Chapter 13: The z-Transform

Problem 13.1

(i) $x_1[k] = 0.5^{k+1}u[k+5]$

By definition, $$0.5^k u[k] \xleftrightarrow{z} \frac{1}{1-0.5z^{-1}} \quad \text{ROC}: |z| > 0.5$$

Using the time shifting property, $$0.5^{k+5} u[k+5] \xleftrightarrow{z} \frac{z^5}{1-0.5z^{-1}} \quad \text{ROC}: |z| > 0.5, z \neq \infty$$

which implies that $$0.5^{k+1} u[k+5] \xleftrightarrow{z} \frac{0.5^{-4}z^5}{1-0.5z^{-1}} \quad \text{ROC}: |z| > 0.5, z \neq \infty .$$

(ii) $x_2[k] = (k+2)0.5^{|k|}$

By definition, $$0.5^k u[k] \xleftrightarrow{z} \frac{1}{1-0.5z^{-1}} \quad \text{ROC}: |z| > 0.5$$

and $$-0.5^{-k} u[-k-1] = -2^k u[-k-1] \xleftrightarrow{z} \frac{1}{1-2z^{-1}} \quad \text{ROC}: |z| < 2.$$

Applying the linearity property,

$$0.5^k u[k] + 0.5^{-k} u[-k-1] \xleftrightarrow{z} \frac{1}{1-0.5z^{-1}} - \frac{1}{1-2z^{-1}} \quad \text{ROC}: \left[|z| > 0.5\right] \cap \left[|z| < 2\right],$$

or, $$0.5^{|k|} \xleftrightarrow{z} \frac{-1.5z^{-1}}{\left(1-0.5z^{-1}\right)\left(1-2z^{-1}\right)} \quad \text{ROC}: 0.5 < |z| < 2.$$

Applying the frequency differentiation property

$$k0.5^{|k|} \xleftrightarrow{z} -z\frac{d}{dz}\frac{-1.5z^{-1}}{\left(1-0.5z^{-1}\right)\left(1-2z^{-1}\right)} \quad \text{ROC}: 0.5 < |z| < 2.$$

We have just proved that

$$2 \times 0.5^{|k|} \xleftrightarrow{z} \frac{3z^{-1}}{\left(1-0.5z^{-1}\right)\left(1-2z^{-1}\right)} \quad \text{ROC}: 0.5 < |z| < 2.$$

Using the linearity property

$$(k+2)0.5^{|k|} \xleftrightarrow{z} \frac{3z^{-1}}{\left(1-0.5z^{-1}\right)\left(1-2z^{-1}\right)} + z\frac{d}{dz}\frac{1.5z^{-1}}{\left(1-0.5z^{-1}\right)\left(1-2z^{-1}\right)} \quad \text{ROC}: 0.5 < |z| < 2.$$

(iii) $x_3[k] = |k+2| \times 0.5^{|k+2|}$

Consider $$|k| 0.5^{|k|} = \begin{cases} k \times 0.5^{k} & k \ge 0 \\ -k \times 0.5^{-k} & k < 0 \end{cases} = k \times 0.5^{k} U[k] - k \times 2^{k} U[-k-1].$$

Now, $$k \times 0.5^{k} U[k] \xleftrightarrow{z} \frac{0.5z^{-1}}{\left(1-0.5z^{-1}\right)^{2}} \quad \text{ROC}: |z| > 0.5$$

and $$-k \times 2^{k} U[-k-1] \xleftrightarrow{z} \frac{2z^{-1}}{\left(1-2z^{-1}\right)^{2}} \quad \text{ROC}: |z| < 2$$

which implies that $$|k| 0.5^{|k|} \xleftrightarrow{z} \frac{0.5z^{-1}}{\left(1-0.5z^{-1}\right)^{2}} + \frac{2z^{-1}}{\left(1-2z^{-1}\right)^{2}} \quad \text{ROC}: 0.5 < |z| < 2.$$

Using the time shifting property

$$|k+2| 0.5^{|k+2|} \xleftrightarrow{z} \frac{0.5z}{\left(1-0.5z^{-1}\right)^{2}} + \frac{2z}{\left(1-2z^{-1}\right)^{2}} \quad \text{ROC}: 0.5 < |z| < 2$$

(iv) $x_4[k] = 3^{k+1}\cos\left(\frac{\pi}{3}k - \frac{\pi}{4}\right) u[-k+5]$

Expressing $$x_4[k] = 3^{k+1}\cos\left(\tfrac{\pi}{3}k\right)\cos\left(\tfrac{\pi}{4}\right) u[-k+5] - 3^{k+1}\sin\left(\tfrac{\pi}{3}k\right)\sin\left(\tfrac{\pi}{4}\right) u[-k+5]$$

or, $$x_4[k] = \frac{3^{k+1}}{\sqrt{2}}\left[\cos\left(\tfrac{\pi}{3}k\right) - \sin\left(\tfrac{\pi}{3}k\right)\right] u[-k+5]$$

or, $$x_4[k] = \frac{3^{k+1}}{\sqrt{2}}\left[\frac{e^{j\pi k/3} + e^{-j\pi k/3}}{2} - \frac{e^{j\pi k/3} - e^{-j\pi k/3}}{2j}\right] u[-k+5]$$

or, $$x_4[k] = \frac{3}{2\sqrt{2}}(1+j1)\left(3e^{j\pi/3}\right)^{k} u[-k+5] + \frac{3}{2\sqrt{2}}(1-j1)\left(3e^{-j\pi/3}\right)^{k} u[-k+5].$$

We know that $$-\alpha^{k} u[-k-1] \xleftrightarrow{z} \frac{1}{1-\alpha z^{-1}} \quad \text{ROC}: |z| < \alpha$$

or, $$-\alpha^{k-6} u[-(k-6)-1] \xleftrightarrow{z} \frac{z^{-6}}{1-\alpha z^{-1}} \quad \text{ROC}: |z| < \alpha$$

or, $$-\alpha^{k} u[-k+5] \xleftrightarrow{z} \frac{(z/\alpha)^{-6}}{1-\alpha z^{-1}} \quad \text{ROC}: |z| < \alpha.$$

Hence, $$X_4(z) = \frac{3^{7}}{2\sqrt{2}}(1+j1)\frac{(ze^{-j\pi/3})^{-6}}{1-3e^{j\pi/3}z^{-1}} + \frac{3^{7}}{2\sqrt{2}}(1-j1)\frac{(ze^{j\pi/3})^{-6}}{1-3e^{-j\pi/3}z^{-1}} \quad \text{ROC}: |z| < 3.$$ ▌

Problem 13.2

(i) The sequence can be expressed as

$$x_1[k] = \delta[k-10] + \delta[k-11] + 2\delta[k-12] + 2\delta[k-15],$$

which has the z-transform

$$X_1(z) = z^{-10} + z^{-11} + 2z^{-12} + 2z^{-15}, \text{ ROC: entire z-plane except } z = 0.$$

(ii) By definition,

$$X_2(z) = \sum_{k=-\infty}^{\infty}\left[3^{-k+2}u[k] + \sum_{m=1}^{4} m\delta[k-m]\right]z^{-k}$$

$$= \sum_{k=0}^{\infty} 3^{-k+2} z^{-k} + \sum_{k=-\infty}^{\infty}\left\{\delta[k-1] + 2\delta[k-2] + 3\delta[k-3] + 4\delta[k-4]\right\} z^{-k}$$

$$= 9\sum_{k=0}^{\infty}(3z)^{-k} + z^{-1} + 2z^{-2} + 3z^{-3} + 4z^{-4} \qquad |z| \neq 0$$

$$= \frac{9}{1-(3z)^{-1}} + z^{-1} + 2z^{-2} + 3z^{-3} + 4z^{-4} \qquad |z| \neq 0, |z| > \tfrac{1}{3}$$

$$= \frac{9}{1-\frac{1}{3}z^{-1}} + z^{-1} + 2z^{-2} + 3z^{-3} + 4z^{-4} \qquad \text{ROC: } |z| > \tfrac{1}{3}$$

(iii) Expressing the sequence as

$$x_3[k] = \sin(\tfrac{\pi k}{5} + \tfrac{\pi}{3})u[k] = \left[\sin(\tfrac{\pi k}{5})\cos(\tfrac{\pi}{3}) + \cos(\tfrac{\pi k}{5})\sin(\tfrac{\pi}{3})\right]u[k] = \left[\tfrac{1}{2}\sin(\tfrac{\pi k}{5}) + \tfrac{\sqrt{3}}{2}\cos(\tfrac{\pi k}{5})\right]u[k]$$

the z-transform is given by

$$X_3(z) = \mathbb{Z}\left\{\left[\tfrac{1}{2}\sin(\tfrac{\pi k}{5}) + \tfrac{\sqrt{3}}{2}\cos(\tfrac{\pi k}{5})\right]u[k]\right\} = \tfrac{1}{2}\mathbb{Z}\left\{\left[\sin(\tfrac{\pi k}{5})\right]u[k]\right\} + \tfrac{\sqrt{3}}{2}\mathbb{Z}\left\{\left[\cos(\tfrac{\pi k}{5})\right]u[k]\right\}$$

$$= \tfrac{1}{2}\frac{\sin(\frac{\pi}{5})z^{-1}}{1-2\cos(\frac{\pi}{5})z^{-1} + z^{-2}} + \tfrac{\sqrt{3}}{2}\frac{1-\cos(\frac{\pi}{5})z^{-1}}{1-2\cos(\frac{\pi}{5})z^{-1} + z^{-2}} \qquad \text{ROC: } |z| > 1$$

$$= \frac{\frac{1}{2}\sin(\frac{\pi}{5})z^{-1} + \frac{\sqrt{3}}{2}\left\{1-\cos(\frac{\pi}{5})z^{-1}\right\}}{1-2\cos(\frac{\pi}{5})z^{-1} + z^{-2}}$$

$$= \frac{\frac{\sqrt{3}}{2} + \frac{1}{2}\sin(\frac{\pi}{5})z^{-1} - \frac{\sqrt{3}}{2}\cos(\frac{\pi}{5})z^{-1}}{1-2\cos(\frac{\pi}{5})z^{-1} + z^{-2}}$$

$$\approx \frac{0.866 - 0.407z^{-1}}{1-1.618z^{-1} + z^{-2}} \qquad \text{ROC: } |z| > 1$$

(iv) We know that

$$(0.5e^{j\pi/5})^k u[k] \overset{Z}{\longleftrightarrow} \frac{1}{1-0.5e^{j\pi/5}z^{-1}} \quad \text{ROC}: |z| > 0.5$$

and

$$(0.5e^{-j\pi/5})^k u[k] \overset{Z}{\longleftrightarrow} \frac{1}{1-0.5e^{-j\pi/5}z^{-1}} \quad \text{ROC}: |z| > 0.5$$

Adding the two transform pairs after multiplying respectively with exp($j\pi/3$) and exp($-j\pi/3$), gives

$$e^{j\pi/3}(0.5e^{j\pi/5})^k u[k] - e^{-j\pi/3}(0.5e^{-j\pi/5})^k u[k] \overset{Z}{\longleftrightarrow} \frac{e^{j\pi/3}}{1-0.5e^{j\pi/5}z^{-1}} - \frac{e^{-j\pi/3}}{1-0.5e^{-j\pi/5}z^{-1}} \quad \text{ROC}: |z| > 0.5$$

or,

$$2j \times 0.5^k \sin(\tfrac{\pi k}{5} + \tfrac{\pi}{3})u[k] \overset{Z}{\longleftrightarrow} \frac{e^{j\pi/3}}{1-0.5e^{j\pi/5}z^{-1}} - \frac{e^{-j\pi/3}}{1-0.5e^{-j\pi/5}z^{-1}} \quad \text{ROC}: |z| > 0.5.$$

(v) By definition

$$X_5(z)=\sum_{k=0}^{\infty}ku[k]z^{-k}=\sum_{k=0}^{\infty}kz^{-k}=\frac{z}{(z-1)^2}\qquad \text{ROC: } |z|>1$$

Problem 13.3

In parts (i)-(vii), the sequences are all causal (right hand sided), and hence the ROC is the outside of the circle with radius equal to the magnitude of the pole furthermost from the origin in the z-plane.

(i) $X_1(z)=\dfrac{z}{z^2-0.9z+0.2}$

By partial fraction expansion, we obtain

$$X_1(z)=\frac{z}{z^2-0.9z+0.2}=\frac{z}{(z-0.5)(z-0.4)}=10z\left[\frac{1}{z-0.5}-\frac{1}{z-0.4}\right]=10\left(\frac{z}{z-0.5}-\frac{z}{z-0.4}\right)$$

Therefore, $$x_1[k]=10\left(0.5^k u[k]-0.4^k u[k]\right)=10\left(0.5^k-0.4^k\right)u[k].$$

(ii) $X_2(z)=\dfrac{z}{z^2-2.1z+0.2}$

By partial fraction expansion,

$$X_2(z)=\frac{z}{z^2-2.1z+0.2}=\frac{z}{(z-2)(z-0.1)}=\tfrac{1}{1.9}z\left[\frac{1}{z-2}-\frac{1}{z-0.1}\right]=\tfrac{10}{19}\left(\frac{z}{z-2}-\frac{z}{z-0.1}\right)$$

Therefore, $$x_2[k]=\tfrac{10}{19}\left(2^k u[k]-0.1^k u[k]\right)=\tfrac{10}{19}\left(2^k-0.1^k\right)u[k].$$

(iii) $X_3(z)=\dfrac{z^2+2}{(z-0.3)(z+0.4)(z-0.7)}$

By partial fraction expansion,

$$X_3(z)=\frac{z^2+2}{(z-0.3)(z+0.4)(z-0.7)}\equiv\frac{k_1}{z-0.3}+\frac{k_2}{z+0.4}+\frac{k_3}{z-0.7}$$

where

$$k_1=\left[\frac{z^2+2}{(z+0.4)(z-0.7)}\right]_{z=0.3}=\frac{2.09}{-0.28}=-7.4643$$

$$k_2=\left[\frac{z^2+2}{(z-0.3)(z-0.7)}\right]_{z=-0.4}=\frac{2.16}{0.77}=2.8052$$

$$k_3=\left[\frac{z^2+2}{(z-0.3)(z+0.4)}\right]_{z=0.7}=\frac{2.49}{0.44}=5.6591$$

In other words,

$$X_3(z)=\frac{z^2+2}{(z-0.3)(z+0.4)(z-0.7)}=-\frac{7.4643}{z-0.3}+\frac{2.8052}{z+0.4}+\frac{5.6591}{z-0.7}=z^{-1}\underbrace{\left[-\frac{7.4643z}{z-0.3}+\frac{2.8052z}{z+0.4}+\frac{5.6591z}{z-0.7}\right]}_{=P(z)}$$

where $$p[k]=z^{-1}\{P(z)\}=[-7.4643\times0.3^k+2.8052\times(-0.4)^k+5.6591\times0.7^k]u[k].$$

Therefore,

$$\begin{aligned} x_3[k] &= z^{-1}\{z^{-1}P(z)\} = p[k-1] \\ &= \left[-7.4643\times 0.3^{k-1} + 2.8052\times(-0.4)^{k-1} + 5.6591\times 0.7^{k-1}\right]u[k-1] \\ &= \left[-24.881\times 0.3^{k} - 7.0130\times(-0.4)^{k} + 8.0844\times 0.7^{k}\right]u[k-1]. \end{aligned}$$

