Appendix E
Sensitivity analysis

This appendix introduces the basic notions of **first-order** and **second-order sensitivity** of a dependent variable with respect to a given independent variable. Our aim is to provide measures that are independent of the scales of both the independent and dependent variables.

E.1 Basic concepts

Let $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ be a scalar-valued function of a scalar-valued variable x. Let $\Delta\phi(x) = \phi(x + \Delta x) - \phi(x)$ denote the change in $\phi(x)$ resulting from a change Δx in x. Then, $\frac{\Delta\phi(x)}{\phi(x)}$ is known as the **relative change** in $\phi(x)$, resulting from the **relative change** $\frac{\Delta x}{x}$ in x. The ratio

$$S_\phi(x) = \frac{\frac{\Delta\phi(x)}{\phi(x)}}{\frac{\Delta x}{x}} \tag{E.1.1}$$

is a measure of the **sensitivity** of the dependent variable ϕ with respect to the independent variable x. Rewriting the right-hand side of (E.1.1) and using the standard finite difference approximation to the derivative of ϕ with respect to x, it follows that

$$S_\phi(x) = \left(\frac{\Delta\phi(x)}{\Delta x}\right)\left(\frac{x}{\phi(x)}\right) \approx \frac{d\phi}{dx}\left(\frac{x}{\phi(x)}\right). \tag{E.1.2}$$

That is, $S_\phi(x)$ is directly proportional to the first derivative of ϕ with respect to x. This is the reason why the gradient of the dependent variable ϕ is usually taken as a measure of its (**first-order**) sensitivity.

Example E.1.1 The power P consumed by a resistor R driven at a voltage V is given by $P = V^2/R$. Clearly,

$$\frac{\mathrm{d}P}{\mathrm{d}V} = \frac{2V}{R}, \quad \text{and} \quad \frac{\mathrm{d}P}{\mathrm{d}R} = -\frac{V^2}{R^2}.$$

Hence, it can be verified that

$$S_P(V) = \left(\frac{\mathrm{d}P}{\mathrm{d}V}\right)\left(\frac{V}{P}\right) = 2,$$

and

$$S_P(R) = \left(\frac{\mathrm{d}P}{\mathrm{d}R}\right)\left(\frac{R}{P}\right) = -1.$$

Thus, the power is twice as sensitive (in magnitude) to changes in voltage compared to changes in the resistor. Also, notice that increase/decrease in voltage results in increase/decrease in power, but an increase/decrease in resistor results in decrease/increase in power.

Let $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}$ be twice continuously differentiable, let $\mathbf{x}^* = (x_1^*, x_2^*, \ldots, x_n^*)^{\mathrm{T}}$ be a vector such that $x_i^* \neq 0$, for $1 \leq i \leq n$. Let $\mathbf{x}$ be a vector close to $\mathbf{x}^*$ and define a vector

$$\mathbf{h} = \left(\frac{x_1 - x_1^*}{x_1}, \frac{x_2 - x_2^*}{x_2}, \ldots, \frac{x_n - x_n^*}{x_n}\right)^{\mathrm{T}}$$

and a diagonal matrix $\mathbf{D}$ as

$$\mathbf{D} = \mathrm{Diag}(x_1^*, x_2^*, \ldots, x_n^*) = \begin{pmatrix} x_1^* & & & \\ & x_2^* & & \\ & & \ddots & \\ & & & x_n^* \end{pmatrix}.$$

Then, using the standard second-order Taylor expansion (Appendix C), we get

$$\phi(\mathbf{x}) = \phi(\mathbf{x}^*) + (\mathbf{x} - \mathbf{x}^*)^{\mathrm{T}} \nabla \phi(\mathbf{x}^*) + \frac{1}{2!}(\mathbf{x} - \mathbf{x}^*)^{\mathrm{T}} \nabla^2 \phi(\mathbf{x}^*)(\mathbf{x} - \mathbf{x}^*) \tag{E.1.3}$$

