')           % Label of X-axis
ylabel('Mag(H4)')             % Label of Y-axis %
axis([-4  4  0.8  1.4])
subplot(2,1,2), plot(omega, PhaseH4), grid
xlabel(' Frequency  (in rad/s)')           % Label of X-axis
ylabel('Phase(H4)')             % Label of Y-axis %
axis([-4  4  -0.3  0.3])
print -dtiff plot.tiff         % Save the figure as a TIFF file
%
%part(v)
omega = [-pi:0.01:pi] ;
H5=exp(-j*5*omega);
MagH5 = abs(H5);
PhaseH5 = phase(H5);
%Unwrapping phase over [-pi,pi]
bin = round(PhaseH5/(2*pi));
PhaseH5 = PhaseH5-bin*2*pi;
subplot(2,1,1), plot(omega, MagH5), grid
xlabel(' Frequency  (in rad/s)')           % Label of X-axis
ylabel('Mag(H5)')             % Label of Y-axis %
axis([-4  4  0.8  1.2])
subplot(2,1,2), plot(omega, PhaseH5), grid
xlabel('Frequency  (in rad/s)')            % Label of X-axis
ylabel('Phase(H5)')             % Label of Y-axis %
print -dtiff plot.tiff         % Save the figure as a TIFF file
```

Problem 11.16:

$$h[k] = 3\delta[k+3] - 2\delta[k+2] + \delta[k+1] + 5\delta[k] - \delta[k-1] - 2\delta[k-2] - 3\delta[k-3] + 4\delta[k-4]$$

$$H(\Omega) = \sum_{k=-\infty}^{\infty} h[k] \cdot e^{-j\Omega k}$$

(i) $H(\Omega)\big|_{\Omega=0} = \sum_{k=-\infty}^{\infty} h[k] = 3 - 2 + 1 + 5 - 1 - 2 - 3 + 4 = 5$

(ii) $H(\Omega)\big|_{\Omega=\pi} = \sum_{k=-\infty}^{\infty} h[k] \cdot e^{-j\pi k} = ... + h[-2] - h[-1] + h[0] - h[1] + h[2] - h[3] + ...$

$$= \sum_{\substack{k=-\infty \\ k=even}}^{\infty} h[k] - \sum_{\substack{k=-\infty \\ k=odd}}^{\infty} h[k] = (-2 + 5 - 2 + 4) - (3 + 1 - 1 - 3) = 5$$

(iii) It is difficult to calculate the $\sphericalangle H(\Omega)$ without calculating the DTFT. The DTFT can be calculated as follows:

$$\begin{aligned}H(\Omega) &= 3e^{j3\Omega} - 2e^{j2\Omega} + e^{j\Omega} + 5 - e^{-j\Omega} - 2e^{-j2\Omega} - 3e^{-j3\Omega} + 4e^{-j4\Omega}\\ &= (3e^{j3\Omega} - 3e^{-j3\Omega}) - (2e^{j2\Omega} + 2e^{-j2\Omega}) + (e^{j\Omega} - e^{-j\Omega}) + 5 + 4e^{-j4\Omega}\\ &= j6\sin(3\Omega) - 4\cos(2\Omega) + j2\sin(\Omega) + 5 + 4\cos(4\Omega) - j4\sin(4\Omega)\\ &= \underbrace{\left[5 - 4\cos(2\Omega) + 4\cos(4\Omega)\right]}_{=>0} + j\left[2\sin(\Omega) + 6\sin(3\Omega) - 4\sin(4\Omega)\right]\end{aligned}$$

Therefore, $\measuredangle H(\Omega) = \tan^{-1}\left(\dfrac{2\sin(\Omega) + 6\sin(3\Omega) - 4\sin(4\Omega)}{5 - 4\cos(2\Omega) + 4\cos(4\Omega)}\right)$.

The phase $\measuredangle H(\Omega)$ is sketched in Fig. S11.16(a).

(iv) From Eq. 11.28b), we know $h[k] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(\Omega) e^{jk\Omega}\, d\Omega$

In other words, $h[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(\Omega) e^{j\times 0\times\Omega}\, d\Omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(\Omega)\, d\Omega$

Therefore, $\int_{-\pi}^{\pi} H(\Omega)\, d\Omega = 2\pi \underbrace{h[0]}_{=\pi} = 10\pi$.

(v) $H(\Omega) = \sum_{k=-\infty}^{\infty} h[k]\cdot e^{-j\Omega k}$.

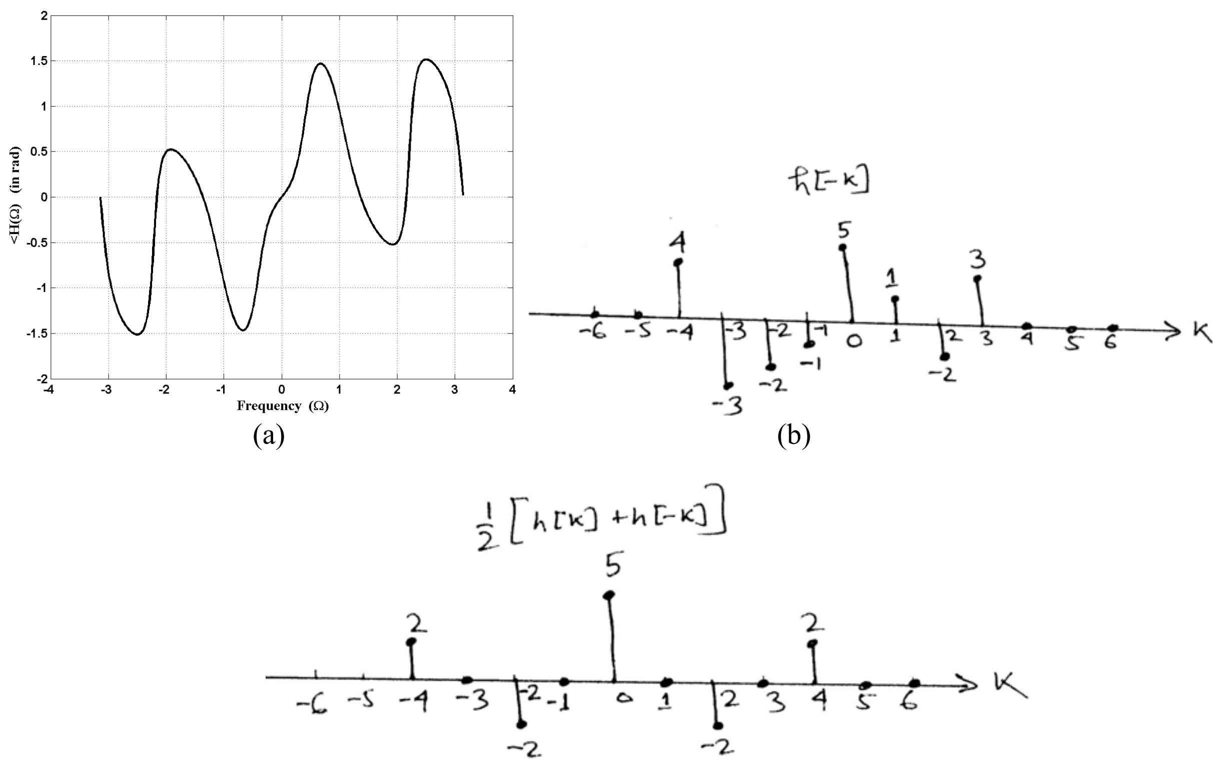
Substituting $\Omega = -\Omega$ on both sides of the DTFT equation, we obtain:

$$\begin{aligned}H(-\Omega) &= \sum_{k=-\infty}^{\infty} h[k]\cdot e^{j\Omega k} = \sum_{k=-\infty}^{\infty} h[-(-k)]\cdot e^{-j\Omega(-k)}\\ &= \sum_{m=-\infty}^{\infty} h[-m]\cdot e^{-j\Omega m} \qquad \left[\text{substituting } m = -k\right]\\ &= \sum_{k=-\infty}^{\infty} h[-k]\cdot e^{-j\Omega k}\\ &= \Im\left\{h[-k]\right\}\end{aligned}$$

Therefore, the DT sequence is given by $h[-k]$, which is sketched in Fig. S11.16(b).