(iv) $X_4(z) = \dfrac{z^2+2}{(z-0.3)(z+0.4)^2}$

By partial fraction expansion,

$$\frac{X_4(z)}{z} = \frac{z^2+2}{z(z-0.3)(z+0.4)^2} \equiv \frac{k_1}{z} + \frac{k_2}{z-0.3} + \frac{k_3}{z+0.4} + \frac{k_4}{(z+0.4)^2}$$

where

$$k_1 = \lim_{z\to 0}\left[\frac{z^2+2}{z(z-0.3)(z+0.4)^2}\times z\right] = \frac{2}{(-0.3)(0.4)^2} = -41.6667\ ,$$

$$k_2 = \lim_{z\to 0.3}\left[\frac{z^2+2}{z(z-0.3)(z+0.4)^2}\times(z-0.3)\right] = \lim_{z\to 0.3}\left[\frac{z^2+2}{z(z+0.4)^2}\right] = \frac{2.09}{(0.3)(0.7)^2} = 14.2177$$

$$\begin{aligned} k_3 &= \lim_{z\to -0.4}\left[\frac{d}{dz}\frac{z^2+2}{z(z-0.3)(z+0.4)^2}\times(z+0.4)^2\right] = \lim_{z\to -0.4}\left[\frac{d}{dz}\frac{z^2+2}{z(z-0.3)}\right] \\ &= \lim_{z\to -0.4}\left[\frac{2z}{z(z-0.3)} - \frac{z^2+2}{z^2(z-0.3)} - \frac{z^2+2}{z(z-0.3)^2}\right] = 27.4490 \end{aligned}$$

$$k_4 = \lim_{z\to -0.4}\left[\frac{z^2+2}{z(z-0.3)(z+0.4)^2}\times(z+0.4)^2\right] = \lim_{z\to -0.4}\left[\frac{z^2+2}{z(z-0.3)}\right] = \frac{2.16}{(-0.4)(-0.7)} = 7.7143$$

Hence,
$$X_4(z) \equiv k_1 + \frac{k_2}{1-0.3z^{-1}} + \frac{k_3}{1+0.4z^{-1}} + \frac{k_4 z^{-1}}{(1+0.4z^{-1})^2}$$

Assuming right hand sequences and taking the inverse z-transform, we get

$$\begin{aligned} x_4[k] &= -41.6667\delta[k] + 14.2177\times 0.3^k u[k] + 27.4490\times(-0.4)^k u[k] - 19.2858k(-0.4)^k u[k] \\ &= -41.6667\delta[k] + \left[14.2177\times 0.3^k - \left(19.2858k - 27.4490\right)\times(-0.4)^k\right]u[k] \end{aligned}$$

(v) $X_5(z) = \dfrac{4z^{-1}}{1-5z^{-1}+6z^{-2}} = \dfrac{4z}{z^2-5z+6}$

By partial fraction expansion,

$$\frac{X_5(z)}{z} = \frac{4}{(z-2)(z-3)} \equiv \frac{k_1}{z-2} + \frac{k_2}{z-3}$$

where $k_1 = \lim\limits_{z\to 2}\left[\dfrac{4}{(z-2)(z-3)}\times(z-2)\right] = \dfrac{4}{(-1)} = -4$,

and $k_2 = \lim\limits_{z\to 3}\left[\dfrac{4}{(z-2)(z-3)}\times(z-3)\right] = 4$.

Hence, $$X_5(z) \equiv \frac{-4}{1-2z^{-1}} + \frac{4}{1-3z^{-1}}$$

Assuming right hand sequences and taking the inverse z-transform, we get

$$x_5[k] \equiv -4\times 2^k u[k] + 4\times 3^k u[k] = 4\left(3^k - 2^k\right)u[k].$$

(vi) $X_6(z) = \dfrac{4z^{-2}}{10-6(z+z^{-1})} = \dfrac{4z^{-2}}{-(6z-10+6z^{-1})} = -\dfrac{2}{3}z^{-2}\times\dfrac{z}{z^2-5z/3+1} = -\dfrac{2}{3}z^{-2}\times P(z)$

Applying the linearity and time shifting property of the z-transform, $x_6[k]$ can be expressed as

$x_6[k] = -\frac{2}{3}p[k-2]$ where $\mathbb{Z}\{p[k]\} = P(z) = \dfrac{z}{z^2-5z/3+1}$

In order to calculate $p[k]$, the transform pair in the Entry 12 of Table 13.1 can be used.

$$P(z) = \frac{z}{z^2-5z/3+1} = \frac{Az+B}{\underbrace{z^2+2\gamma z+\alpha^2}_{A=0,B=1,\gamma=-5/6,\alpha=1}}.$$

The function $p[k]$ can be expressed as, $p[k] = r\cdot\alpha^k\sin\left(\Omega_0 k+\theta\right)\cdot u[k]$ where

$r = \sqrt{\frac{A^2\alpha^2+B^2-2AB\gamma}{\alpha^2-\gamma^2}} = \sqrt{\frac{1}{1-25/36}} = \sqrt{\frac{1}{11/36}} \approx 1.809$, $\Omega_0 = \cos^{-1}\left(\frac{-\gamma}{\alpha}\right) = \cos^{-1}\left(\frac{5}{6}\right) = 0.5857$, and

$\theta = \tan^{-1}\left(\frac{A\sqrt{\alpha^2-\gamma^2}}{B-A\gamma}\right) = \tan^{-1}(0) = 0$.

In other words, $p[k] = 1.809\sin\left(0.5857k\right)u[k]$, and the function $x_6[k]$ can be expressed as

$$\begin{aligned} p[k] &= -\tfrac{2}{3}\times 1.809\sin\left(0.5857(k-2)\right)u[k-2] \\ &= 1.206\sin\left(0.5857k - 67.1163^0\right)u[k-2] \\ &\equiv -1.11108\delta[k] - 0.6667\delta[k-1] - 1.206\sin\left(0.5857k-67.1163^0\right)u[k] \end{aligned}$$

(vii) $X_7(z) = \dfrac{2z^{-2}}{\left(1-4z^{-1}\right)^2\left(1-2z^{-1}\right)} = \dfrac{2z}{(z-4)^2(z-2)}$

By partial fraction expansion,

$$\frac{X_7(z)}{z} = \frac{2}{(z-4)^2(z-2)} \equiv -\frac{0.5}{z-4} + \frac{1}{(z-4)^2} + \frac{0.5}{z-2}$$

Assuming right hand sequences and taking the inverse z-transform, we get

$$\begin{aligned} x_7[k] &= -0.5\times 4^k u[k] + 0.25k\times 4^k u[k] + 0.5\times 2^k u[k] \\ &= \left[(0.25k-0.5)\times 4^k + 0.5\times 2^k\right]u[k]. \end{aligned}$$

The first 10 sample values of each x[k]'s are shown in MATLAB Program 13.3. ▮

Program 13.3: MATLAB program for Problem 13.3.

```
% MATLAB Code to calculate x[k]'s in Problem 13.3. This code
% generates first 10 samples of each time-domain function.
%
k=0:9 ;
%
% part (i)
x1=10*((0.5).^k-(0.4).^k) ;
% x1 = [0,  1,  0.9,  0.61,  0.369, 0.2101, 0.11529, 0.0617,
0.0325, 0.0169]
% part (ii)
x2=(10/19)*(2.^k-(0.1).^k) ;
% x2 = [0  1.00  2.10  4.2100 8.4210 16.8421  33.6842  67.3684  134.7368
269.4737]


%part (iii)
k=[0:10] ;
p1 = (0.3).^k ; p2 = (-0.4).^k; p3 = (0.7).^k;
x3 = (-24.881*p1-7.0130*p2+8.0844*p3).*(k>=1) ;
% x3 = [0   1.0  0.60  2.55  1.56  1.3701  0.9043  0.6718  0.4598  0.3276
0.2275]

%part (iv)
k=[0:10] ;
x4 = 14.2177*((0.3).^k) - (19.2858*k-27.4490).*((-0.4).^k);
x4(1) = x4(1)-41.6667;
%x4 = [0, 1, -0.5, 2.33, -1.157, 0.741  -0.351, 0.179, -0.0822, 0.0386, -
0.0173]

%part (v)
k=[0:10] ;
x5 = 4*((3.^k) - (2.^k));
%x5 = 0   4   20  76  260   844   2660  8236  25220   76684   232100


%part (vi)
k=[0:10] ;
x6 = -1.206*sin(0.5857*k-0.5857*2);
x6(1) = x6(1)- x6(1);   % =1.11108;
x6(2) = x6(2)- x6(2);   % = 0.6666561;
x6
% x6= [0, 0, 0, -0.6666, -1.1111, -1.1851, -0.864, -0.2550, 0.439, 0.9867,
1.2055]

%part (vii)
k=[0:10] ;
x7 = (0.25*k-0.5).*4.^k ;
x7 = x7 + 0.5*(2.^k);
% x7=  [0    0   2  20  136  784  4128  20544  98432  459008   2097664]
```

Problem 13.4

(i) $X_1(z) = \dfrac{z}{z^2 - 0.9z + 0.2}$

$$
\begin{array}{r|l}
 & z^{-1} + 0.9z^{-2} + 0.61z^{-3} + 0.369z^{-4} + 0.2101z^{-5} \\
\hline
z^2 - 0.9z + 0.2 & z \\
 & \underline{z \mp 0.9 \pm 0.2z^{-1}} \\
 & \quad +0.9 - 0.2z^{-1} \\
 & \quad \underline{\pm 0.9 \mp 0.81z^{-1} + 0.18z^{-2}} \\
 & \qquad +0.61z^{-1} - 0.18z^{-2} \\
 & \qquad \underline{\pm 0.61z^{-1} \mp 0.549z^{-2} \pm 0.122z^{-3}} \\
 & \qquad\qquad +0.369z^{-2} - 0.122z^{-3} \\
 & \qquad\qquad \underline{\pm 0.369z^{-2} \mp 0.3321z^{-3} \pm 0.0738z^{-4}} \\
 & \qquad\qquad\qquad +0.2101z^{-3} - 0.0738z^{-4} \\
 & \qquad\qquad\qquad \underline{\pm 0.2101z^{-3} \mp 0.1891z^{-4} \pm 0.0420}
\end{array}
$$

Hence,

$$X_1(z) = \frac{z}{z^2 - 0.9z + 0.2} = z^{-1} + 0.9z^{-2} + 0.61z^{-3} + 0.369z^{-4} + 0.2101z^{-5} + \cdots$$

Taking the inverse transform gives the following values for the first five samples of $x_1[k]$

$$x_1[0] = 0, x_1[1] = 1, x_1[2] = 0.9, x_1[3] = 0.61, x_1[4] = 0.369, x_1[5] = 0.2101, \cdots.$$

Note that the above values are consistent with those in Problem 13.3(i).

(ii) $X_2(z) = \dfrac{z}{z^2 - 2.1z + 0.2}$

$$
\begin{array}{r|l}
 & z^{-1} + 2.1z^{-2} + 4.21z^{-3} + 8.421z^{-4} + 16.8421z^{-5} \\
\hline
z^2 - 2.1z + 0.2 & z \\
 & \underline{z \mp 2.1 \pm 0.2z^{-1}} \\
 & \quad +2.1 - 0.2z^{-1} \\
 & \quad \underline{\pm 2.1 \mp 4.41z^{-1} \pm 0.42z^{-2}} \\
 & \qquad +4.21z^{-1} - 0.42z^{-2} \\
 & \qquad \underline{\pm 4.21z^{-1} \mp 8.841z^{-2} \pm 0.842z^{-3}} \\
 & \qquad\qquad +8.421z^{-2} - 0.842z^{-3} \\
 & \qquad\qquad \underline{\pm 8.421z^{-2} \mp 17.6841z^{-3} \pm 1.6841z^{-4}} \\
 & \qquad\qquad\qquad +16.8421z^{-3} - 1.6841z^{-4} \\
 & \qquad\qquad\qquad \underline{\pm 16.8421z^{-3} \mp 35.3684z^{-4} \pm 3.36842z^{-5}}
\end{array}
$$

Hence,

$$X_2(z) = \frac{z}{z^2 - 2.1z + 0.2} = z^{-1} + 2.1z^{-2} + 4.21z^{-3} + 8.421z^{-4} + 16.8421z^{-5} + \cdots$$

Taking the inverse transform gives the following values for the first five samples of $x_2[k]$

$$x_2[0]=0, x_2[1]=1, x_2[2]=2.1, x_2[3]=4.21, x_2[4]=8.421, x_2[5]=16.8421,\cdots.$$

Note that the above values are consistent with those in Problem 13.3(ii).

(iii) $X_3(z)=\dfrac{z^2+2}{(z-0.3)(z+0.4)(z-0.7)}=\dfrac{z^2+2}{z^3-0.6z^2-0.19z+0.084}$

By power division, we get

$$\begin{array}{r|l}
 & z^{-1}+0.6z^{-2}+2.55z^{-3}+1.560z^{-4}+1.3685z^{-5} \\
\hline
z^3-0.6z^2-0.19z+0.084 & z^2+0z\ \ +2 \\
 & z^2 \mp 0.6z \mp 0.19 \pm 0.084z^{-1} \\
\hline
 & +0.6z+2.19-0.084z^{-1} \\
 & \pm 0.6z \mp 0.36 \mp 0.114z^{-1} \pm 0.048z^{-2} \\
\hline
 & +2.55+0.030z^{-1}-0.048z^{-2} \\
 & \pm 2.55 \mp 1.530z^{-1} \mp 0.4845z^{-2} \pm 0.2142z^{-3} \\
\hline
 & +1.560z^{-1}+0.4325z^{-2}-0.2142z^{-3} \\
 & \pm 1.560z^{-1} \mp 0.9360z^{-2} \mp 0.2964z^{-3} \pm 0.1310z^{-4} \\
\hline
 & +1.3685z^{-2}+0.0822z^{-3}-0.1310z^{-4} \\
 & \pm 1.3685z^{-2} \mp 0.8211z^{-3} \mp 0.2600z^{-4} \pm 0.1150z^{-5} \\
\hline
\end{array}$$

Hence,

$$X_3(z)=\frac{z^2+2}{(z-0.3)(z+0.4)(z-0.7)}=z^{-1}+0.6z^{-2}+2.55z^{-3}+1.560z^{-4}+1.3685z^{-5}+\cdots$$

Taking the inverse transform gives the following values for the first five samples of $x_3[k]$

$$x_3[0]=0, x_3[1]=1, x_3[2]=0.6, x_3[3]=2.55, x_3[4]=1.560, x_3[5]=1.3685,\cdots.$$

Note that the above values are consistent with those in Problem 13.3(iii).

(iv) $X_4(z)=\dfrac{z^2+2}{(z-0.3)(z+0.4)^2}=\dfrac{z^2+2}{z^3+0.5z^2-0.08z-0.048}$

By power division, we obtain

$$
\begin{array}{r|l}
 & z^{-1} - 0.5z^{-2} + 2.33z^{-3} - 1.157z^{-4} + 0.7409z^{-5} \\
\hline
z^3 + 0.5z^2 - 0.08z - 0.048 & z^2 + 0z \quad + 2 \\
 & z^2 \pm 0.5z \mp 0.08 \mp 0.048z^{-1} \\
\hline
 & -0.5z + 2.08 + 0.048z^{-1} \\
 & \mp 0.5z \mp 0.25 \pm 0.040z^{-1} \pm 0.024z^{-2} \\
\hline
 & +2.33 + 0.008z^{-1} - 0.024z^{-2} \\
 & \pm 2.33 \pm 1.165z^{-1} \mp 0.1864z^{-2} \mp 0.1118z^{-3} \\
\hline
 & -1.157z^{-1} + 0.1624z^{-2} + 0.1118z^{-3} \\
 & \mp 1.157z^{-1} \mp 0.5785z^{-2} \pm 0.0926z^{-3} \pm 0.0555z^{-4} \\
\hline
 & +0.7409z^{-2} + 0.0192z^{-3} + 0.0555z^{-4} \\
 & \pm 0.7409z^{-2} \pm 0.3705z^{-3} \mp 0.0593z^{-4} \mp 0.0356z^{-5} \\
\hline
\end{array}
$$

Hence,

$$X_4(z) = \frac{z^2 + 2}{(z - 0.3)(z + 0.4)^2} = z^{-1} - 0.5z^{-2} + 2.33z^{-3} - 1.157z^{-4} + 0.7409z^{-5} - + \cdots$$

Taking the inverse transform gives the following values for the first five samples of $x_4[k]$

$$x_4[0] = 0, x_4[1] = 1, x_4[2] = -0.5, x_4[3] = 2.33, x_4[4] = -1.157, x_4[5] = 0.7409, \cdots.$$

Note that the above values are consistent with those in Problem 13.3(iv).

