Chapter 6: Laplace Transform

Problem 6.1

(a) $$X(s)=\int_{-\infty}^{\infty}x(t)e^{-st}dt=\int_{-\infty}^{\infty}e^{-5t}u(t)e^{-st}dt+\int_{-\infty}^{\infty}e^{4t}u(-t)e^{-st}dt=\underbrace{\int_{0}^{\infty}e^{-(s+5)t}dt}_{I}+\underbrace{\int_{-\infty}^{0}e^{(4-s)t}dt}_{II}.$$

Integral I reduces to

$$I=\int_{0}^{\infty}e^{-(s+5)t}dt=\left.\frac{e^{-(s+5)t}}{-(s+5)}\right|_{0}^{\infty}=\frac{-1}{(s+5)}\left[0-1\right]=\frac{1}{s+5}\quad \text{provided } \mathrm{Re}\{(s+5)\}>0\Rightarrow ROC\ R_1:\mathrm{Re}\{s\}>-5,$$

while integral II reduces to

$$II=\int_{-\infty}^{0}e^{(4-s)t}dt=\left.\frac{e^{(4-s)t}}{(4-s)}\right|_{-\infty}^{0}=\frac{1}{(4-s)}\left[1-0\right]=\frac{-1}{s-4}\quad \text{provided } \mathrm{Re}\{(4-s)\}>0\Rightarrow ROC\ R_1:\mathrm{Re}\{s\}<4\cdot$$

The Laplace transform is therefore given by

$$X(s)=I+II=\frac{1}{s+5}-\frac{1}{s-4}=\frac{-9}{(s+5)(s-4)}\quad with\ ROC:R=R_1\cap R_2\ or\ R:(-5<\mathrm{Re}\{s\}<4)\cdot$$

(b) $$X(s)=\int_{-\infty}^{\infty}x(t)e^{-st}dt=\int_{-\infty}^{\infty}e^{-3|t|}e^{-st}dt=\int_{-\infty}^{0}e^{3t}e^{-st}dt+\int_{0}^{\infty}e^{-3t}e^{-st}dt=\underbrace{\int_{-\infty}^{0}e^{(3-s)t}dt}_{I}+\underbrace{\int_{0}^{\infty}e^{-(s+3)t}dt}_{II}.$$

Integral I reduces to

$$I=\int_{-\infty}^{0}e^{(3-s)t}dt=\left.\frac{e^{(3-s)t}}{(3-s)}\right|_{-\infty}^{0}=\frac{1}{(3-s)}\left[1-0\right]=\frac{-1}{s-3}\quad provided\ \mathrm{Re}\{(3-s)\}>0\Rightarrow ROC\ R_1:\mathrm{Re}\{s\}<3$$

,

while integral II reduces to

$$II=\int_{0}^{\infty}e^{-(s+3)t}dt=\left.\frac{e^{-(s+3)t}}{-(s+3)}\right|_{0}^{\infty}=\frac{-1}{(s+3)}\left[0-1\right]=\frac{1}{s+3}\quad provided\ \mathrm{Re}\{(s+3)\}>0\Rightarrow ROC\ R_1:\mathrm{Re}\{s\}>-3$$

The Laplace transform is therefore given by

$$X(s)=I+II=\frac{1}{s+3}-\frac{1}{s-3}=\frac{-6}{s^2-9}\quad with\ ROC:R=R_1\cap R_2\ or\ R:(-3<\mathrm{Re}\{s\}<3)\cdot$$

(c) $X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt = \int_{-\infty}^{\infty} t^2\cos(10t)u(-t)e^{-st}dt = \frac{1}{2j}\int_{-\infty}^{0} t^2\left(e^{j10t} - e^{-j10t}\right)e^{-st}dt$

$$= \frac{1}{2j}\int_{-\infty}^{0} t^2 e^{-(s-j10)t}dt - \frac{1}{2j}\int_{-\infty}^{0} t^2 e^{-(s+j10)t}dt$$

$$= \frac{0.5j}{(s-j10)^3}\left[e^{-(s-j10)t}\left((s-j10)^2t^2 + 2(s-j10)t + 2\right)\right]_{-\infty}^{0}$$

$$- \frac{0.5j}{(s+j10)^3}\left[e^{-(s+j10)t}\left((s+j10)^2t^2 + 2(s+j10)t + 2\right)\right]_{-\infty}^{0} \qquad s \neq \pm j10$$

$$= \frac{0.5j}{(s-j10)^3}[2-0] - \frac{0.5j}{(s+j10)^3}[2-0] \qquad \mathrm{Re}\ \{s \pm j10\} < 0$$

$$= j\left[\frac{1}{(s-j10)^3} - \frac{1}{(s+j10)^3}\right] \qquad \text{ROC: Re}\{s\} < 0$$

$$= j\frac{(s+j10)^3 - (s-j10)^3}{(s-j10)^3(s+j10)^3} = j\frac{6s^2j10 - j2000}{(s^2+100)^3} = \frac{-60s^2+2000}{(s^2+100)^3} \qquad \text{ROC: Re}\{s\} < 0$$

(d) $X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt = \int_{-\infty}^{\infty} e^{-3|t|}\cos(5t)e^{-st}dt == \underbrace{\int_{-\infty}^{0} e^{(3-s)t}\cos(5t)dt}_{I} + \underbrace{\int_{0}^{\infty} e^{-(s+3)t}\cos(5t)dt}_{II}$.

Integral I reduces to

$$I = \int_{-\infty}^{0} e^{(3-s)t}\cos(5t)dt = \frac{1}{(3-s)^2+5^2}\left[(3-s)e^{(3-s)t}\cos(5t) + 5e^{(3-s)t}\sin(bt)\right]\Bigg|_{-\infty}^{0}$$

$$= \frac{1}{(3-s)^2+5^2}\left[(3-s+0)-(0+0)\right] = \frac{-(s-3)}{(s-3)^2+5^2} \quad \textit{provided} \quad \mathrm{Re}\{(3-s)\} > 0 \Rightarrow ROC\ R_1 : \mathrm{Re}\{s\} < 3$$

while integral II reduces to

$$II = \int_{0}^{\infty} e^{-(s+3)t}\cos(5t)dt = \frac{1}{(s+3)^2+5^2}\left[-(s+3)e^{-(s+3)t}\cos(5t) + 5e^{-(s+3)t}\sin(5t)\right]\Bigg|_{0}^{\infty}$$

$$= \frac{1}{(s+3)^2+5^2}\left[(0+0)-(-(s+3)+0)\right] = \frac{(s+3)}{(s+3)^2+5^2} \quad \textit{provided}\ \mathrm{Re}\{(s+3)\} > 0 \Rightarrow ROC\ R_1 : \mathrm{Re}\{s\} > -3$$

The Laplace transform is therefore given by

$$X(s) = I + II = \frac{s+3}{(s+3)^2+5^2} - \frac{s-3}{(s-3)^2+5^2} \quad \textit{with ROC} : R = R_1 \cap R_2\ \textit{or}\ R : (-3 < \mathrm{Re}\{s\} < 3).$$

(e) $$X(s)=\int_{-\infty}^{\infty} x(t)e^{-st}dt=\int_{-\infty}^{\infty} e^{7t}\cos(9t)u(t)e^{-st}dt=\int_{0}^{\infty} e^{-(s-7)t}\cos(9t)dt$$

$$=\frac{1}{(s-7)^2+9^2}\left[-(s-7)e^{-(s-7)t}\cos(9t)+9e^{-(s-7)t}\sin(9t)\right]\Big|_0^{\infty} \qquad \text{provided } \operatorname{Re}\{(s-7)\}>0$$

$$=\frac{1}{(s-7)^2+9^2}\left[(0+0)-(-(s-7)+0)\right]$$

$$=\frac{(s-7)}{(s-7)^2+9^2} \qquad ROC\ R:\quad \operatorname{Re}\{s\}>7$$

(e) $$X(s)=\int_{-\infty}^{\infty} x(t)e^{-st}dt=\int_{-\infty}^{\infty} e^{7t}\cos(9t)u(-t)e^{-st}dt=\int_{-\infty}^{0} e^{-(s-7)t}\cos(9t)dt$$

$$=\frac{1}{(s-7)^2+9^2}\left[-(s-7)e^{-(s-7)t}\cos(9t)+9e^{-(s-7)t}\sin(9t)\right]\Big|_{-\infty}^{0} \qquad \text{provided } \operatorname{Re}\{(s-7)\}<0$$

$$=\frac{1}{(s-7)^2+9^2}\left[-(s-7)+0-(0+0)\right]$$

$$=\frac{-(s-7)}{(s-7)^2+9^2} \qquad ROC\ R:\quad \operatorname{Re}\{s\}<7$$

(f) $$X(s)\big|_{s=0}=\int_{-\infty}^{\infty} x(t)dt=2\int_{0}^{1}(1-t)dt=2\left[t-0.5t^2\right]_0^1=1$$

$$X(s)\big|_{s\neq 0}=\int_{-\infty}^{\infty} x(t)e^{-st}dt=\underbrace{\int_{-1}^{0}(1+t)e^{-st}dt}_{I}+\underbrace{\int_{0}^{1}(1-t)e^{-st}dt}_{II}$$

where

$$I=\int_{-1}^{0}(1+t)e^{-st}dt=\int_{-1}^{0}e^{-st}dt+\int_{-1}^{0}te^{-st}dt=\tfrac{-1}{s}\left[e^{-st}\right]_{-1}^{0}+\tfrac{1}{s^2}\left[e^{-st}(-st-1)\right]_{-1}^{0}$$

$$=\tfrac{1}{s}\left(e^{s}-1\right)+\tfrac{1}{s^2}\left[-1-e^{s}(s-1)\right]=\tfrac{1}{s^2}\left[se^{s}-s-1-e^{s}(s-1)\right]=\tfrac{1}{s^2}\left(e^{s}-s-1\right)$$

and

$$II=\int_{0}^{1}(1-t)e^{-st}dt=\int_{0}^{1}e^{-st}dt-\int_{0}^{1}te^{-st}dt=\tfrac{-1}{s}\left[e^{-st}\right]_{0}^{1}-\tfrac{1}{s^2}\left[e^{-st}(-st-1)\right]_{0}^{1}$$

$$=\tfrac{1}{s}\left(1-e^{-s}\right)-\tfrac{1}{s^2}\left[e^{-s}(-s-1)+1\right]=\tfrac{1}{s^2}\left[s-se^{-s}+e^{-s}(s+1)-1\right]=\tfrac{1}{s^2}\left(e^{-s}+s-1\right)$$

The Laplace transform is therefore given by

$$X(s)=\begin{cases}1 & s=0\\ \tfrac{1}{s^2}\left(e^{s}+e^{-s}-2\right) & s\neq 0\end{cases} \qquad ROC:\text{Entire s-plane}$$

▌

Problem 6.2

(a) $X(s)=\int_0^\infty t^5 e^{-st}dt=\left[t^5\frac{e^{-st}}{(-s)}-5t^4\frac{e^{-st}}{(-s)^2}+20t^3\frac{e^{-st}}{(-s)^3}-60t^2\frac{e^{-st}}{(-s)^4}+120t\frac{e^{-st}}{(-s)^5}-120\frac{e^{-st}}{(-s)^6}\right]_0^\infty$.

Applying the limits, we obtain

$X(s)=\frac{120}{s^6}=\frac{5!}{s^6}$ with ROC: $\text{Re}\{s\}>0$.

(b) $X(s)=\int_{0^-}^\infty \sin(6t)u(t)e^{-st}dt=\int_0^\infty \sin(6t)e^{-st}dt=\frac{1}{2j}\int_0^\infty\left(e^{j6t}-e^{-j6t}\right)e^{-st}dt$

$$=\frac{1}{2j}\int_0^\infty e^{-(s-j6)t}dt-\frac{1}{2j}\int_0^\infty e^{-(s+j6)t}dt$$

$$=\frac{0.5j}{s-j6}\left[e^{-(s-j6)t}\right]_0^\infty-\frac{0.5j}{s+j6}\left[e^{-(s+j6)t}\right]_0^\infty \qquad s\neq\pm j\omega_0$$

$$=-\frac{0.5j}{s-j6}+\frac{0.5j}{s+j6} \qquad \text{Re }\{s\pm j6\}>0$$

$$=0.5j\left[\frac{1}{s+j6}-\frac{1}{s-j6}\right]=0.5j\times\frac{-j12}{s^2+36}$$

$$=\frac{6}{s^2+36} \qquad \text{ROC: Re}\{s\}>0$$

c) $X(s)=\int_0^\infty \cos^2(6t)e^{-st}dt=\frac{1}{2}\int_0^\infty e^{-st}dt+\frac{1}{2}\int_0^\infty e^{-st}\cos(12t)dt=\frac{1}{2s}+\left[\frac{1}{2}\left(\frac{-s\cos(12t)+12\sin(12t)}{s^2+12^2}\right)e^{-st}\right]_0^\infty$

which reduces to

$X(s)=\frac{1}{2s}+\frac{1}{2}\times\frac{s}{s^2+12^2}=\frac{(s^2+72)}{s(s^2+144)}$ with ROC: $\text{Re}\{s\}>0$.

(d) $X(s)=\int_{-\infty}^\infty x(t)e^{-st}dt=\int_{-\infty}^\infty e^{-3t}\cos(9t)u(t)e^{-st}dt=\int_0^\infty e^{-(s+3)t}\cos(9t)dt$

$$=\frac{1}{(s+3)^2+9^2}\left[-(s+3)e^{-(s+3)t}\cos(9t)+9e^{-(s+3)t}\sin(9t)\right]\Big|_0^\infty \qquad \text{provided Re}\{(s+3\}<0$$

$$=\frac{1}{(s+3)^2+9^2}\left[(0+0)-(-(s+3)+0)\right]$$

$$=\frac{s+3}{(s+3)^2+9^2} \qquad ROC\ R:\quad \text{Re}\{s\}<-3$$

(e) $X(s)=\int_0^\infty t^2\cos(10t)e^{-st}dt=\frac{1}{2}\underbrace{\int_0^\infty t^2e^{-(s-j10)}dt}_{I}+\frac{1}{2}\underbrace{\int_0^\infty t^2e^{-(s+j10)}dt}_{II}$.

Integral I reduces to

$$I=\left[t^2\frac{e^{-(s-j10)t}}{-(s-j10)}-2t\frac{e^{-(s-j10)t}}{(s-j10)^2}+2\frac{e^{-(s-j10)t}}{-(s-j10)^3}\right]_0^{\infty}=\frac{2}{(s-j10)^3}\ \text{with ROC: Re\{s\}} > 0.$$

Integral II reduces to

$$II=\left[t^2\frac{e^{-(s+j10)t}}{-(s+j10)}-2t\frac{e^{-(s+j10)t}}{(s+j10)^2}+2\frac{e^{-(s+j10)t}}{-(s+j10)^3}\right]_0^{\infty}=\frac{2}{(s+j10)^3}\ \text{with ROC: Re\{s\}} > 0.$$

The Laplace transform is, therefore, given by

$$X(s)=\frac{1}{(s-j10)^3}+\frac{1}{(s+j10)^3}\ \text{with ROC: Re\{s\}} > 0.$$

(f) $X(s)\big|_{s=0}=\int_{0^-}^{\infty}x(t)dt=\int_0^1(1-t)dt=\left[t-0.5t^2\right]_0^1=\frac{1}{2}$

$$X(s)\big|_{s\neq 0}=\int_{0^-}^{\infty}x(t)e^{-st}dt=\int_0^1(1-t)e^{-st}dt=\int_0^1 e^{-st}dt-\int_0^1 te^{-st}dt$$

$$=\tfrac{-1}{s}\left[e^{-st}\right]_0^1-\tfrac{1}{s^2}\left[e^{-st}(-st-1)\right]_0^1=\tfrac{1}{s}\left(1-e^{-s}\right)-\tfrac{1}{s^2}\left[e^{-s}(-s-1)+1\right]$$

$$=\tfrac{1}{s^2}\left[s-se^{-s}+e^{-s}(s+1)-1\right]=\tfrac{1}{s^2}\left(e^{-s}+s-1\right)$$

Therefore, $X(s)=\begin{cases}\frac{1}{2} & s=0\\ \frac{1}{s^2}\left(e^{-s}+s-1\right) & s\neq 0\end{cases}$ $\qquad ROC:\text{Entire s-plane}$ ▮

Problem 6.3

(a) Using partial fraction expansion and associating the ROC to individual terms, yields

$$X(s)=\frac{s^2+2s+1}{(s+1)(s^2+5s+6)}=\frac{(s+1)^2}{(s+1)(s+2)(s+3)}=\frac{s+1}{(s+2)(s+3)}=\underbrace{\frac{A}{s+2}}_{\text{ROC:Re}\{s\}>-2}+\underbrace{\frac{B}{s+3}}_{\text{ROC:Re}\{s\}>-3}$$

where $A=\left[\frac{s+1}{s+3}\right]_{s=-2}=-1,\quad B=\left[\frac{s+1}{s+2}\right]_{s=-3}=2$.