(vi) $$\begin{aligned}\mathrm{Re}\left\{H(\Omega)\right\} &= \mathrm{Re}\left\{\sum_{k=-\infty}^{\infty} h[k]\cdot e^{-j\Omega k}\right\} = \sum_{k=-\infty}^{\infty} h[k]\cos(\Omega k)\\ &= \frac{1}{2}\sum_{k=-\infty}^{\infty} h[k]\cdot\left[e^{j\Omega k} + e^{-j\Omega k}\right] = \frac{1}{2}\sum_{k=-\infty}^{\infty} h[k]e^{j\Omega k} + \frac{1}{2}\sum_{k=-\infty}^{\infty} h[k]e^{-j\Omega k}\\ &= \frac{1}{2}\Im\left\{h[-k]\right\} + \frac{1}{2}\Im\left\{h[k]\right\} = \frac{1}{2}\Im\left\{h[k] + h[-k]\right\}\end{aligned}$$

Therefore, the DT sequence is given by $\frac{1}{2}\left(h[k] + h[-k]\right)$, which is sketched in Fig. S11.16(c). ∎



(a) (b) (c)

Fig. S11.16: The DT sequences obtained in Problem 11.16.

```
% P11.16, part (iii) Phase plotting
omega = [-pi:0.01:pi] ;
R=5-4*cos(2*omega)+4*cos(4*omega);
I=2*sin(omega) + 6*sin(3*omega)-4*sin(4*omega);
H = R + j*I;
PhaseH = phase(H);
%Unwrapping phase over [-pi,pi] %P=unwrap(PhaseH,pi);
bin = round(PhaseH/(2*pi));
PhaseH = PhaseH-bin*2*pi;    %mod(PhaseH3,pi)
plot(omega, PhaseH), grid
xlabel(' Frequency  (\Omega)')           % Label of X-axis
ylabel(' <H(\Omega)')             % Label of Y-axis %
%axis([-4  4  0.2  1.2])
print -dtiff plot.tiff
```

Problem 11.17:

There is a typo in the question. This question can be easily solved using the Convolution (and not Parseval's theorem) property of the DTFT as follows.

Note that the sum is of the form: $\sum_{k=-\infty}^{\infty} x[k] = \sum_{k=-\infty}^{\infty} \frac{\sin(\frac{\pi}{5}k)\sin(\frac{\pi}{7}k)}{k^2}$. The sum of a discrete sequence can easily calculated if the DTFT is known. Substituing Ω=0 on both sides of the DTFT expression $X(\Omega) = \sum_{k=-\infty}^{\infty} x[k] \cdot e^{-j\Omega k}$, we obtain

$$\sum_{k=-\infty}^{\infty} x[k] = X(\Omega)\Big|_{\Omega=0}.$$

Noting that $x[k]$ is a product of two sinc functions, we expect that the $X(\Omega)$ will be the convolution of two rectangular fiunctions, and in this case, it will be a trapezoidal shape. It can be shown that (see the solution of P11.4(a)(vi))

$$\Im\left[\frac{\sin(\frac{\pi}{5}k)\sin(\frac{\pi}{7}k)}{k^2}\right] = X(\Omega)\Big|_{|\Omega|\le\pi} = \pi^2 \begin{cases} \frac{1}{2\pi}\left(\Omega+\frac{12\pi}{35}\right) & -\frac{12\pi}{35} \le \Omega < -\frac{2\pi}{35} \\ \frac{1}{7} & -\frac{2\pi}{35} \le \Omega < \frac{2\pi}{35} \\ \frac{1}{2\pi}\left(\Omega-\frac{12\pi}{35}\right) & \frac{2\pi}{35} \le \Omega < \frac{12\pi}{35} \\ 0 & otherwise. \end{cases}$$

Therefore, $\displaystyle\sum_{k=-\infty}^{\infty} \frac{\sin\left(\frac{\pi k}{5}\right)\sin\left(\frac{\pi k}{7}\right)}{k^2} = X(\Omega)\Big|_{\Omega=0} = \frac{\pi^2}{7}.$ ▮

Problem 11.18:

$h[k] = \text{sinc}\left(\frac{3k}{4}\right) = \text{sinc}\left(\frac{(3\pi/4)k}{\pi}\right).$

Using the DTFT pair (16) in Table 11.2, the transfer function $H(\Omega)$ is obtained as follows.

$$H(\Omega)\Big|_{|\Omega|\le\pi} = \frac{\pi}{3\pi/4} \times \begin{cases} 1 & |\Omega| \le 3\pi/4 \\ 0 & 3\pi/4 < |\Omega| \le \pi \end{cases} = \begin{cases} 4/3 & |\Omega| \le 3\pi/4 \\ 0 & 3\pi/4 < |\Omega| \le \pi \end{cases}$$

Note that the DT system is a lowpass filter, which passes the DT frequency components below $3\pi/4$ rad/s, and stops the frequency above it.

(i) $x[k] = \cos\left(\dfrac{11\pi k}{16}\right)\cos\left(\dfrac{3\pi k}{16}\right) = \dfrac{1}{2}\left[\cos\left(\dfrac{7\pi k}{8}\right) + \cos\left(\dfrac{\pi k}{2}\right)\right]$

$$X(\Omega)\Big|_{|\Omega|\le\pi} = \frac{\pi}{2}\left[\delta\left(\Omega+\tfrac{\pi}{2}\right) + \delta\left(\Omega-\tfrac{\pi}{2}\right) + \delta\left(\Omega+\tfrac{7\pi}{8}\right) + \delta\left(\Omega-\tfrac{7\pi}{8}\right)\right]$$

$$Y(\Omega)\Big|_{|\Omega|\le\pi} = X(\Omega)\Big|_{|\Omega|\le\pi} \times H(\Omega)\Big|_{|\Omega|\le\pi} = (2\pi/3)\left[\delta\left(\Omega+\tfrac{\pi}{2}\right) + \delta\left(\Omega-\tfrac{\pi}{2}\right)\right]$$

Using the DTFT pair (10) in Table 11.2, the output sequence is obtained as

$$y[k] = \frac{2}{3}\cos\left(\frac{\pi k}{2}\right)$$

(ii) The DTFT $X(\Omega)$ is given by (see solution of P11.4(b)(i))

$$X(\Omega)=2\pi\sum_{n=-\infty}^{\infty}D_n\delta\left(\Omega-\tfrac{2\pi n}{6}\right)$$

$$=2\pi\begin{bmatrix}....+D_{-3}\delta(\Omega+\pi)+D_{-2}\delta\left(\Omega+\tfrac{2\pi}{3}\right)+D_{-1}\delta\left(\Omega+\tfrac{\pi}{3}\right)+D_0\delta(\Omega)+\\ D_1\delta\left(\Omega-\tfrac{\pi}{3}\right)+D_2\delta\left(\Omega-\tfrac{2\pi}{3}\right)+D_3\delta(\Omega-\pi)+...\end{bmatrix}$$

where $D_n=\dfrac{1}{6}\dfrac{e^{-jn\pi/3}-6e^{-j6n\pi/3}+5e^{-j7n\pi/3}}{(1-e^{-jn\pi/3})^2}$. Substituting different values of n, we obtain,

$D_0=\tfrac{5}{2};D_{\pm1}=-\tfrac{1}{2}\pm j\tfrac{\sqrt{3}}{2};D_{\pm2}\approx-\tfrac{1}{2}\pm j0.2887$.