(v) $X_5(z) = \dfrac{4z^{-1}}{1 - 5z^{-1} + 6z^{-2}} = \dfrac{4z}{z^2 - 5z + 6}$

By power division, we get

$$
\begin{array}{r|l}
 & 4z^{-1} + 20z^{-2} + 76z^{-3} + 260z^{-4} + 844z^{-5} \\
\hline
z^2 - 5z + 6 & 4z \\
 & 4z \mp 20 \pm 24z^{-1} \\
\hline
 & +20 - 24z^{-1} \\
 & \pm 20 \mp 100z^{-1} \pm 120z^{-2} \\
\hline
 & +76z^{-1} - 120z^{-2} \\
 & \pm 76z^{-1} \mp 380z^{-2} \pm 456z^{-3} \\
\hline
 & +260z^{-2} - 456z^{-3} \\
 & \pm 260z^{-2} \mp 1300z^{-3} \pm 1560z^{-4} \\
\hline
 & +844z^{-3} - 1560z^{-4} \\
 & \pm 844z^{-3} \mp 4220z^{-4} \pm 5064z^{-5} \\
\hline
\end{array}
$$

Hence,

$$X_5(z) = \frac{4z^{-1}}{(1 - 5z^{-1} + 6z^{-2})} = 4z^{-1} + 20z^{-2} + 76z^{-3} + 260z^{-4} + 844z^{-5} + \cdots$$

Taking the inverse transform gives the following values for the first five samples of $x_5[k]$

$$x_5[0] = 0, x_5[1] = 4, x_5[2] = 20, x_5[3] = 76, x_5[4] = 260, x_5[5] = 844, \cdots.$$

Note that the above values are consistent with those in Problem 13.3(v).

(vi) $X_6(z) = \dfrac{4z^{-2}}{10-6(z^1+z^{-1})} = \dfrac{-4z^{-1}}{6z^2-10z+6}$

By power division, we get,

$$
\begin{array}{r|l}
 & -\frac{2}{3}z^{-3}-\frac{10}{9}z^{-4}-\frac{32}{27}z^{-5}-\frac{70}{81}z^{-6}-\frac{62}{243}z^{-7} \\
\hline
6z^2-10z+6 & -4z^{-1} \\
 & \underline{-4z^{-1}+\frac{20}{3}z^{-2}-4z^{-3}} \\
 & \qquad -\frac{20}{3}z^{-2}+4z^{-3} \\
 & \qquad \underline{-\frac{20}{3}z^{-2}+\frac{100}{9}z^{-3}-\frac{20}{3}z^{-4}} \\
 & \qquad\qquad -\frac{64}{9}z^{-3}+\frac{20}{3}z^{-4} \\
 & \qquad\qquad \underline{-\frac{64}{9}z^{-3}+\frac{320}{27}z^{-4}-\frac{64}{9}z^{-5}} \\
 & \qquad\qquad\qquad -\frac{140}{27}z^{-4}+\frac{64}{9}z^{-5} \\
 & \qquad\qquad\qquad \underline{-\frac{140}{27}z^{-4}+\frac{700}{81}z^{-5}-\frac{140}{27}z^{-6}} \\
 & \qquad\qquad\qquad\qquad -\frac{124}{81}z^{-5}+\frac{140}{27}z^{-6} \\
 & \qquad\qquad\qquad\qquad \underline{-\frac{124}{81}z^{-5}+\frac{620}{243}z^{-6}+\frac{124}{81}z^{-7}}
\end{array}
$$

Hence,

$$X_6(z) = \frac{4z^{-2}}{10-6(z^1+z^{-1})} = \frac{-4z^{-1}}{6z^2-10z+6} = -\tfrac{2}{3}z^{-3}-\tfrac{10}{9}z^{-4}-\tfrac{32}{27}z^{-5}-\tfrac{70}{81}z^{-6}-\tfrac{62}{243}z^{-7}-\ldots\ldots$$

Taking the inverse transform gives the following values for the first five samples of $x_6[k]$.

$$x_6[0]=0, x_6[1]=0, x_6[2]=0, x_6[3]=-\tfrac{2}{3}, x_6[4]=-\tfrac{10}{9}, x_6[5]=-\tfrac{32}{27},$$
$$x_6[6]=-\tfrac{70}{81}, x_6[7]=-\tfrac{62}{243}, \cdots \quad .$$

Note that the above values are consistent with those in Problem 13.3(vi).

(vii) $X_7(z) = \dfrac{2z^{-2}}{\left(1-4z^{-1}\right)^2\left(1-2z^{-1}\right)} = \dfrac{2z}{(z-4)^2(z-2)} = \dfrac{2z}{z^3-10z^2+32z-32}$

By power division, we get,

$$
\begin{array}{r|l}
 & 2z^{-2}+20z^{-3}+136z^{-4}+784z^{-5}+4128z^{-6} \\
\hline
z^3-10z^2+32z-32 & 2z \\
 & \underline{2z-20+64z^{-1}-64z^{-2}} \\
 & \quad 20-64z^{-1}+64z^{-2} \\
 & \quad \underline{20-200z^{-1}+640z^{-2}-640z^{-3}} \\
 & \qquad 136z^{-1}-576z^{-2}+640z^{-3} \\
 & \qquad \underline{136z^{-1}-1360z^{-2}+4352z^{-3}-4352z^{-4}} \\
 & \qquad\quad 784z^{-2}-3712z^{-3}+4352z^{-4} \\
 & \qquad\quad \underline{784z^{-2}-7840z^{-3}+25088z^{-4}-25088z^{-5}} \\
 & \qquad\qquad 4128z^{-3}-20736z^{-4}+25088z^{-5} \\
 & \qquad\qquad \underline{4128z^{-3}-41280z^{-4}+132096z^{-5}-132096z^{-6}}
\end{array}
$$

Hence, $X_7(z)=\dfrac{2z^{-2}}{\left(1-4z^{-1}\right)^2\left(1-2z^{-1}\right)}=2z^{-2}+20z^{-3}+136z^{-4}+784z^{-5}+4128z^{-6}+\ldots\ldots$

Taking the inverse transform gives the following values for the first five samples of $x_7[k]$.

$$x_7[0]=0, x_7[1]=0, x_7[2]=2, x_7[3]=20, x_7[4]=136, x_7[5]=784, x_7[6]=4128, \cdots.$$

Note that the above values are consistent with those in Problem 13.3(vii).

Problem 13.5

(a) We know that (Entry 4 of Table 13.1, with $\alpha=\alpha e^{\pm j\Omega_0}$)

$$(\alpha e^{j\Omega_0})^k U[k] \xleftrightarrow{z} \frac{1}{1-\alpha e^{j\Omega_0}z^{-1}} \quad \text{ROC}: |z|>\alpha$$

and

$$(\alpha e^{-j\Omega_0})^k U[k] \xleftrightarrow{z} \frac{1}{1-\alpha e^{-j\Omega_0}z^{-1}} \quad \text{ROC}: |z|>\alpha$$

Adding the two transform pairs after multiplying with $re^{j\theta}$ and $re^{-j\theta}$, respectively, gives

$$re^{j\theta}(\alpha e^{j\Omega_0})^k U[k]-re^{-j\theta}(\alpha e^{-j\Omega_0})^k U[k] \xleftrightarrow{z} \frac{re^{j\theta}}{1-\alpha e^{j\Omega_0}z^{-1}}-\frac{re^{-j\theta}}{1-\alpha e^{-j\Omega_0}z^{-1}} \quad \text{ROC}: |z|>\alpha$$

or, $$r\alpha^k\left[e^{j(\Omega_0 k+\theta)}-e^{-j(\Omega_0 k+\theta)}\right]u[k] \xleftrightarrow{z} \frac{re^{j\theta}(1-\alpha e^{-j\Omega_0}z^{-1})-re^{-j\theta}(1-\alpha e^{j\Omega_0}z^{-1})}{(1-\alpha e^{j\Omega_0}z^{-1})(1-\alpha e^{-j\Omega_0}z^{-1})} \quad \text{ROC}: |z|>\alpha$$

or,

$$2jr\times\alpha^k\sin(\Omega_0 k+\theta)u[k] \xleftrightarrow{z} \frac{re^{j\theta}-re^{-j\theta}+\alpha r\left(e^{j(\Omega_0-\theta)}-e^{-j(\Omega_0-\theta)}\right)z^{-1}}{1-\left(\alpha e^{j\Omega_0}z^{-1}+\alpha e^{-j\Omega_0}z^{-1}\right)+\alpha^2z^{-2}} \quad \text{ROC}: |z|>\alpha,$$

$$=\frac{r\times 2j\sin\theta+\alpha r\times 2j\sin(\Omega_0-\theta)z^{-2}}{1-2\alpha\cos(\Omega_0)z^{-1}+\alpha^2z^{-2}}$$

or,

$$r\times\alpha^k \sin(\Omega_0 k+\theta)u[k] \xleftrightarrow{z} \frac{r\sin\theta+\alpha r\sin(\Omega_0-\theta)z^{-1}}{1-2\alpha\cos\Omega_0 z^{-1}+\alpha^2 z^{-2}} \quad \text{ROC}: |z|>\alpha$$
$$=\frac{A+Bz^{-1}}{1+2\gamma z^{-1}+\alpha^2 z^{-2}}.$$

where $A=r\sin\theta, B=\alpha r\sin(\Omega_0-\theta), \gamma=-\alpha\cos\Omega_0$. Alternatively, the parameters r,Ω_0,θ can be expressed in terms of A,B,γ,α as follows (as given in Entry 12).

$$r=\sqrt{\frac{A^2\alpha^2+B^2-2AB\gamma}{\alpha^2-\gamma^2}},\ \Omega_0=\cos^{-1}\left(\frac{-\gamma}{\alpha}\right), \text{ and } \theta=\tan^{-1}\left(\frac{A\sqrt{\alpha^2-\gamma^2}}{B-A\gamma}\right).$$

Proof for the expressions of r,Ω_0,θ in Problem 13.5

$$\sqrt{\frac{A^2\alpha^2+B^2-2AB\gamma}{\alpha^2-\gamma^2}}=\sqrt{\frac{r^2\alpha^2\sin^2\theta+r^2\alpha^2\sin^2(\Omega_0-\theta)+2r^2\alpha^2\sin\theta\cos(\Omega_0)\sin(\Omega_0-\theta)}{\alpha^2-\alpha^2\cos^2(\Omega_0)}}$$
$$=r\sqrt{\frac{\sin^2\theta+\sin^2(\Omega_0-\theta)+2\sin\theta\cos(\Omega_0)\sin(\Omega_0-\theta)}{1-\cos^2(\Omega_0)=\sin^2(\Omega_0)}}=r$$

because

$$\sin^2\theta+\sin^2(\Omega_0-\theta)+2\sin\theta\cos(\Omega_0)\sin(\Omega_0-\theta)$$
$$=\sin^2\theta+\left[\sin(\Omega_0)\cos\theta-\cos(\Omega_0)\sin\theta\right]^2+2\sin\theta\cos(\Omega_0)\left[\sin(\Omega_0)\cos\theta-\cos(\Omega_0)\sin\theta\right]$$
$$=\sin^2\theta+\sin^2(\Omega_0)\cos^2\theta+\cos^2(\Omega_0)\sin^2\theta-2\sin(\Omega_0)\cos(\Omega_0)\cos\theta\sin\theta+$$
$$2\sin(\Omega_0)\cos(\Omega_0)\sin\theta\cos\theta-2\cos^2(\Omega_0)\sin^2\theta$$
$$=\sin^2\theta+\sin^2(\Omega_0)\cos^2\theta-\cos^2(\Omega_0)\sin^2\theta$$
$$=\sin^2(\Omega_0)\cos^2\theta+\sin^2\theta\left(1-\cos^2(\Omega_0)\right)$$
$$=\sin^2(\Omega_0)\cos^2\theta+\sin^2\theta\sin^2(\Omega_0)$$
$$=\sin^2(\Omega_0)$$

$$\cos^{-1}\left(\frac{-\gamma}{\alpha}\right)=\cos^{-1}\left(\frac{\alpha\cos\Omega_0}{\alpha}\right)=\Omega_0$$

$$\tan^{-1}\left(\frac{A\sqrt{\alpha^2-\gamma^2}}{B-A\gamma}\right)=\tan^{-1}\left(\frac{r\sin\theta\sqrt{\alpha^2-\alpha^2\cos^2\Omega_0}}{\alpha r\sin(\Omega_0-\theta)+\alpha r\sin\theta\cos\Omega_0}\right)=\tan^{-1}\left(\frac{\alpha r\sin\Omega_0\sin\theta}{\alpha r\sin\Omega_0\cos\theta}\right)=\tan^{-1}\left(\tan\theta\right)=\theta$$

(b) By comparing $X(z)=\dfrac{1}{1-z^{-1}+z^{-2}}$ with the above transform pair, we get $A = 1$, $B = 0$, $\gamma = -0.5$, and $\alpha = 1$. Substituting the values to compute α, θ, and Ω_0, gives

$$r=\sqrt{\frac{A^2\alpha^2+B^2-2AB\gamma}{\alpha^2-\gamma^2}}=\sqrt{\frac{1}{1-0.25}}=\frac{2}{\sqrt{3}},$$

$$\Omega_0=\cos^{-1}\left(\frac{-\gamma}{\alpha}\right)=\cos^{-1}(0.5)=\frac{\pi}{3},$$

and $$\theta = \tan^{-1}\left(\tfrac{A\sqrt{\alpha^2-\gamma^2}}{B-A\gamma}\right) = \tan^{-1}\left(\tfrac{\sqrt{0.75}}{0.5}\right) = \tan^{-1}\left(\sqrt{3}\right) = \tfrac{\pi}{3}$$

Hence, the inverse z-transform of $X(z)$ is given by

$$x[k] = r\alpha^k \sin(\Omega_0 k + \theta)\, u[k] = \tfrac{2}{\sqrt{3}} \sin\left(\tfrac{\pi k}{3} + \tfrac{\pi}{3}\right) u[k].$$

Alternative Solution of Part (b)

$$\begin{aligned}
X(z) &= \mathbb{Z}\left\{\tfrac{2}{\sqrt{3}}\sin\left(\tfrac{\pi k}{3}+\tfrac{\pi}{3}\right)u[k]\right\} = \tfrac{2}{\sqrt{3}}\mathbb{Z}\left\{\sin\left(\tfrac{\pi k}{3}+\tfrac{\pi}{3}\right)u[k]\right\} \\
&= \tfrac{2}{\sqrt{3}}\mathbb{Z}\left\{\sin\left(\tfrac{\pi k}{3}\right)\cos\left(\tfrac{\pi}{3}\right)u[k] + \cos\left(\tfrac{\pi k}{3}\right)\sin\left(\tfrac{\pi}{3}\right)u[k]\right\} \\
&= \tfrac{2}{\sqrt{3}}\mathbb{Z}\left\{\tfrac{1}{2}\sin\left(\tfrac{\pi k}{3}\right)u[k] + \tfrac{\sqrt{3}}{2}\cos\left(\tfrac{\pi k}{3}\right)u[k]\right\} \\
&= \tfrac{1}{\sqrt{3}}\mathbb{Z}\left\{\sin\left(\tfrac{\pi k}{3}\right)u[k]\right\} + \mathbb{Z}\left\{\cos\left(\tfrac{\pi k}{3}\right)u[k]\right\} \\
&= \tfrac{1}{\sqrt{3}}\frac{\sin(\tfrac{\pi}{3})z^{-1}}{1-2\cos(\tfrac{\pi}{3})z^{-1}+z^{-2}} + \frac{1-\cos(\tfrac{\pi}{3})z^{-1}}{1-2\cos(\tfrac{\pi}{3})z^{-1}+z^{-2}} \qquad \text{ROC: } |z|>1 \\
&= \frac{\tfrac{1}{\sqrt{3}}\cdot\tfrac{\sqrt{3}}{2}z^{-1}}{1-z^{-1}+z^{-2}} + \frac{1-\tfrac{1}{2}z^{-1}}{1-z^{-1}+z^{-2}} \qquad \text{ROC: } |z|>1 \\
&= \frac{1}{1-z^{-1}+z^{-2}}
\end{aligned}$$

