One sphere in Stokes flow

Élisabeth Guazzelli and Jeffrey F. Morris
with illustrations by Sylvie Pic

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Cambridge Texts in Applied Mathematics

1. Three single sphere flows
   - Rotation
   - Translation
   - Straining

2. Hydrodynamic force moments
   - Force
   - Torque
   - Stresslet
   - Computing the hydrodynamic force

3. Faxén laws for the sphere

4. A sphere in simple shear flow

Decomposition of the relative motion about a point

Fluid motion near a point (Taylor series to linear order) = uniform translation + linearly varying field

$$\begin{aligned}\mathbf{u}^{\infty}(\mathbf{x}) &= \mathbf{u}^{\infty}(\mathbf{x_0}) + \nabla\mathbf{u}^{\infty}(\mathbf{x_0})\cdot(\mathbf{x}-\mathbf{x_0}) \\ &= \mathbf{U}^{\infty} + \mathbf{\Omega}^{\infty}\cdot\mathbf{x} + \mathbf{E}^{\infty}\cdot\mathbf{x}\end{aligned}$$

Antisymmetric rate-of-rotation tensor

$$\Omega_{ij}^{\infty} = \frac{1}{2}\left[\frac{\partial u_i^{\infty}}{\partial x_j} - \frac{\partial u_j^{\infty}}{\partial x_i}\right]$$

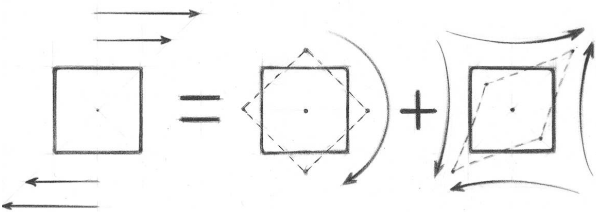
can be expressed as rotation vector: $\omega_i^{\infty} = -\frac{1}{2}\epsilon_{ijk}\Omega_{jk}^{\infty} = \frac{1}{2}(\nabla\times\mathbf{u}^{\infty})_i = \frac{1}{2}\tilde{\omega}_i^{\infty}$

Symmetric rate-of-strain tensor

$$E_{ij}^{\infty} = \frac{1}{2}\left[\frac{\partial u_i^{\infty}}{\partial x_j} + \frac{\partial u_j^{\infty}}{\partial x_i}\right]$$

Decomposition of shear into rotation and strain

Simple shear: contributions of rotation and strain equal in magnitude



$$\nabla \mathbf{u}^{\infty} = \boldsymbol{\Omega}^{\infty} + \mathbf{E}^{\infty}$$

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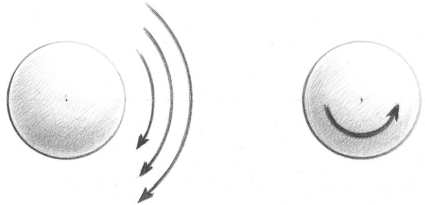
Sphere in a rotational field and sphere rotating



Disturbance problem for the sphere in a rotational field

Disturbance fields=velocity and pressure differences from those existing in the imposed flow in the absence of the body

$$\begin{aligned}\mathbf{u}(\mathbf{x}) &= \mathbf{u}^{\mathbf{actual}}(\mathbf{x}) - \mathbf{u}^{\infty}(\mathbf{x}) \\ p(\mathbf{x}) &= p^{actual}(\mathbf{x}) - p^{\infty}(\mathbf{x})\end{aligned}$$

Homogeneous Stokes equations for the disturbance fields

$$\begin{aligned}\nabla \cdot \mathbf{u} &= 0 \\ \mu \nabla^2 \mathbf{u} &= \nabla p\end{aligned}$$

Boundary conditions

$$\begin{array}{ccc} \mathbf{u} = -\boldsymbol{\omega}^{\infty} \times \mathbf{x} & \text{at} & r = |\mathbf{x}| = a \\ \mathbf{u} \text{ and } p \to 0 & \text{as} & r = |\mathbf{x}| \to \infty \end{array}$$