Calculating the inverse transform of $X(s)$, yields

$$x(t)=-e^{-2t}u(t)+2e^{-3t}u(t)=\left(2e^{-3t}-e^{-2t}\right)u(t).$$

(b) Using partial fraction expansion and associating the ROC to individual terms, yields

$$X(s)=\frac{s^2+2s+1}{(s+1)(s^2+5s+6)}=\frac{s+1}{(s+2)(s+3)}=\underbrace{\frac{A}{s+2}}_{\text{ROC:Re}\{s\}<-2}+\underbrace{\frac{B}{s+3}}_{\text{ROC:Re}\{s\}<-3}$$

where constants A, and B were computed in part (a) as $A=-1$, and $B=2$.

In other words, $X(s)=\underbrace{\frac{-1}{s+2}}_{\text{ROC:Re}\{s\}<-2}+\underbrace{\frac{2}{s+3}}_{\text{ROC:Re}\{s\}<-3}$

Using the transform pair $-e^{-at}u(-t) \xleftrightarrow{L} \dfrac{1}{(s+a)}$ with ROC: $\text{Re}\{s\} < -a$, the inverse transform of $X(s)$ yields

$$x(t) = e^{-2t}u(-t) - e^{-3t}u(-t) = \left(e^{-2t} - e^{-3t}\right)u(-t).$$

Note that the same rational fraction for $X(s)$ yields different time domain representations if the associated ROC is different.

(c) $X(s) = \frac{s^2+3s-4}{(s+1)(s^2+5s+6)} = \frac{s^2+3s-4}{(s+1)(s+2)(s+3)} = \underbrace{\frac{A}{s+1}}_{\text{ROC:Re}\{s\}>-1} + \underbrace{\frac{B}{s+2}}_{\text{ROC:Re}\{s\}>-2} + \underbrace{\frac{C}{s+3}}_{\text{ROC:Re}\{s\}>-3}$

where $A = \left.\frac{s^2+3s-4}{(s+2)(s+3)}\right|_{s=-1} = -3,\ B = \left.\frac{s^2+3s-4}{(s+1)(s+3)}\right|_{s=-2} = 6,\ C = \left.\frac{s^2+3s-4}{(s+1)(s+3)}\right|_{s=-2} = -2$.

$X(s)$ can be expressed as: $X(s) = \underbrace{\frac{-3}{s+1}}_{\text{ROC:Re}\{s\}>-1} + \underbrace{\frac{6}{s+2}}_{\text{ROC:Re}\{s\}>-2} + \underbrace{\frac{-2}{s+3}}_{\text{ROC:Re}\{s\}>-3}$.

The inverse transform of $X(s)$ yields

$$x(t) = \left(-3e^{-t} + 6e^{-2t} - 2e^{-3t}\right)u(t)$$

(d) $X(s) = \frac{s^2+3s-4}{(s+1)(s^2+5s+6)} = \frac{s^2+3s-4}{(s+1)(s+2)(s+3)} = \underbrace{\frac{A}{s+1}}_{\text{ROC:Re}\{s\}<-1} + \underbrace{\frac{B}{s+2}}_{\text{ROC:Re}\{s\}<-2} + \underbrace{\frac{C}{s+3}}_{\text{ROC:Re}\{s\}<-3}$

where constants A, B, and C were computed in part (a) as A = -3, B = 6, and C =-2.

$X(s)$ can be expressed as: $X(s) = \underbrace{\frac{-3}{s+1}}_{\text{ROC:Re}\{s\}<-1} + \underbrace{\frac{6}{s+2}}_{\text{ROC:Re}\{s\}<-2} + \underbrace{\frac{-2}{s+3}}_{\text{ROC:Re}\{s\}<-3}$.

Using the transform pair $-e^{-at}u(-t) \xleftrightarrow{L} \dfrac{1}{(s+a)}$ with ROC: $\text{Re}\{s\} < -a$, the inverse transform of $X(s)$ yields

$$x(t) = \left(3e^{-t} - 6e^{-2t} + 2e^{-3t}\right)u(-t).$$

(e) Using partial fraction expansion and associating the ROC to individual terms, yields

$$X(s) = \frac{s^2+1}{s(s+1)(s^2+2s+17)} = \underbrace{\frac{A}{s}}_{\text{ROC:Re}\{s\}>0} + \underbrace{\frac{B}{(s+1)}}_{\text{ROC:Re}\{s\}>-1} + \underbrace{\frac{Cs+D}{(s^2+2s+17)}}_{\text{ROC:Re}\{s\}>\text{Re}\{-1\pm j4\}}$$

where

$$A = \left[\frac{s^2+1}{s(s+1)(s^2+2s+17)}s\right]_{s=0} = \left[\frac{s^2+1}{(s+1)(s^2+2s+17)}\right]_{s=0} = \frac{1}{17}$$

and $B = \left[\frac{s^2+1}{s(s+1)(s^2+2s+17)}(s+1)\right]_{s=-1} = \left[\frac{s^2+1}{s(s^2+2s+17)}\right]_{s=-1} = -\frac{1}{8}$.

To evaluate C and D, expand $X(s)$ as

$$s^2+1 = A(s+1)(s^2+2s+17)+Bs(s^2+2s+17)+(Cs+D)s(s+1)$$

and compare the coefficients of s^3 and s^2. We get

$$0 = A+B+C$$
$$1 = 3A+2B+C+D$$

which has a solution C = 9/136 and D = 137/136. The Laplace transform may be expressed as

$$X(s) = \underbrace{\tfrac{1}{17s}}_{\text{ROC:Re}\{s\}>0} - \underbrace{\tfrac{1}{8(s+1)}}_{\text{ROC:Re}\{s\}>-1} + \underbrace{\tfrac{9(s+1)}{136((s+1)^2+4^2)}}_{\text{ROC:Re}\{s\}>-1} + \underbrace{\tfrac{32\times 4}{136((s+1)^2+4^2)}}_{\text{ROC:Re}\{s\}>-1}$$

Taking the inverse transform of $X(s)$ yields

$$x(t) = \tfrac{1}{17}u(t) - \tfrac{1}{8}e^{-t}u(t) + \tfrac{9}{136}e^{-t}\cos(4t)u(t) + \tfrac{4}{17}e^{-t}\sin(4t)u(t)$$
$$= \left(\tfrac{1}{17} - \tfrac{1}{8}e^{-t} + \tfrac{9}{136}e^{-t}\cos(4t) + \tfrac{4}{17}e^{-t}\sin(4t)\right)u(t).$$

(f) Using partial fraction expansion and associating the ROC to individual terms yields

$$X(s) = \tfrac{s+1}{(s+2)^2(s+3)(s+4)} = \underbrace{\tfrac{A}{(s+2)}}_{\text{Re}\{s\}>-2} + \underbrace{\tfrac{B}{(s+2)^2}}_{\text{Re}\{s\}>-2} + \underbrace{\tfrac{C}{(s+3)}}_{\text{Re}\{s\}>-3} + \underbrace{\tfrac{D}{(s+4)}}_{\text{Re}\{s\}>-4}$$

where

$$B = \left[\tfrac{s+1}{(s+2)^2(s+3)(s+4)}(s+2)^2\right]_{s=-1} = \left[\tfrac{s+1}{(s+3)(s+4)}\right]_{s=-2} = -\frac{1}{2}$$

$$C = \left[\tfrac{s+1}{(s+2)^2(s+3)(s+4)}(s+3)\right]_{s=-1} = \left[\tfrac{s+1}{(s+2)^2(s+4)}\right]_{s=-3} = -2$$

and $D = \left[\tfrac{s+1}{(s+2)^2(s+3)(s+4)}(s+4)\right]_{s=-4} = \left[\tfrac{s+1}{(s+2)^2(s+3)}\right]_{s=-4} = \dfrac{3}{4}.$

To evaluate A, expand $X(s)$ as

$$s+1 = A(s+2)(s+3)(s+4)+B(s+3)(s+4)+C(s+2)^2(s+4)+D(s+2)^2(s+3)$$

and compare the coefficients of s^3. We get

$$0 = A+C+D$$

which has a solution A = 5/4.

Taking the inverse transform of $X(s)$ yields

$$x(t) = \tfrac{5}{4}e^{-2t}u(t) - \tfrac{1}{2}te^{-2t}u(t) - 2e^{-3t}u(t) + \tfrac{3}{4}e^{-3t}u(t)$$
$$= \left(\tfrac{5}{4}e^{-2t} - \tfrac{1}{2}te^{-2t} - 2e^{-3t} + \tfrac{3}{4}e^{-3t}\right)u(t).$$

(g) Using partial fraction expansion and associating the ROC to individual terms, gives

$$X(s) = \tfrac{s^2-2s+1}{(s+1)^3(s^2+16)} = \underbrace{\tfrac{A}{(s+1)}}_{\text{ROC:Re}\{s\}<-1} + \underbrace{\tfrac{B}{(s+1)^2}}_{\text{ROC:Re}\{s\}<-1} + \underbrace{\tfrac{C}{(s+1)^3}}_{\text{ROC:Re}\{s\}<-1} + \underbrace{\tfrac{(Ds+E)}{(s^2+16)}}_{\text{ROC:Re}\{s\}<0}$$

where

$$C=\left[\frac{s^2-2s+1}{(s+1)^3(s^2+16)}(s+1)^3\right]_{s=-1}=\left[\frac{s^2-2s+1}{(s^2+16)}\right]_{s=-2}=\frac{4}{17}.$$

To evaluate A, B, and C expand $X(s)$ as

$$s^2-2s+1=A(s+1)^2(s^2+16)+B(s+1)(s^2+16)+C(s^2+16)+(Ds+E)(s+1)^3$$

and compare the coefficients of s^4, s^3, s^2, and s. We get

$$\begin{array}{rl} 0=A+D & (\text{coefficients of } s^4)\\ 0=2A+B+3D+E & (\text{coefficients of } s^3)\\ 1=17A+B+C+3D+3E & (\text{coefficients of } s^2)\\ -2=32A+16B+D+3E & (\text{coefficients of } s)\end{array},\ \text{or,}\ \begin{array}{rl} 0=A+D\\ 0=2A+B+3D+E\\ \frac{13}{17}=17A+B+3D+3E\\ -2=32A+16B+D+3E\end{array}.$$

which has a solution of $A = 0.0206$, $B = -0.2076$, $D = -0.0206$, and $E = 0.2282$.

Taking the inverse transform of $X(s)$ yields

$$x(t)=-0.0206e^{-t}u(-t)+0.2076te^{-t}u(-t)-0.1176t^2e^{-t}u(-t)+0.0206\cos(4t)u(-t)-0.057\sin(4t)u(-t)$$
$$=\left[-0.0206e^{-t}+0.2076te^{-t}-0.1176t^2e^{-t}+0.0206\cos(4t)-0.057\sin(4t)\right]u(-t).$$

Problem 6.4

The Laplace transform of the combined signal $x_1(t)+2x_2(t)$ is given by

$$x_1(t)+2x_2(t)\xleftrightarrow{L}\frac{s}{s^2+5s+6}+\frac{2}{s^2+5s+6}=\frac{s+2}{s^2+5s+6}=\frac{s+2}{(s+2)(s+3)}.$$

The ROC of $L\{x_1(t)+2x_2(t)\}$ includes the region ($R_1 \cap R_2$), or, $\text{Re}\{s\} > -2$. However, simplifying the expression of $L\{x_1(t)+2x_2(t)\}$, we obtain

$$x_1(t)+2x_2(t)\xleftrightarrow{L}\frac{1}{s+3}.$$

Since the pole at $s = -2$ cancels out, the overall ROC is greater than the intersection of the two individual ROC's and is given by R: $\text{Re}\{s\} > -3$.

Problem 6.5

Using partial fraction expansion, the Laplace transform is given by

$$X(s)=-\frac{0.025}{(s+1)}+\frac{0.025}{(s-1)}+\frac{-0.375s+0.125}{(s^2-4s+5)}+\frac{0.375s+0.125}{(s^2+4s+5)}.$$

(a) For ROC R: $\text{Re}\{s\} < -2$, the ROC's associated with individual terms are given by

$$X(s)=-\underbrace{\frac{0.025}{(s+1)}}_{\text{Re}\{s\}<-1}+\underbrace{\frac{0.025}{(s-1)}}_{\text{Re}\{s\}<1}+\underbrace{\frac{-0.375s+0.125}{(s^2-4s+5)}}_{\text{Re}\{s\}<2}+\underbrace{\frac{0.375s+0.125}{(s^2+4s+5)}}_{\text{Re}\{s\}<-2}.$$

Taking the inverse Laplace transform, the time domain representation is obtained as

$$x(t)=0.025e^{-t}u(-t)-0.025e^{t}u(-t)+0.375e^{2t}\cos tu(-t)-0.125e^{2t}\sin tu(-t)$$
$$-0.375e^{-2t}\cos tu(-t)-0.125e^{-2t}\sin tu(-t).$$

(b) For ROC R: $-2 < \mathrm{Re}\{s\} < -1$, the ROC's associated with individual terms are given by

$$X(s) = -\underbrace{\frac{0.025}{(s+1)}}_{\mathrm{Re}\{s\}<-1} + \underbrace{\frac{0.025}{(s-1)}}_{\mathrm{Re}\{s\}<1} + \underbrace{\frac{-0.375s+0.125}{(s^2-4s+5)}}_{\mathrm{Re}\{s\}<2} + \underbrace{\frac{0.375s+0.125}{(s^2+4s+5)}}_{\mathrm{Re}\{s\}>-2}.$$

Note that the ROC associated with the last term is changed. Taking the inverse Laplace transform, the time domain representation is obtained as

$$x(t) = 0.025e^{-t}u(-t) - 0.025e^{t}u(-t) + 0.375e^{2t}\cos tu(-t) - 0.125e^{2t}\sin tu(-t) + 0.375e^{-2t}\cos tu(t) + 0.125e^{-2t}\sin tu(t).$$

(c) For ROC R: $-1 < \mathrm{Re}\{s\} < 1$, the ROC's associated with individual terms are given by

$$X(s) = -\underbrace{\frac{0.025}{(s+1)}}_{\mathrm{Re}\{s\}>-1} + \underbrace{\frac{0.025}{(s-1)}}_{\mathrm{Re}\{s\}<1} + \underbrace{\frac{-0.375s+0.125}{(s^2-4s+5)}}_{\mathrm{Re}\{s\}<2} + \underbrace{\frac{0.375s+0.125}{(s^2+4s+5)}}_{\mathrm{Re}\{s\}>-2}.$$

Taking the inverse Laplace transform, the time domain representation is obtained as

$$x(t) = -0.025e^{-t}u(t) - 0.025e^{t}u(-t) + 0.375e^{2t}\cos tu(-t) - 0.125e^{2t}\sin tu(-t) + 0.375e^{-2t}\cos tu(t) + 0.125e^{-2t}\sin tu(t).$$