Assuming that $\phi(x^*) \neq 0$, we can rewrite (E.1.3) as

$$\frac{\phi(\mathbf{x}) - \phi(\mathbf{x}^*)}{\phi(\mathbf{x}^*)} = \mathbf{h}^{\mathrm{T}} \left[\frac{\mathbf{D}\nabla\phi(\mathbf{x}^*)}{\phi(\mathbf{x}^*)}\right] + \mathbf{h}^{\mathrm{T}} \left[\frac{\mathbf{D}\nabla^2\phi(\mathbf{x}^*)\mathbf{D}}{2\phi(\mathbf{x}^*)}\right]\mathbf{h} \tag{E.1.4}$$

where the vector $\mathbf{h}$ and the diagonal matrix $\mathbf{D}$ are defined above. Define a vector

$$S_\phi^1(\mathbf{x}^*) = \frac{1}{\phi(\mathbf{x}^*)}\left(\mathbf{D}\nabla\phi(x^*)\right) \tag{E.1.5}$$

and a matrix

$$S_\phi^2(\mathbf{x}^*) = \frac{1}{2\phi(\mathbf{x}^*)}\left(\mathbf{D}\nabla^2\phi(x^*)\mathbf{D}\right). \tag{E.1.6}$$

Using these, (E.1.4) can be rewritten as

$$\frac{\phi(\mathbf{x}) - \phi(\mathbf{x}^*)}{\phi(\mathbf{x}^*)} = \mathbf{h}^{\mathrm{T}} S_\phi^1(\mathbf{x}^*) + \mathbf{h}^{\mathrm{T}} S_\phi^2(\mathbf{x}^*)\mathbf{h}. \tag{E.1.7}$$

The vector $S^1_\phi(\mathbf{x}^*)$ is called the **first-order sensitivity vector** and $S^2_\phi(\mathbf{x}^*)$ is called the **second-order sensitivity matrix**.

Remark E.1.1 Let $\mathbf{x}^*$ be the minimum of $\phi(\mathbf{x})$. Then since $\nabla\phi(x^*) = \mathbf{O}$, so is $S^1_\phi(\mathbf{x}^*)$. Hence, in such cases, $S^2_\phi(\mathbf{x}^*)$, which is directly related to the Hessian, holds the key. In particular, the diagonal elements of $\nabla^2\phi(x^*)$ and hence of $S^2_\phi(\mathbf{x}^*)$ relate to the sensitivity with respect to the individual variables.

Example E.1.2 Consider a closed cylindrical tank with x_1 as the radius of its circular base and x_2 as its height. This structure is commonly used for storing products like grain, gasoline, etc. The problem is to maximize the volume when the total surface area is fixed. Let V be the volume and A be the total surface area. Then $V = \pi x_1^2 x_2$ and $A = 2\pi x_1^2 + 2\pi x_1 x_2$. Let $\mathbf{x} = (x_1, x_2)^{\mathrm{T}}$. The problem may be restated equivalently as

$$\text{minimize } \frac{V}{\pi} = \phi(x) = -x_1^2 x_2,$$

subject to the constraint $A/2\pi = f(x) = x_1^2 + x_1 x_2 - a = 0$ for some given fixed area a. Consider the Lagrangian

$$L(x, \lambda) = -x_1^2 x_2 + \lambda(x_1^2 + x_1 x_2 - a)$$

where λ is the undetermined Lagrangian multiplier. Then

$$\frac{\partial L}{\partial x_1} = -2x_1 x_2 + 2\lambda x_1 + \lambda x_2$$
$$\frac{\partial L}{\partial x_2} = -x_1^2 + \lambda x_1$$

and

$$\frac{\partial L}{\partial \lambda} = x_1^2 + x_1 x_2 - a.$$

Setting these derivatives to zero and solving we obtain

$$x_1^* = \sqrt{\frac{a}{3}}\ x_2^* = 2\sqrt{\frac{a}{3}}\ \lambda = \sqrt{\frac{a}{3}}.$$

It can be verified that

$$\nabla\phi(x) = \begin{bmatrix} -2x_1 x_2 \\ -x_1^2 \end{bmatrix} \quad \text{and} \quad \nabla^2\phi(x) = \begin{bmatrix} -2x_2 & -2x_1 \\ -2x_1 & 0 \end{bmatrix}.$$