Solution for the pressure

$p = \sum$ decaying harmonics

spherical solid harmonics: $\dfrac{1}{r}$ and its gradients,

$$\frac{x_i}{r^3},\ \frac{\delta_{ij}}{r^3} - \frac{3x_ix_j}{r^5},\ \frac{\delta_{ij}x_k + \delta_{ik}x_j + \delta_{kj}x_i}{r^5} - \frac{5x_ix_jx_k}{r^7}, \ldots$$

$p =$ scalar and linear in $-\boldsymbol{\omega}^\infty$

$$p \propto \frac{\omega_i^\infty x_i}{r^3}$$

But $\boldsymbol{\omega}^\infty$ pseudo-vector whereas p frame independent

$$p = 0$$

Solution for the velocity

$\mathbf{u} = \sum$ decaying harmonics

$$\frac{1}{r}, \frac{x_i}{r^3}, \frac{\delta_{ij}}{r^3} - \frac{3x_i x_j}{r^5}, \frac{\delta_{ij}x_k + \delta_{ik}x_j + \delta_{kj}x_i}{r^5} - \frac{5x_i x_j x_k}{r^7}, \ldots$$

$\mathbf{u} =$ vector and linear in $-\boldsymbol{\omega}^\infty$

$$\mathbf{u}(\mathbf{x}) = \lambda_1 \, \boldsymbol{\omega}^\infty \times \frac{\mathbf{x}}{r^3}$$

Velocity boundary condition $\Rightarrow \lambda_1 = -a^3$

$$\mathbf{u}(\mathbf{x}) = -\boldsymbol{\omega}^\infty \times \mathbf{x} \left(\frac{a}{r}\right)^3$$

Sphere in a rotational field and sphere rotating

Sphere in a rotational field $\boldsymbol{\omega}^{\infty}$

- No pressure induced by the presence of the sphere
- Disturbance velocity:

$$\mathbf{u}(\mathbf{x}) = -\boldsymbol{\omega}^{\infty} \times \mathbf{x} \left(\frac{a}{r}\right)^3$$

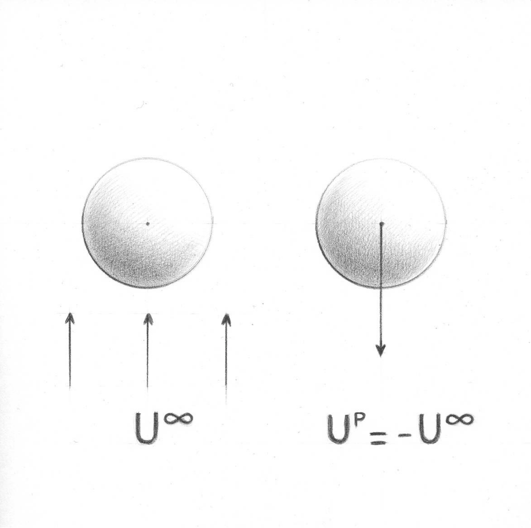
decays as r^{-2} and retains the symmetry of the boundary condition on the sphere

Sphere rotating at $\boldsymbol{\omega}^{p}$

- No induced pressure
- Velocity:

$$\mathbf{u}(\mathbf{x}) = \boldsymbol{\omega}^{p} \times \mathbf{x} \left(\frac{a}{r}\right)^3$$

Sphere in a translational field and translating sphere



Disturbance problem for the sphere fixed in the uniform stream $\mathbf{u}^\infty = \mathbf{U}^\infty$

Homogeneous Stokes equations for the disturbance fields

$$\begin{aligned} \nabla \cdot \mathbf{u} &= 0 \\ \mu \nabla^2 \mathbf{u} &= \nabla p \end{aligned}$$

Boundary conditions

$$\begin{array}{ccc} \mathbf{u} = -\mathbf{U}^\infty & \text{at} & r = |\mathbf{x}| = a \\ \mathbf{u} \text{ and } p \to 0 & \text{as} & r = |\mathbf{x}| \to \infty \end{array}$$