(d) For ROC R: $1 < \mathrm{Re}\{s\} < 2$, the ROC's associated with individual terms are given by

$$X(s) = -\underbrace{\frac{0.025}{(s+1)}}_{\mathrm{Re}\{s\}>-1} + \underbrace{\frac{0.025}{(s-1)}}_{\mathrm{Re}\{s\}>1} + \underbrace{\frac{-0.375s+0.125}{(s^2-4s+5)}}_{\mathrm{Re}\{s\}<2} + \underbrace{\frac{0.375s+0.125}{(s^2+4s+5)}}_{\mathrm{Re}\{s\}>-2}.$$

Taking the inverse Laplace transform, the time domain representation is obtained as

$$x(t) = -0.025e^{-t}u(t) + 0.025e^{t}u(t) + 0.375e^{2t}\cos tu(-t) - 0.125e^{2t}\sin tu(-t) + 0.375e^{-2t}\cos tu(t) + 0.125e^{-2t}\sin tu(t).$$

(e) For ROC R: $\mathrm{Re}\{s\} > 2$, the ROC's associated with individual terms are given by

$$X(s) = -\underbrace{\frac{0.025}{(s+1)}}_{\mathrm{Re}\{s\}>-1} + \underbrace{\frac{0.025}{(s-1)}}_{\mathrm{Re}\{s\}>1} + \underbrace{\frac{-0.375s+0.125}{(s^2-4s+5)}}_{\mathrm{Re}\{s\}>2} + \underbrace{\frac{0.375s+0.125}{(s^2+4s+5)}}_{\mathrm{Re}\{s\}>-2}.$$

Calculating the inverse Laplace transform, the time domain representation is obtained as

$$\begin{aligned} x(t) &= -0.025e^{-t}u(t) + 0.025e^{t}u(t) - 0.375e^{2t}\cos tu(t) + 0.125e^{2t}\sin tu(t) \\ &\quad + 0.375e^{-2t}\cos tu(t) + 0.125e^{-2t}\sin tu(t) \\ &= \left[-0.025e^{-t} + 0.025e^{t} - 0.375e^{2t}\cos t + 0.125e^{2t}\sin t + 0.375e^{-2t}\cos t + 0.125e^{-2t}\sin t\right]u(t). \end{aligned}$$ ▌

Problem 6.6

Assume that

$$x(t) \overset{L}{\longleftrightarrow} X(s) \qquad \text{with ROC}: \ R.$$

By definition

$$e^{s_0 t}x(t) \overset{L}{\longleftrightarrow} \int_{-\infty}^{\infty} e^{s_0 t}x(t)e^{-st}\,dt = \int_{-\infty}^{\infty} x(t)e^{-(s-s_0)t}\,dt = X(s-s_0).$$

with ROC: $R + \text{Re}\{s_0\}$ because the new transform is a shifted version of $X(s)$. For any s in the ROC R of $x(t)$, the values of $s + \text{Re}\{s_0\}$ are in the ROC of $\exp[s_0 t]\, x(t)$. ▌

Problem 6.7

Unilateral Laplace Transform: By definition, the unilateral Laplace transform is given by

$$X(s) = \int_{0^-}^{\infty} x(t)e^{-st}\,dt .$$

Dividing both sides with s and integrating the right hand side by parts, we get

$$\frac{X(s)}{s} = \int_{0^-}^{\infty} x(t)\frac{e^{-st}}{s}\,dt = \underbrace{\int_{0^-}^{t} x(\alpha)d\alpha\, \frac{e^{-st}}{s}\Bigg|_{0^-}^{\infty}}_{Term\ I} - \int_{0^-}^{\infty}\left[\int_{0^-}^{t} x(\alpha)d\alpha\right]\left[-e^{-st}\right]dt .$$

The time integration property for unilateral Laplace transform follows directly from the above relationship by noting that Term I is zero at both the upper and lower limits.

Bilateral Laplace Transform: The proof for the bilateral Laplace transform is similar except that Term I is nonzero. ▌

Problem 6.8

To prove the initial value theorem, we expand $x(t)$ using the Taylor series expansion about $t = 0^+$ as follows.

$$x(t) = x(0^+) + tx^{(1)}(0^+) + \tfrac{t^2}{2!}x^{(2)}(0^+) + \cdots + \tfrac{t^n}{n!}x^{(n)}(0^+) + \cdots$$

where $x^{(n)}$ denotes the n'th order derivative of $x(t)$.

Because, $x(t)$ is a causal function, it can be expressed as follows.

$$x(t) = x(t)u(t) = x(0^+)u(t) + x^{(1)}(0^+)tu(t) + x^{(2)}(0^+)\tfrac{t^2}{2!}u(t) + \cdots + x^{(n)}(0^+)\tfrac{t^n}{n!}u(t) + \cdots,$$

Taking the Laplace transform of both sides of the above equation, we get

$$X(s) = x(0^+)\tfrac{1}{s} + x^{(1)}(0^+)\tfrac{1}{s^2} + x^{(2)}(0^+)\tfrac{1}{s^3} + \cdots + x^{(n)}(0^+)\tfrac{1}{s^{n+1}} + \cdots,$$

Multiplying both sides of the above equation with s and applying the limit, $t \longrightarrow \infty$, gives

$$\begin{aligned}
\lim_{s\to\infty} sX(s) &= \lim_{s\to\infty}\left[x(0^+) + x^{(1)}(0^+)\tfrac{1}{s} + x^{(2)}(0^+)\tfrac{1}{s^2} + \cdots + x^{(n)}(0^+)\tfrac{1}{s^n} + \cdots\right] \\
&= \lim_{s\to\infty} x(0^+) + \underbrace{\lim_{s\to\infty}\left[x^{(1)}(0^+)\tfrac{1}{s} + x^{(2)}(0^+)\tfrac{1}{s^2} + \cdots + x^{(n)}(0^+)\tfrac{1}{s^n} + \cdots\right]}_{=0,\quad \text{assuming } x^{(r)}(0^+)<\infty \text{ for } r=1,2,3,\ldots} \\
&= \lim_{s\to\infty} x(0^+)
\end{aligned}$$

In proving the theorem, we assumed $x(t)u(t) = x(t)$ which is valid only for causal signals, and therefore, the initial value theorem holds true for the unilateral Laplace transform and <u>not</u> for the bilateral Laplace transform. In addition, $x(t)$ should not contain an impulse function or any other discontinuity at $t=0$ so that $x^{(r)}(0^+)<\infty$ for $r=1,2,3,\ldots$.

Problem 6.9

From the time differentiation property, we know

$$\frac{dx}{dt} \xleftrightarrow{L} sX(s)-x(0^-) \qquad \text{with ROC}: \; R$$

or,
$$\int_{0^-}^{\infty} \tfrac{dx}{dt} e^{-st}\,dt = sX(s)-x\left(0^-\right).$$

Applying the limit, $s \to 0$, on both sides of the equation, we get

or,
$$\lim_{s\to 0}\left(\int_{0^-}^{\infty} \tfrac{dx}{dt} e^{-st}\,dt\right) = \lim_{s\to 0}\left[sX(s)-x(0^-)\right]$$

which simplifies to
$$\int_{0^-}^{\infty} \frac{dx}{dt}\left[\lim_{s\to 0} e^{-st}\right]dt = \lim_{s\to 0}\left[sX(s)-x(0^-)\right]$$

or,
$$\int_{0^-}^{\infty} \tfrac{dx}{dt}dt = \lim_{s\to 0}\left[sX(s)-x(0^-)\right] \qquad \left[\because \; \lim_{s\to 0} e^{-st}=1\right].$$

Applying the limits to
$$x(t)\big|_{0^-}^{\infty} = \lim_{s\to 0} sX(s)-x(0^-),$$

we get
$$x(\infty)=\lim_{s\to 0} sX(s),$$

which proves the final value theorem.

Problem 6.10

(a) From Table 6.1,

$$\cos(\omega_0 t)u(t) \xleftrightarrow{L} \tfrac{s}{\omega_0^2+s^2}.$$

Using the s-domain differentiation property (see Table 6.2), we get

$$-t\cos(\omega_0 t)u(t) \xleftrightarrow{L} \frac{d}{ds}\frac{s}{\omega_0^2+s^2} = \frac{(\omega_0^2+s^2)-s(2s)}{(\omega_0^2+s^2)^2}$$

or,
$$t\cos(\omega_0 t)u(t) \xleftrightarrow{L} \tfrac{s^2-\omega_0^2}{(s^2+\omega_0^2)^2}.$$

(b) From Table 6.1,

$$\sin(\omega_0 t)u(t) \xleftrightarrow{L} \tfrac{\omega_0}{\omega_0^2+s^2}.$$

Using the s-domain differentiation property (see Table 6.2), we get

$$-t\sin(\omega_0 t)u(t) \xleftrightarrow{L} \frac{d}{ds}\frac{\omega_0}{\omega_0^2+s^2} = -\frac{\omega_0(2s)}{(\omega_0^2+s^2)^2}$$

or, $$t\sin(\omega_0 t)u(t) \xleftrightarrow{L} \tfrac{2\omega_0 s}{(s^2+\omega_0^2)^2}.$$

(c) Using the results from part (b) and using the linearity property, we get

$$\tfrac{1}{2a^3}\left(\sin(at) - at\cos(at)\right)u(t) \xleftrightarrow{L} \tfrac{1}{2a^3}\left[\tfrac{a}{(a^2+s^2)} - a\tfrac{(s^2-a^2)}{(a^2+s^2)^2}\right]$$

which reduces

$$\tfrac{1}{2a^3}\left(\sin(at) - at\cos(at)\right)u(t) \xleftrightarrow{L} \tfrac{1}{\left(s^2+a^2\right)^2}.$$

Problem 6.11

(a) $f_1(t) = \cos(10t)x(t) = \tfrac{1}{2}e^{j5t}x(t) + \tfrac{1}{2}e^{-j5t}x(t).$

Using the s-shifting property, we obtain

$F_1(s) = \tfrac{1}{2}X(s-5) + \tfrac{1}{2}X(s+5) \quad \text{ROC}: \left[R_x \text{shifted by } s=5\right] \cap \left[R_x \text{shifted by } s=5\right].$

(b) $f_2(t) = e^{-5t}x(4t-3)$.

Using the time-shifting property,

$$x(t-3) \xleftrightarrow{L} e^{-3s}X(s) \quad \text{ROC}: R_x.$$

Using the scaling property,

$$x(4t-3) \xleftrightarrow{L} \tfrac{1}{4}e^{-\frac{3}{4}s}X(\tfrac{s}{4}) \quad \text{ROC}: 2R_x.$$

Using the s-shifting property, we obtain

$$f_2(t) = e^{-5t}\, x(4t-3) \xleftrightarrow{L} F_2(s) = \tfrac{1}{4}e^{-\frac{3}{4}(s+5)}X(\tfrac{s+5}{4}) \quad \text{ROC}: 2R_x \text{ shifted by } s=-5.$$

(c) Using the time-shifting property,

$$x(t-4) \xleftrightarrow{L} e^{-4s}X(s) \quad \text{ROC}: R_x.$$

Using the time differentiation property,

$$\tfrac{d}{dt}\left[x(t)\right] \xleftrightarrow{L} sX(s) - x(0^-) \quad \text{ROC}: R_x.$$

We now use the s-plane differentiation property

$$\text{If } x(t) \xleftrightarrow{L} X(s) \quad \text{ROC}: R_x \quad \text{then} \quad -tx(t) \xleftrightarrow{L} \tfrac{d}{ds}\left[X(s)\right] \quad \text{ROC}: R_x.$$

Using the s-plane differentiation property,

$$(-t)^4\, \tfrac{d}{dt}\left[x(t)\right] \xleftrightarrow{L} \tfrac{d^4}{ds^4}\left[sX(s)\right] \quad \text{ROC}: R_x.$$

Using the time shifting property,

$$(t-4)^4\, \tfrac{d}{dt}\left[x(t-4)\right] \xleftrightarrow{L} e^{-4s}\, \tfrac{d^4}{ds^4}\left[sX(s)\right] \quad \text{ROC}: R_x.$$

(d) $f_4(t) = \left[x(t)+2\right]^2 = x^2(t)+4x(t)+4$

Using the s-convolution property, we obtain

$$x^2(t) \xleftrightarrow{L} \frac{1}{2\pi} X(s) * X(s) \qquad \text{ROC}: \{R_x\}.$$

$$x(t) \xleftrightarrow{L} X(s) \qquad \text{ROC}: \{R_x\}$$

$$1 \xleftrightarrow{L} \frac{1}{s} \qquad \text{ROC}: \quad \text{Re}\{s\} > 0$$

Using the linearity property, we obtain

$$F_4(s) = \frac{1}{2\pi} X(s) * X(s) + 4X(s) + \frac{4}{s} \qquad \text{ROC}: \left[R_x \cap \{\text{Re}(s) > 0\}\right].$$

(e) Using the s-shifting property, we obtain

$$e^{-s_0 t} x(t) \xleftrightarrow{L} X(s+s_0) \qquad \text{ROC}: \{R_x - \text{Re}(s_0)\}.$$

Using the time integration property,

$$\int_{-\infty}^{t} e^{-\alpha s_0} x(\alpha) d\alpha \xleftrightarrow{L} \tfrac{X(s+s_0)}{s} + \tfrac{1}{s} \int_{-\infty}^{0^-} e^{-s_0 \alpha} x(\alpha) d\alpha \qquad \text{ROC}: \left[\{R_x - \text{Re}(s_0)\} \cap \{\text{Re}\{s\} > 0\}\right].$$

<u>Problem 6.12</u>

To determine the ROC, we use Property 2 that states:

For a right sided (causal) function, the ROC takes the form Re$\{s\} > \sigma_0$ and consists of the right side of the complex s-plane

(a) Poles lie at $s = -6.8541$ and $s = -0.1459$, and hence the ROC is given by R: $s > -0.1459$. Since the ROC contains both $s = 0$ and $s = \infty$, the initial and final value theorems can be applied.

Initial value: $\lim\limits_{t \to 0} x(t) = \lim\limits_{s \to \infty} sX(s) = \lim\limits_{s \to \infty} \frac{s^2}{s^2+7s+1} = \lim\limits_{s \to \infty} \frac{1}{1+\frac{7}{s}+\frac{1}{s^2}} = 1.$

Final value: $\lim\limits_{t \to \infty} x(t) = \lim\limits_{s \to 0} sX(s) = \lim\limits_{s \to 0} \frac{s^2}{s^2+7s+1} = 0.$

Note that $\qquad x(t) = 1.0217e^{-6.8541t}u(t) - 0.0217e^{-0.1459t}u(t),$

and therefore the initial and final value theorems compute the correct answer.

(b) Poles lie at $s = -5.7016$ and $s = 0.7016$, and hence the ROC is given by R: $s > 0.7016$. Since the ROC contains $s = \infty$, the initial value theorem can be applied. The final value theorem may give an incorrect answer.

Initial value: $\lim\limits_{t \to 0} x(t) = \lim\limits_{s \to \infty} sX(s) = \lim\limits_{s \to \infty} \frac{s^2}{s^2+5s+4} = \lim\limits_{s \to \infty} \frac{1}{1+\frac{5}{s}-\frac{4}{s^2}} = 1.$

Final value: $\lim\limits_{t \to \infty} x(t) = \lim\limits_{s \to 0} sX(s) = \lim\limits_{s \to 0} \frac{s^2}{s^2+5s-4} = 0.$

Note that $$x(t) = 0.8904e^{-5.7016t}u(t) + 0.1096e^{0.1096t}u(t),$$

and therefore the value of $x(\infty)$ obtained from the final value theorem is incorrect. The initial value theorem computes the correct answer.