▌

Program 13.5. MATLAB Program to calculate and verify solutions in Problem 13.5.

```
k=[0:10] ;
x = (2/sqrt(3))*sin(pi*k/3 + pi/3)
% x =   [1  1  0  -1  -1  0  1  1  0  -1  -1]
stem(k, x, 'filled'), grid
xlabel('k')              % Label of X-axis
ylabel('x[k]')           % Label of Y-axis
print -dtiff plot.tiff        % Save the figure as a TIFF file
%MATLAB Verification
%  X(z) = 1/(1 - z^-1 + z^-2)
sys = filt([1],[1  -1  1])
output = impulse(sys,10)
% output=    y1=    [1  1  0  -1  -1  0  1  1  0  -1  -1]
```

Problem 13.6

(a) (i) From pair 9 in Table 13.1, we know $\mathbb{Z}\{\sin(\Omega_0 k)u[k]\} = \dfrac{z\sin(\Omega_0)}{z^2 - 2z\cos(\Omega_0)+1}$ ROC: $|z|>1$.

$$\mathbb{Z}\left\{k\sin(\Omega_0 k)u[k]\right\} = -z\frac{d}{dz}\left[\mathbb{Z}\left\{\sin(\Omega_0 k)u[k]\right\}\right] \qquad \left[\text{using frequency differentiation property}\right]$$

$$= -z\frac{d}{dz}\left[\frac{z\sin(\Omega_0)}{z^2 - 2z\cos(\Omega_0)+1}\right] \qquad \text{ROC: } |z|>1$$

$$= -z\frac{(z^2-2z\cos(\Omega_0)+1)\sin(\Omega_0) - z\sin(\Omega_0)\times(2z-2\cos(\Omega_0))}{(z^2-2z\cos(\Omega_0)+1)^2} \qquad \text{ROC: } |z|>1$$

$$= -z\frac{-z^2\sin(\Omega_0)+\sin(\Omega_0)}{(z^2-2z\cos(\Omega_0)+1)^2} = \frac{z\sin(\Omega_0)(z^2-1)}{(z^2-2z\cos(\Omega_0)+1)^2} \qquad \text{ROC: } |z|>1$$

(ii) Substituting $r=\alpha=1$, in Entry 12 of Table 13.1, we get the following pair.

$$\mathbb{Z}\left\{\sin(\Omega_0 k+\theta)u[k]\right\} = \frac{z\left[z\sin\theta + \sin(\Omega_0-\theta)\right]}{z^2-2z\cos\Omega_0+1}, \quad \text{ROC: } |z|>1$$

$$\mathbb{Z}\left\{k\sin(\Omega_0 k+\theta)u[k]\right\}$$

$$= -z\frac{d}{dz}\left[\mathbb{Z}\left\{\sin(\Omega_0 k+\theta)u[k]\right\}\right] \qquad \left[\text{using frequency differentiation property}\right]$$

$$= -z\frac{d}{dz}\left[\frac{z\left[z\sin\theta+\sin(\Omega_0-\theta)\right]}{z^2-2z\cos\Omega_0+1}\right] \qquad \text{ROC: } |z|>1$$

$$= -z\frac{(z^2-2z\cos(\Omega_0)+1)\left[2z\sin\theta+\sin(\Omega_0-\theta)\right] - z\left\{z\sin\theta+\sin(\Omega_0-\theta)\right\}\times(2z-2\cos(\Omega_0))}{(z^2-2z\cos(\Omega_0)+1)^2} \qquad \text{ROC: } |z|>1$$

$$= \frac{-z}{(z^2-2z\cos(\Omega_0)+1)^2}\left[-\left\{2\cos(\Omega_0)\sin\theta+\sin(\Omega_0-\theta)\right\}z^2 + 2z\sin\theta + \sin(\Omega_0-\theta)\right]$$

$$= \frac{z\left[z^2\sin(\Omega_0+\theta) - 2z\sin\theta - \sin(\Omega_0-\theta)\right]}{(z^2-2z\cos(\Omega_0)+1)^2} \qquad \text{ROC: } |z|>1$$

(b) Substituting $\Omega_0 = \frac{\pi}{3}, \theta = \frac{\pi}{6}$, in pair a(ii) derived above, we get the following pair

$$\mathbb{Z}\left\{k\sin(\tfrac{\pi}{3}k+\tfrac{\pi}{6})u[k]\right\} = \frac{z\left[z^2\sin(\frac{\pi}{2}) - 2z\sin(\frac{\pi}{6}) - \sin(\frac{\pi}{6})\right]}{(z^2-2z\cos(\frac{\pi}{3})+1)^2} \qquad \text{ROC: } |z|>1$$

$$= \frac{z\left[z^2-z-0.5\right]}{(z^2-z+1)^2} \qquad \text{ROC: } |z|>1$$

On the other hand, from pair 9 in Table 13.1, we know the following z-transform pair.

$$\mathbb{Z}\left\{\sin(\tfrac{\pi}{3}k)u[k]\right\} = \frac{z\sin(\frac{\pi}{3})}{z^2-2z\cos(\frac{\pi}{3})+1} = \frac{\frac{\sqrt{3}}{2}z}{z^2-z+1} \qquad \text{ROC: } |z|>1 .$$

The given z-transform pair can now be proved as follows.

$$\mathbb{Z}\left\{\left[\tfrac{2}{3}\sin(\tfrac{\pi}{3}k)-\tfrac{k}{\sqrt{3}}\sin(\tfrac{\pi}{3}k+\tfrac{\pi}{6})\right]u[k]\right\}$$

$$=\tfrac{2}{3}\mathbb{Z}\left\{\sin(\tfrac{\pi}{3}k)u[k]\right\}-\tfrac{1}{\sqrt{3}}\mathbb{Z}\left\{k\sin(\tfrac{\pi}{3}k+\tfrac{\pi}{6})u[k]\right\}$$

$$=\tfrac{2}{3}\times\frac{\frac{\sqrt{3}}{2}z}{z^2-z+1}-\tfrac{1}{\sqrt{3}}\times\frac{z\left[z^2-z-0.5\right]}{(z^2-z+1)^2} \qquad \text{ROC: } |z|>1$$

$$=\tfrac{z}{\sqrt{3}}\left[\frac{1}{z^2-z+1}-\frac{z^2-z-0.5}{(z^2-z+1)^2}\right]=\tfrac{z}{\sqrt{3}}\left[\frac{z^2-z+1-(z^2-z-0.5)}{(z^2-z+1)^2}\right] \qquad \text{ROC: } |z|>1$$

$$=\tfrac{z}{\sqrt{3}}\left[\frac{1.5}{(z^2-z+1)^2}\right]=\frac{\frac{\sqrt{3}}{2}z}{(z^2-z+1)^2} \qquad \text{ROC: } |z|>1$$

∎

Alternative (Direct) Solution of Part (b):

Direct calculation of $y[k]$ using partial fraction method:

First express, $\dfrac{1}{(z^2-z+1)^2}$ as follows.

$$\frac{1}{(z^2-z+1)^2}=\left[\frac{1}{(z-0.5-j\frac{\sqrt{3}}{2})(z-0.5+j\frac{\sqrt{3}}{2})}\right]^2=\left[\frac{1}{j\sqrt{3}}\left\{\frac{1}{z-0.5-j\frac{\sqrt{3}}{2}}-\frac{1}{z-0.5+j\frac{\sqrt{3}}{2}}\right\}\right]^2$$

$$=-\frac{1}{3}\left[\frac{1}{z-0.5-j\frac{\sqrt{3}}{2}}-\frac{1}{z-0.5+j\frac{\sqrt{3}}{2}}\right]^2$$

$$=-\frac{1}{3}\left[\frac{1}{\left(z-0.5-j\frac{\sqrt{3}}{2}\right)^2}+\frac{1}{\left(z-0.5+j\frac{\sqrt{3}}{2}\right)^2}-\frac{2}{\left(z-0.5-j\frac{\sqrt{3}}{2}\right)\left(z-0.5+j\frac{\sqrt{3}}{2}\right)}\right]$$

$$=-\frac{1}{3}\left[\frac{1}{\left(z-0.5-j\frac{\sqrt{3}}{2}\right)^2}+\frac{1}{\left(z-0.5+j\frac{\sqrt{3}}{2}\right)^2}-\frac{2}{z^2-z+1}\right]$$

The function $y[k]$ can then be calculated as follows.

$$
\begin{aligned}
y[k] &= \mathbb{Z}^{-1}\left\{\frac{\frac{\sqrt{3}}{2}z}{(z^2-z+1)^2}\right\} \\
&= \mathbb{Z}^{-1}\left\{-\frac{1}{2\sqrt{3}}\left[\frac{z}{\left(z-0.5-j\frac{\sqrt{3}}{2}\right)^2}+\frac{z}{\left(z-0.5+j\frac{\sqrt{3}}{2}\right)^2}-\frac{2z}{z^2-z+1}\right]\right\} \\
&= -\frac{1}{2\sqrt{3}}\left[\mathbb{Z}^{-1}\left\{\frac{z}{\left(z-0.5-j\frac{\sqrt{3}}{2}\right)^2}\right\}+\mathbb{Z}^{-1}\left\{\frac{z}{\left(z-0.5+j\frac{\sqrt{3}}{2}\right)^2}\right\}\right]+\frac{1}{\sqrt{3}}\times\frac{2}{\sqrt{3}}\frac{\frac{\sqrt{3}}{2}z}{z^2-z+1} \\
&= -\frac{1}{2\sqrt{3}}\left[k\left(0.5+j\tfrac{\sqrt{3}}{2}\right)^{k-1}+k\left(0.5-j\tfrac{\sqrt{3}}{2}\right)^{k-1}\right]u[k]+\frac{2}{3}\sin(\tfrac{\pi}{3}k)u[k] \\
&= -\frac{k}{2\sqrt{3}}\left[e^{\frac{j\pi}{3}(k-1)}+e^{-\frac{j\pi}{3}(k-1)}\right]u[k]+\frac{2}{3}\sin(\tfrac{\pi}{3}k)u[k] \\
&= -\frac{k}{2\sqrt{3}}\times 2\cos\left(\tfrac{\pi}{3}(k-1)\right)u[k]+\frac{2}{3}\sin(\tfrac{\pi}{3}k)u[k] \\
&= \tfrac{2}{3}\sin(\tfrac{\pi}{3}k)u[k]-\tfrac{k}{\sqrt{3}}\cos\left((\tfrac{\pi}{3}k+\tfrac{\pi}{6})-\tfrac{\pi}{2}\right)u[k] \\
&= \left[\tfrac{2}{3}\sin(\tfrac{\pi}{3}k)-\tfrac{k}{\sqrt{3}}\sin(\tfrac{\pi}{3}k+\tfrac{\pi}{6})\right]u[k]
\end{aligned}
$$

▮

Program 13.6. MATLAB Program (for verification)

```
k=0:9 ;
y1=(2/3)*sin(pi*k/3).*(k>=0);
y2=-(k/sqrt(3)).*sin(pi*k/3+pi/6).*(k>=0);
y= y1+y2
% y =  [0  0  0  0.8660  1.7321  0.8660  -1.7321  -3.4641  -1.7321  2.5981]
%MATLAB Verification:
```

$$\% \frac{\frac{\sqrt{3}}{2}z^{-3}}{\left(1-z^{-1}+z^{-2}\right)^2}=\frac{\frac{\sqrt{3}}{2}z^{-3}}{1-2z^{-1}+3z^{-2}-2z^{-3}+z^{-4}}$$

```
sys = filt([0  0  0  sqrt(3)/2],[1  -2  3  -2  1])
h = impulse(sys,10)
plot(k,y,k,h)
% h =  [0  0  0  0.8660  1.7321  0.8660  -1.7321  -3.4641  -1.7321  2.5981]'
```

Problem 13.7

Example 13.3(v) showed that

$$x_5[k]=\begin{cases}1, & k=0,1\\ 2, & k=2,5\\ 0, & \text{otherwise.}\end{cases} \xleftrightarrow{z} X(z)=1+z^{-1}+2z^{-2}+2z^{-5} \quad \text{ROC: entire z-plane except } (z\neq 0).$$

Since $g[k]=x_5[k-10]$, $G(z)=z^{-10}X_5(z)$, which yields

$$G(z) = z^{-10} + z^{-11} + 2z^{-12} + 2z^{-15} \quad \text{ROC: entire z-plane except } (z \neq 0).$$

Problem 13.8

For a causal sequence $x[k]$, the z-transform is given by

$$X(z) = x[0] + x[1]z^{-1} + x[2]z^{-2} + x[3]z^{-3} + \cdots + x[k]z^{-k} + \cdots$$

Applying the limit, $z \to \infty$, the above equation reduces to

$$x[0] = \lim_{z\to\infty} X(z).$$

Problem 13.9

Using the time shifting property, $\mathbb{Z}\left(x[k+1] - x[k]\right) = \mathbb{Z}\left(x[k+1]\right) - \mathbb{Z}\left(x[k]\right) = (z-1)X(z)$.

However, from definition, $\mathbb{Z}\left(x[k+1] - x[k]\right) = \lim_{N\to\infty} \sum_{k=0}^{N} \left(x[k+1] - x[k]\right) z^{-n}$

Therefore, $(z-1)X(z) = \lim_{N\to\infty} \sum_{k=0}^{N} \left(x[k+1] - x[k]\right) z^{-n}$.

Applying the limit, $z \to 1$, the above equation reduces to

$$\begin{aligned}
\lim_{z\to 1}(z-1)X(z) &= \lim_{N\to\infty} \sum_{k=0}^{N} \left(x[k+1] - x[k]\right) \\
&= \lim_{N\to\infty} \left[\left(x[1] - x[0]\right) + \left(x[2] - x[1]\right) + \ldots\ldots + \left(x[N+1] - x[N]\right)\right]. \\
&= \lim_{N\to\infty} x[N+1] \\
&= \lim_{k\to\infty} x[k]
\end{aligned}$$

which is the final value theorem.

Problem 13.10

(i) $x[k] = (5/6)^k u[k-6]$. We know that

$$(5/6)^k u[k] \stackrel{z}{\longleftrightarrow} \frac{1}{1-(5/6)z^{-1}} \quad \text{ROC}: |z| > 5/6$$

Using the time shifting property,

$$(5/6)^{k-6} u[k-6] \stackrel{z}{\longleftrightarrow} \frac{z^{-6}}{1-(5/6)z^{-1}} \quad \text{ROC}: |z| > 5/6$$

or,
$$(5/6)^{k} u[k-6] \stackrel{z}{\longleftrightarrow} (5/6)^6 \frac{z^{-6}}{1-(5/6)z^{-1}} \quad \text{ROC}: |z| > 5/6.$$

(ii) $x[k] = k(2/9)^k u[k]$

We know that

$$(2/9)^k u[k] \xleftrightarrow{z} \frac{1}{1-(2/9)z^{-1}} \quad \text{ROC}: |z| > 2/9$$

Using the frequency differentiation property,

$$k(2/9)^k u[k] \xleftrightarrow{z} -z\frac{d}{dz}\frac{1}{1-(2/9)z^{-1}} \quad \text{ROC}: |z| > 2/9$$

or, $$k(2/9)^k u[k] \xleftrightarrow{z} -z\frac{1}{\left(1-(2/9)z^{-1}\right)^2}(-1)(-2/9)(-1)(z^{-2}) \quad \text{ROC}: |z| > 2/9 .$$

or, $$k\,(2/9)^k u[k] \xleftrightarrow{z} \frac{(2/9)z^{-1}}{\left(1-(2/9)z^{-1}\right)^2} \quad \text{ROC}: |z| > 2/9 .$$

(iii) $x[k]$ = ramp(k) = $k\underline{u}[k]$.

We know that $$k\,u[k] = \sum_{m=0}^{k} u[m] - u[k] .$$

Calculating the z-transform of both sides and applying the time-accumulation property, we get

$$k\,u[k] \xleftrightarrow{z} \frac{z}{z-1}\frac{1}{1-z^{-1}} - \frac{1}{1-z^{-1}} ,$$

which reduces to $$k\,u[k] \xleftrightarrow{z} \frac{z^{-1}}{\left(1-z^{-1}\right)^2} ,$$

which can be expressed in the alternate form

$$k\,u[k] \xleftrightarrow{z}, \frac{z}{(z-1)^2} .$$

(iv) $x[k] = e^k \sin(k)u[k]$.