The DTFT of the output is given by

$$Y(\Omega)\Big|_{|\Omega|\le\pi}=X(\Omega)\Big|_{|\Omega|\le\pi}\times H(\Omega)\Big|_{|\Omega|\le\pi}=(8\pi/3)\begin{bmatrix}D_{-2}\delta\left(\Omega+\tfrac{2\pi}{3}\right)+D_{-1}\delta\left(\Omega+\tfrac{\pi}{3}\right)+D_0\delta(\Omega)+\\ D_1\delta\left(\Omega-\tfrac{\pi}{3}\right)+D_2\delta\left(\Omega-\tfrac{2\pi}{3}\right)\end{bmatrix}$$

Using the DTFT pair (8) in Table 11.2, the output sequence is obtained as

$$y[k]=(4\pi/3)\left[D_{-2}e^{-\frac{j2\pi k}{3}}+D_{-1}e^{-\frac{j\pi k}{3}}+D_0+D_1e^{\frac{j\pi k}{3}}+D_2e^{\frac{j2\pi k}{3}}\right]$$

$$=(4\pi/3)\begin{bmatrix}(-\tfrac{1}{2}-j0.2287)e^{-\frac{j2\pi k}{3}}+(-\tfrac{1}{2}-j\tfrac{\sqrt{3}}{2})e^{-\frac{j\pi k}{3}}+\tfrac{5}{2}+\\ (-\tfrac{1}{2}+j\tfrac{\sqrt{3}}{2})e^{\frac{j\pi k}{3}}+(-\tfrac{1}{2}+j0.2287)e^{\frac{j2\pi k}{3}}\end{bmatrix}$$

$$=(4\pi/3)\left[-\cos(\tfrac{2\pi k}{3})-0.4574\sin(\tfrac{2\pi k}{3})-\cos(\tfrac{\pi k}{3})-\sqrt{3}\sin(\tfrac{\pi k}{3})+\tfrac{5}{2}\right]$$

$$=(4\pi/3)\left[\tfrac{5}{2}-\cos(\tfrac{\pi k}{3})-\sqrt{3}\sin(\tfrac{\pi k}{3})-\cos(\tfrac{2\pi k}{3})-0.4574\sin(\tfrac{2\pi k}{3})\right].$$

(iii) The DTFT $X(\Omega)$ is given by (see solution of P11.4(b)(ii))

$$X(\Omega)=2\pi\sum_{n=-\infty}^{\infty}D_n\delta\left(\Omega-\tfrac{2\pi n}{9}\right)$$

$$=2\pi\begin{bmatrix}....+D_{-4}\delta\left(\Omega+\tfrac{8\pi}{9}\right)+D_{-3}\delta\left(\Omega+\tfrac{6\pi}{9}\right)+D_{-2}\delta\left(\Omega+\tfrac{4\pi}{9}\right)+D_{-1}\delta\left(\Omega+\tfrac{2\pi}{9}\right)+\\ D_0\delta(\Omega)+D_1\delta\left(\Omega-\tfrac{2\pi}{9}\right)+D_2\delta\left(\Omega-\tfrac{4\pi}{9}\right)+D_3\delta\left(\Omega-\tfrac{6\pi}{9}\right)+D_4\delta\left(\Omega-\tfrac{8\pi}{9}\right)+...\end{bmatrix}$$

where $D_n=\dfrac{1}{9}\left(1+0.5e^{-j6n\pi/9}\right)\left(1+e^{-j2n\pi/9}+e^{-j4n\pi/9}\right)$. Substituting different values of *n*, we obtain,

$D_0=0.5;D_{\pm1}\approx0.0833\mp j0.2290;D_{\pm2}\approx0.0833\mp j0.0993,D_{\pm3}=0$.

The DTFT of the output is given by

$$Y(\Omega)\Big|_{|\Omega|\le\pi}=X(\Omega)\Big|_{|\Omega|\le\pi}\times H(\Omega)\Big|_{|\Omega|\le\pi}$$

$$=(8\pi/3)\begin{bmatrix}D_{-3}\delta\left(\Omega+\tfrac{6\pi}{9}\right)+D_{-2}\delta\left(\Omega+\tfrac{4\pi}{9}\right)+D_{-1}\delta\left(\Omega+\tfrac{2\pi}{9}\right)+D_0\delta(\Omega)+\\ D_1\delta\left(\Omega-\tfrac{2\pi}{9}\right)+D_2\delta\left(\Omega-\tfrac{4\pi}{9}\right)+D_3\delta\left(\Omega-\tfrac{6\pi}{9}\right)\end{bmatrix}$$

$$=(8\pi/3)\left[D_{-2}\delta\left(\Omega+\tfrac{4\pi}{9}\right)+D_{-1}\delta\left(\Omega+\tfrac{2\pi}{9}\right)+D_0\delta(\Omega)+D_1\delta\left(\Omega-\tfrac{2\pi}{9}\right)+D_2\delta\left(\Omega-\tfrac{4\pi}{9}\right)\right]$$

Using the DTFT pair (8) in Table 11.2, the output sequence is obtained as

$$y[k]=(4\pi/3)\left[D_{-2}e^{-\frac{j4\pi k}{9}}+D_{-1}e^{-\frac{j2\pi k}{9}}+D_0+D_1e^{\frac{j2\pi k}{9}}+D_2e^{\frac{j4\pi k}{9}}\right]$$

$$\approx(4\pi/3)\begin{bmatrix}(0.0833+j0.0993)e^{-\frac{j4\pi k}{9}}+(0.0833+j0.2290)e^{-\frac{j2\pi k}{9}}+0.5+\\(0.0833-j0.2290)e^{\frac{j2\pi k}{9}}+(0.0833-j0.0993)e^{\frac{j4\pi k}{9}}\end{bmatrix}$$

$$=(4\pi/3)\left[0.1666\cos(\tfrac{4\pi k}{9})+0.1986\sin(\tfrac{4\pi k}{9})+0.1666\cos(\tfrac{2\pi k}{9})+0.1986\sin(\tfrac{2\pi k}{9})+0.5\right]$$

$$=(2\pi/3)\left[1+0.3332\cos(\tfrac{4\pi k}{9})+0.3972\sin(\tfrac{4\pi k}{9})+0.3332\cos(\tfrac{2\pi k}{9})+0.3972\sin(\tfrac{2\pi k}{9})\right].$$

(iv) The DTFT $X(\Omega)$ is given by (see solution of P11.4(b)(v))

$$X(\Omega)=\frac{2\pi}{5}\sum_{n=-\infty}^{\infty}\delta\left(\Omega-\tfrac{2\pi n}{5}\right)$$

$$=\frac{2\pi}{5}\left[....+\delta\left(\Omega+\tfrac{4\pi}{5}\right)+\delta\left(\Omega+\tfrac{2\pi}{5}\right)+\delta(\Omega)+\delta\left(\Omega-\tfrac{2\pi}{5}\right)+\delta\left(\Omega-\tfrac{4\pi}{5}\right)+...\right]$$

The DTFT of the output is given by

$$Y(\Omega)\big|_{|\Omega|\le\pi}=X(\Omega)\big|_{|\Omega|\le\pi}\times H(\Omega)\big|_{|\Omega|\le\pi}=(8\pi/15)\left[\delta\left(\Omega+\tfrac{2\pi}{5}\right)+\delta(\Omega)+\delta\left(\Omega-\tfrac{2\pi}{5}\right)\right]$$

$$=\left(\tfrac{4}{15}\right)\times2\pi\delta(\Omega)+\left(\tfrac{8}{15}\right)\times\pi\left[\delta\left(\Omega+\tfrac{2\pi}{5}\right)+\delta\left(\Omega-\tfrac{2\pi}{5}\right)\right]$$

Using the DTFT pairs (1) and (10) in Table 11.2, the output sequence is obtained as

$$y[k]=\tfrac{4}{15}+\tfrac{8}{15}\cos(\tfrac{2\pi}{5}k).$$

(v) $x[k]=\mathrm{sinc}\left(\frac{k}{3}\right)=\mathrm{sinc}\left(\frac{(\pi/3)k}{\pi}\right)$. The DTFT $X(\Omega)$ is given by.