(c) Poles lie at $s = -5$ and $s = 5$, and hence the ROC is given by R: $s > 5$. Since the ROC contains $s = \infty$, the initial value theorem can be applied. The final value theorem may give an incorrect answer.

Initial value: $\lim_{t \to 0} x(t) = \lim_{s \to \infty} sX(s) = \lim_{s \to \infty} \frac{s(s^2+9)}{s^2-25} = \lim_{s \to \infty} \frac{s\left(1+\frac{9}{s^2}\right)}{1-\frac{25}{s^2}} = \infty.$

Final value: $\lim_{t \to \infty} x(t) = \lim_{s \to 0} sX(s) = \lim_{s \to 0} \frac{s(s^2+9)}{s^2-25} = 0.$

Note that $$x(t) = \delta(t) + 3.4e^{5t}u(t) - 3.4e^{-5t}u(t),$$

and therefore the value of $x(\infty)$ obtained from the final value theorem is incorrect. The initial value theorem computes the correct answer of $x(0) = \infty$.

(d) Poles lie at $s = -1.5 + j1.323$ and $s = -1.5 - j1.323$, and hence the ROC is given by R: $s > -1.5$. Since the ROC contains both $s = 0$ and $s = \infty$, the initial and final value theorems can be applied.

Initial value: $\lim_{t \to 0} x(t) = \lim_{s \to \infty} sX(s) = \lim_{s \to \infty} \frac{s\left(s^2+2s+1\right)}{s^2+3s+4} = \lim_{s \to \infty} \frac{s\left(1+2/s+1/s^2\right)}{1+3/s+4/s^2} = \infty.$

Final value: $\lim_{t \to \infty} x(t) = \lim_{s \to 0} sX(s) = \lim_{s \to 0} \frac{s\left(s^2+2s+1\right)}{s^2+3s+4} = 0.$

Note that $$x(t) = \delta(t) - e^{-1.5t}\cos(\sqrt{2.75}t)u(t) - \frac{3}{\sqrt{2.75}}e^{-1.5t}\sin(\sqrt{2.75}t)u(t,$$

and therefore the initial and final value theorems compute the correct answer.

(e) Poles lie at $s = 0, -1$, $s = -2$, and $s = -3$, and hence the ROC is given by R: $s > 0$. Since the ROC contains $s = \infty$, the initial value theorem can be applied. The final value theorem may give an incorrect answer.

Initial value: $\lim_{t\to 0} x(t) = \lim_{s\to\infty} sX(s) = \lim_{s\to\infty} e^{-5s}\frac{(s^2+4)}{(s+1)(s+2)(s+3)} = \lim_{s\to\infty} \frac{(1+4/s^2)}{e^{5s}(1+1/s)(1+2/s)(s+3)} = 0.$

Final value: $\lim_{t \to \infty} x(t) = \lim_{s \to 0} sX(s) = \lim_{s \to 0} e^{-5s}\frac{(s^2+4)}{(s+1)(s+2)(s+3)} = \frac{2}{3}.$

Note that $x(t) = \frac{2}{3}\delta(t-4) - \frac{5}{2}e^{-(t-5)}u(t-5) + 4e^{-2(t-5)}u(t-5) - \frac{13}{3}e^{-3(t-5)}u(t-5)$,

and therefore the initial value theorem computes the correct answer. The value of $x(\infty)$ obtained from the final value theorem is incorrect. ▌

Problem 6.13

(a) Calculating the Laplace transform of both sides, we obtain

$$\left[s^2Y(s) - s\underbrace{y(0^-)}_{=0} - \underbrace{\dot{y}(0^-)}_{=0}\right] + 3\left[sY(s) - \underbrace{y(0^-)}_{=0}\right] + 2Y(s) = 1$$

or, $(s^2 + 3s + 2)Y(s) = 1$ or $Y(s) = \frac{1}{(s^2+3s+2)} = \frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}$.

Calculating the inverse Laplace transform, we obtain

$$y(t) = e^{-t}u(t) - e^{-2t}u(t) = \left(e^{-t} - e^{-2t}\right)u(t).$$

(b) Calculating the Laplace transform of both sides, we obtain

$$\left[s^2Y(s) - s\underbrace{y(0^-)}_{=0} - \underbrace{\dot{y}(0^-)}_{=0}\right] + 4\left[sY(s) - \underbrace{y(0^-)}_{=0}\right] + 4Y(s) = \tfrac{1}{s}$$

or, $(s^2 + 4s + 4)Y(s) = \frac{1}{s}$ or $Y(s) = \frac{1}{s(s+2)^2} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{(s+2)^2}$,

where the partial fraction coefficients are calculated as

$$A = \left.\tfrac{1}{(s+2)^2}\right|_{s=0} = \frac{1}{4}, \text{ and } C = \left[\tfrac{1}{s}\right]_{s=-2} = -\frac{1}{2}$$

Expanding $Y(s)$, and comparing the numerator of both sides, we get

$$\begin{aligned}1 &= A(s+2)^2 + Bs(s+2) + Cs \\ &= (A+B)s^2 + (4A + 2B + C) + 4A\end{aligned}$$

Comparing the coefficients of s^2 in both sides, we get $(A + B) = 0$ or $B = -1/4$.

In other words, $Y(s) = \frac{1}{s(s+2)^2} = \frac{1/4}{s} - \frac{1/4}{s+2} - \frac{1/2}{(s+2)^2}$

Calculating the inverse Laplace transform of Y(s) yields

$$y(t) = \tfrac{1}{4}\left[1 - e^{-2t} - 2te^{-2t}\right]u(t) = \tfrac{1}{4}\left[1 - (2t+1)e^{-2t}\right]u(t).$$

(c) Calculating the Laplace transform of both sides, we obtain

$$\left[s^2Y(s) - s\underbrace{y(0^-)}_{=1} - \underbrace{\dot{y}(0^-)}_{=1}\right] + 6\left[sY(s) - \underbrace{y(0^-)}_{=1}\right] + 8Y(s) = \tfrac{1}{(s+3)^2}$$

or, $(s^2 + 6s + 8)Y(s) = \frac{1}{(s+3)^2} + (s + 1 + 6)$ or $Y(s) = \frac{1}{(s+2)(s+3)^2(s+4)} + \frac{s+7}{(s+2)(s+4)}$.

Calculating the partial fraction expansion of the two terms separately, we obtain

$$\frac{1}{(s+2)(s+3)^2(s+4)} = \frac{1/2}{s+2} + \frac{0}{s+3} - \frac{1}{(s+3)^2} - \frac{1/2}{s+4}$$

$$\text{and} \quad \frac{s+7}{(s+2)(s+4)} = \frac{5/2}{s+2} - \frac{3/2}{s+4}$$

Expanding $Y(s)$ as

$$Y(s) = \tfrac{1/2}{s+2} - \tfrac{1}{(s+3)^2} - \tfrac{1/2}{s+4} + \tfrac{5/2}{s+2} - \tfrac{3/2}{s+4} = \tfrac{3}{s+2} - \tfrac{1}{(s+3)^2} - \tfrac{2}{s+4}.$$

Calculating the inverse Laplace transform of Y(s) yields

$$y(t) = \left(3e^{-2t} - te^{-3t} - 2e^{-4t}\right)u(t).$$

(d) Calculating the Laplace transform of both sides, we obtain

$$\left[s^3Y(s)-s^2\underbrace{y(0^-)}_{=1}-s\underbrace{\dot{y}(0^-)}_{=0}-\underbrace{\ddot{y}(0^-)}_{=0}\right]+8\left[s^2Y(s)-s\underbrace{y(0^-)}_{=1}-\underbrace{\dot{y}(0^-)}_{=0}\right]+19\left[sY(s)-\underbrace{y(0^-)}_{=1}\right]+12Y(s)=\tfrac{1}{s^2}$$

or, $(s^3+8s^2+19s+12)Y(s)=\frac{1}{s^2}+(s^2+8s+19)$ or

$$Y(s)=\frac{1}{s^2(s^3+8s^2+19s+12)}+\frac{s^2+8s+19}{s^3+8s^2+19s+12}=\frac{s^4+8s^3+19s^2+1}{s^2(s+1)(s+3)(s+4)}=\frac{k_1}{s}+\frac{k_2}{s^2}+\frac{k_3}{s+1}+\frac{k_4}{s+3}+\frac{k_5}{s+4}$$

where

$$k_1=\left[\frac{d}{ds}\left(\frac{s^4+8s^3+19s^2+1}{(s+1)(s+3)(s+4)}\right)\right]_{s=0}$$

$$=\left[\frac{1}{(s+1)(s+3)(s+4)}\frac{d}{ds}\left(s^4+8s^3+19s^2+1\right)-\frac{s^4+8s^3+19s^2+1}{(s+1)^2(s+3)^2(s+4)^2}\frac{d}{ds}\left((s+1)(s+3)(s+4)\right)\right]_{s=0}$$

$$=\left[\frac{1}{12}\times 0-\frac{1}{144}\times\frac{d}{ds}\left(s^3+8s^2+19s+12\right)\right]_{s=0}=-\frac{1}{144}\times 19=-\frac{19}{144}$$

$$k_2=\left.\frac{s^4+8s^3+19s^2+1}{(s+1)(s+3)(s+4)}\right|_{s=0}=\frac{1}{12}$$

$$k_3=\left.\frac{s^4+8s^3+19s^2+1}{s^2(s+3)(s+4)}\right|_{s=-1}=\frac{1-8+19+1}{1\times2\times3}=\frac{13}{6}$$

$$k_4=\left.\frac{s^4+8s^3+19s^2+1}{s^2(s+1)(s+4)}\right|_{s=-3}=\frac{81-216+171+1}{9\times(-2)\times1}=\frac{37}{-18}$$

$$k_5=\left.\frac{s^4+8s^3+19s^2+1}{s^2(s+1)(s+3)}\right|_{s=-4}=\frac{256-512+304+1}{16\times(-3)\times(-1)}=\frac{49}{48}$$

In other words, $Y(s)=-\frac{19/144}{s}+\frac{1/12}{s^2}+\frac{13/6}{s+1}-\frac{37/18}{s+3}+\frac{49/48}{s+4}$.

Therefore, $y(t)=\left(-\frac{19}{144}+\frac{1}{12}t+\frac{13}{6}e^{-t}-\frac{37}{18}e^{-3t}+\frac{49}{48}e^{-4t}\right)u(t)$.

An Alternative (Equivalent) Solution of (d)

Calculating the Laplace transform of both sides, we get

$$\left[s^3Y(s)-s^2\underbrace{y(0^-)}_{=1}-s\underbrace{\dot{y}(0^-)}_{=0}-\underbrace{\ddot{y}(0^-)}_{=0}\right]+8\left[s^2Y(s)-s\underbrace{y(0^-)}_{=1}-\underbrace{\dot{y}(0^-)}_{=0}\right]+19\left[sY(s)-\underbrace{y(0^-)}_{=1}\right]+12Y(s)=\tfrac{1}{s^2}$$

or, $(s^3+8s^2+19s+12)Y(s)=\frac{1}{s^2}+(s^2+8s+19)$ or

$$Y(s)=\frac{1}{s^2(s^3+8s^2+19s+12)}+\frac{s^2+8s+19}{s^3+8s^2+19s+12}=\frac{1}{s^2(s+1)(s+3)(s+4)}+\frac{s^2+8s+19}{(s+1)(s+3)(s+4)}$$

Taking the partial fraction expansion of the two terms separately

$$\frac{1}{s^2(s^3+8s^2+19s+12)}=\frac{0.0208}{(s+4)}-\frac{0.0556}{(s+3)}+\frac{0.1667}{(s+1)}-\frac{0.1319}{s}+\frac{0.0833}{s^2}$$

and $$\frac{s^2+8s+19}{(s^3+8s^2+19s+12)}=\frac{1}{(s+4)}-\frac{2}{(s+3)}+\frac{2}{(s+1)}$$

Expanding $Y(s)$, we get

$$Y(s) = \frac{0.0208}{(s+4)} - \frac{0.0556}{(s+3)} + \frac{0.1667}{(s+1)} - \frac{0.1319}{s} + \frac{0.0833}{s^2} + \frac{1}{(s+4)} - \frac{2}{(s+3)} + \frac{2}{(s+1)}$$
$$= \frac{1.0208}{(s+4)} - \frac{2.0556}{(s+3)} + \frac{2.1667}{(s+1)} - \frac{0.1319}{s} + \frac{0.0833}{s^2}.$$

Taking the inverse Laplace transform of Y(s) gives

$$y(t) = 1.0208e^{-4t}u(t) - 2.0556e^{-3t}u(t) + 2.1667e^{-t}u(t) - 0.1319u(t) + 0.0833tu(t)$$
$$= \left[-0.1319 + 0.0833t + 2.1667e^{-t} - 2.0556e^{-3t} + 1.0208e^{-4t}\right]u(t)$$

(e) $\dfrac{d^4y}{dt^4} + 2\dfrac{d^2y}{dt^2} + y(t) = u(t); \qquad y(0^-) = \dot{y}(0^-) = \ddot{y}(0^-) = \dddot{y}(0^-) = 0.$

Calculating the Laplace transform of both sides, we get

$$\left[s4Y(s) - \underbrace{s^3y(0^-) - s^2\dot{y}(0^-) - s\ddot{y}(0^-) - \dddot{y}(0^-)}_{=0}\right] + 2\left[s^2Y(s) - \underbrace{sy(0^-) - \dot{y}(0^-)}_{=0}\right] + Y(s) = \tfrac{1}{s}$$

or, $(s^4 + 2s^2 + 1)Y(s) = \frac{1}{s}$ or $Y(s) = \frac{1}{s(s^4+2s^2+1)} = \frac{A}{s} + \frac{Bs+C}{(s^2+1)} + \frac{Ds+E}{(s^2+1)^2}$.

where $A = \left[\frac{1}{s(s^4+2s^2+1)}s\right]_{s=0} = \left[\frac{1}{(s^4+2s^2+1)}\right]_{s=0} = 1.$

Equating numerator of $Y(s)$ in both sides, we get (Note: A=1)

$$1 = (s^2+1)^2 + (Bs+C)s(s^2+1) + (Ds+E)s$$
$$= (1+B)s^4 + Cs^3 + (2+B+D)s^2 + (C+E)s + 1$$

Comparing the coefficients of polynomials of different order we get

Coefficients of s^4: $1 + B = 0 \Rightarrow B = -1$

Coefficients of s^3: $C = 0$

Coefficients of s^2: $2 + B + D = 0 \Rightarrow 1 + D = 0 \Rightarrow D = -1$

Coefficients of s: $C + E = 0 \Rightarrow E = 0$

The partial fraction expansion of $Y(s)$ is given by

$$Y(s) = \frac{1}{s} - \frac{s}{s^2+1} - \frac{s}{(s^2+1)^2}$$

Noting that (see Problem 6.10(b)) $t\sin(\omega_0 t)\,u(t) \xleftrightarrow{L} \frac{2\omega_0 s}{\left(s^2+\omega_0^2\right)^2}$, the inverse transform is obtained as

$$y(t) = \left[1 - \cos(t) + 0.5t\sin(t)\right]u(t).$$

▮

Problem 6.14

(a) (i) $X(s)=L\{4u(t)\}=\frac{4}{s}$ and $Y(s)=L\left\{tu(t)+e^{-2t}u(t)\right\}=\frac{1}{s^2}+\frac{1}{s+2}=\frac{s^2+s+2}{s^2(s+2)}$.

The transfer function is given by $H(s)=\frac{Y(s)}{X(s)}=\frac{s^2+s+2}{s^2(s+2)}\times\frac{s}{4}=\frac{s^2+s+2}{4s(s+2)}$.

