Solutions to the Tutorial Problems in
the book "Magnetohydrodynamics of the Sun"
by ER Priest (2014)
CHAPTER 5

PROBLEM 5.1. Hydrodynamic Shock Wave.
Show that the shock relations for a hydrodynamic shock, namely,

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)M_1^2}{2+(\gamma-1)M_1^2}, \tag{1}$$

$$\frac{v_2}{v_1} = \frac{2+(\gamma-1)M_1^2}{(\gamma+1)M_1^2}, \tag{2}$$

$$\frac{p_2}{p_1} = \frac{2\gamma M_1^2-(\gamma-1)}{\gamma+1}, \tag{3}$$

together with the entropy condition

$$s_2 \geqslant s_1; \tag{4}$$

imply that the Mach number exceeds unity, i.e.,

$$M_1 \geqslant 1. \tag{5}$$

SOLUTION.
The equation

$$s_2 \geqslant s_1$$

is equivalent from the definition of s to

$$\frac{p_2}{\rho_2^\gamma} \geqslant \frac{p_1}{\rho_1^\gamma}$$

or

$$\frac{p_2}{p_1} \geqslant \frac{\rho_2^\gamma}{\rho_1^\gamma}.$$

The easiest way to prove that this implies $M_1 \geqslant 1$ is graphically, namely, to sketch the left and right hand sides of this inequality as functions of M_1.

Thus, for the graph of p_2/p_1 as a function of M_1, note that, when $M_1 = 0$, $p_2/p_1 = -(\gamma-1)/(\gamma+1)$, which is negative since $\gamma > 1$, and its gradient with

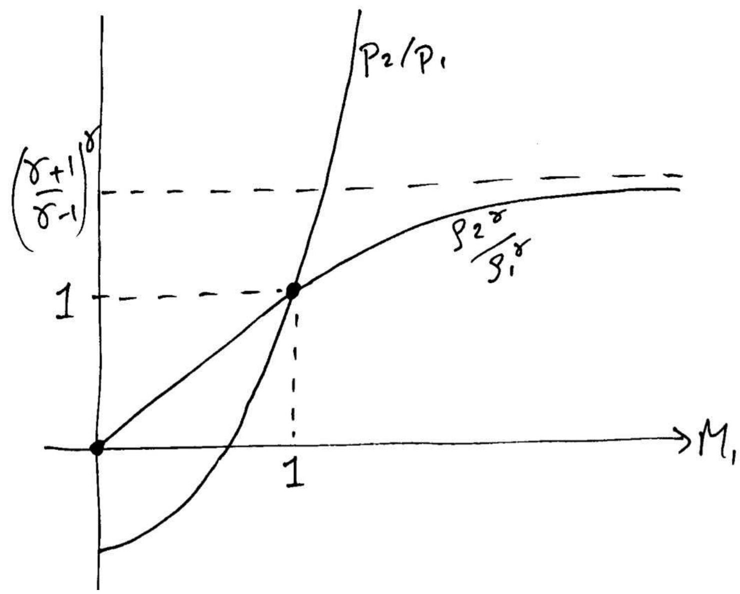

Figure 1: Sketches of p_2/p_1 and $\rho_2^\gamma/\rho_1^\gamma$ as functions of M_1 assuming $\gamma > 1$.

respect to M_1 vanishes there. Furthermore, $p_2/p_1 = 0$ when $M_1^2 = (\gamma-1)/2\gamma$ and $p_2/p_1 = 1$ when $M_1 = 1$. At $M_1 = 1$ the gradient of p_2/p_1 is equal to $4\gamma/(\gamma+1)$ and for larger values of M_1 it exceeds this value, increasing like M_1.

On the other hand, the graph of $\rho_2^\gamma/\rho_1^\gamma$ shows that it vanishes when $M_1 = 0$ and increases to the value of unity when $M_1 = 1$, where its gradient is $[4/(\gamma+1)]^\gamma$, which is less than that of p_2/p_1 when $M_1 = 1$ and decreases with M_1 for larger values of M_1.

Thus, the graphs of these two functions are as shown in Figure **??**, from which we deduce that

$$\frac{p_2}{p_1} \geqslant \frac{\rho_2^\gamma}{\rho_1^\gamma}$$

implies $M_1 \geqslant 1$, as required.