We know that

$$e^{(1+j)k}u[k] \xleftrightarrow{z} \frac{1}{1-e^{(1+j)}z^{-1}} \quad \text{ROC}: |z| > e$$

and $$e^{(1-j)k}u[k] \xleftrightarrow{z} \frac{1}{1-e^{(1-j)}z^{-1}} \quad \text{ROC}: |z| > e$$

Adding the two transform pairs,

$$e^{(1+j)k}u[k] + e^{(1-j)k}u[k] \xleftrightarrow{z} \frac{1}{1-e^{(1+j)}z^{-1}} - \frac{1}{1-e^{(1-j)}z^{-1}} \quad \text{ROC}: |z| > e$$

or, $$2j\sin(k)e^k u[k] \xleftrightarrow{z} \frac{1-e^{(1-j)}z^{-1} - 1 + e^{(1+j)}z^{-1}}{\left(1-e^{(1+j)}z^{-1}\right)\left(1-e^{(1-j)}z^{-1}\right)} \quad \text{ROC}: |z| > e .$$

or, $$\sin(k)e^k u[k] \xleftrightarrow{z} \frac{e\sin(1)z^{-1}}{1-2e\cos(1)z^{-1}+e^2z^{-2}} \quad \text{ROC}: |z| > e$$ ▌

Problem 13.11

The four possible regions of convergences are:

1. $|z| < 0.5$.
2. $0.5 < |z| < 0.75$
3. $0.75 < |z| < 1.25$
4. $|z| > 1.25$

Since a stable system must have the unit circle within its ROC, we choose $0.75 < |z| < 1.25$ as the ROC. This ROC corresponds to a double sided sequence, and hence the system is not causal.

To determine the impulse response, we perform the partial fraction expansion

$$\frac{H(z)}{z} = \frac{z(z-1)}{(z-0.5)(z-0.75)(z-1.25)} \equiv \frac{-4/3}{z-0.5} + \frac{3/2}{z-0.75} + \frac{25/3}{z-1.25}$$

or, $$H(z) = \underbrace{\frac{-4/3}{1-0.5z^{-1}}}_{ROC:|z|>0.5} + \underbrace{\frac{3/2}{1-0.75z^{-1}}}_{ROC:|z|>0.75} + \underbrace{\frac{5/6}{1-1.25z^{-1}}}_{ROC:|z|<1.25}.$$

Calculating the inverse z-transform, we obtain

$$h[k] = -\frac{4}{3}\times 0.5^k u[k] + \frac{3}{2}\times 0.75^k u[k] - \frac{5}{6}\times 1.25^k u[-k-1].$$ ▌

Problem 13.12

(i) Calculating the z-transform of $h_{\text{inv}}[k] * h[k] = \delta[k]$, yields

$$H_{\text{inv}}(z)\, H(z) = 1, \text{ or, } H_{\text{inv}}(z) = 1/H(z).$$

The z-transfer function of the LTID system is given by

$$h[k] = 5^{-k}u[k] = 0.2^k u[k] \xleftrightarrow{z} H(z) = \frac{1}{1-0.2z^{-1}} \quad \text{ROC:} |z| > 0.2\,.$$

The z-transfer function of the inverse LTID system is given by

$$H_{\text{inv}}(z) = 1 - 0.2z^{-1} \quad \text{ROC: Entire } z\text{-plane except z = 0.}$$

Calculating the inverse z-transform, we obtain

$$h_{inv}[k] = \delta[k] - 0.2\delta[k-1].$$

(ii)

(a) Input: $x_1[k] = u[k]$. Using the transform pair

$$u[k] \xleftrightarrow{z} \frac{1}{1-z^{-1}} \quad \text{ROC:} |z| > 1,$$

the z-transform of the output is given by

$$Y_1(z) = X_1(z)H(z) = \frac{1}{(1-0.2z^{-1})(1-z^{-1})} \quad \text{ROC:}|z|>1.$$

Taking the partial fraction expansion, we obtain

$$\frac{Y_1(z)}{z} = \underbrace{\frac{1.25}{z-1}}_{ROC:|z|>1} - \underbrace{\frac{0.25}{z-0.2}}_{ROC:|z|>0.2}$$

Calculating the inverse z-transform, we obtain

$$y_1[k] = (1.25 - 0.25 \times 0.2^k)u[k].$$

(b) Input: $x_2[k] = 5\delta[k-4] - 2\delta[k+4]$.

Using the time domain convolution method, the output is obtained as

$$y_2[k] = 5^{-k}u[k] * (5\delta[k-4] - 2\delta[k+4]) = 5 \times 5^{-(k-4)}u[k-4] - 2 \times 5^{-(k+4)}u[k+4].$$

(c) Input: $x_3[k] = e^{k+2}u[-k+2] = e^5(e)^{k-3}u[-k+2]$.

From Example 13.2, we know the transform pair

$$-e^k u[-k-1] \xleftrightarrow{z} \frac{1}{1-ez^{-1}} \quad \text{ROC:}|z|<e.$$

Using the time shifting property, we obtain

$$-e^{k-3}u[-(k-3)-1] = -e^{k-3}u[-k+2] \xleftrightarrow{z} \frac{z^{-3}}{1-ez^{-1}} \quad \text{ROC:}|z|<e.$$

In other words, $X_3(z) = \mathbb{Z}\left\{e^5(e)^{k-3}u[-k+2]\right\} = -e^5 \dfrac{z^{-3}}{1-ez^{-1}} \quad \text{ROC:}|z|<e.$

The z-transform of the output is then given by

$$Y_3(z) = X_3(Z)H(z) = -e^5 \frac{z^{-3}}{(1-0.2z^{-1})(1-ez^{-1})} \quad \text{ROC:}0.2<|z|<e$$

or, $$\frac{Y_3(z)}{z} = -e^5 \frac{1}{z^2(z-0.2)(z-e)} \qquad \text{ROC: } 0.2<|z|<e$$

Calculating the partial fraction expansion, we obtain

$$\frac{Y_3(z)}{z} = -e^5 \left[\underbrace{\frac{9.8737}{z}}_{ROC:|z|>0} + \underbrace{\frac{1.8394}{z^2}}_{ROC:|z|>0} - \underbrace{\frac{9.9274}{(z-0.2)}}_{ROC:|z|>0.2} + \underbrace{\frac{0.0537}{(z-e)}}_{ROC:|z|<e} \right] \quad \text{ROC:}0.2<|z|<e,$$

or, $$Y_3(z) = -e^5 \left[\underbrace{9.8737}_{ROC:|z|>0} + \underbrace{1.8394z^{-1}}_{ROC:|z|>0} - \underbrace{\frac{9.9274}{(1-0.2z^{-1})}}_{ROC:|z|>0.2} + \underbrace{\frac{0.0537}{(1-ez^{-1})}}_{ROC:|z|<e} \right].$$

Calculating the inverse z-transform, we obtain

$$y_3[k] = -e^5\left[9.8737\delta[k] + 1.8394\delta[k-1] - 9.9274(1/5)^k u[k] - 0.0537u[-k-1]\right].$$

Problem 13.13

(i) Taking the z-transform of the input and output, we get

$$\begin{aligned} X(z) &= \frac{1}{1-(1/3)z^{-1}} - \frac{z^{-1}}{1-(1/4)z^{-1}} \qquad \text{ROC}: |z| > (1/3) \\ &= \frac{1-(5/4)z^{-1} + (1/3)z^{-2}}{\left(1-(1/3)z^{-1}\right)\left(1-(1/4)z^{-1}\right)} \end{aligned}$$

and

$$Y(z) = \frac{1}{1-(1/4)z^{-1}} \qquad \text{ROC}: |z| > (1/4).$$

The transfer function of the system is given by

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1-(1/3)z^{-1}}{1-(5/4)z^{-1} + (1/3)z^{-2}} \qquad \text{ROC}: |z| > 0.8694$$

(ii) By expressing the transfer function as

$$\frac{H(z)}{z} = \frac{z-(1/3)}{(z-0.8644)(z-0.3856)} \qquad \text{ROC}: |z| > 0.8694.$$

and taking the partial fraction expansion, we get

$$\frac{H(z)}{z} = \underbrace{\frac{1.1093}{(z-0.8644)}}_{ROC:|z|>0.8644} - \underbrace{\frac{0.1093}{(z-0.3856)}}_{ROC:|z|>0.3856}$$

Taking the inverse z-transform, we get

$$h[k] = (1.1093 \times 0.8644^k - 0.1093 \times 0.3856^k)U[k].$$

(iii) By expressing the transfer function as

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1-(1/3)z^{-1}}{1-(5/4)z^{-1} + (1/3)z^{-2}}$$

and cross multiplying, we get

$$Y(z) - (5/4)z^{-1}Y(z) + (1/3)z^{-2}Y(z) = X(z) - (1/3)z^{-1}X(z)$$

Taking the inverse z-transform, we get

$$y[k] - (5/4)y[k-1] + (1/3)y[k-2] = x[k] - (1/3)x[k-1]$$

with initial conditions: $y[-2] = y[-1] = 0$.

Problem 13.14

(i) We express the transfer function as

$$\frac{H(z)}{z} = \frac{z^2}{(z-0.3)(z-0.5)(z-0.7)} \quad \text{ROC:}|z| > 0.7$$

where the ROC is selected to obtain a causal LTID system. Taking the partial fraction expansion, we get

$$\frac{H(z)}{z} = \underbrace{\frac{1.125}{z-0.3}}_{ROC:|z|>0.3} - \underbrace{\frac{6.25}{z-0.5}}_{ROC:|z|>0.5} + \underbrace{\frac{6.125}{z-0.7}}_{ROC:|z|>0.7}$$

Taking the inverse z-transform, we get

$$h[k] = \left(1.125 \times 0.3^k - 6.25 \times 0.5^k + 6.125 \times 0.7^k\right) u[k].$$

(ii) By expressing the transfer function as

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - 1.5z^{-1} + 0.71z^{-2} - 0.105z^{-3}}$$

and cross multiplying, we obtain

$$Y(z) - 1.5z^{-1}Y(z) + 0.71z^{-2}Y(z) - 0.105z^{-3}Y(z) = X(z)$$

Taking the inverse z-transform, we obtain

$$y[k] - 1.5y[k-1] + 0.71y[k-2] - 0.105y[k-3] = x[k]$$

with initial conditions: $y[-3] = y[-2] = y[-1] = 0$.

(iii) Using the convolution property, the output for the unit step function is given by

$$Y(z) = H(z)X(z) = \frac{z^4}{(z-1)(z-0.3)(z-0.5)(z-0.7)}.$$

Taking the partial fraction expansion, we obtain

$$\frac{Y(z)}{z} = \underbrace{\frac{200/21}{z-1}}_{ROC:|z|>1} - \underbrace{\frac{27/56}{z-0.3}}_{ROC:|z|>0.3} + \underbrace{\frac{25/4}{z-0.5}}_{ROC:|z|>0.5} - \underbrace{\frac{343/24}{z-0.7}}_{ROC:|z|>0.7}$$

Taking the inverse z-transform, we get

$$y[k] = (9.524 - 0.4821 \times 0.3^k + 6.25 \times 0.5^k - 14.2917 \times 0.7^k)u[k].$$

(iv) Using linear convolution, the output is given by

$$y[k] = u[k] * (1.125 \times 0.3^k - 6.25 \times 0.5^k + 6.125 \times 0.7^k)u[k].$$

Note that $u[k] * \alpha^k u[k] = \sum_{n=-\infty}^{\infty} u[k-n]\alpha^n u[n] = \sum_{n=0}^{k} \alpha^n = \dfrac{\alpha^{k+1} - 1}{\alpha - 1}.$

Hence, the output is given by

$$y[k] = \left[1.125 \times \frac{0.3^{k+1} - 1}{-0.7} - 6.25 \times \frac{0.5^{k+1} - 1}{-0.5} + 6.125 \times \frac{0.7^{k+1} - 1}{-0.3}\right] u[k]$$

which reduces to

$$y[k] = (9.524 - 0.4821 \times 0.3^k + 6.25 \times 5^k - 14.2917 \times 7^k) u[k].$$

Problem 13.15

(i) Using Eq. (13.35), the transfer function of the system (assuming a causal system) is calculated as follows:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - z^{-2}}{1 + z^{-1} + \frac{1}{4} z^{-2}} = \frac{1 - z^{-2}}{\left(1 + 0.5 z^{-1}\right)^2} \qquad \text{ROC:} |z| > 0.5$$

(ii) In order to calculate the impulse response, we represent the $H(z)$ as follows.

$$H(z) = \frac{1 - z^{-2}}{\left(1 + 0.5 z^{-1}\right)^2} = \frac{1}{\left(1 + 0.5 z^{-1}\right)^2} + \frac{-z^{-2}}{\left(1 + 0.5 z^{-1}\right)^2} = (-2z) . \frac{-0.5 z^{-1}}{\left(1 - (-0.5) z^{-1}\right)^2} + 2 z^{-1} \frac{-0.5 z^{-1}}{\left(1 - (-0.5) z^{-1}\right)^2} .$$

From Entry 7 of Table 13.1, we get the following z-transform pair.

$$p[k] = k(-0.5)^k u[k] \xleftrightarrow{\;z\;} P(z) = \frac{-0.5 z^{-1}}{\left(1 - (-0.5) z^{-1}\right)^2} \qquad \text{ROC:} |z| > 0.5$$

Therefore, applying the time shifting property, we get

$$\begin{aligned} h[k] &= -2 p[k+1] + 2 p[k-1] \\ &= -2(k+1)(-0.5)^{k+1} u[k+1] + 2(k-1)(-0.5)^{k-1} u[k-1] \\ &= \underbrace{(k+1)(-0.5)^k}_{=0,\ at\ k=-1} u[k+1] - 4(k-1)(-0.5)^k u[k-1] \\ &= (k+1)(-0.5)^k u[k] - \underbrace{4(k-1)(-0.5)^k}_{=-4\ at\ k=0} u[k-1] \\ &= -4\delta[k] + (k+1)(-0.5)^k u[k] - 4(k-1)(-0.5)^k u[k] \\ &= -4\delta[k] - (3k-5)(-0.5)^k u[k] \end{aligned}$$

The impulse response function is shown in the Fig. S13.15(a).

(iii) $X(z) = \frac{1}{1 - 0.5 z^{-1}}$. Using the convolution property, the output for the unit step function is given by

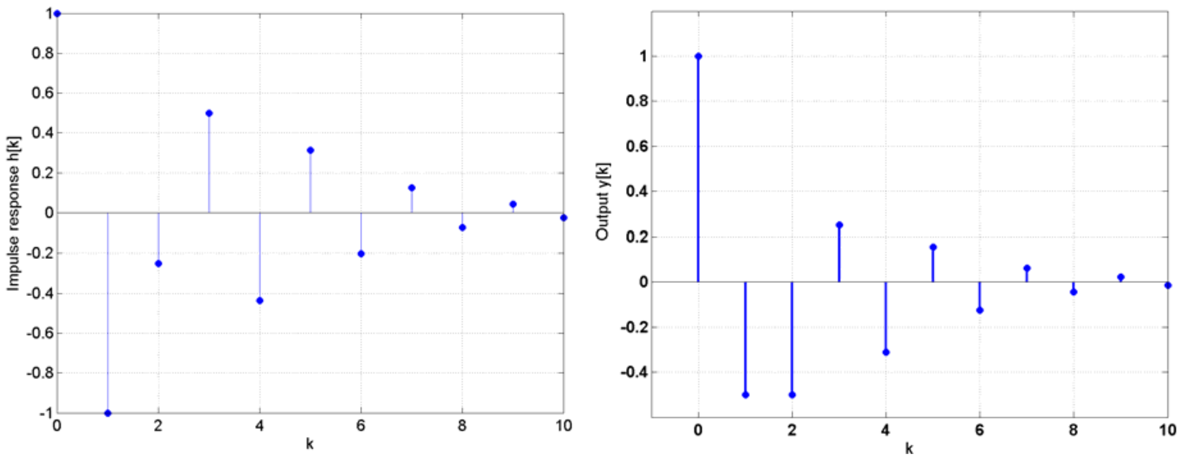
$$Y(z) = X(z) H(z) = \frac{1 - z^{-2}}{\left(1 - 0.5 z^{-1}\right)\left(1 + 0.5 z^{-1}\right)^2} \equiv \frac{k_1}{1 - 0.5 z^{-1}} + \frac{k_2}{1 + 0.5 z^{-1}} + \frac{k_3}{\left(1 + 0.5 z^{-1}\right)^2}$$

where

$$k_1 = \left[\frac{1 - z^{-2}}{\left(1 + 0.5 z^{-1}\right)^2} \right]_{z^{-1} = 2} = -\frac{3}{4}$$

$$k_2 = \left[\tfrac{d}{du} \left\{ \frac{1-u^2}{1-0.5u} \right\} \right]_{u=-2} = \left[\frac{(1-0.5u)(-2u)-(1-u^2)(-0.5)}{(1-0.5u)^2} \right]_{u=-2} = \frac{-19}{8}$$

$$k_3 = \left[\frac{1-z^{-2}}{1-0.5z^{-1}} \right]_{z^{-1}=-2} = -\frac{3}{2}$$



(a) (b)

Fig. S13.15. (a) Impulse response function and (b) output response in Problem 13.15.