(ii) The impulse response is given by

$$h(t)=L^{-1}\left\{\frac{s^2+s+2}{4s(s+2)}\right\}=\frac{1}{4}L^{-1}\left\{1+\frac{2-s}{s(s+2)}\right\}=\frac{1}{4}L^{-1}\left\{1+\frac{1}{s}-\frac{2}{s+2}\right\}=\frac{1}{4}\left(\delta(t)+u(t)-2e^{-2t}u(t)\right)$$

(iii) In order to calculate the input-output relationship in the form of a differential equation, we represent the transfer function as follows.

$$H(s)=\frac{s^2+s+2}{4s(s+2)}=\frac{Y(s)}{X(s)}$$

or, $(4s^2+8s)Y(s)=(s^2+s+2)X(s)$

Calculating the inverse Laplace transform and assuming zero initial conditions, the differential equation representing the system is given by

$$4\frac{d^2y}{dt^2}+8\frac{dy}{dt}=\frac{d^2x}{dt^2}+\frac{dx}{dt}+2x(t).$$

(b) (i) The Laplace transform of the input and output signals are given by

$$X(s)=\frac{1}{(s+2)} \quad \text{and} \quad Y(s)=3e^{-4s}\frac{1}{(s+2)}.$$

Dividing $Y(s)$ with $X(s)$, the transfer function is given by

$$H(s)=\frac{Y(s)}{X(s)}=3e^{-4s}.$$

(ii) The impulse response is obtained by calculating the inverse Laplace transform. The impulse response is given by

$$h(t)=3\delta(t-4).$$

(iii) In order to calculate the input-output relationship in the form of a differential equation, we represent the transfer function as

$$H(s)=3e^{-4s}=\frac{Y(s)}{X(s)}.$$

Cross multiplying, we get $\quad Y(s)=3e^{-4s}X(s)$.

Calculating the inverse Laplace transform, the input-output relationship of the system is given by

$$y(t)=3x(t-4).$$

(c) (i) The Laplace transform of the input and output signals are given by

$$X(s)=\frac{1}{s^2} \quad \text{and} \quad Y(s)=\frac{2}{s^3}-3\frac{1}{(s+4)}.$$

Dividing $Y(s)$ with $X(s)$, the transfer function is given by

$$H(s)=\frac{Y(s)}{X(s)}=\frac{2}{s}-\frac{3s^2}{(s+4)}.$$

(ii) The impulse response is obtained by calculating the inverse Laplace transform. The impulse response is given by

$$h(t) = 2u(t) - 3\tfrac{d^2}{dt^2}\left[e^{-4t}u(t)\right].$$

(iii) In order to calculate the input-output relationship in the form of a differential equation, we represent the transfer function as

$$H(s) = \tfrac{2(s+4)-3s^3}{s(s+4)} = \tfrac{Y(s)}{X(s)}.$$

Cross multiplying, we get $s^2Y(s) + 4sY(s) = -3s^3X(s) + 2sX(s) + 8X(s)$.

Calculating the inverse Laplace transform, the input-output relationship of the system is given by

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} = -3\frac{d^3x}{dt^3} + 2\frac{dx}{dt} + 8x(t).$$

(d) (i) The Laplace transform of the input and output signals are given by

$$X(s) = \tfrac{1}{s+2} \quad \text{and} \quad Y(s) = \tfrac{1}{s+1} + \tfrac{1}{s+3}.$$

Dividing $Y(s)$ with $X(s)$, the transfer function is given by

$$H(s) = \tfrac{Y(s)}{X(s)} = \tfrac{(s+2)}{(s+1)} + \tfrac{(s+2)}{(s+3)} \equiv 2 + \tfrac{1}{s+1} - \tfrac{1}{s+3}.$$

(ii) The impulse response is obtained by calculating the inverse Laplace transform. The impulse response is given by

$$h(t) = 2u(t) + e^{-t}u(t) - e^{-3t}u(t).$$

(iii) In order to calculate the input-output relationship in the form of a differential equation, we represent the transfer function as

$$H(s) = \tfrac{(s+2)(s+1+s+3)}{(s+1)(s+3)} = \tfrac{Y(s)}{X(s)}.$$

Cross multiplying, we get $2s^2Y(s) + 8sY(s) + 8Y(s) = s^2X(s) + 4sX(s) + 3X(s)$.

Calculating the inverse Laplace transform, the input-output relationship of the system is given by

$$2\frac{d^2y}{dt^2} + 8\frac{dy}{dt} + 8y(t) = \frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 3x(t).$$

(e) Note that there is no overlap between the ROC's of the two terms $\exp(t)u(-t)$ and $\exp(-3t)u(t)$, and hence the Laplace transform for $y(t)$ does not exist. ▌

Problem 6.15

(a) $H(s) = \frac{s^2+1}{s^2+2s+1} = \frac{(s+j)(s-j)}{(s+1)^2}$

Two zeros: at $s = j, -j$

Two poles: at $s = -1, -1$. The zeros and poles are shown in Fig. S6.15(a).

Because both poles are in the left side of the s-plane, the system is always BIBO stable.

(b) $H(s) = \frac{2s+5}{s^2+s-6} = \frac{2(s+2.5)}{(s+3)(s-2)}$

One zero: at $s = -2.5$

Two poles: at $s = -3, +2$. The zero and poles are shown in Fig. S6.15(b).

Because one pole is located at the right-hand side of the s-plane, the system is NOT stable.

(c) $H(s) = \frac{3s+10}{s^2+9s+18} = \frac{3(s+10/3)}{(s+6)(s+3)}$

One zero: at $s = -\frac{10}{3}$

Two poles: at $s = -6, -3$. The zero and poles are shown in Fig. S6.15(c).

Because both poles are in the left side of the s-plane, the system is always BIBO stable.

(d) $H(s) = \frac{s+2}{s^2+9} = \frac{s+2}{(s+j3)(s-j3)}$

One zero: at $s = -2$

Two poles: at $s = +j3, -j3$. The zero and poles are shown in Fig. S6.15(d).

There are only two poles, and both poles are located on the imaginary axis. Therefore the system is a marginally stable system.

(e) $H(s) = \frac{s^2+3s+2}{s^3+3s^2+2s} = \frac{s^2+3s+2}{s(s^2+3s+2)} = \frac{1}{s}$

The system does not have any zero.

One pole: at $s = 0$. The pole is shown in Fig. S6.15(e).

There is only one pole, which is located on the imaginary axis. Therefore the system is a marginally stable system. ∎

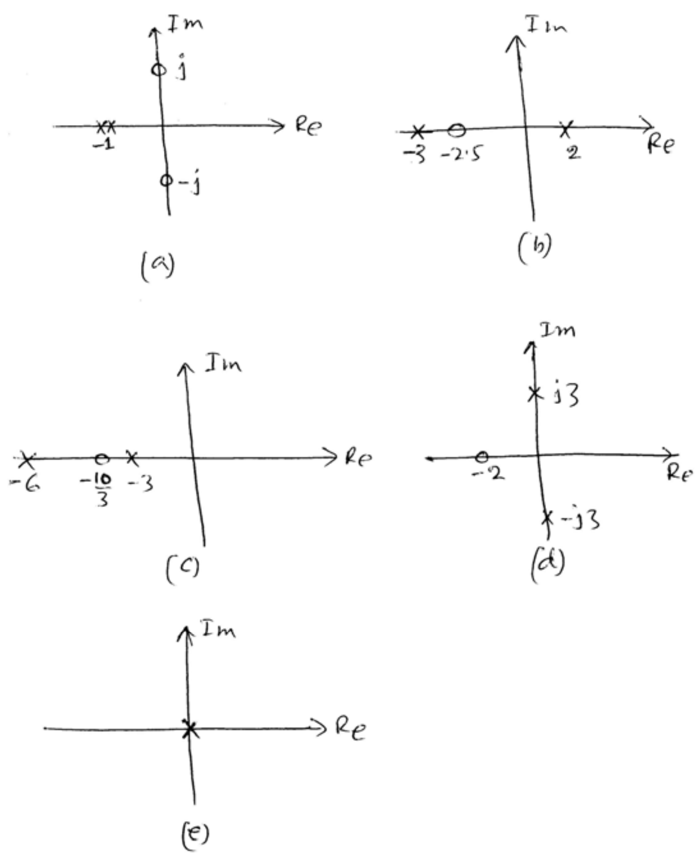


Figure S6.15

<u>Problem 6.16</u>

Note that the poles of the LTIC system are located at s = –5, $j2$, $-j2$, $j3$, $-j3$, $-2+j1.5$, and $-2-j1.5$. Since there are unrepeated poles at the imaginary axis, the system is a marginally stable system. It produces bounded output for bounded inputs provided the input signal does not include a complex exponential term of the form exp($j2t$), exp($-j2t$), exp($j3t$), or exp($-j3t$).

(a) The input signal includes $e^{-j2t}u(t)$. The system will produce an unbounded output.

(b) The input signal does not include a complex exponential term of the form exp($j2t$), exp($-j2t$), exp($j3t$), or exp($-j3t$). Therefore, the system will produce a bounded output.

(c) $x(t)=\left[\cos(t)+\sin(4t)\right]u(t)=\left[0.5(e^{jt}+e^{-jt})-0.5j(e^{j4t}-e^{-j4t})\right]u(t)$

The input signal does not include a complex exponential term of the form exp($j2t$), exp($-j2t$), exp($j3t$), or exp($-j3t$). Therefore, the system will produce a bounded output.

(d) $x(t)=\left[\cos(2t)+\sin(3t)\right]u(t)=\left[0.5(e^{j2t}+e^{-j2t})-0.5j(e^{j3t}-e^{-j3t})\right]u(t)$

The input signal includes $e^{j2t}u(t)$, $e^{-j2t}u(t)$, $e^{j3t}u(t)$, and $e^{-j3t}u(t)$. Therefore, the system will produce an unbounded output.

(e) $x(t)=\left[e^{-(1+j2)t}\sin(3t)\right]u(t)=-0.5j\left[e^{-(1+j2)t}(e^{j3t}-e^{-j3t})\right]u(t)=-0.5j\left[e^{(-1+j)t}-e^{-(1+j5)t}\right]u(t)$

The input signal does not include a complex exponential term of the form exp($j2t$), exp($-j2t$), exp($j3t$), or exp($-j3t$). Therefore, the system will produce a bounded output. ▌

Problem 6.17

(a) There are two zeros at s = 0, 1 and two poles at s = –2, –4. Since all poles lie in the left half of the s-plane, and therefore System (a) is stable.

The transfer function of System (a) is given by

$$H(s)=K\frac{s(s-1)}{(s+2)(s+4)} \text{ with ROC: Re}\{s\}>-2.$$

Substituting H(4) = 1, we get $H(4)=1=K\frac{4(3)}{(6)(8)}$, or, K = 4.

The transfer function of System (a) is given by

$$H(s)=4\frac{s(s-1)}{(s+2)(s+4)} \text{ with ROC: Re}\{s\}>-2.$$

(b) There are two zeros at s = 0, –4 and three poles at s = 1, –2, –3. Since a pole lies in the right half of the s-plane, and therefore System (b) is NOT stable.

The transfer function of System (b) is given by

$$H(s)=K\frac{s(s+4)}{(s-1)(s+2)(s+3)} \text{ with ROC: Re}\{s\}>1.$$

Substituting H(4) = 1, we get $H(4)=1=K\frac{4(8)}{(3)(6)(7)}$, or, K = 63/16.

The transfer function of System (b) is given by

$$H(s)=\frac{63}{16}\frac{s(s+4)}{(s-1)(s+2)(s+3)} \text{ with ROC: Re}\{s\}>1.$$

(c) There are two zeros at $s = 2 + j2$, $2 - j2$ and two poles at s = $-2 - j2$, $-2 + j2$. Since both poles lie in the left half of the s-plane, and therefore System (c) is stable.

The transfer function of System (c) is given by

$$H(s)=K\frac{(s-2+j2)(s-2-j2)}{(s+2+j2)(s+2-j2)}=K\frac{(s^2-4s+8)}{(s^2+4s+8)} \text{ with ROC: Re}\{\text{s}\}>-2.$$

Substituting H(4) = 1, we get $H(4)=1=K\frac{(8)}{(40)}$, or, K = 5.

The transfer function of System (b) is given by

$$H(s)=5\frac{(s^2-4s+8)}{(s^2+4s+8)} \text{ with ROC: Re}\{\text{s}\}>-2.$$

(d) There are two zeros at s = 1, 2 and four poles at s = $-3, -3, -2 - j3, -2 + j3$. Since all poles lie in the left half of the s-plane, and therefore System (d) is stable.

The transfer function of System (c) is given by

$$H(s)=K\frac{(s-1)(s-2)}{(s+3)(s+3)(s+2+j3)(s+2-j3)}=K\frac{(s^2-3s+2)}{(s^4+10s^3+46s^2+114s+117)}$$

with ROC: $\mathrm{Re}\{s\} > -2$.

Substituting $H(4) = 1$, we get $H(4) = 1 = K\frac{(6)}{(2205)}$, or, $K = 735/2$.

The transfer function of System (b) is given by

$$H(s) = \frac{735}{2}\frac{(s^2-3s+2)}{(s^4+10s^3+46s^2+114s+117)} \text{ with ROC: } \mathrm{Re}\{s\} > -2.$$

Problem 6.18

The transfer functions for noncausal implementations stay the same as P6.17. Only the ROC changes as shown below:

(a) $H(s) = 4\frac{s(s-1)}{(s+2)(s+4)}$ with ROC: $\mathrm{Re}\{s\} < -4$.

A second possible transfer function for a noncausal implementation is obtained by considering the ROC: $-4 < \mathrm{Re}\{s\} < -2$.

(b) $H(s) = \frac{63}{16}\frac{s(s+4)}{(s-1)(s+2)(s+3)}$ with ROC: $\mathrm{Re}\{s\} > -3$.

Two possible transfer functions for noncausal implementations are obtained by considering the ROC: $-3 < \mathrm{Re}\{s\} < -2$ and ROC: $-2 < \mathrm{Re}\{s\} < 1$.

(c) $H(s) = 5\frac{(s^2-4s+8)}{(s^2+4s+8)}$ with ROC: $\mathrm{Re}\{s\} < -2$.

(d) $H(s) = \frac{735}{2}\frac{(s^2-3s+2)}{(s^4+10s^3+46s^2+114s+117)}$ with ROC: $\mathrm{Re}\{s\} > -3$.

A second possible transfer function for a noncausal implementation is obtained by considering the ROC: $-3 < \mathrm{Re}\{s\} < -2$.

Problem 6.19

Since $$H(s) \times H_{inv}(s) = 1, \text{ or, } H_{inv}(s) = \frac{1}{H(s)},$$

the poles of $H(s)$ must map as zeros of the inverse system. Similarly, the zeros of $H(s)$ must map as poles of the inverse system. Fig. S6.19 shows the locations of poles and zeros of the LTIC systems of Fig. P6.17.

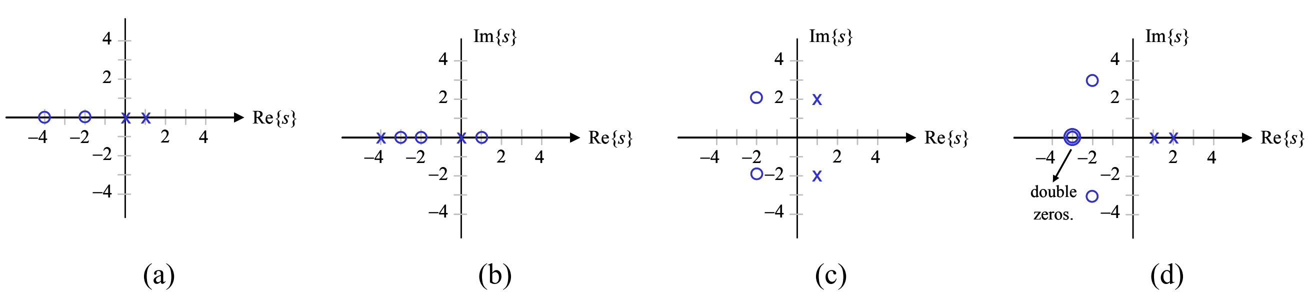


Fig. S6.19: Poles and zero plots for the inverse systems.