Thus,

$$\phi(x^*) = \frac{-2a\sqrt{a}}{3\sqrt{3}}, \quad \nabla\phi(x^*) = -\frac{a}{3}\begin{pmatrix} 4 \\ 1 \end{pmatrix}$$
$$\mathbf{D} = \sqrt{\frac{a}{3}}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad \nabla^2\phi(x^*) = -2\sqrt{\frac{a}{3}}\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}.$$

From these and the definition, we obtain

$$S_\phi^1(\mathbf{x}^*) = \frac{1}{\phi(\mathbf{x}^*)}\mathbf{D}\nabla\phi(x^*) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Since this is a constrained problem, not every arbitrary perturbation is admissible. From (E.1.4), we readily obtain the conditions for the admissibility of perturbations as

$$\left.\begin{aligned} (\Delta x_1, \Delta x_2)\nabla f(\mathbf{x}^*) &= (\Delta x_1, \Delta x_2)\begin{pmatrix} 2x_1^* \\ x_1^* \end{pmatrix} \\ &= \sqrt{\tfrac{a}{3}}(4\Delta x_1 + \Delta x_2) = 0 \end{aligned}\right\} \tag{E.1.8}$$

which in turn implies that $\Delta x_2 = -4\Delta x_1$. We now verify that in the space of admissible perturbations, the Hessian $\nabla^2\phi(x^*)$ is positive definite. Thus,

$$\left.\begin{aligned} &(\Delta x_1, \Delta x_2)\nabla^2\phi(x^*)\begin{pmatrix} \Delta x_1 \\ \Delta x_2 \end{pmatrix} \\ &\qquad = -2\sqrt{\tfrac{a}{3}}(\Delta x_1)^2(1, -4)\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}\begin{pmatrix} 1 \\ 4 \end{pmatrix} \\ &\qquad = 12\sqrt{\tfrac{a}{3}}(\Delta x_1)^2 > 0 \end{aligned}\right\} \tag{E.1.9}$$

Hence, x^* is a constrained minimum. It can be verified that

$$S_\phi^2(\mathbf{x}^*) = \frac{1}{2\phi(\mathbf{x}^*)}\mathbf{D}\nabla^2\phi(x^*)\mathbf{D} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}. \tag{E.1.10}$$

Letting $x_i - x_i^* = \Delta x_i$ for $i = 1, 2$, and combining all the above computations, we obtain

$$\begin{aligned} \frac{\phi(\mathbf{x})-\phi(\mathbf{x}^*)}{\phi(\mathbf{x}^*)} &= \left(\frac{\Delta x_1}{x_1^*}, \frac{\Delta x_2}{x_2^*}\right)\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \left(\frac{\Delta x_1}{x_1^*}, \frac{\Delta x_2}{x_2^*}\right)\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{\Delta x_1}{x_1^*} \\ \frac{\Delta x_2}{x_2^*} \end{pmatrix} \\ &= \left(2\frac{\Delta x_1}{x_1^*}, \frac{\Delta x_2}{x_2^*}\right) + \frac{(\Delta x_1)^2}{(x_1^*)^2} + \frac{2\Delta x_1 \Delta x_2}{x_1^* x_2^*}. \end{aligned}$$

Thus, after finding the optimum value, if we make an error of, say, 10% in x_1 (namely $\Delta x_1/x_1^* = 0.1$), neglecting the second-order terms, the percentage changes in ϕ is **twice** as large compared to a similar 10% error ($\Delta x_2/x_2 = 0.1$) in x_2. That is, the volume is twice as sensitive to changes in x_1, the radius, compared to those in x_2, the height.

E.2 Sensitivity with respect to perturbations in the constraints

Let $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}$; $f_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ for $1 \le i \le k$, where k is an integer less than or equal to n. Consider the problem of minimizing $\phi(\mathbf{x})$ subject to a set of equality constraints $f_i(\mathbf{x}) = a_i$ for $1 \le i \le k$. Consider the Lagrangian

$$L(\mathbf{x}, \lambda) = \phi(\mathbf{x}) + \sum_{i=1}^{k} \lambda_i [f_i(\mathbf{x}) - a_i] \tag{E.2.1}$$