Substituting the above values, we obtain

$$\begin{aligned} Y(z) &= \frac{-3/4}{1-0.5z^{-1}} + \frac{13/4}{1+0.5z^{-1}} + \frac{-3/2}{\left(1+0.5z^{-1}\right)^2} \\ &= \frac{-3/4}{1-0.5z^{-1}} + \frac{13/4}{1+0.5z^{-1}} + 3z\frac{-0.5z^{-1}}{\left(1-(-0.5)z^{-1}\right)^2} \end{aligned}$$

Calculating the inverse z-transform, we get

$$
\begin{aligned}
y[k] &= -\tfrac{3}{4}(0.5)^k u[k] + \tfrac{13}{4}(-0.5)^k u[k] + 3(k+1)(-0.5)^{k+1} u[k+1] \\
&= -\tfrac{3}{4}(0.5)^k u[k] + \tfrac{13}{4}(-0.5)^k u[k] - \tfrac{3}{2}\underbrace{(k+1)(-0.5)^k}_{=0,\ at\ k=-1} u[k+1] \\
&= -\tfrac{3}{4}(0.5)^k u[k] + \tfrac{13}{4}(-0.5)^k u[k] - \tfrac{3}{2}(k+1)(-0.5)^k u[k] \\
&= -\left\{\tfrac{3}{4}(0.5)^k + (\tfrac{3}{2}k - \tfrac{7}{4})(-0.5)^k\right\} u[k] \\
&= -\tfrac{1}{4}\left\{3(0.5)^k + (6k-7)(-0.5)^k\right\} u[k] \\
&= \begin{cases} -\tfrac{1}{4}\left\{3(0.5)^k + (6k-7)(0.5)^k\right\} u[k] & k = even \\ -\tfrac{1}{4}\left\{3(0.5)^k - (6k-7)(0.5)^k\right\} u[k] & k = odd \end{cases} \\
&= \begin{cases} -\tfrac{1}{4}\left\{3 + (6k-7)\right\}(0.5)^k u[k] & k = even \\ -\tfrac{1}{4}\left\{3 - (6k-7)\right\}(0.5)^k u[k] & k = odd \end{cases} \\
&= \begin{cases} -\tfrac{1}{4}(6k-4)(0.5)^k u[k] & k = even \\ -\tfrac{1}{4}(10-6k)(0.5)^k u[k] & k = odd \end{cases} \\
&= \begin{cases} -(3k-2)(0.5)^{k+1} u[k] & k = even \\ (3k-5)(0.5)^{k+1} u[k] & k = odd \end{cases}
\end{aligned}
$$

The output response is shown in the Fig. S13.15(b).

(iv) The impulse response of the system, $h[k] = -4\delta[k] - (3k-5)(-0.5)^k u[k]$, and the input signal, $x[k] = (\tfrac{1}{2})^k u[k] = (0.5)^k u[k]$. The output signal can be calculated by applying linear convolution as follows.

$$
\begin{aligned}
y[k] = h[k] * x[k] &= \left\{-4\delta[k] - (3k-5)(-0.5)^k u[k]\right\} * x[k] \\
&= -4\delta[k] * x[k] - \left\{(3k-5)(-0.5)^k u[k]\right\} * x[k] \\
&= -4x[k] - \sum_{m=-\infty}^{\infty} \left\{(3m-5)(-0.5)^m u[k]\right\} \cdot x[k-m] \\
&= -4x[k] - \sum_{m=0}^{\infty} \left\{(3m-5)(-0.5)^m\right\} \cdot (0.5)^{k-m} u[k-m] \\
&= -4(0.5)^k u[k] - (0.5)^k u[k] \sum_{m=0}^{k} (3m-5)(-0.5)^m (0.5)^{-m} \\
&= (0.5)^k u[k] \left[-4 - \sum_{m=0}^{k} (3m-5)(-1)^m\right]
\end{aligned}
$$

Note: $\sum_{m=0}^{k}(3m-5)(-1)^m = 3\underbrace{\sum_{m=0}^{k} m(-1)^m}_{=\begin{cases} k/2 & k=even \\ -(k+1)/2 & k=odd \end{cases}} -5\underbrace{\sum_{m=0}^{k}(-1)^m}_{=\begin{cases} 1 & k=even \\ 0 & k=odd \end{cases}} = \begin{cases} \frac{1}{2}(3k-10) & k=even \\ -\frac{3}{2}(k+1) & k=odd \end{cases}.$

Therefore,

$$\begin{aligned} y[k] &= (0.5)^k u[k]\left[-4-\sum_{m=0}^{k}(3m-5)(-1)^m\right] \\ &= (0.5)^k u[k]\times\begin{cases} -4-\frac{1}{2}(3k-10) & k=even \\ -4+\frac{3}{2}(k+1) & k=odd \end{cases} \\ &= (0.5)^k u[k]\times\begin{cases} -\frac{1}{2}(3k-2) & k=even \\ \frac{1}{2}(3k-5) & k=odd \end{cases} \\ &= \begin{cases} -(3k-2)(0.5)^{k+1}u[k] & k=even \\ (3k-5)(0.5)^{k+1}u[k] & k=odd \end{cases} \end{aligned}$$

It is observed that the output obtained above by applying time-domain convolution is identical to that obtained in step (iii) using the z-transform approach. ▌

Program 13.15. MATLAB Program

```
%Plotting the impulse response
k=[0:10] ;
h = -(3*k-5).*((-0.5).^k) ;
h(1)=h(1)-4 ;
stem(k, h, 'filled'), grid
xlabel('k')                 % Label of X-axis
ylabel('h[k]')              % Label of Y-axis
print -dtiff plot.tiff           % Save the figure as a TIFF file
%MATLAB Verification
sys = filt([1  0  -1],[1  1  0.25])
h1 = impulse(sys,10)
%i_response = [1.0 -1.0 -0.25 0.50 -0.4375 0.3125 -0.2031 0.1250 -0.0742
0.0430]

% Part (iii) – Output response
k=[0:10] ;
y1 = (0.5).^k ;
y2 = (-0.5).^k;
y = -0.25*(3*y1+(6*k-7).*y2)
% y=    [1.0000   -0.5000   -0.5000    0.2500   -0.3125    0.1563   -0.1250
0.0625   -0.0430    0.0215]
stem(k, y, 'filled'), grid
xlabel('k')                 % Label of X-axis
ylabel('y[k]')              % Label of Y-axis
print -dtiff plot.tiff           % Save the figure as a TIFF file
%MATLAB Verification
%
```

$$Y(z) = \frac{1-z^{-2}}{\left(1-0.5z^{-1}\right)\left(1+0.5z^{-1}\right)^2} = \frac{1-z^{-2}}{1+0.5z^{-1}-0.25z^{-2}-0.125z^{-3}}$$

```
sys = filt([1  0  -1],[1  0.5  -0.25  -0.125])
output = impulse(sys,10)
% output=    [1.0000   -0.5000   -0.5000    0.2500   -0.3125    0.1563   -
0.1250    0.0625   -0.0430    0.0215]
```

Problem 13.16

(i) $x[k]=u[k+2]-u[k-3]$ and $h[k]=u[k-5]-u[k-6]$.

The z-transform of the input and impulse response is given by

$$X(z)=\frac{z^2-z^{-3}}{z-1}\ \ \text{ROC}:|z|>1 \quad \text{and} \quad H(z)=\frac{z^{-5}-z^{-6}}{z-1}=z^{-6}\ \ \text{ROC}:|z|>1.$$

The output is then given by

$$Y(z)=\frac{z^{-4}-z^{-9}}{z-1}\ \ \text{ROC}:|z|>1.$$

Calculating the inverse z-transform, we obtain the output

$$y[k]=u[k-4]-u[k-9].$$

(ii) $x[k]=u[k]-u[k-9]$ and $h[k]=3^{-k}u[k-4]=3^{-4}\times 3^{-(k-4)}u[k-4]$.

The z-transform of the input and impulse response is given by

$$X(z)=\frac{z-z^{-8}}{z-1}\ \ \text{ROC}:|z|>1 \quad \text{and} \quad H(z)=\frac{3^{-4}z^{-4}}{z-\frac{1}{3}}\text{ROC}:|z|>(1/3).$$

The output is then given by

$$Y(z)=\frac{3^{-4}\left(z^{-3}-z^{-12}\right)}{(z-1)(z-\frac{1}{3})}=3^{-4}\left(z^{-3}-z^{-12}\right)\times\frac{3}{2}\left[\frac{1}{z-1}-\frac{1}{z-\frac{1}{3}}\right]\ \ \text{ROC}:|z|>1$$

or,
$$Y(z)=\frac{1}{54}\left(z^{-4}-z^{-13}\right)\left[\frac{z}{z-1}-\frac{z}{z-\frac{1}{3}}\right]\ \ \text{ROC}:|z|>1$$

Noting that $\left[1-3^{-k}\right]u[k]\leftrightarrow\left[\frac{z}{z-1}-\frac{z}{z-\frac{1}{3}}\right]$, the output is obtained as

$$y[k]=\frac{1}{54}\left[1-3^{-(k-4)}\right]u[k-4]-\frac{1}{54}\left[1-3^{-(k-13)}\right]u[k-13].$$

(iii) $x[k]=2^{-k}u[k]=0.5^k u[k]$ and $h[k]=k(u[k]-u[k-4])$

The z-transform of the input signal is given by

$$X(z)=\frac{z}{z-0.5}\ \ \text{ROC}:|z|>0.5\ .$$

The z-transform of the impulse response is given by

$$H(z)=\sum_{k=0}^{\infty}k(u[k]-u[k-4])z^{-k}=\sum_{k=0}^{3}kz^{-k}=z^{-1}+2z^{-2}+3z^{-3}\qquad \text{ROC}:|z|>0$$

The z-transform $Y(z)$ of the output response is then given by

$$Y(z) = \left(z^{-1} + 2z^{-2} + 3z^{-3}\right)\mathrm{X}(z) \qquad \mathrm{ROC}: |z| > 0.5$$

Calculating the inverse z-transform, we obtain the output

$$\begin{aligned} y[k] &= x[k-1] + 2x[k-2] + 3x[k-3] \\ &= 0.5^{k-1}u[k-1] + 2\times 0.5^{k-2}u[k-2] + 3\times 0.5^{k-3}u[k-3]. \end{aligned}$$

(iv) $x[k] = u[k]$ and $h[k] = 4^{-|k|}$.

Recall that $$\alpha^{|k|} \xleftrightarrow{z} \frac{(\alpha - 1/\alpha)z}{(z-\alpha)(z-1/\alpha)} \quad \mathrm{ROC}: \alpha < |z| < (1/\alpha).$$

The z-transform of the input and impulse response is given by

$$X(z) = \frac{z}{z-1}\mathrm{ROC}: |z| > 1 \quad \text{and} \quad H(z) = \frac{-3.75z}{(z-0.25)(z-4)}\mathrm{ROC}: 0.25 < |z| < 4.$$

The output is then given by

$$Y(z) = \frac{-3.75z^2}{(z-1)(z-0.25)(z-4)}\mathrm{ROC}: 1 < |z| < 4.$$

By partial fraction expansion, we obtain

$$\frac{Y(z)}{z} = \underbrace{\frac{1.6667}{z-1}}_{ROC:|z|>1} - \underbrace{\frac{0.3333}{z-0.25}}_{ROC:|z|>0.25} - \underbrace{\frac{1.3333}{z-4}}_{ROC:|z|<4}$$

which results in the output

$$y[k] = 1.6667u[k] - 0.3333\times 0.25^k u[k] + 1.3333\times 4^k u[-k-1].$$

(v) $x[k] = 2^{-k}u[k]$ and $h[k] = 2^k u[-k-1]$.

The z-transform of the input and impulse response is given by

$$X(z) = \frac{z}{z-0.5}\mathrm{ROC}: |z| > 0.5 \quad \text{and} \quad H(z) = \frac{-z}{z-2}\mathrm{ROC}: |z| < 2.$$

The output is then given by

$$Y(z) = \frac{-z^2}{(z-0.5)(z-2)}\mathrm{ROC}: 0.5 < |z| < 2.$$

By partial fraction expansion, we get

$$\frac{Y(z)}{z} = \underbrace{\frac{0.3333}{z-0.5}}_{ROC:|z|>0.5} - \underbrace{\frac{1.3333}{z-2}}_{ROC:|z|<2}$$

which results in the output

$$y[k] = 0.3333\times 0.5^k u[k] + 1.3333\times 2^k u[-k-1].$$ ▮

Problem 13.17

(i) Calculating the z-transform of the input and output, we obtain

$$X(z) = \frac{1}{1-(1/4)z^{-1}} + \frac{1}{1-(1/3)z^{-1}} = \frac{2-(7/12)z^{-1}}{\left(1-(1/3)z^{-1}\right)\left(1-(1/4)z^{-1}\right)} \qquad \text{ROC:} |z| > (1/3)$$

$$Y(z) = \frac{2}{1-(1/4)z^{-1}} - \frac{4}{1-(3/4)z^{-1}} = -\frac{2+(1/2)z^{-1}}{\left(1-(1/3)z^{-1}\right)\left(1-(3/4)z^{-1}\right)} \qquad \text{ROC:} |z| > (3/4)$$

The transfer function of the system is given by

$$H(z) = \frac{Y(z)}{X(z)} = -\frac{\left[2+(1/2)z^{-1}\right]\left[1-(1/3)z^{-1}\right]\left[1-(1/4)z^{-1}\right]}{\left[2-(7/12)z^{-1}\right]\left[1-(1/3)z^{-1}\right]\left[1-(3/4)z^{-1}\right]} \qquad \text{ROC:} |z| > (3/4)$$

$$= -\frac{\left[2+(1/2)z^{-1}\right]\left[1-(1/4)z^{-1}\right]}{\left[2-(7/12)z^{-1}\right]\left[1-(3/4)z^{-1}\right]}$$

$$= -\frac{1-(1/16)z^{-1}}{1-(25/24)z^{-1}+(21/96)z^{-2}} \qquad \text{ROC:} |z| > (3/4)$$

(ii) Calculating the partial fraction expansion of $H(z)$, we obtain

$$\frac{H(z)}{z} = -\frac{z-(1/16)}{z^2-(25/24)z+(21/96)} \equiv -\underbrace{\frac{1.5}{(z-3/4)}}_{ROC:|z|>(3/4)} + \underbrace{\frac{0.5}{(z-7/12)}}_{ROC:|z|>(7/12)}.$$

Calculating the inverse z-transform, we obtain

$$h[k] = (-1.5\times 0.75^k - 0.5\times 0.2917^k)u[k].$$

(iii) By expressing the transfer function as

$$H(z) = \frac{Y(z)}{X(z)} = -\frac{1-(1/16)z^{-1}}{1-(25/24)z^{-1}+(21/96)z^{-2}}$$

and cross multiplying, we obtain

$$Y(z)-(25/24)z^{-1}Y(z)+(21/96)z^{-2}Y(z) = -X(z)+(1/16)z^{-1}X(z)$$

Calculating the inverse z-transform, we obtain

$$y[k]-(25/24)y[k-1]+(21/96)y[k-2] = -x[k]+(1/16)x[k-1]$$

with initial conditions: $y[-2] = y[-1] = 0$. ▮

Problem 13.18

The impulse response of the LTIC system is given by

$$h(t) = \sum_{k=0}^{\infty} 0.3^{kT}\,\delta(t-kT)$$

The impulse response includes a series of causal CT impulse functions with decaying amplitude. Note that the input also is a train of causal impulse functions with decaying magnitude. It appears that both the input and the system are CT representation of discrete signal and system, respectively. Therefore, it will be easier to calculate the output in the DT domain.