Problem 6.20

Property (c) states that $H(s)$ has four poles but no zeros, therefore, the transfer function can be expressed as

$$H(s) = \frac{K}{s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0}. \qquad \text{(S6.20.1)}$$

Property (a) states that the impulse response $h(t)$ is even and real-valued. Using the even property $h(t) = h(-t)$, we show that $H(s) = H(-s)$ as follows:

By definition, $$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}\,dt,$$

implying that $$H(-s) = \int_{-\infty}^{\infty} h(t)e^{st}\,dt.$$

Substituting $t = -\alpha$, we get $$H(-s) = \int_{\infty}^{-\infty} h(-\alpha)e^{-s\alpha}(-d\alpha) = \int_{-\infty}^{\infty} h(\alpha)e^{-s\alpha}\,d\alpha = H(s). \qquad \text{(S6.20.2)}$$

Combining Eq. (S6.20.1) with (S6.20.2), we can see that $a_3 = a_1 = 0$ and the transfer function takes the form

$$H(s) = \frac{K}{s^4 + a_2 s^2 + a_0} \qquad \text{(S6.20.3)}$$

with coefficients $a_2 = a_0$ real valued. The transfer function of the form (S6.20.3) with real coefficients has poles that occurs in conjugate symmetry. In other words, if $s = 0.5\exp(j\pi/4)$ is a pole, then $s = 0.5\exp(j3\pi/4)$, $s = 0.5\exp(-j3\pi/4)$, and $s = 0.5\exp(-j\pi/4)$ should also be poles. The resulting transfer function is

$$H(s) = \frac{K}{(s-0.5e^{j3\pi/4})(s-0.5e^{-j3\pi/4})(s-0.5e^{j\pi/4})(s-0.5e^{j\pi/4})} = \frac{K}{s^4 + 0.0625}.$$

Using property (a), we substitute $$H(0) = \int_{-\infty}^{\infty} h(t)\,dt = 8$$

to get $K = 0.5$. The transfer function is given by

$$H(s) = \frac{0.5}{s^4 + 0.0625}.$$

▮

Problem 6.21

Calculating the Laplace transform of both sides of the input-output relationship, we get

$$s^2 W(s) + \tfrac{R}{L} s W(s) + \tfrac{1}{L} W(s) = \tfrac{1}{LC} X(s)$$

The transfer function is given by $H(s) = \frac{Y(s)}{X(s)} = \frac{\frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{L}}$. Note that the transfer function has two poles, which are the roots of the characteristic equation $s^2 + \frac{R}{L}s + \frac{1}{L} = 0$, and are given by

$$\begin{cases} p_1 = \dfrac{-\frac{R}{L} + \sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{L}}}{2} = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{L}} \\ p_2 = \dfrac{-\frac{R}{L} - \sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{L}}}{2} = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{L}} \end{cases}$$

Depending on the values of R, L, and C, two cases can occur.

Case 1: $\left(\frac{R}{2L}\right)^2 - \frac{1}{L} \geq 0 \;\; or, \; R^2 \geq 4L$

In this case, $\sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{L}}$ is real, and both poles will be real poles. Because $\sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{L}} < \frac{R}{2L}$, both p_1 and p_2 will have negative values. In other words, the poles are in the left side of the s-plane, and therefore, the system is always stable.

Case 2: $\left(\frac{R}{2L}\right)^2 - \frac{1}{L} < 0 \;\; or, \; R^2 < 4L$

In this case, $\sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{L}}$ is imaginary, and both poles will be complex poles. The real part of both these complex conjugate poles will be $-\frac{R}{2L}$. In other words, the poles are in the left side of the s-plane, and therefore, the system is always stable.

Therefore, the R-L-C circuit is always stable. ▌

Problem 6.22

(a) Factorizing $H(s)$ yields the following expression for the transfer function

$$H(s) = \frac{(s-3)(s+2)}{(s+1)(s+2)(s+3)(s+4)} = \frac{s-3}{(s+1)(s+3)(s+4)}.$$

The poles of $H(s)$ are located at s = –1, –3, –4. Possible choices of the ROC are:

Choice 1: ROC: $\text{Re}\{s\} > -1$.

Choice 2: ROC: $-3 < \text{Re}\{s\} < -1$.

Choice 3: ROC: $-4 < \text{Re}\{s\} < -3$.

Choice 4: ROC: $\text{Re}\{s\} < -4$.

(b) For a causal implementation of $H(s)$, the ROC must cover most of the rightmost side of the s-plane to ensure that $h_1(t)$ is a right hand sided sequence. The overall ROC is therefore given by ROC: $\text{Re}\{s\} > -1$.

Taking the partial fraction expansion of $H(s)$ yields

$$H(s) = \frac{(s-3)}{(s+1)(s+3)(s+4)} = -\underbrace{\frac{2/3}{(s+1)}}_{\text{ROC:Re}\{s\}>-1} + \underbrace{\frac{3}{(s+3)}}_{\text{ROC:Re}\{s\}>-3} - \underbrace{\frac{7/3}{(s+4)}}_{\text{ROC:Re}\{s\}>-4}.$$

Calculating the inverse Laplace transform, we obtain

$$h_1(t) = -\tfrac{2}{3}e^{-t}u(t) + 3e^{-3t}u(t) - \tfrac{7}{3}e^{-4t}u(t).$$

Since all three terms in $h_1(t)$ decay to 0 as $t \to \infty$, $h_1(t)$ is stable.

(c) For a left hand sided implementation of $H(s)$, the ROC must cover the leftmost side of the s-plane. The overall ROC is therefore given by ROC: $\mathrm{Re}\{s\} < -4$.

Taking the partial fraction expansion of $H(s)$ yields

$$H(s) = \frac{s-3}{(s+1)(s+3)(s+4)} = -\underbrace{\frac{2/3}{(s+1)}}_{\mathrm{ROC:Re}\{s\}<-1} + \underbrace{\frac{3}{(s+3)}}_{\mathrm{ROC:Re}\{s\}<-3} - \underbrace{\frac{7/3}{(s+4)}}_{\mathrm{ROC:Re}\{s\}<-4}.$$

Calculating the inverse Laplace transform, we obtain

$$h_2(t) = \tfrac{2}{3}e^{-t}u(-t) - 3e^{-3t}u(-t) + \tfrac{7}{3}e^{-4t}u(-t).$$

Note that $h_2(t)$ is not stable because all three terms $e^{-t}u(-t)$, $e^{-3t}u(-t)$, and $e^{-4t}u(-t)$ are unstable.

(d) For a double sided implementation of $H(s)$, the ROC must consist of a narrow strip within the s-plane. The overall ROC is therefore given by ROC: $(-3 < \mathrm{Re}\{s\} < -1)$, or, ROC: $(-4 < \mathrm{Re}\{s\} < -3)$.

(i) If ROC: $(-3 < \mathrm{Re}\{s\} < -1)$, then $H(s)$ is expressed as

$$H(s) = \frac{(s-3)}{(s+1)(s+3)(s+4)} = -\underbrace{\frac{2/3}{(s+1)}}_{\mathrm{ROC:Re}\{s\}<-1} + \underbrace{\frac{3}{(s+3)}}_{\mathrm{ROC:Re}\{s\}>-3} - \underbrace{\frac{7/3}{(s+4)}}_{\mathrm{ROC:Re}\{s\}>-4}.$$

Calculating the inverse Laplace transform, we obtain

$$h_3(t) = \tfrac{2}{3}e^{-t}u(-t) + 3e^{-3t}u(t) - \tfrac{7}{3}e^{-4t}u(t).$$

Note that such $h_3(t)$ is not stable because the term $e^{-t}u(-t)$ is not stable.

(ii) On the other hand, if ROC: $(-4 < \mathrm{Re}\{s\} < -3)$, then $H(s)$ is expressed as

$$H(s) = \frac{(s-3)}{(s+1)(s+3)(s+4)} = -\underbrace{\frac{2/3}{(s+1)}}_{\mathrm{ROC:Re}\{s\}<-1} + \underbrace{\frac{3}{(s+3)}}_{\mathrm{ROC:Re}\{s\}<-3} - \underbrace{\frac{7/3}{(s+4)}}_{\mathrm{ROC:Re}\{s\}>-4}.$$

Calculating the inverse Laplace transform, we obtain

$$h_4(t) = \tfrac{2}{3}e^{-t}u(-t) - 3e^{-3t}u(-t) - \tfrac{7}{3}e^{-4t}u(t).$$

Note that such $h_4(t)$ is not stable because the terms $e^{-t}u(-t)$ and $3e^{-3t}u(-t)$ are not stable.

(e) As shown above, the implementation $h_1(t)$ with the overall ROC given by ROC: $\mathrm{Re}\{s\} > -1$ is stable. The remaining implementations $h_2(t)$, $h_3(t)$, and $h_4(t)$ are unstable. ▌

Problem 6.23

(a) Factorizing $H(s)$ gives the following expression for the transfer function

$$H(s) = \frac{(s-12)(s+7)}{(s+4)(s+5)^2(s-7)}.$$

The poles of $H(s)$ are located at s = –4, –5, –5, and 7. Possible choices of the ROC are:

Choice 1: ROC: $\mathrm{Re}\{s\} > 7$.

Choice 2: ROC: $-4 < \mathrm{Re}\{s\} < 7$.

Choice 3: ROC: $-5 < \mathrm{Re}\{s\} < -4$.

Choice 4: ROC: $\mathrm{Re}\{s\} < -5$.

(b) For a causal implementation of $H(s)$, the ROC must cover most of the right half of the s-plane to ensure that $h_1(t)$ is a right hand sided sequence. The overall ROC is therefore given by ROC: $\mathrm{Re}\{s\} > 7$.

Taking the partial fraction expansion of $H(s)$ gives

$$H(s) = \frac{(s-12)(s+7)}{(s+4)(s+5)^2(s-7)} = \underbrace{\frac{48/11}{(s+4)}}_{\mathrm{ROC:Re}\{s\}>-4} - \underbrace{\frac{3421/792}{(s+5)}}_{\mathrm{ROC:Re}\{s\}>-5} - \underbrace{\frac{17/6}{(s+5)^2}}_{\mathrm{ROC:Re}\{s\}>-5} - \underbrace{\frac{35/792}{(s-7)}}_{\mathrm{ROC:Re}\{s\}>7}.$$

Taking the inverse Laplace transform gives

$$h_1(t) = \tfrac{48}{11}e^{-4t}u(t) - \tfrac{3421}{792}e^{-5t}u(t) - \tfrac{17}{12}te^{-5t}u(t) - \tfrac{35}{792}e^{7t}u(t).$$

Since the last term $\tfrac{35}{792}e^{7t}u(t)$ in $h_1(t)$ reaches ∞ as $t \to \infty$, $h_1(t)$ is NOT stable.

(c) For a left hand sided implementation of $H(s)$, the ROC must cover most of the left half of the s-plane. The overall ROC is therefore given by ROC: $\mathrm{Re}\{s\} < -5$.

Taking the partial fraction expansion of $H(s)$ gives

$$H(s) = \frac{(s-12)(s+7)}{(s+4)(s+5)^2(s-7)} = \underbrace{\frac{48/11}{(s+4)}}_{\mathrm{ROC:Re}\{s\}<-4} - \underbrace{\frac{3421/792}{(s+5)}}_{\mathrm{ROC:Re}\{s\}<-5} - \underbrace{\frac{17/6}{(s+5)^2}}_{\mathrm{ROC:Re}\{s\}<-5} - \underbrace{\frac{35/792}{(s-7)}}_{\mathrm{ROC:Re}\{s\}<7}.$$

Taking the inverse Laplace transform gives

$$h_2(t) = -\tfrac{48}{11}e^{-4t}u(-t) + \tfrac{3421}{792}e^{-5t}u(-t) + \tfrac{17}{12}te^{-5t}u(-t) + \tfrac{35}{792}e^{7t}u(-t).$$

Note that $h_2(t)$ is not stable because of the first three terms which are all unstable.

(d) For a double sided implementation of $H(s)$, the ROC must consist of a narrow strip within the s-plane. The overall ROC is therefore given by ROC: $(-4 < \mathrm{Re}\{s\} < 7)$, or, ROC: $(-5 < \mathrm{Re}\{s\} < -4)$.

If ROC is $(-4 < \mathrm{Re}\{s\} < 7)$, then $H(s)$ is expressed as

$$H(s) = \frac{(s-12)(s+7)}{(s+4)(s+5)^2(s-7)} = \underbrace{\frac{48/11}{(s+4)}}_{\mathrm{ROC:Re}\{s\}>-4} - \underbrace{\frac{3421/792}{(s+5)}}_{\mathrm{ROC:Re}\{s\}>-5} - \underbrace{\frac{17/6}{(s+5)^2}}_{\mathrm{ROC:Re}\{s\}>-5} - \underbrace{\frac{35/792}{(s-7)}}_{\mathrm{ROC:Re}\{s\}<7}.$$

Taking the inverse Laplace transform gives

$$h_3(t) = \tfrac{48}{11}e^{-4t}u(t) - \tfrac{3421}{792}e^{-5t}u(t) - \tfrac{17}{12}te^{-5t}u(t) + \tfrac{35}{792}e^{7t}u(-t).$$

Note that such $h_3(t)$ is stable.

On the other hand, if ROC: $(-5 < \mathrm{Re}\{s\} < -4)$, then $H(s)$ is expressed as

$$H(s) = \frac{(s-12)(s+7)}{(s+4)(s+5)^2(s-7)} = \underbrace{\frac{48/11}{(s+4)}}_{\mathrm{ROC:Re}\{s\}<-4} - \underbrace{\frac{3421/792}{(s+5)}}_{\mathrm{ROC:Re}\{s\}>-5} - \underbrace{\frac{17/6}{(s+5)^2}}_{\mathrm{ROC:Re}\{s\}>-5} - \underbrace{\frac{35/792}{(s-7)}}_{\mathrm{ROC:Re}\{s\}<7}.$$

Calculating the inverse Laplace transform gives

$$h_4(t) = -\tfrac{48}{11}e^{-4t}u(-t) - \tfrac{3421}{792}e^{-5t}u(t) - \tfrac{17}{12}te^{-5t}u(t) + \tfrac{35}{792}e^{7t}u(-t).$$

Note that such $h_4(t)$ is not stable because the first term is not stable.

(f) As shown above, the implementation $h_3(t)$ with the overall ROC given by ROC: $(-4 < \text{Re}\{s\} < 7)$ is stable. The remaining implementations are unstable. ▌

Problem 6.24

(a) The causal implementation of $H(s) = \frac{s^2+1}{s^2+2s+1} = \frac{(s+j)(s-j)}{(s+1)^2}$

is always BIBO stable.

(b) The causal implementation of

$$H(s) = \frac{2s+5}{s^2+s-6} = 2\frac{(s+2.5)}{(s+3)(s-2)}$$

is not stable because of the pole at $s = 2$. The all-pass system

$$H_{ap}(s) = \frac{s-2}{s+2}$$

cascaded with the original system would cancel out the pole at $s = 2$, making the overall system with transfer function

$$H_{overall}(s) = H(s) \times H_{ap}(s) = 2\frac{s+2.5}{(s+3)(s-2)} \times \frac{s-2}{s+2} = 2\frac{s+2.5}{(s+3)(s+2)}$$

stable. Note that the overall system has the same magnitude spectrum as the original system. The phase spectrum will be different.

(c) The causal implementation of $H(s) = \frac{3s+10}{s^2+9s+18} = 3\frac{s+10/3}{(s+3)(s+6)}$

is always BIBO stable.

(d) The causal implementation of $H(s) = \frac{s+2}{s^2+9} = \frac{s+2}{(s+j3)(s-j3)}$

is marginally stable.