$$X(\Omega)\big|_{|\Omega|\le\pi}=\frac{\pi}{\pi/3}\times\begin{cases}1 & |\Omega|\le\pi/3\\0 & \pi/3<|\Omega|\le\pi\end{cases}=\begin{cases}3 & |\Omega|\le\pi/3\\0 & \pi/3<|\Omega|\le\pi\end{cases}$$

The DTFT of the output is given by

$$Y(\Omega)\big|_{|\Omega|\le\pi}=X(\Omega)\big|_{|\Omega|\le\pi}\times H(\Omega)\big|_{|\Omega|\le\pi}=\begin{cases}4 & |\Omega|\le\pi/3\\0 & \pi/3<|\Omega|\le\pi\end{cases}=\left(\tfrac{4}{3}\right)X(\Omega)\big|_{|\Omega|\le\pi}$$

Calculating the inverse DTFT, the output sequence is obtained as

$$y[k]=\tfrac{4}{3}x[k]=\tfrac{4}{3}\mathrm{sinc}\left(\tfrac{k}{3}\right).$$

▌

Problem 11.19:

$$X(\Omega)=\Im\left\{4^{-k}u[k]+3^{-k}u[k]\right\}=\frac{1}{1-\frac{1}{4}e^{-j\Omega}}+\frac{1}{1-\frac{1}{3}e^{-j\Omega}}=\frac{2\left(1-\frac{7}{24}e^{-j\Omega}\right)}{\left(1-\frac{1}{4}e^{-j\Omega}\right)\left(1-\frac{1}{3}e^{-j\Omega}\right)}$$

$$Y(\Omega)=\Im\left\{2\left(\tfrac{1}{4}\right)^k u[k]-4\left(\tfrac{3}{4}\right)^k u[k]\right\}=\frac{2}{1-\frac{1}{4}e^{-j\Omega}}-\frac{4}{1-\frac{3}{4}e^{-j\Omega}}=\frac{-2\left(1+\frac{1}{4}e^{-j\Omega}\right)}{\left(1-\frac{1}{4}e^{-j\Omega}\right)\left(1-\frac{3}{4}e^{-j\Omega}\right)}.$$

(i) The transfer function of the system is calculated as follows.

$$H(\Omega)=\frac{Y(\Omega)}{X(\Omega)}=\frac{-\left(1+\frac{1}{4}e^{-j\Omega}\right)\left(1-\frac{1}{3}e^{-j\Omega}\right)}{\left(1-\frac{3}{4}e^{-j\Omega}\right)\left(1-\frac{7}{24}e^{-j\Omega}\right)}=\frac{-1+\frac{1}{12}e^{-j\Omega}+\frac{1}{12}e^{-j2\Omega}}{1-\frac{25}{24}e^{-j\Omega}+\frac{7}{32}e^{-j2\Omega}}$$

(ii) Using the partial fraction approach, the impulse response can be obtained.

$$H(\Omega)=\frac{-1+\frac{1}{12}e^{-j\Omega}+\frac{1}{12}e^{-j2\Omega}}{1-\frac{25}{24}e^{-j\Omega}+\frac{7}{32}e^{-j2\Omega}}=\frac{\frac{8}{21}\left(1-\frac{25}{24}e^{-j\Omega}+\frac{7}{32}e^{-j2\Omega}\right)-(1+\frac{8}{21})+(\frac{1}{12}+\frac{25}{63})e^{-j\Omega}}{1-\frac{25}{24}e^{-j\Omega}+\frac{7}{32}e^{-j2\Omega}}$$

$$=\frac{8}{21}+\frac{-\frac{29}{21}+\frac{121}{252}e^{-j\Omega}}{\left(1-\frac{3}{4}e^{-j\Omega}\right)\left(1-\frac{7}{24}e^{-j\Omega}\right)}=\frac{8}{21}+\frac{k_1}{1-\frac{3}{4}e^{-j\Omega}}+\frac{k_2}{1-\frac{7}{24}e^{-j\Omega}}$$

where $k_1=\dfrac{-\frac{29}{21}+\frac{121}{252}\times\frac{4}{3}}{1-\frac{7}{24}\times\frac{4}{3}}=-\dfrac{40}{33}$ and $k_2=\dfrac{-\frac{29}{21}+\frac{121}{252}\times\frac{24}{7}}{1-\frac{3}{4}\times\frac{24}{7}}=-\dfrac{13}{77}$

Substituting the values of k_1 and k_2, we get

$$H(\Omega)=\frac{8}{21}-\frac{40/33}{1-\frac{3}{4}e^{-j\Omega}}-\frac{13/77}{1-\frac{7}{24}e^{-j\Omega}}.$$

Calculating the inverse DTFT of $H(\Omega)$, the impulse response $h[k]$ is obtained as follows.

$$h[k]=\frac{8}{21}\delta[k]-\left[\frac{40}{33}\times(3/4)^k+\frac{13}{77}\times(7/24)^k\right]u[k].$$

(iii) $H(\Omega)=\dfrac{Y(\Omega)}{X(\Omega)}=\dfrac{-1+\frac{1}{12}e^{-j\Omega}+\frac{1}{12}e^{-j2\Omega}}{1-\frac{25}{24}e^{-j\Omega}+\frac{7}{32}e^{-j2\Omega}}$

By cross-multiplication, we obtain:

$$\left(1-\frac{25}{24}e^{-j\Omega}+\frac{7}{32}e^{-j2\Omega}\right)Y(\Omega)=\left(-1+\frac{1}{12}e^{-j\Omega}+\frac{1}{12}e^{-j2\Omega}\right)X(\Omega)$$

By calculating the inverse DTFT of both sides, we obtain the following difference equation:

$$y[k]-\frac{25}{24}y[k-1]+\frac{7}{32}y[k-2]=-x[k]+\frac{1}{12}x[k-1]+\frac{1}{12}x[k-2].$$

(iv) In part (ii), the impulse response was obtained as

$$h[k]=\frac{8}{21}\delta[k]-\left[\frac{40}{33}\times(3/4)^k+\frac{13}{77}\times(7/24)^k\right]u[k]$$

Because, $h[k]=0$ for all $k<0$, the system is causal. ▌

Problem 11.20:

The three LTID systems have the transfer functions as follows.