To solve the problem in the discrete domain, we use the impulse transformation.

The equivalent DT input is given by $f[k] = 0.2^k u[k]$, with $F(z) = \dfrac{z}{z-0.2}$.

The transfer function of the equivalent DT system is given by

$$H(z) = \frac{z}{z-0.3} \qquad ROC: |z| > 0.3.$$

The z-transform of the DT output is therefore given by

$$Y(z) = H(z)F(z) = \frac{z^2}{(z-0.3)(z-0.2)}.$$

Using partial fraction expansion, the output is expressed as

$$\frac{Y(z)}{z} = \frac{z}{(z-0.3)(z-0.2)} = \frac{3}{z-0.3} - \frac{2}{z-0.2},$$

or, $$y[k] = 3\times 0.3^k u[k] - 2\times 0.3^k u[k] = \left(3\times 0.3^k - 2\times 0.3^k\right)u[k].$$

The equivalent CT output is given by, $y(t) = \sum_{k=0}^{\infty}\left(3\times 0.3^{kT} - 2\times 0.2^{kT}\right)\delta(t-kT)$.

The above answer can also be obtained by using the Laplace transform in the CT domain. ▌

Problem 13.19

To determine the stability, we will assume that the systems are physically realizable, i.e., causal.

(i) $$H(z) = \frac{z-2}{(z-0.6+j0.8)(z^2+0.25)}$$

The pole-zero plot is shown in Fig. S13.21(i). There is one zero, at $z = 2$, and three poles at $z = 0.6 - j0.8$, $\pm j0.5$. Two poles at $z = \pm 0.5$ are inside the unit circle ($|z| = 0.5$), and the pole at $z = 0.6 - j0.8$ is on the unit circle ($|z| = 1$). Therefore, the system is a **marginally stable** system.

(ii) $$H(z) = \frac{(z-2)(z-1)}{(z^2-2.5z+1)(z^2+0.25)} = \frac{(z-2)(z-1)}{(z-2)(z-0.5)(z^2+0.25)} = \frac{z-1}{(z-0.5)(z^2+0.25)}$$

The pole-zero plot is shown in Fig. S13.21(ii). There is one zero at $z = 1$ and three poles at $z = 0.5$, $\pm j0.5$. As all poles are inside the unit circle ($|z| = 0.5$), the system is **absolutely stable**.

(iii) $$H(z) = \frac{z-0.2}{(z+0.1)(z^2+4)}$$

The pole-zero plot is shown in Fig. S13.21(iii). There is one zero at $z = 0.2$, and three poles at $z = 0.1, \pm j2$. The pole at $z = 0.1$ is inside the unit circle. However, the two poles at $z = \pm j2$ are outside the unit circle ($|z| = 2$).Therefore, the system is an **unstable system**.

For system (iii), by selecting the ROC, $0.1 < |z| < 2$, a stable implementation of $H(z)$ can be obtained (as the ROC includes the unit circle). However, such an implementation will not be causal (physically realizable). A stable and causal implementation is not possible for this transfer function.

(iv) $$H(z) = z^{-1} - 2z^{-2} + z^{-3} = \frac{z^2 - 2z + 1}{z^3} = \frac{(z-1)^2}{z^3}$$

The pole-zero plot is shown in Fig. S13.21(iv). There are two zeros at $z = 1$, and three poles at $z = 0$. As all three poles are inside the unit circle ($|z| = 0$), the system is **absolutely stable**.

(v) $$H(z) = \frac{(z^2 + 2.5z + 0.9 + j0.15)z}{z^3 + (1.8 + j0.3)z^2 + (0.6 + j0.6)z - 0.2 + j0.3} = \frac{(z^2 + 2.5z + 0.9 + j0.15)z}{(z+1)^2(z - 0.2 + j0.3)}$$

$$= \frac{z(z + 0.4309 + j0.0916)(z + 2.0691 - j0.0916)}{(z+1)^2(z - 0.2 + j0.3)}$$

The pole-zero plot is shown in Fig. S13.21(v). There is a double pole on the unit circle. Therefore, the system is **unstable**. Note that if it was a single pole on the unit circle, the system would have been marginally stable.

(vi) $$H(z) = \frac{z^3 - 1.2z^2 + 2.5z + 0.8}{z^6 + 0.3z^5 + 0.23z^4 + 0.209z^3 + 0.1066z^2 - 0.04162z - 0.07134}$$

$$= \frac{(z + 0.2753)(z - 0.7376 - j1.5369)(z - 0.7376 + j1.5369)}{(z - 0.5)(z + 0.6)(z - 0.3 - j0.7)(z - 0.3 + j0.7)(z + 0.4 - j0.5)(z + 0.4 + j0.5)}$$

The pole-zero plot is shown in Fig. S13.21(vi). All six poles are inside the unit circle. Therefore, the system is **stable**. ▌

Problem 13.20

The frequency response of the system is given by

$$H(\Omega) = H(z)\big|_{z=\exp(j\Omega)} = \frac{e^{j\Omega}}{e^{j\Omega} + 0.1} = \frac{1}{1 + 0.1e^{-j\Omega}}.$$

The frequency response is plotted in Figure S13.20 using the following MATLAB code. A blown-up is also included to calculate the amplitude and phase gain at $\Omega = \pi/10$.

```
omega = [-pi:pi/20:pi] ;
H = 1./(1+0.1*exp(-j*omega)) ;
subplot(2,1,1), plot(omega, abs(H)), grid
xlabel('Omega')                    % Label of X-axis
ylabel('|H(Omega)|')               % Label of Y-axis %
axis([-3.2  3.2  0.9  1.2])
print -dtiff plot.tiff             % Save figure as a TIFF file
%
subplot(2,1,2), plot(omega, angle(H)), grid
xlabel('Omega')                    % Label of X-axis
ylabel('<H(Omega) (in rad)')       % Label of Y-axis %
axis([-3.2  3.2  -0.15  0.15])
print -dtiff plot.tiff             % Save figure as a TIFF file
```

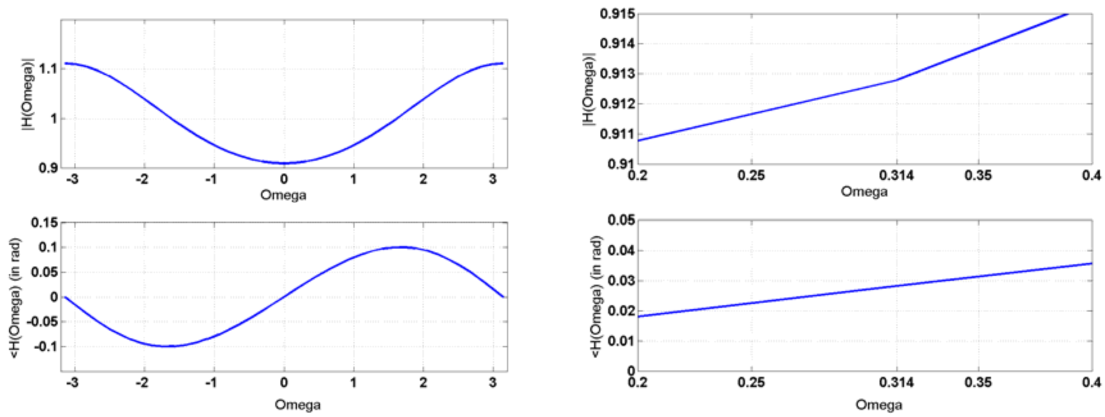


Fig. S13.20: Amplitude and phase spectrums of the LTID system in Problem 13.20.

It is observed from the plot that $|H(\pi/10)| = |H(0.314)| \approx 0.913$ and $<H(\pi/10)| = <H(0.314)| \approx 0.03$ radians. Therefore, the steady state output for the signal $x[k] = 5\cos(\pi k/10)$ is given by

$$y[k] \approx (5\times 0.913)\cos\left(\tfrac{\pi k}{10}+0.03^r\right) = 4.565\cos\left(\tfrac{\pi k}{10}+0.03^r\right).$$

■

Problem 13.21

(i) $H(z) = \dfrac{z-2}{(z-0.6+j0.8)(z^2+0.25)}$

One zero at $z = 2$ and three poles at $z = 0.6 - j0.8, j0.5, -j0.5$

```
>>    Z = [2];
>>    P = [0.6-j*0.8  ;   j*0.5  ;  -j*0.5 ] ;
>>    zplane(Z,P)
>>    print -dtiff plot.tiff
```

The pole-zero plot is shown in Fig. S13.21(i). Two poles are inside the unit circle and one pole is on the unit circle. Therefore, the system is a marginally stable system.

(ii) $H(z) = \dfrac{(z-2)(z-1)}{(z^2-2.5z+1)(z^2+0.25)} = \dfrac{(z-2)(z-1)}{(z-2)(z-0.5)(z^2+0.25)} = \dfrac{z-1}{(z-0.5)(z^2+0.25)}$

Note that the pole and zero at z=2 cancel each other. Therefore, there is one zero at $z = 1$ and three poles at z = 0.5, $j0.5$, $-j0.5$.

```
>>    Z = [1];
>>    P = [ 0.5;  -j*0.5;   j*0.5];
>>    zplane(Z,P);
>>    print -dtiff plot.tiff;
```

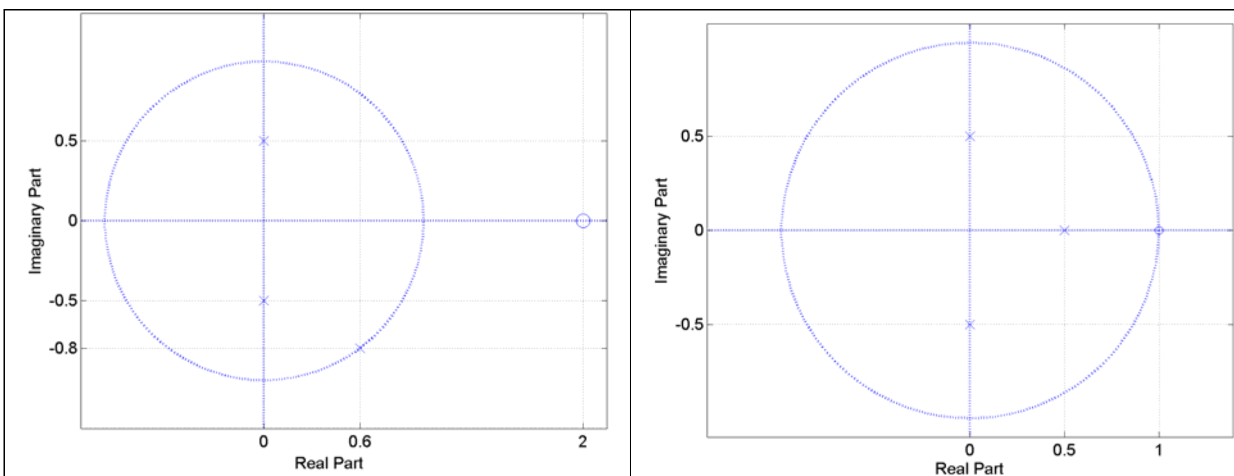
The pole-zero plot is shown in Fig. S13.21(ii). All three poles are inside the unit circle. Therefore, the system is absolutely BIBO stable and causal at the same time.

(iii) $H(z) = \dfrac{z-0.2}{(z+0.1)(z^2+4)}$

One zero at $z = 0.2$ and three poles at $z = -0.1$, $j2$, $-j2$. The pole-zero plot can be sketched using the following MATLAB code.

```
>>    Z = [0.2];
>>    P = [-0.1; j*2; -j*2];
>>    zplane(Z,P);
>>    print -dtiff plot.tiff;
```

The pole-zero plot is shown in Fig. S13.21(iii). Two poles are outside the unit circle. Therefore, the system cannot be stable and causal at the same time.



(i) (ii)

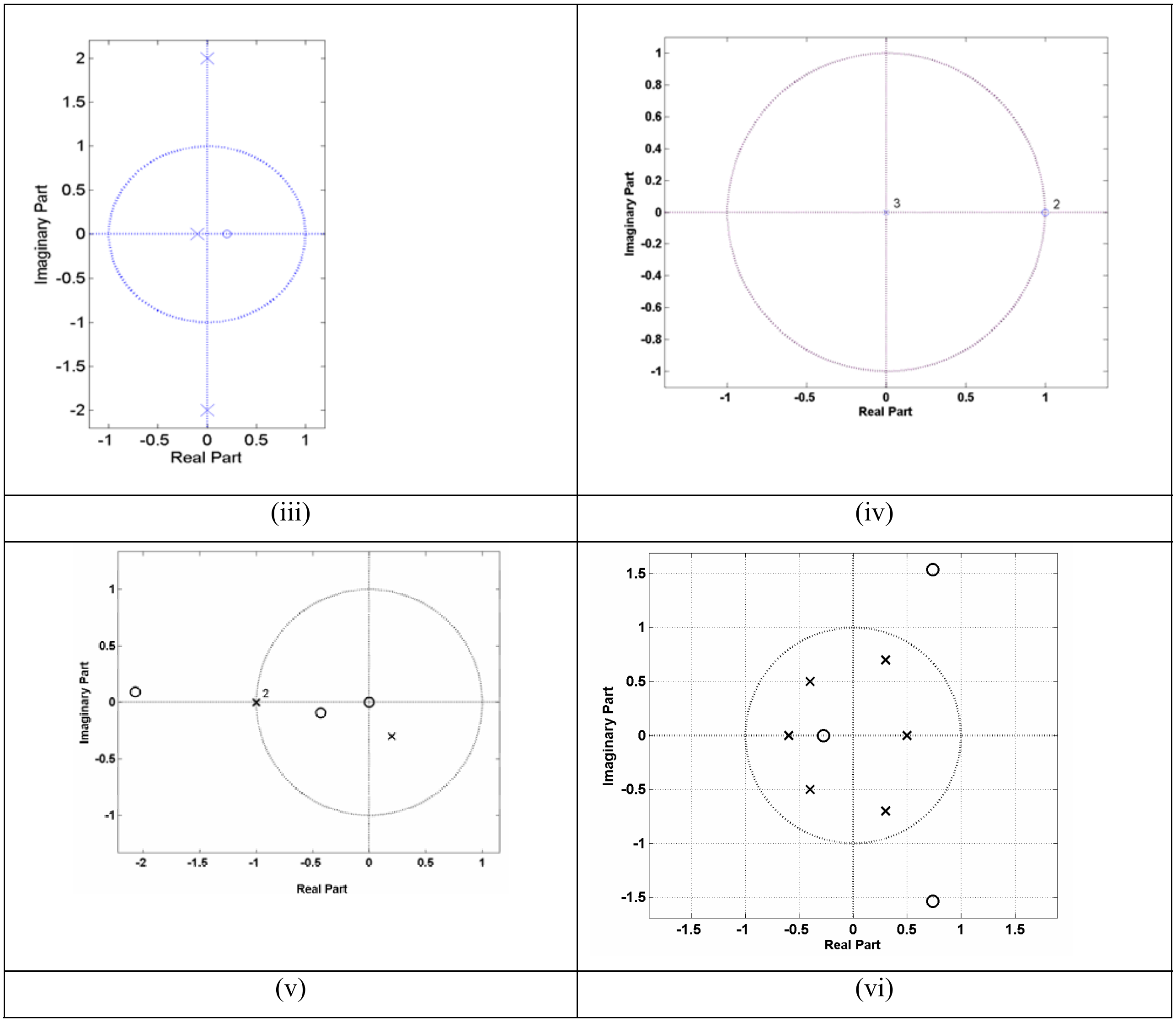


Figure S13.21: Pole-zero plots of the transfer functions in Problem 13.21.

(iv) $H(z) = z^{-1} - 2z^{-2} + z^{-3} = \dfrac{z^2 - 2z + 1}{z^3}$

```
% MATLAB Prog.
Z = [1; 1];
P = [0; 0; 0];
zplane(Z,P);
print -dtiff plot.tiff;
```

The pole-zero plot is shown in Fig. S13.21(iv). All poles are inside the unit circle. Therefore, the system is stable.