(e) The causal implementation of $H(s) = \frac{s^2+3s+2}{s^3+3s^2+2s} = \frac{1}{s}$

is marginally stable. ▌

Problem 6.25

(a) Based on the feedback configuration,

$$H_1(s) = \frac{\frac{1}{(s+5)}}{1+\frac{1}{(s+3)(s+5)}} = \frac{s+3}{(s+3)(s+5)+1} = \frac{s+3}{(s+3)(s+5)+1} = \frac{s+3}{(s+4)^2}.$$

Using the cascaded configuration,

$$H_2(s) = \frac{1}{(s+1)(s+2)} \times H_1(s) = \frac{s+3}{(s+1)(s+2)(s+4)^2}.$$

Finally, using the feedback configuration, the overall transfer function of the system is expressed as

$$H_a(s) = \frac{\frac{1}{(s+6)}}{1+\frac{H_2(s)}{(s+6)}} = \frac{(s+1)(s+2)(s+4)^2}{(s+1)(s+2)(s+4)^2(s+6)+(s+3)} = \frac{s^4+11s^3+42s^2+64s+32}{s^5+17s^4+108s^3+316s^2+417s+195}.$$

(b) Expressing the system in the form shown in Fig. S6.25(b), the response of the feedback configured subsystem enclosed in dotted rectangle is given by

$$H_1(s) = \frac{\frac{1}{(s+1)}}{1+\frac{1}{(s+1)(s+2)(s+4)}} = \frac{(s+2)(s+4)}{(s+1)(s+2)(s+4)+1} = \frac{s^2+6s+8}{s^3+7s^2+14s+9}.$$

The overall transfer function of the system is expressed as

$$H_b(s) = H_1(s) \times \left[\frac{1}{(s+2)(s+3)} - \frac{1}{(s+5)}\right] = -\frac{(s^2+4s+1)H_1(s)}{s^3+10s^2+31s+30},$$

or, $$H_b(s) = -\frac{(s^2+4s+1)(s^2+6s+8)}{(s^3+10s^2+31s+30)(s^3+7s^2+14s+9)},$$

or, $$H_b(s) = -\frac{s^4+10s^3+33s^2+38s+8}{s^6+17s^5+115s^4+396s^3+734s^2+699s+270}.$$

(c) The outputs of the two summers are given by

$$V(s) = X(s) - \frac{1}{(s+4)(s+5)(s+6)} W(s)$$

and $$W(s) = X(s) - \frac{1}{(s+1)(s+2)(s+7)} V(s).$$

Substituting the values of $W(s)$ in the first equation, we get

$$V(s) = X(s) - \frac{1}{(s+4)(s+5)(s+6)} X(s) + \frac{1}{(s+1)(s+2)(s+4)(s+5)(s+6)(s+7)} V(s),$$

or, $$V(s) = \frac{(s+1)(s+2)(s+4)(s+5)(s+6)(s+7)}{1-(s+1)(s+2)(s+4)(s+5)(s+6)(s+7)} \times \frac{(s+4)(s+5)(s+6)-1}{(s+4)(s+5)(s+6)} X(s),$$

or, $$V(s) = \frac{(s+1)(s+2)(s+7)[(s+4)(s+5)(s+6)-1]}{(s+1)(s+2)(s+4)(s+5)(s+6)(s+7)-1} X(s).$$

The system's output is given by

$$Y(s) = \frac{1}{(s+1)(s+2)(s+3)} V(s) = \frac{1}{(s+3)} \times \frac{(s+7)[(s+4)(s+5)(s+6)-1]}{(s+1)(s+2)(s+4)(s+5)(s+6)(s+7)-1} X(s),$$

which results in the overall transfer function

$$H_c(s) = \frac{s^4 + 22s^3 + 179s^2 + 637s + 833}{s^7 + 28s^6 + 322s^5 + 1960s^4 + 6769s^3 + 13132s^2 + 13067s + 5037}.$$

Problem 6.26

The following MATLAB code computes the partial fraction coefficients for the transfer functions specified in parts (a) to (g). The coefficients and the roots are specified after the MATLAB code.

```
% part (a)
num_a = [1 2 1];
denum_a = poly([-1; roots([1 5 6])]);
[coeff_a,roots_a,K_a] = residue(num_a,denum_a);
% part (b) is same as part (a)
% part (c)
num_c = [1 3 -4];
denum_c = poly([-1; roots([1 5 6])]);
[coeff_c,roots_c,K_c] = residue(num_c,denum_c)
% part (d) is same as part (c)
% part (e)
num_e = [1 0 1];
denum_e = poly([0; -1; roots([1 2 17])]);
[coeff_e,roots_e,K_e] = residue(num_e,denum_e);
% part (f)
num_f = [1 1];
denum_f = poly([-2; -2; -3; -4]);
[coeff_f,roots_f,K_f] = residue(num_f,denum_f);
% part (g)
num_g = [1 -2 1];
denum_g = poly([-1; -1; -1; roots([1 0 16])]);
[coeff_g,roots_g,K_g] = residue(num_g,denum_g);
```

(a) The MATLAB code produces the following output

`coeff_a = [2  -1  0]; roots_a = [-3  -2  -1]; and K_a = [];`

leading to the partial fraction expansion

$$X(s) = \frac{2}{s+3} - \frac{1}{s+2}.$$

(b) The results for Part (b) are the same as the results for Part (a).

(c) The MATLAB code produces the following output

`coeff_c = [-2  6  -3]; roots_c = [-3  -2  -1]; and K_c = [];`

leading to the partial fraction expansion

$$X(s) = -\frac{2}{s+3} + \frac{6}{s+2} - \frac{3}{s+1}.$$

(d) The results for Part (d) are the same as the results for Part (c).

(e) The MATLAB code produces the following output

`coeff_e = [0.0331-j0.1176  0.0331+j0.1176  -0.125   0.0588];`

```
roots_e = [-1+j4  -1-j4  -1   0]; and K_e = [];
```

leading to the partial fraction expansion

$$X(s)=\frac{0.0331-j0.1176}{(s+1-j4)}+\frac{0.0331+j0.1176}{(s+1+j4)}-\frac{0.125}{(s+1)}+\frac{0.0588}{s}.$$

Note that the term containing the complex poles can be combined as

$$X(s)=\frac{2\times 0.0331(s+1)+2\times 4\times 0.1176}{(s+1)^2+16}-\frac{0.125}{(s+1)}+\frac{0.0588}{s},$$

or,

$$X(s)=\frac{0.0662s+1.007}{(s+2s+17)^2+16}-\frac{0.125}{(s+1)}+\frac{0.0588}{s}$$

(f) The MATLAB code produces the following output

```
coeff_f = [0.75  -2  1.25   -0.5];
roots_f = [-4  -3  -2   -2]; and K_f = [];
```

leading to the partial fraction expansion

$$X(s)=\frac{0.75}{s+4}-\frac{2}{s+3}+\frac{1.25}{s+2}-\frac{0.5}{(s+2)^2}.$$

(g) The MATLAB code produces the following output

```
coeff_f = [-0.0103-j0.0285  -0.0103+j0.0285  0.02056  -0.2076  0.2352]
roots_f = [j4  -j4  -1   -1  -1]; and K_f = [];
```

leading to the partial fraction expansion

$$X(s)=-\frac{0.0103+j0.0285}{s-j4}-\frac{0.0103-j0.0285}{s+j4}+\frac{0.02056}{s+1}-\frac{0.2076}{(s+1)^2}+\frac{0.2352}{(s+1)^3}.$$

Note that the term containing the complex poles can be combined as

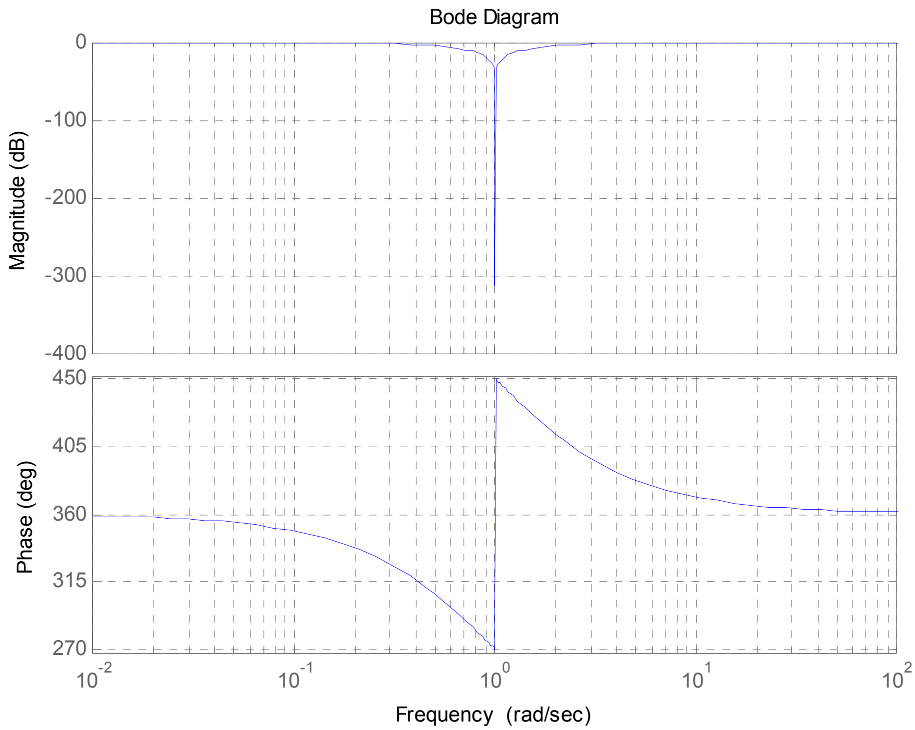
$$X(s)=-\frac{0.0206s-0.228}{s^2+16}+\frac{0.02056}{s+1}-\frac{0.2076}{(s+1)^2}+\frac{0.2352}{(s+1)^3}.$$

Problem 6.27

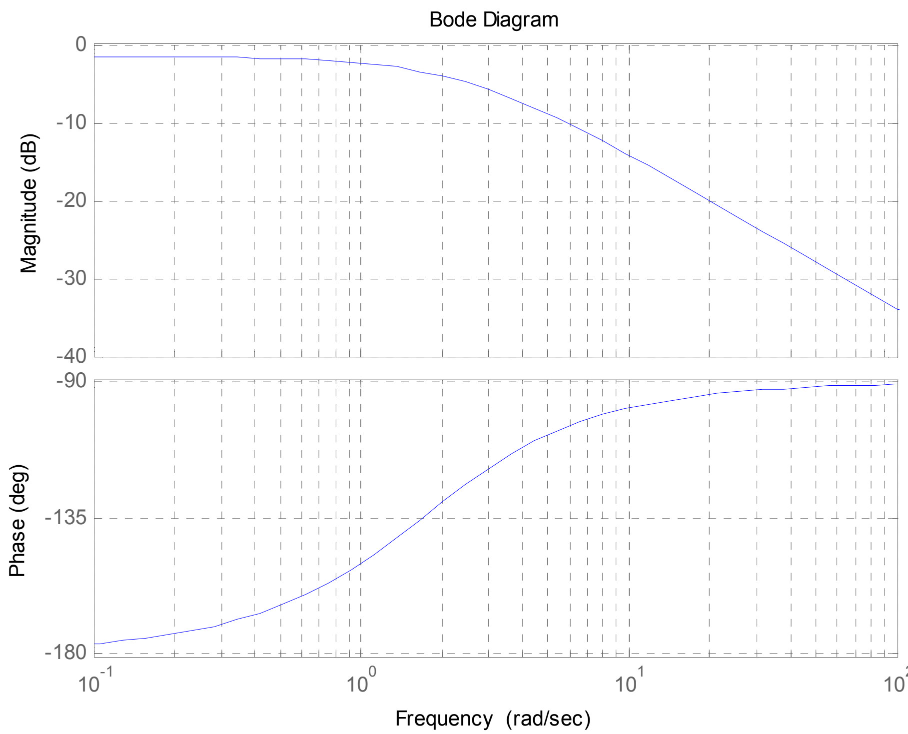
The MATLAB code for generating the Bode plots is given below followed by the plots in Fig. S6.27.

```
% part (a)
figure (1); num = [1 0 1]; denum = [1 2 1];
S = tf(num,denum); bode(S); grid on
% part (b)
figure (2); num = [2 5]; denum = [1 1 -6];
S = tf(num,denum); bode(S); grid on
% part (c)
figure (3); num = [3 10]; denum = [1 9 18];
S = tf(num,denum); bode(S); grid on
% part (d)
```

```
figure (4); num = [1 2]; denum = [1 0 9];
S = tf(num,denum); bode(S); grid on
% part (e)
figure (5); num = [1 3 2]; denum = [1 3 2 0];
S = tf(num,denum); bode(S); grid on
```

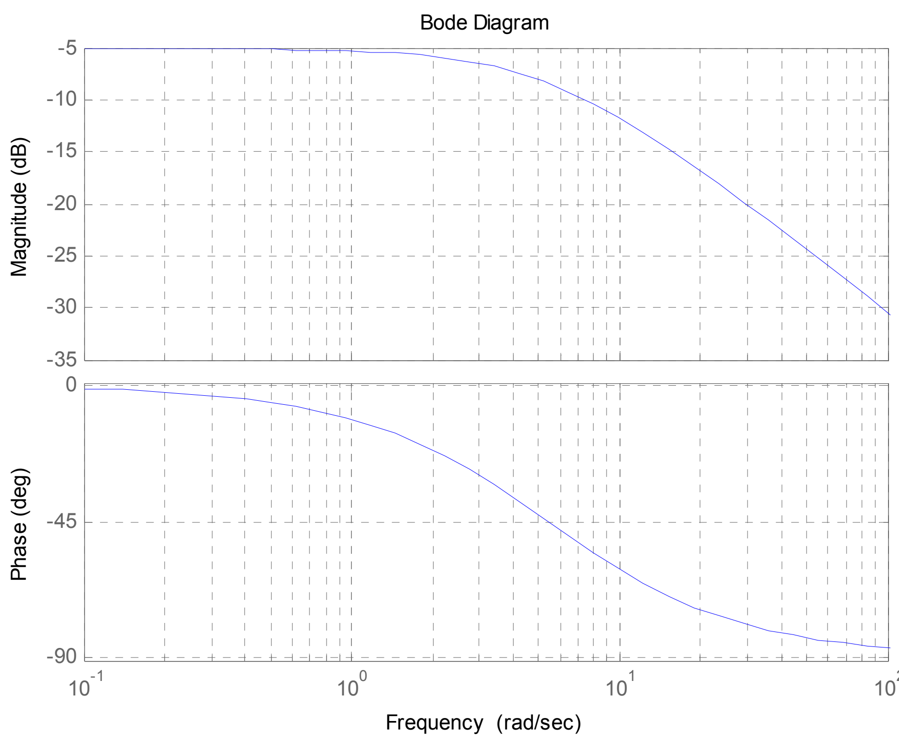


(a)