where $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)^{\mathrm{T}}$ is a set of undetermined Lagrangian multipliers. Let $\mathbf{x}^*$, λ^* be the value of $\mathbf{x}$ and λ that minimizes $L(\mathbf{x}, \lambda)$ in (E.2.1). From Appendix D, it follows that such an $\mathbf{x}^*$ and λ^* are the solutions of

$$\left.\begin{array}{ll} & \nabla_{\mathbf{x}} L(\mathbf{x}, \lambda) = \nabla_{\mathbf{x}} \phi(\mathbf{x}) + \sum_{i=1}^{k} \lambda_i \nabla_{\mathbf{x}} f_i(\mathbf{x}) = 0 \\ \text{and} & \\ & \nabla_{\lambda} L(\mathbf{x}, \lambda) = f_i(\mathbf{x}) = a_i \text{ for } 1 \le i \le k. \end{array}\right\} \tag{E.2.2}$$

We are interested in analyzing the changes in $\phi(\mathbf{x}^*)$ resulting from the changes in a_i's. Let $\Delta\mathbf{x}$ be the changed in $\mathbf{x}$ induced by the changes Δa_i in a_i. Then

$$f_i(\mathbf{x}^* + \Delta\mathbf{x}) = a_i + \Delta a_i, \; 1 \le i \le k.$$

Neglecting the second- and higher-order terms we obtain

$$\Delta\phi = \phi(\mathbf{x}^* + \Delta\mathbf{x}) - \phi(\mathbf{x}^*) = (\Delta\mathbf{x})^{\mathrm{T}} \nabla\phi(x^*) \tag{E.2.3}$$

$$\Delta a_i = f_i(\mathbf{x}^* + \Delta\mathbf{x}) - f_i(\mathbf{x}^*) = (\Delta\mathbf{x})^{\mathrm{T}} \nabla f_i(\mathbf{x}^*) \tag{E.2.4}$$

Then, from (E.2.4), we have

$$\sum_{i=1}^{k} \lambda_i^* \Delta a_i = (\Delta x)^{\mathrm{T}} \sum_{i=1}^{k} \lambda_i^* \nabla f_i(\mathbf{x}^*). \tag{E.2.5}$$

Adding (E.2.4) and (E.2.5), we have

$$\begin{aligned} \Delta\phi &= (\Delta\mathbf{x})^{\mathrm{T}} \left[\nabla\phi(x^*) + \textstyle\sum_{i=1}^{k} \lambda_i^* \nabla f_i(\mathbf{x}^*)\right] - \sum i = 1^k \lambda_i^* \Delta a_i \\ &= -\textstyle\sum_{i=1}^{k} \lambda_i^* \Delta a_i \end{aligned} \tag{E.2.6}$$

since from (E.2.2) the first term within the square bracket on the right-hand side is zero.

In other words, the change in ϕ resulting from those in a_i are decided by the values of the Lagrangian multiplier arising from the solution of (E.2.2). In this sense, Lagrangian multipliers are sometimes called the **sensitivity parameters**.

Example E.2.1 Continuing the Example E.1.2, recall that $V/\pi = \phi(\mathbf{x}) = -x_1^2 x_2$ and $A/2\pi = f(\mathbf{x}) = x_1^2 + x_1 x_2 = a$. Recall that at the optimum, we have

$$x_1^* = \sqrt{\frac{a}{3}}, \; x_2^* = 2\sqrt{\frac{a}{3}}, \quad \text{and} \quad \lambda^* = \sqrt{\frac{a}{3}}.$$

$$a = x_1^2 + x_1 x_2 = (\lambda^*)^2 + 2(\lambda^*)^2 = 3(\lambda^*)^2.$$

Hence a 10% change in a induces a change

$$\Delta\phi = -\lambda^* \Delta a = -\lambda^*(0.1a) = -\lambda^*(0.3(\lambda^*)^2) = -0.3(\lambda^*)^3$$

in $\phi(\mathbf{x}^*) = -(x_1^*)^2 x_2^* = -2(\lambda^*)^3$, which translates into $\Delta\phi/\phi(\mathbf{x}^*) = 0.15$, that is, a 15% change.

Notes and references

The discussion related to the basic sensitivity analysis is adapted from Pierre and Lowe (1975). Sensitivity with respect to the constraints is discussed extensively in Luenberger (1973).