$$H_1(\Omega)=\frac{2}{1-\frac{3}{4}e^{-j\Omega}+\frac{1}{8}e^{-j2\Omega}}.$$

$$H_2(\Omega) = \begin{cases} 1 & |\Omega| \le \frac{\pi}{6} \\ 0 & \frac{\pi}{6} < |\Omega| \le \pi. \end{cases}$$

$$H_3(\Omega) = \begin{cases} 0 & |\Omega| \le \frac{\pi}{6} \\ 1 & \frac{\pi}{6} < |\Omega| \le \pi. \end{cases}$$

The problem includes several CT sinusoidal functions which are discretized using a sampling frequency of 8000 Hz. Let us assume that a sinusoidal signal with frequency f_0 Hz is passed through the three systems. After sampling the CT sinusoid, the corresponding DT frequency of the DT signal is given by $\Omega_0 = \dfrac{2\pi f_0}{8000}$. The gains of the three systems will be as follows.

$$H_1(\Omega)\Big|_{\Omega=\Omega_0} = \frac{2}{1 - \frac{3}{4}e^{-j\Omega_0} + \frac{1}{8}e^{-j2\Omega_0}}.$$

$$H_2(\Omega)\Big|_{\Omega=\Omega_0} = \begin{cases} 1 & \Omega_0 \le \frac{\pi}{6} \\ 0 & otherwise \end{cases}$$

$$H_3(\Omega)\Big|_{\Omega=\Omega_0} = \begin{cases} 0 & \Omega_0 \le \frac{\pi}{6} \\ 1 & otherwise \end{cases}$$

(i) $x_1(t) = 2 + 3\cos(400\pi t) + 7\cos(800\pi t)$

The DT sequence is given by, $x_1[k] = x_1(kT_s) = x_1(\frac{k}{8000}) = \underbrace{2}_{\Omega_0=0} + 3\underbrace{\cos\left(\frac{\pi k}{20}\right)}_{\Omega_1=\frac{\pi}{20}} + 5\underbrace{\cos\left(\frac{\pi k}{10}\right)}_{\Omega_2=\frac{\pi}{10}}$.

(a) The gains of the system at $\Omega = 0, \frac{\pi}{20}, \frac{\pi}{10}$ are given by:

$$H_1(\Omega)\Big|_{\Omega=0} = \frac{2}{1 - \frac{3}{4} + \frac{1}{8}} = \frac{16}{3};$$

$$H_1(\Omega)\Big|_{\Omega=\frac{\pi}{20}} = \frac{2}{1 - \frac{3}{4}e^{-j\frac{\pi}{20}} + \frac{1}{8}e^{-j\frac{\pi}{10}}} \approx \frac{2}{0.3781 + j0.0787} \approx 5.0698 - j1.0552 = 5.1784\sphericalangle -0.2052$$

$$H_1(\Omega)\Big|_{\Omega=\frac{\pi}{10}} = \frac{2}{1 - \frac{3}{4}e^{-j\frac{\pi}{10}} + \frac{1}{8}e^{-j\frac{\pi}{5}}} \approx \frac{2}{0.3878 + j0.1583} \approx 4.4205 - j1.8042 = 4.7745\sphericalangle -0.3875$$

The output signal can then be expressed as:

$$\begin{aligned} y_{1,H1}[k] &= 2\times\tfrac{16}{3} + 3\times 5.1784\cos\left(\tfrac{\pi k}{20} - 0.2052\right) + 5\times 4.7745\cos\left(\tfrac{\pi k}{10} - 0.3875\right) \\ &= \tfrac{32}{3} + 15.5352\cos\left(\tfrac{\pi k}{20} - 0.2052\right) + 23.8725\cos\left(\tfrac{\pi k}{10} - 0.3875\right). \end{aligned}$$

(b) The gains of the system at $\Omega = 0, \frac{\pi}{20}, \frac{\pi}{10}$ are given by:

$$H_2(\Omega)\Big|_{\Omega=0} = 1; \quad H_2(\Omega)\Big|_{\Omega=\frac{\pi}{20}} = 1; \quad H_2(\Omega)\Big|_{\Omega=\frac{\pi}{10}} = 1$$

As all three components will pass with a gain of 1, the output signal can then be expressed as:

$$y_{1,H2}[k] = 2 + 3\cos\left(\tfrac{\pi k}{20}\right) + 5\cos\left(\tfrac{\pi k}{10}\right).$$

(c) The gains of the system at $\Omega = 0, \frac{\pi}{20}, \frac{\pi}{10}$ are given by:

$$H_3(\Omega)\Big|_{\Omega=0} = 0; \qquad H_3(\Omega)\Big|_{\Omega=\frac{\pi}{20}} = 0; \qquad H_3(\Omega)\Big|_{\Omega=\frac{\pi}{10}} = 0.$$

As all three components will pass with a gain of 0, the output signal can then be expressed as:

$$y_{1,H3}[k] = 0.$$

(ii) $x_2(t) = 2\cos(4000\pi t) + 5\cos(6000\pi t)$

The DT sequence is given by, $x_2[k] = x_2(kT_s) = x_2(\frac{k}{8000}) = 2\underbrace{\cos\left(\frac{\pi k}{2}\right)}_{\Omega_0=\frac{\pi}{2}} + 5\underbrace{\cos\left(\frac{3\pi k}{4}\right)}_{\Omega_1=\frac{3\pi}{4}}$.

(a) The gains of the system at $\Omega = \frac{\pi}{2}, \frac{3\pi}{4}$ are given by:

$$H_1(\Omega)\Big|_{\Omega=\frac{\pi}{2}} = \frac{2}{1-\frac{3}{4}e^{-j\frac{\pi}{2}}+\frac{1}{8}e^{-j\pi}} \approx \frac{2}{0.875+j0.75} \approx 1.3176 - j1.1294 = 1.7354\measuredangle -0.7086$$

$$H_1(\Omega)\Big|_{\Omega=\frac{3\pi}{4}} = \frac{2}{1-\frac{3}{4}e^{-j\frac{3\pi}{4}}+\frac{1}{8}e^{-j\frac{3\pi}{2}}} \approx \frac{2}{1.5303+j0.6553} \approx 1.1044 - j0.4729 = 1.2014\measuredangle -0.4046$$

The output signal can then be expressed as:

$$\begin{aligned} y_{2,H1}[k] &= 2\times 1.7354\cos\left(\tfrac{\pi k}{2} - 0.7086\right) + 5\times 1.2014\cos\left(\tfrac{3\pi k}{4} - 0.4046\right) \\ &= 3.4708\cos\left(\tfrac{\pi k}{2} - 0.7086\right) + 6.007\cos\left(\tfrac{3\pi k}{4} - 0.4046\right). \end{aligned}$$

(b) The gains of the system at $\Omega = \frac{\pi}{2}, \frac{3\pi}{4}$ are given by:

$$H_2(\Omega)\Big|_{\Omega=\frac{\pi}{2}} = 0; \qquad H_2(\Omega)\Big|_{\Omega=\frac{3\pi}{4}} = 0.$$

As all three components will pass with a gain of 0, the output signal can then be expressed as:

$$y_{2,H2}[k] = 0.$$

(c) The gains of the system at $\Omega = \frac{\pi}{2}, \frac{3\pi}{4}$ are given by:

$$H_3(\Omega)\Big|_{\Omega=\frac{\pi}{2}} = 1; \qquad H_3(\Omega)\Big|_{\Omega=\frac{3\pi}{4}} = 1.$$

As all three components will pass with a gain of 1, the output signal can then be expressed as:

$$y_{2,H3}[k] = 2\cos\left(\tfrac{\pi k}{2}\right) + 5\cos\left(\tfrac{3\pi k}{4}\right).$$

(iii) $x_3(t) = 5\cos(600\pi t) + 9\cos(900\pi t) + 2\cos(3000\pi t)$

The DT sequence is given by, $x_3[k] = x_3(kT_s) = x_3(\frac{k}{8000}) = 5\underbrace{\cos\left(\frac{3\pi k}{40}\right)}_{\Omega_0=\frac{3\pi}{40}} + 9\underbrace{\cos\left(\frac{9\pi k}{80}\right)}_{\Omega_1=\frac{9\pi}{80}} + 2\underbrace{\cos\left(\frac{3\pi k}{8}\right)}_{\Omega_1=\frac{3\pi}{8}}$.