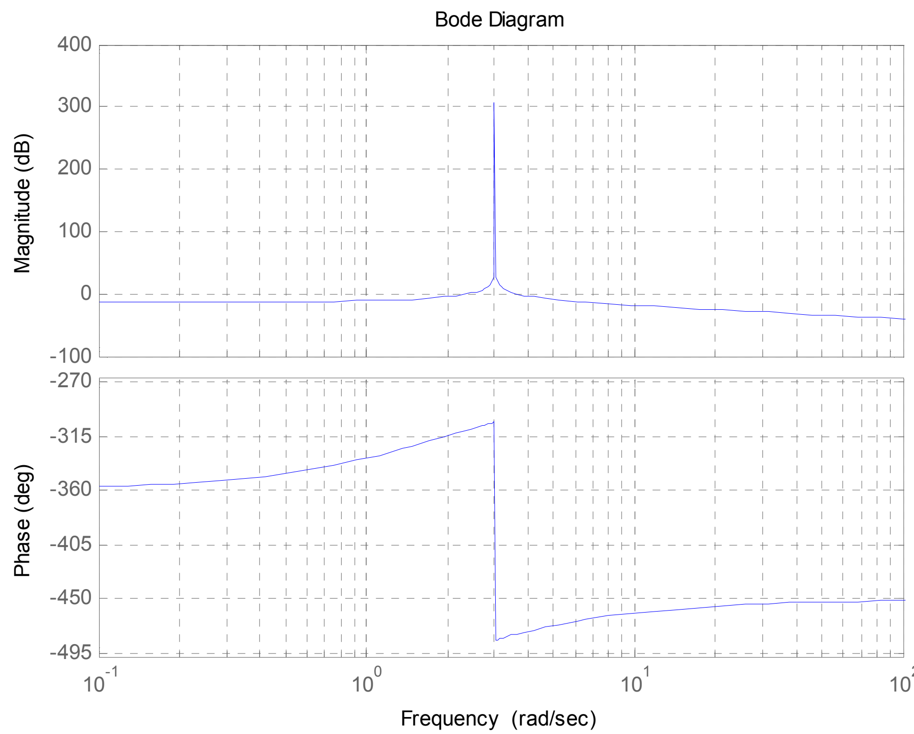
Bode Diagram
Magnitude (dB)
0
-10
-20
-30
-40
Phase (deg)
-90
-135
-180
10^{-1}
10^{0}
10^{1}
10^{2}
Frequency (rad/sec)

(b)

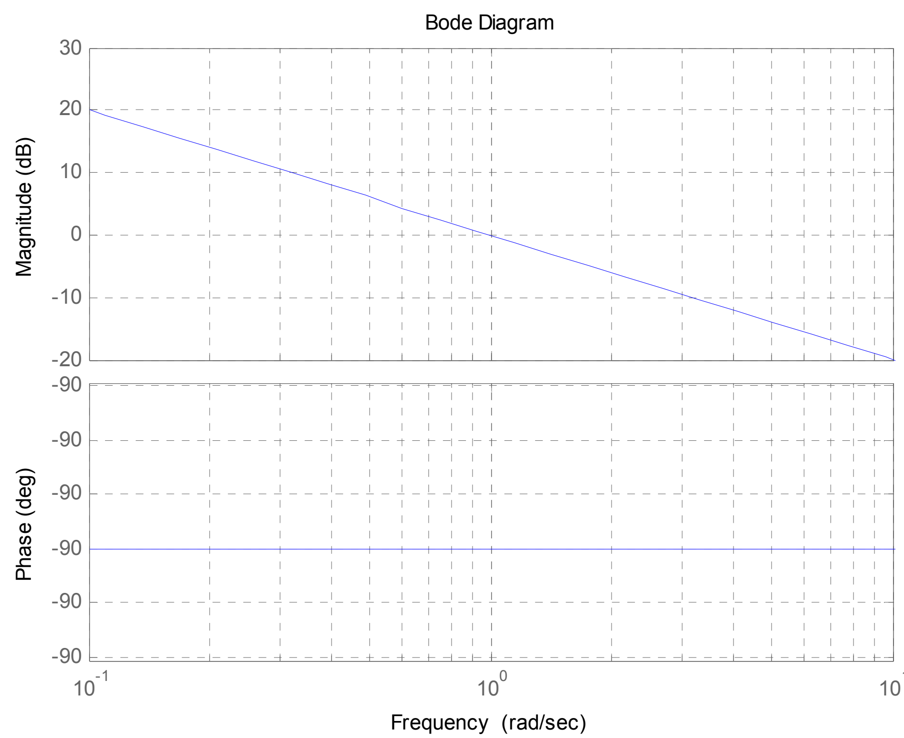


Bode Diagram
Magnitude (dB)
-5
-10
-15
-20
-25
-30
-35
Phase (deg)
0
-45
-90
10^{-1}
10^{0}
10^{1}
10^{2}
Frequency (rad/sec)

(c)



(d)



(e)

Fig. S6.27: Bode plots for Problem 6.27.

Problem 6.28

The MATLAB code for generating the Bode plots is given below.

```
% part (a)
figure (1); num = [1 0 1]; denum = [1 2 1];
w = [0.01:0.001:100]; [H,w] = freqs(num,denum,w);
subplot(211); semilogx(w,20*log10(abs(H))); grid on;
subplot(212); semilogx(w,angle(H)); grid on

% part (b)
figure (2); num = [2 5]; denum = [1 1 -6];
w = [0.1:0.001:100]; [H,w] = freqs(num,denum,w);
subplot(211); semilogx(w,20*log10(abs(H))); grid on;
subplot(212); semilogx(w,angle(H)); grid on

% part (c)
figure (3); num = [3 10]; denum = [1 1 -6];
w = [0.1:0.001:100]; [H,w] = freqs(num,denum,w);
subplot(211); semilogx(w,20*log10(abs(H))); grid on;
subplot(212); semilogx(w,angle(H)); grid on

% part (d)
figure (4); num = [1 2]; denum = [1 0 9];
w = [0.1:0.001:100]; [H,w] = freqs(num,denum,w);
subplot(211); semilogx(w,20*log10(abs(H))); grid on;
subplot(212); semilogx(w,angle(H)); grid on

% part (e)
figure (5); num = [1 3 2]; denum = [1 3 2 0];
w = [0.1:0.001:10]; [H,w] = freqs(num,denum,w);
subplot(211); semilogx(w,20*log10(abs(H))); grid on;
subplot(212); semilogx(w,angle(H)); grid on
```

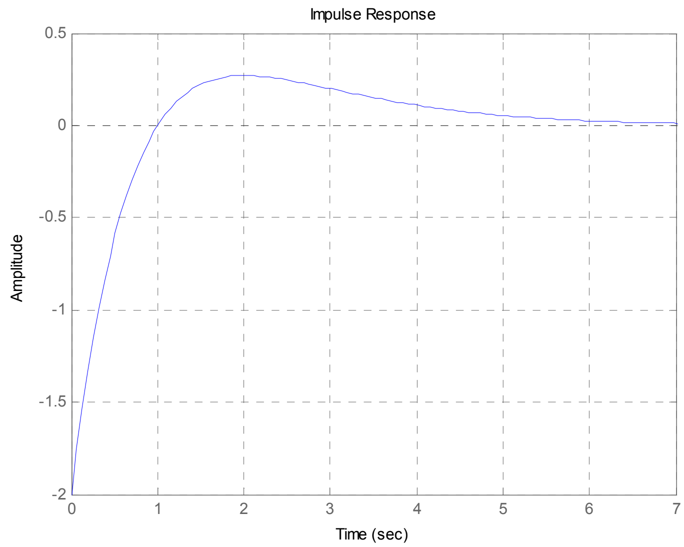
The plots are similar to the plots obtained in Problem 6.27 except the phase that needs to be converted to degrees. ▮

Problem 6.29

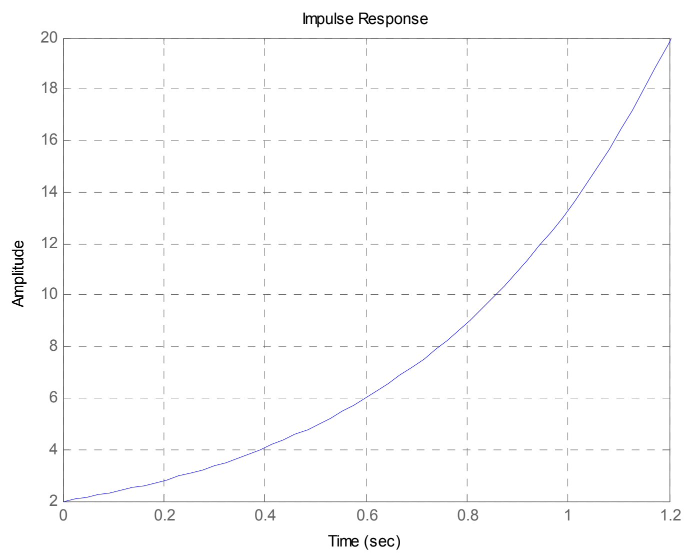
The MATLAB code for generating the impulse is given below followed by the plots in Fig. S6.29.

```
% part (a)
figure (1); num = [1 0 1]; denum = [1 2 1];
S = tf(num,denum); impulse(S);
% part (b)
figure (2); num = [2 5]; denum = [1 1 -6];
S = tf(num,denum); impulse(S);
% part (c)
figure (3); num = [3 10]; denum = [1 9 18];
S = tf(num,denum); impulse(S);
% part (d)
figure (4); num = [1 2]; denum = [1 0 9];
S = tf(num,denum); impulse(S);
% part (e)
```

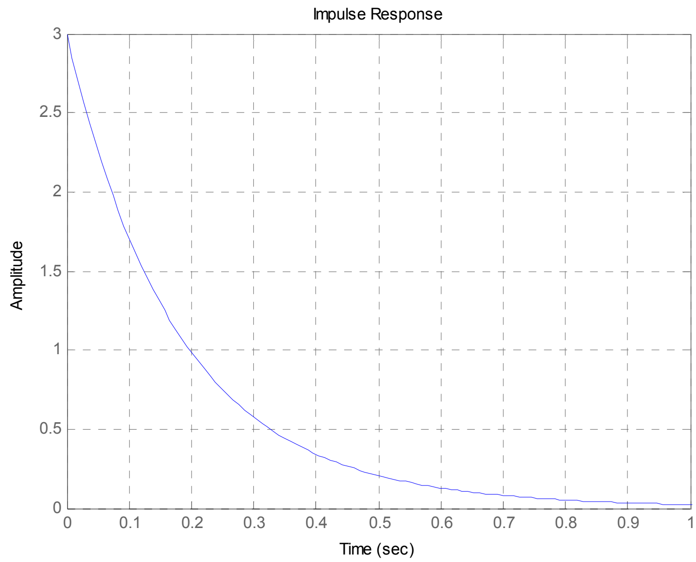
```
figure (5); num = [1 3 2]; denum = [1 3 2 0];
S = tf(num,denum); impulse(S);
```



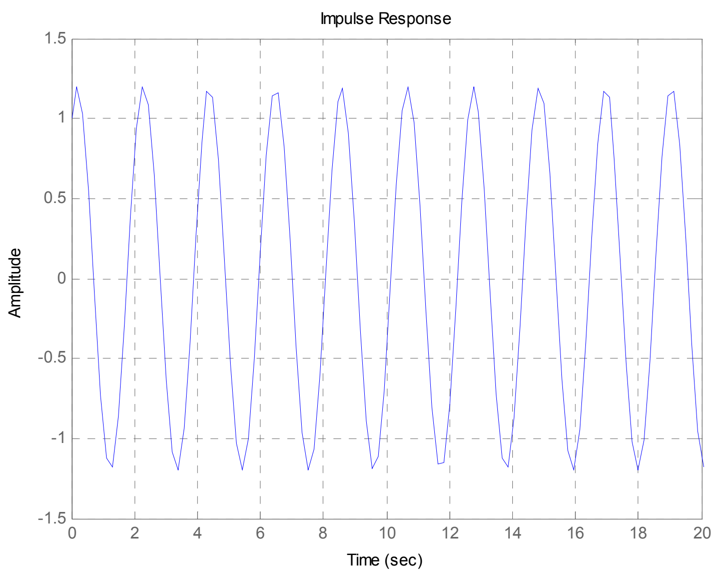
(a)



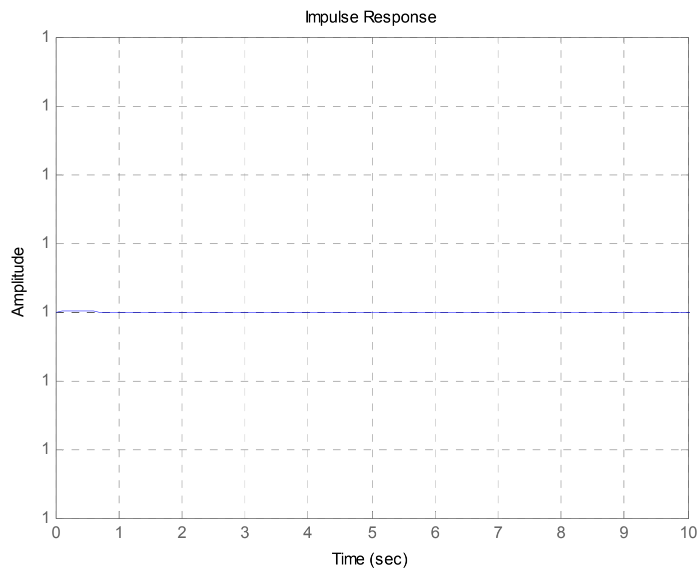
(b)


Impulse Response
Amplitude
Time (sec)
3
2.5
2
1.5
1
0.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9

(c)


Impulse Response
Amplitude
Time (sec)
1.5
1
0.5
0
-0.5
-1
-1.5
2
4
6
8
10
12
14
16
18
20

(d)



(e)

Fig. S6.29: Impulse response for Problem 6.29.

Problem 6.30

(a) The MATLAB code for computing the roots of the functions is given below:

(i) $H_1(s) = \frac{s^2 - 5s - 84}{s^4 + 7s^3 - 33s^2 - 355s - 700}$

```
>> roots([1  -5  -84])
>> roots([1  7  -33  -355  -700])
```

which computes two zeros at $s = 2.5 + j8.8176, -7$ and four poles at $s = 7, -5, -5, -4$.

(ii) $H_2(s) = \frac{s^2 - 19s + 84}{s^4 + 7s^3 - 33s^2 - 355s - 700}$

```
>> roots([1  -19  84])
>> roots([1  7  -33  -355  -700])
```

which computes two zeros at s = 12, 7 and four poles at $s = 7, -5, -5, -4$. Note that the zero at s = 7 cancels the pole at s = 7. Therefore, in effect there is one zero at s = 12, and three poles at $s = -5, -5, -4$.

(iii) $H_3(s) = \frac{s^3 + 20s^2 + 15s + 61}{s^4 + 5s^3 + 31s^2 + 125s + 150}$

```
>> roots([1  20  15  61])
>> roots([1  5  31  125  150])
```

which computes three zeros at $s = -19.3886, -0.3057 \pm j1.7472$ and four poles at $s = j5, -j5, -3, -2$.

(iv) $H_4(s) = \frac{s^3 - 10s^2 + 25s + 7}{s^6 + 6s^5 + 42s^4 + 48s^3 + 288s^2 + 96s + 544}$

```
>> roots([1  -10  25  7])
>> roots([1  6  42  48  288  96  544])
```

which computes three zeros at $s = -0.2536, 5.1268 \pm j1.1473$ and six poles at $s = \pm j2, \pm j2, -3 \pm j5$.

(v) $H_5(s) = \frac{s^2 + 3s + 7}{s^3 + (6 - j7)s^2 + (11 - j28)s + (6 - j21)}$

```
>> roots([1  3  7])
>> roots([1  6-j*7  11-j*28  6-j*21])
```

which computes two zeros at $s = -1.5 \pm j2.1794$ and three poles at $s = -3, -1, -2 + j7$.

(b)

(i) One pole (at s=7) is in the right-half of the s-plane. Therefore the system is NOT stable.

(ii) All three uncanceled poles $s = -5, -5, -4$ are located in the left-half of the s-plane. Therefore the system is absolutely stable.

(iii) No pole is located in the right-half of the s-plane, and two non-repeated poles $(s = \pm j5)$ are located on the imaginary axis. Therefore the system is marginally stable.

(iv) There are two repeated set of poles $(s = j2, j2, -j2, -j2)$ on the imaginary axis. Therefore the system is NOT stable.

(v) All three poles at $s = -3, -1, -2 + j7$ are located in the left-half of the s-plane. Therefore the system is absolutely stable.