(v) $H(z) = \dfrac{(z^2 + 2.5z + 0.9 + j0.15)z}{z^3 + (1.8 + j0.3)z^2 + (0.6 + j0.6)z - 0.2 + j0.3}$

```
Z = roots([1, 2.5, 0.9+j*0.15, 0])
P = roots([1, 1.8+j*0.3, 0.6+j*0.6, -0.2+j*0.3])
zplane(Z,P);
print -dtiff plot.tiff;
```

The pole-zero plot is shown in Fig. S13.21(v). There is a double pole on the unit circle. Therefore, the system is **unstable**. Note that if it was a single pole on the unit circle, the system would have been marginally stable.

(vi) $H(z)=\dfrac{z^3-1.2z^2+2.5z+0.8}{z^6+0.3z^5+0.23z^4+0.209z^3+0.1066z^2-0.04162z-0.07134}$

```
Z = roots([1, -1.2, 2.5, 0.8])
P = roots([1, 0.3,0.23,0.209,0.1066,-0.04162,-0.07134])
zplane(Z,P);
print -dtiff plot.tiff;
```

The pole-zero plot is shown in Fig. S13.21(vi). All six poles are inside the unit circle. Therefore, the system is **stable**. ▌

Problem 13.22

(i) $\dfrac{H(z)}{z}=\dfrac{z-2}{z(z-0.6+j0.8)(z^2+0.25)}=\dfrac{z-2}{z(z-0.6+j0.8)(z-j0.5)(z+j0.5)}$

The partial fraction expansion obtained by

```
>>    num = [0; 0; 1; -2]
>>    denum = poly([0;  0.6-j*0.8; j*0.5; -j*0.5])
>>    [R,P,K] = residue(num, denum)
```

which gives the following values for R, P, and K

```
R=[1.6715 - 0.1561i  -1.8049 - 2.2439i  -4.6667 - 4.0000i   4.8000 + 6.4000i

P=[0.6000 - 0.8000i   0.0000 + 0.5000i  -0.0000 - 0.5000i   0]
K = []
```

The partial fraction expansion is, therefore, given by

$$H(z)=z\left[\frac{1.6715-j0.1561}{(z-0.6+j0.8)}+\frac{-1.8049-j2.2439}{(z-j0.5)}+\frac{-4.6667-j4}{(z+j0.5)}+\frac{4.8-j6.4}{z}\right].$$

Calculating the inverse Laplace transform yields

$$h[k]=(4.8-j6.4)\delta[k]+\begin{bmatrix}(1.6715-j0.1561)(0.6-j0.8)^k-(1.8049+j2.2439)(j0.5)^k\\ -(4.6667+j4)(-j0.5)^k\end{bmatrix}u[k].$$

(ii) $\dfrac{H(z)}{z}=\dfrac{(z-2)(z-1)}{z(z^2-2.5z+1)(z^2+0.25)}=\dfrac{(z-2)(z-1)}{z(z-2)(z-0.5)(z^2+0.25)}=\dfrac{z-1}{z(z-0.5)(z^2+0.25)}$

The partial fraction expansion obtained by

```
>>    num = [0; 0; 0; 1; -1]
>>    denum = poly([0; 0.5; j*0.5; -j*0.5])
>>    [R,P,K] = residue(num, denum)
```

which gives the following values for R, P, and K

```
R = [-2.0000   -3.0000 - 1.0000i  -3.0000 + 1.0000i   8.0000]
P = [ 0.5000; 0.0000 + 0.5000i; 0.0000 - 0.5000i   0]
K = []
```

The partial fraction expansion is, therefore, given by

$$H(z) = z\left[-\frac{2}{(z-0.5)} - \frac{3+j}{(z-j0.5)} - \frac{3-j}{(z+j0.5)} + \frac{8}{z}\right].$$

Calculating the inverse Laplace transform yields

$$h[k] = 8\delta[k] + \left[-2\times 0.5^k - (3+j)(j0.5)^k - (3-j)(-j0.5)^k\right]u[k].$$

(iii) $\dfrac{H(z)}{z} = \dfrac{z-0.2}{z(z+0.1)(z^2+4)}$

The partial fraction expansion obtained by

```
>>    num = [0; 0; 0; 1; -0.2]
>>    denum = poly([0; -0.1; j*2; -j*2])
>>    [R,P,K] = residue(num, denum)
```

which gives the following values for R, P, and K

```
R = [-0.1241 - 0.0187i   -0.1241 + 0.0187i   0.7481   -0.5000]
P = [0.0000 + 2.0000i; 0.0000 - 2.0000i;      -0.1000;     0]
K = []
```

The partial fraction expansion is, therefore, given by

$$H(z) = z\left[-\frac{0.1241+j0.0187}{z-j2} - \frac{0.1241-j0.0187}{z+j2} + \frac{0.7481}{z+0.1} - \frac{0.5}{z}\right].$$

Calculating the inverse Laplace transform yields

$$h[k] = -0.5\delta[k] + \left[-(0.1241+j0.0187)\times(j2)^k - (0.1241-j0.0187)(-j2)^k + 0.7481(-0.1)^k\right]u[k].$$

(iv) $H(z) = z^{-1} - 2z^{-2} + z^{-3} = \dfrac{z^2-2z+1}{z^3}$

The partial fraction calculation is unnecessary here. The inverse Laplace transform can be directly calculated, which yields

$$h[k] = \delta[k-1] - 2\delta[k-2] + \delta[k-3].$$

(v) $\dfrac{H(z)}{z} = \dfrac{z^2+2.5z+0.9+j0.15}{z^3+(1.8+j0.3)z^2+(0.6+j0.6)z-0.2+j0.3}$

The partial fraction expansion obtained by

```
>>    num = [0; 1; 2.5; 0.9+j*0.15]
>>    denum = [1; 1.8+j*0.3; 0.6+j*0.6; -0.2+j*0.3]
>>    [R,P,K] = residue(num, denum)
```

which gives the following values for R, P, and K

```
R = [0.0000 + 0.0000i   0.5000     1.0000 - 0.0000i]
P = [-1.0000 - 0.0000i  -1.0000 - 0.0000i   0.2000 - 0.3000i]
K = []
```

The partial fraction expansion is, therefore, given by

$$H(z) = z\left[\frac{0}{z+1} + \frac{0.5}{(z+1)^2} + \frac{1}{z-0.2+j0.3}\right] = z\left[\frac{0.5}{(z+1)^2} + \frac{1}{z-0.2+j0.3}\right].$$

Calculating the inverse Laplace transform yields

$$h[k] = \left[-0.5k(-1)^k + (0.2-j0.3)^k\right]u[k].$$

(vi) $\dfrac{H(z)}{z} = \dfrac{z^3 - 1.2z^2 + 2.5z + 0.8}{z(z^6 + 0.3z^5 + 0.23z^4 + 0.209z^3 + 0.1066z^2 - 0.04162z - 0.07134)}$

The partial fraction expansion obtained by

```
>>    num = [0; 0; 0; 0; 1; -1.2; 2.5; 0.8]
>>    denum = [1; 0.3; 0.23; 0.209; 0.1066; -0.04162; -0.07134; 0]
>>    [R,P,K] = residue(num, denum)
```

which gives the following values for R, P, and K

```
R = [2.0754 + 0.6638i   2.0754 - 0.6638i   3.2063 + 4.0354i   3.2063 -
4.0354i  -5.4176   6.0682   -11.2139]

P = [0.3000 + 0.7000i   0.3000 - 0.7000i  -0.4000 + 0.5000i  -0.4000 -
0.5000i  -0.6000   0.5000  0]

K = []
```

The partial fraction expansion is, therefore, given by

$$H(z) = z\left[\begin{array}{l}\dfrac{2.0754+j0.6638}{z-0.3-j0.7} + \dfrac{2.0754-j0.6638}{z-0.3+j0.7} + \dfrac{3.2063+j4.0354}{z+0.4-j0.5} + \dfrac{3.2063-j4.0354}{z+0.4+j0.5} \\ -\dfrac{5.4176}{z+0.6} + \dfrac{6.0682}{z-0.5} - \dfrac{11.2139}{z}\end{array}\right].$$

Calculating the inverse Laplace transform yields

$$h[k] = -11.2139\delta[k] + \left[\begin{array}{l}(2.0754+j0.6638)(0.3+j0.7)^k + (2.0754-j0.6638)(0.3-j0.7)^k \\ +(3.2063+j4.0354)(-0.4+j0.5)^k + (3.2063-j4.0354)(-0.4-j0.5)^k \\ -5.4176(-0.6)^k + 6.0682(0.5)^k\end{array}\right]u[k].$$

Problem 13.23

The transfer functions are first expressed as a ratio of two polynomials of z^{-1} as follows.

$$X(z) = \frac{b_m + b_{m-1}z^{-1} + \cdots + b_1 z^{-m+1} + b_0 z^{-m}}{1 + a_{n-1}z^{-1} + \cdots + a_1 z^{-n+1} + a_0 z^{-n}}.$$

(i) $X_1(z) = \dfrac{z}{z^2 - 0.9z + 0.2} = \dfrac{z^{-1}}{1 - 0.9z^{-1} + 0.2z^{-2}}$

(ii) $X_2(z) = \dfrac{z}{z^2 - 2.1z + 0.2} = \dfrac{z^{-1}}{1 - 2.1z^{-1} + 0.2z^{-2}}$

(iii) $X_3(z) = \dfrac{z^2 + 2}{(z-0.3)(z+0.4)(z-0.7)} = \dfrac{z^2 + 2}{z^3 - 0.6z^2 - 0.19z + 0.084} = \dfrac{z^{-1} + 2z^{-3}}{1 - 0.6z^{-1} - 0.19z^{-2} + 0.084z^{-3}}$

(iv) $X_4(z) = \dfrac{z^2 + 2}{(z-0.3)(z+0.4)^2} = \dfrac{z^2 + 2}{z^3 + 0.5z^2 - 0.08z - 0.048} = \dfrac{z^{-1} + 2z^{-3}}{1 + 0.5z^{-1} - 0.08z^{-2} - 0.048z^{-3}}$

(v) $X_5(z) = \dfrac{4z^{-1}}{1 - 5z^{-1} + 6z^{-2}}$

(vi) $X_6(z) = \dfrac{4z^{-2}}{10 - 6(z^1 + z^{-1})} = \dfrac{4z^{-2}}{-(6z - 10 + 6z^{-1})} = \dfrac{-4z^{-3}}{6 - 10z^{-1} + 6z^{-2}}$

(vii) $X_7(z) = \dfrac{2z^{-2}}{\left(1 - 4z^{-1}\right)^2 \left(1 - 2z^{-1}\right)} = \dfrac{2z^{-2}}{1 - 10z^{-1} + 32z^{-2} - 32z^{-3}}$

The MATLAB code is shown in Program 13.23. The impulse responses are plotted in Fig. S13.23.

Program 13.24: MATLAB Program

```
% MATLAB code for Problem 13.24
clf          % clear figure
k=0:9 ;
sys1 = filt([0  1],[1  -0.9  0.2])
sys2 = filt([0  1],[1  -2.1  0.2])
sys3 = filt([0  1  0  2],[1  -0.6  -0.19  0.084])
sys4 = filt([0  1  0  2],[1  0.5  -0.08  -0.048])
sys5 = filt([0  4],[1  -5  6])
sys6 = filt([0  0  0  -4],[6  -10  6])
sys7 = filt([0  0  2],[1  -10  32  -32])
h1 = impulse(sys1,10)
h2 = impulse(sys2,10)
h3 = impulse(sys3,10)
h4 = impulse(sys4,10)
h5 = impulse(sys5,10)
h6 = impulse(sys6,10)
h7 = impulse(sys7,10)
%
% signal defined in part (i)
subplot(4,2,1), stem(k, h1, 'filled'), grid on
xlabel('k')                         % Label of X-axis
ylabel('h1[k]')                     % Label of Y-axis
axis([0  10  0  1.2])

% signal defined in part (ii)
subplot(4,2,2), stem(k, h2, 'filled'), grid on
xlabel('k')                         % Label of X-axis
ylabel('h2[k]')                     % Label of Y-axis
axis([0  10  0  300])
```

```
% signal defined in part (iii)
subplot(4,2,3), stem(k, h3, 'filled'), grid on
xlabel('k')                          % Label of X-axis
ylabel('h3[k]')                      % Label of Y-axis
axis([0  10  0  3])

% signal defined in part (iv)
subplot(4,2,4), stem(k, h4, 'filled'), grid on
xlabel('k')                                % Label of X-axis
ylabel('h4[k]')                            % Label of Y-axis
axis([0  10  -2  3])

% signal defined in part (v)
subplot(4,2,5), stem(k, h5, 'filled'), grid on
xlabel('k')                                % Label of X-axis
ylabel('h5[k]')                      % Label of Y-axis

% signal defined in part (vi)
subplot(4,2,6), stem(k, h6, 'filled'), grid
xlabel('k')                                % Label of X-axis
ylabel('h6[k]')                            % Label of Y-axis
axis([0  10  -1.5  1.5])

% signal defined in part (vii)
subplot(4,2,7), stem(k, h7, 'filled'), grid
xlabel('k')                                % Label of X-axis
ylabel('h7[k]')                            % Label of Y-axis
%axis([0  10  -1.5  1.5])
print -dtiff plot.tiff
```

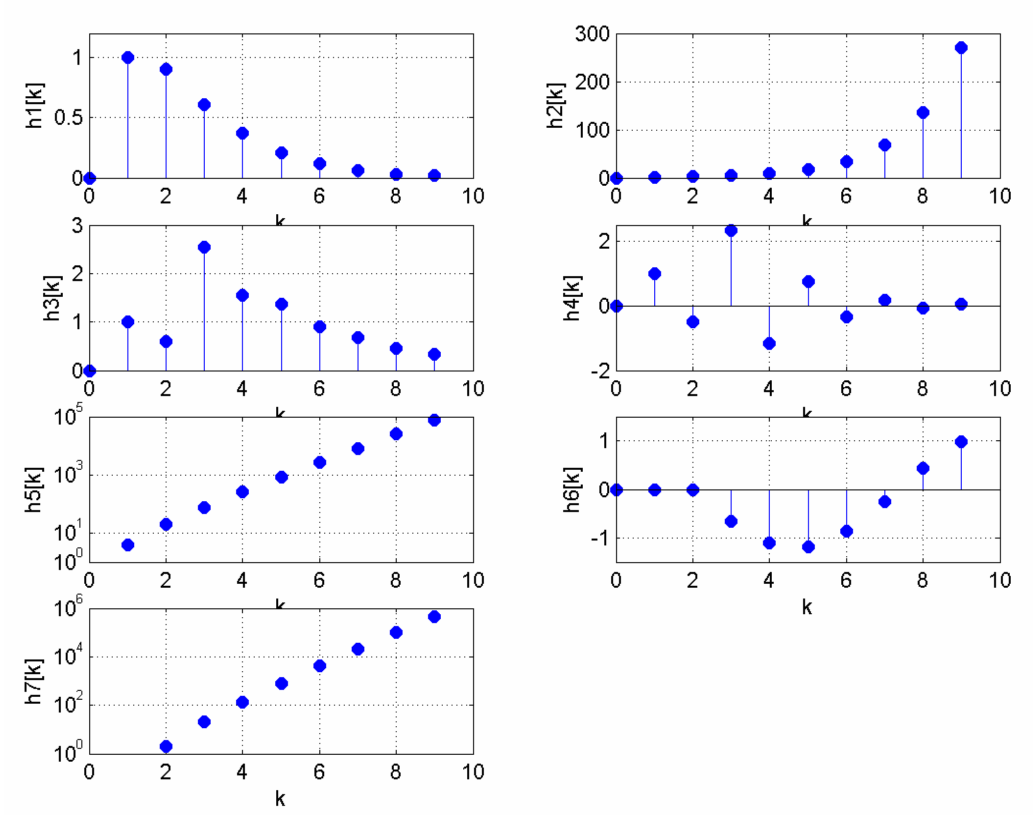


Figure S13.23: Impulse responses, obtained using MATLAB, in Problem 13.23.