(a) The gains of the system at $\Omega = \frac{3\pi}{40}, \frac{9\pi}{80}, \frac{3\pi}{8}$ are given by:

$$H_1(\Omega)\Big|_{\Omega=\frac{3\pi}{40}} = \frac{2}{1-\frac{3}{4}e^{-j\frac{3\pi}{40}}+\frac{1}{8}e^{-j\frac{3\pi}{20}}} \approx \frac{2}{0.3821+j0.1183} \approx 4.7762 - j1.4792 = 5\measuredangle -0.3003$$

$$H_1(\Omega)\Big|_{\Omega=\frac{9\pi}{80}} = \frac{2}{1-\frac{3}{4}e^{-j\frac{9\pi}{80}}+\frac{1}{8}e^{-j\frac{9\pi}{80}}} \approx \frac{2}{0.3914+j0.1784} \approx 4.2308-j1.9284 = 4.6495\measuredangle -0.4277$$

$$H_1(\Omega)\Big|_{\Omega=\frac{3\pi}{8}} = \frac{2}{1-\frac{3}{4}e^{-j\frac{3\pi}{8}}+\frac{1}{8}e^{-j\frac{3\pi}{4}}} \approx \frac{2}{0.6246+j0.6045} \approx 1.6533-j1.6002 = 2.3009\measuredangle -0.7691$$

The output signal can then be expressed as:

$$\begin{aligned} y_{3,H1}[k] &= 5\times 5\cos\left(\tfrac{3\pi k}{40}-0.3003\right)+9\times 4.6495\cos\left(\tfrac{9\pi k}{80}-0.4277\right)+2\times 2.3009\cos\left(\tfrac{3\pi k}{8}-0.7691\right) \\ &= 25\cos\left(\tfrac{3\pi k}{40}-0.3003\right)+41.8455\cos\left(\tfrac{9\pi k}{80}-0.4277\right)+4.6018\cos\left(\tfrac{3\pi k}{8}-0.7691\right) \end{aligned}$$

(b) The gains of the system at $\Omega=\frac{3\pi}{40},\frac{9\pi}{80},\frac{3\pi}{8}$ are given by:

$$H_2(\Omega)\Big|_{\Omega=\frac{3\pi}{40}}=1; \qquad H_2(\Omega)\Big|_{\Omega=\frac{9\pi}{80}}=1; \qquad H_2(\Omega)\Big|_{\Omega=\frac{3\pi}{8}}=0.$$

As the first two components will pass with a gain of 1, and the third component will pass with a gain of 0, the output signal can then be expressed as:

$$y_{3,H2}[k]=5\cos\left(\tfrac{3\pi k}{40}\right)+9\cos\left(\tfrac{9\pi k}{80}\right).$$

(c) The gains of the system at $\Omega=\frac{3\pi}{40},\frac{9\pi}{80},\frac{3\pi}{8}$ are given by:

$$H_3(\Omega)\Big|_{\Omega=\frac{3\pi}{40}}=0; \qquad H_3(\Omega)\Big|_{\Omega=\frac{9\pi}{80}}=0; \qquad H_3(\Omega)\Big|_{\Omega=\frac{3\pi}{8}}=1.$$

As the first two components will pass with a gain of 0 and the third component will pass with a gain of 1, the output signal can then be expressed as:

$$y_{3,H3}[k]=2\cos\left(\tfrac{3\pi k}{8}\right).$$

(iv) $x_4(t)=4\cos(600\pi t)+6\cos(12000\pi t)$

The DT sequence is given by,

$$x_4[k]=x_4(kT_s)=x_4\left(\tfrac{k}{8000}\right)=4\cos\left(\tfrac{3\pi k}{40}\right)+6\cos\left(\tfrac{3\pi k}{2}\right)=4\underbrace{\cos\left(\tfrac{3\pi k}{40}\right)}_{\Omega_0=\frac{3\pi}{40}}+6\underbrace{\cos\left(\tfrac{\pi k}{2}\right)}_{\Omega_1=\frac{\pi}{2}}.$$

(a) The gains of the system at $\Omega=\frac{3\pi}{40},\frac{\pi}{2}$ are given by:

$$H_1(\Omega)\Big|_{\Omega=\frac{3\pi}{40}} = \frac{2}{1-\frac{3}{4}e^{-j\frac{3\pi}{40}}+\frac{1}{8}e^{-j\frac{3\pi}{20}}} \approx \frac{2}{0.3821+j0.1183} \approx 4.7762-j1.4792 = 5\measuredangle -0.3003$$

$$H_1(\Omega)\Big|_{\Omega=\frac{\pi}{2}} = \frac{2}{1-\frac{3}{4}e^{-j\frac{\pi}{2}}+\frac{1}{8}e^{-j\pi}} = \frac{2}{1+\frac{3}{4}j-\frac{1}{8}} \approx \frac{2}{\frac{7}{8}+j0.75} \approx 1.3176-j1.1294 = 1.7354\measuredangle -0.7086$$

The output signal can then be expressed as:

$$\begin{aligned} y_{4,H1}[k] &= 5\times 4\cos\left(\tfrac{3\pi k}{40}-0.3003\right)+6\times 1.7354\cos\left(\tfrac{\pi k}{2}-0.7086\right) \\ &= 20\cos\left(\tfrac{3\pi k}{40}-0.3003\right)+10.4124\cos\left(\tfrac{\pi k}{2}-0.7086\right). \end{aligned}$$

(b) The gains of the system at $\Omega=\frac{3\pi}{40},\frac{\pi}{2}$ are given by:

$$H_2(\Omega)\big|_{\Omega=\frac{3\pi}{40}} = 1; \qquad H_2(\Omega)\big|_{\Omega=\frac{\pi}{2}} = 0.$$

As the first component will pass with a gain of 1, and the second component will pass with a gain of 0, the output signal can be expressed as:

$$y_{4,H2}[k] = 4\cos\left(\tfrac{3\pi k}{40}\right).$$

(c) The gains of the system at $\Omega = \frac{3\pi}{40}, \frac{\pi}{2}$ are given by:

$$H_3(\Omega)\big|_{\Omega=\frac{3\pi}{40}} = 0; \qquad H_3(\Omega)\big|_{\Omega=\frac{\pi}{2}} = 1.$$

As the first component will pass with a gain of 0 and the second component will pass with a gain of 1, the output signal can be expressed as:

$$y_{4,H3}[k] = 6\cos\left(\tfrac{\pi k}{2}\right).$$ ▌

Problem 11.21:

The three LTID systems have the transfer functions as follows.

$$H_1(\Omega) = \frac{2}{1 - \frac{3}{4}e^{-j\Omega} + \frac{1}{8}e^{-j2\Omega}}.$$

$$H_2(\Omega) = \begin{cases} 1 & |\Omega| \le \frac{\pi}{6} \\ 0 & \frac{\pi}{6} < |\Omega| \le \pi. \end{cases}$$

$$H_3(\Omega) = \begin{cases} 0 & |\Omega| \le \frac{\pi}{6} \\ 1 & \frac{\pi}{6} < |\Omega| \le \pi. \end{cases}$$

The problem includes several CT sinusoidal functions which are discretized using a sampling frequency of 22000 Hz. Let us assume that a sinusoidal signal with frequency f_0 Hz is passed through the three systems. After sampling the CT sinusoid, the corresponding DT frequency of the DT signal is given by $\Omega_0 = \dfrac{2\pi f_0}{22000}$. The gains of the three systems will be as follows.

$$H_1(\Omega)\big|_{\Omega=\Omega_0} = \frac{2}{1 - \frac{3}{4}e^{-j\Omega_0} + \frac{1}{8}e^{-j2\Omega_0}}.$$

$$H_2(\Omega)\big|_{\Omega=\Omega_0} = \begin{cases} 1 & \Omega_0 \le \frac{\pi}{6} \\ 0 & otherwise \end{cases}$$

$$H_3(\Omega)\big|_{\Omega=\Omega_0} = \begin{cases} 0 & \Omega_0 \le \frac{\pi}{6} \\ 1 & otherwise \end{cases}$$

(i) $x_1(t) = 2 + 3\cos(8000\pi t) + 7\cos(18000\pi t)$

The DT sequence is given by, $x_1[k] = x_1(kT_s) = x_1(\frac{k}{22000}) = \underbrace{2}_{\Omega_1=0} + 3\underbrace{\cos(\tfrac{4\pi}{11}k)}_{\Omega_2=\frac{4\pi}{11}} + 7\underbrace{\cos(\tfrac{9\pi}{11}k)}_{\Omega_3=\frac{9\pi}{11}}$.

(a) The gains of the system at $\Omega = 0, \frac{4\pi}{11}, \frac{9\pi}{11}$ are given by:

$$H_1(\Omega)\big|_{\Omega=0} = \frac{2}{1-\frac{3}{4}+\frac{1}{8}} = \frac{16}{3};$$

$$H_1(\Omega)\big|_{\Omega=\frac{4\pi}{11}} = \frac{2}{1-\frac{3}{4}e^{-j\frac{4\pi}{11}}+\frac{1}{8}e^{-j\frac{8\pi}{11}}} \approx \frac{2}{0.6066+j0.5878} \approx 1.7005-j1.6478 = 2.3679\measuredangle-0.7696$$

$$H_1(\Omega)\big|_{\Omega=\frac{9\pi}{11}} = \frac{2}{1-\frac{3}{4}e^{-j\frac{9\pi}{11}}+\frac{1}{8}e^{-j\frac{18\pi}{11}}} \approx \frac{2}{1.6829+j0.5192} \approx 1.0852-j0.3348 = 1.1356\measuredangle-0.2992$$

The output signal can then be expressed as:

$$\begin{aligned} y_{1,H1}[k] &= \tfrac{16}{3}\times 2 \ + \ 3\times 2.3679\cos(\tfrac{4\pi}{11}k-0.7696) \ + \ 7\times 1.1356\cos(\tfrac{9\pi}{11}k-0.2992) \\ &= \tfrac{32}{3} + \ 7.1037\cos(\tfrac{4\pi}{11}k-0.7696) \ + \ 7.9492\cos(\tfrac{9\pi}{11}k-0.2992). \end{aligned}$$

(b) The gains of the system at $\Omega = 0, \frac{4\pi}{11}, \frac{9\pi}{11}$ are given by:

$$H_2(\Omega)\big|_{\Omega=0} = 1; \qquad H_2(\Omega)\big|_{\Omega=\frac{4\pi}{11}} = 0; \qquad H_2(\Omega)\big|_{\Omega=\frac{9\pi}{11}} = 0.$$

As the first two components will pass with a gain of 1, and the third component will pass with a gain of 0, the output signal can be expressed as:

$$y_{1,H2}[k] = 2.$$

(c) The gains of the system at $\Omega = 0, \frac{4\pi}{11}, \frac{9\pi}{11}$ are given by:

$$H_3(\Omega)\big|_{\Omega=0} = 0; \qquad H_3(\Omega)\big|_{\Omega=\frac{4\pi}{11}} = 1; \qquad H_3(\Omega)\big|_{\Omega=\frac{9\pi}{11}} = 1.$$

As the first component will pass with a gain of 0 and the next two components will pass with a gain of 1, the output signal can be expressed as:

$$y_{1,H3}[k] = \ 3\cos(\tfrac{4\pi}{11}k) \ + \ 7\cos(\tfrac{9\pi}{11}k).$$

(ii) $x_2(t) = 2\cos(10000\pi t) + 5\cos(30000\pi t)$

The DT sequence is given by,

$$x_2[k] = x_2(kT_s) = x_2(\tfrac{k}{22000}) = 2\cos(\tfrac{5\pi}{11}k) + 5\cos(\tfrac{15\pi}{11}k) = 2\underbrace{\cos(\tfrac{5\pi}{11}k)}_{\Omega_2=\frac{5\pi}{11}} + 5\underbrace{\cos(\tfrac{7\pi}{11}k)}_{\Omega_3=\frac{7\pi}{11}}.$$

(a) The gains of the system at $\Omega = \frac{5\pi}{11}, \frac{7\pi}{11}$ are given by:

$$H_1(\Omega)\big|_{\Omega=\frac{5\pi}{11}} = \frac{2}{1-\frac{3}{4}e^{-j\frac{5\pi}{11}}+\frac{1}{8}e^{-j\frac{10\pi}{11}}} \approx \frac{2}{0.7733+j0.7071} \approx 1.4085-j1.2880 = 1.9086\measuredangle-0.7407$$

$$H_1(\Omega)\big|_{\Omega=\frac{7\pi}{11}} = \frac{2}{1-\frac{3}{4}e^{-j\frac{7\pi}{11}}+\frac{1}{8}e^{-j\frac{14\pi}{11}}} \approx \frac{2}{1.2297+j0.7767} \approx 1.1626-j0.7343 = 1.3751\measuredangle-0.5633$$

The output signal can then be expressed as:

$$\begin{aligned} y_{2,H1}[k] &= \ 2\times 1.9086\cos(\tfrac{5\pi}{11}k-0.7407) \ + \ 5\times 1.3751\cos(\tfrac{7\pi}{11}k-0.5633) \\ &= 3.8172\cos(\tfrac{5\pi}{11}k-0.7407) \ + \ 6.8755\cos(\tfrac{7\pi}{11}k-0.5633). \end{aligned}$$

(b) The gains of the system at $\Omega = \frac{5\pi}{11}, \frac{7\pi}{11}$ are given by:

$$H_2(\Omega)\big|_{\Omega=\frac{5\pi}{11}} = 0; \qquad H_2(\Omega)\big|_{\Omega=\frac{7\pi}{11}} = 0.$$

As both the components will pass with a gain of 0, the output $y_{2,H2}[k] = 0$.

(c) The gains of the system at $\Omega = \frac{5\pi}{11}, \frac{7\pi}{11}$ are given by:

$$H_3(\Omega)\big|_{\Omega=\frac{5\pi}{11}} = 1; \qquad H_3(\Omega)\big|_{\Omega=\frac{7\pi}{11}} = 1.$$

As both the components will pass with a gain of 1, the output

$$y_{2,H1}[k] = 2\cos(\tfrac{5\pi}{11}k - 0.7407) + 5\cos(\tfrac{7\pi}{11}k - 0.5633).$$

(iii) $x_3(t) = 5\cos(600\pi t) + 9\cos(900\pi t) + 2\cos(3000\pi t)$

The DT sequence is given by, $x_3[k] = x_3(kT_s) = x_3(\frac{k}{22000}) = 5\underbrace{\cos(\tfrac{3\pi}{110}k)}_{\Omega_0=\frac{3\pi}{110}} + 9\underbrace{\cos(\tfrac{9\pi}{220}k)}_{\Omega_1=\frac{9\pi}{220}} + 2\underbrace{\cos(\tfrac{3\pi}{22}k)}_{\Omega_2=\frac{3\pi}{22}}$.

(a) The gains of the system at $\Omega = \frac{3\pi}{110}, \frac{9\pi}{220}, \frac{3\pi}{22}$ are given by:

$$H_1(\Omega)\big|_{\Omega=\frac{3\pi}{110}} = \frac{2}{1-\frac{3}{4}e^{-j\frac{3\pi}{110}} + \frac{1}{8}e^{-j\frac{3\pi}{55}}} \approx \frac{2}{0.3759 + j0.0429} \approx 5.2520 - j0.5989 = 5.2860\measuredangle -0.1135;$$

$$H_1(\Omega)\big|_{\Omega=\frac{9\pi}{220}} = \frac{2}{1-\frac{3}{4}e^{-j\frac{9\pi}{220}} + \frac{1}{8}e^{-j\frac{9\pi}{110}}} \approx \frac{2}{0.3771 + j0.0643} \approx 5.1538 - j0.8795 = 5.2283\measuredangle -0.1690$$

$$H_1(\Omega)\big|_{\Omega=\frac{3\pi}{22}} = \frac{2}{1-\frac{3}{4}e^{-j\frac{3\pi}{22}} + \frac{1}{8}e^{-j\frac{3\pi}{11}}} \approx \frac{2}{0.3996 + j0.2171} \approx 3.8642 - j2.0992 = 4.3976\measuredangle -0.4976$$

The output signal can then be expressed as:

$$\begin{aligned} y_{3,H1}[k] &= 5\times 5.2860\cos(\tfrac{3\pi}{110}k - 0.1135) + 9\times 5.2283\cos(\tfrac{9\pi}{220}k - 0.1690) + 2\times 4.3976\cos(\tfrac{3\pi}{22}k - 0.4976) \\ &= 26.43\cos(\tfrac{3\pi}{110}k - 0.1135) + 47.0547\cos(\tfrac{9\pi}{220}k - 0.1690) + 8.7952\cos(\tfrac{3\pi}{22}k - 0.4976). \end{aligned}$$

(b) The gains of the system at $\Omega = \frac{3\pi}{110}, \frac{9\pi}{220}, \frac{3\pi}{22}$ are given by:

$$H_2(\Omega)\big|_{\Omega=\frac{3\pi}{110}} = 1; \qquad H_2(\Omega)\big|_{\Omega=\frac{9\pi}{220}} = 1; \qquad H_2(\Omega)\big|_{\Omega=\frac{3\pi}{22}} = 1.$$

As all there components will pass with a gain of 1, the output is given by:

$$y_{3,H2}[k] = x_3[k] = 5\cos(\tfrac{3\pi}{110}k) + 9\cos(\tfrac{9\pi}{220}k) + 2\cos(\tfrac{3\pi}{22}k).$$

(c) The gains of the system at $\Omega = \frac{3\pi}{110}, \frac{9\pi}{220}, \frac{3\pi}{22}$ are given by:

$$H_3(\Omega)\big|_{\Omega=\frac{3\pi}{110}} = 0; \qquad H_3(\Omega)\big|_{\Omega=\frac{9\pi}{220}} = 0; \qquad H_3(\Omega)\big|_{\Omega=\frac{3\pi}{22}} = 0.$$

As all there components will pass with a gain of 0, the output is given by:

$$y_{3,H3}[k] = 0.$$

(iv) $x_4(t) = 4\cos(28000\pi t) + 6\cos(18000\pi t)$

The DT sequence is given by,

$$x_4[k] = x_4(kT_s) = x_4(\tfrac{k}{22000}) = 4\cos(\tfrac{14\pi}{11}k) + 9\cos(\tfrac{9\pi}{11}k) = 4\underbrace{\cos(\tfrac{8\pi}{11}k)}_{\Omega_0=\frac{8\pi}{11}} + 9\underbrace{\cos(\tfrac{9\pi}{11}k)}_{\Omega_1=\frac{9\pi}{11}}.$$

(a) The gains of the system at $\Omega = \frac{8\pi}{11}, \frac{9\pi}{11}$ are given by:

$$H_1(\Omega)\Big|_{\Omega=\frac{8\pi}{11}} = \frac{2}{1-\frac{3}{4}e^{-j\frac{8\pi}{11}} + \frac{1}{8}e^{-j\frac{16\pi}{11}}} \approx \frac{2}{1.4734 + j0.6905} \approx 1.1130 - j0.5216 = 1.2291\measuredangle -0.4383;$$

$$H_1(\Omega)\Big|_{\Omega=\frac{9\pi}{11}} = \frac{2}{1-\frac{3}{4}e^{-j\frac{9\pi}{11}} + \frac{1}{8}e^{-j\frac{18\pi}{11}}} \approx \frac{2}{1.6829 + j0.5192} \approx 1.0852 - j0.3348 = 1.1356\measuredangle -0.2992$$

The output signal can then be expressed as:

$$\begin{aligned} y_{4,H1}[k] &= 5\times 5.2860\cos(\tfrac{3\pi}{110}k - 0.1135) + 9\times 5.2283\cos(\tfrac{9\pi}{220}k - 0.1690) + 2\times 4.3976\cos(\tfrac{3\pi}{22}k - 0.4976) \\ &= 26.43\cos(\tfrac{3\pi}{110}k - 0.1135) + 47.0547\cos(\tfrac{9\pi}{220}k - 0.1690) + 8.7952\cos(\tfrac{3\pi}{22}k - 0.4976). \end{aligned}$$

(b) The gains of the system at $\Omega = \frac{8\pi}{11}, \frac{9\pi}{11}$ are given by:

$$H_2(\Omega)\Big|_{\Omega=\frac{8\pi}{11}} = 0; \qquad H_2(\Omega)\Big|_{\Omega=\frac{9\pi}{11}} = 0.$$

As both the components will pass with a gain of 0, the output is given by:

$$y_{4,H2}[k] = 0.$$

(c) The gains of the system at $\Omega = \frac{8\pi}{11}, \frac{9\pi}{11}$ are given by:

$$H_3(\Omega)\Big|_{\Omega=\frac{8\pi}{11}} = 1; \qquad H_3(\Omega)\Big|_{\Omega=\frac{9\pi}{11}} = 1.$$

As both the components will pass with a gain of 1, the output is given by:

$$y_{4,H3}[k] = x_4[k] = 4\cos(\tfrac{8\pi}{11}k) + 9\cos(\tfrac{9\pi}{11}k). \qquad \blacksquare$$