Solution for the pressure

$p = \sum$ decaying harmonics

$$\frac{1}{r}, \frac{x_i}{r^3}, \frac{\delta_{ij}}{r^3} - \frac{3x_i x_j}{r^5}, \frac{\delta_{ij}x_k + \delta_{ik}x_j + \delta_{kj}x_i}{r^5} - \frac{5x_i x_j x_k}{r^7}, \ldots$$

$p =$ scalar and linear in $-\mathbf{U}^\infty$

$$p = \lambda_1 \mathbf{U}^\infty \cdot \frac{\mathbf{x}}{r^3}$$

Solution for the velocity

Particular solution

$$\mathbf{u}^{(\mathbf{p})} = p\,\mathbf{x}/2\mu = \frac{\lambda_1}{2\mu}\frac{\mathbf{xx}}{r^3}\cdot\mathbf{U}^\infty$$

Homogeneous solution

- $\mathbf{u}^{(\mathbf{h})} = \sum$ decaying harmonics:

$$\frac{1}{r}, \frac{x_i}{r^3}, \frac{\delta_{ij}}{r^3} - \frac{3x_ix_j}{r^5}, \frac{\delta_{ij}x_k + \delta_{ik}x_j + \delta_{kj}x_i}{r^5} - \frac{5x_ix_jx_k}{r^7}, \ldots$$

- $\mathbf{u}^{(\mathbf{h})}$ = vector and linear in $-\mathbf{U}^\infty$

$$\mathbf{u}^{(\mathbf{h})} = \lambda_2\frac{1}{r}\mathbf{U}^\infty + \lambda_3\left(\frac{\mathbf{I}}{r^3} - \frac{3\mathbf{xx}}{r^5}\right)\cdot\mathbf{U}^\infty$$

Determination of constants

Continuity: $\nabla \cdot \mathbf{u} = 0$ yields (after some work)

$$\frac{\partial u_i}{\partial x_i} = \left(\frac{\lambda_1}{2\mu} - \lambda_2\right)\frac{\mathbf{U}^\infty \cdot \mathbf{x}}{r^3} = 0$$

$\Rightarrow \lambda_2 = \lambda_1/(2\mu)$

Boundary condition $\mathbf{u} = -\mathbf{U}^\infty$ at $r = a$

$$\frac{\lambda_1}{2\mu a}\left[n_i U_j^\infty n_j + U_i^\infty\right] + \frac{\lambda_3 U_j^\infty}{a^3}\left[\delta_{ij} - 3n_i n_j\right] = -U_i^\infty$$

$\Rightarrow \lambda_1 = -3\mu a/2$ and $\lambda_3 = -a^3/4$

Boundary conditions at ∞

Automatically satisfied by decaying harmonics

Disturbance velocity and pressure

Disturbance velocity

$$u_i = -\frac{3a}{4} U_j^\infty \left(\frac{\delta_{ij}}{r} + \frac{x_i x_j}{r^3} \right) - \frac{3a^3}{4} U_j^\infty \left(\frac{\delta_{ij}}{3r^3} - \frac{x_i x_j}{r^5} \right)$$

Disturbance pressure

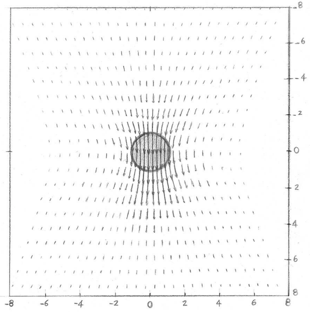
$$p - p_\infty = -\frac{3\mu a}{2} \frac{U_j^\infty x_j}{r^3}$$

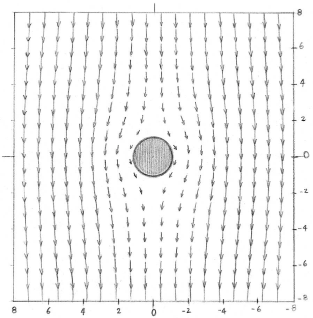
Disturbance solution

$\equiv$ velocity induced by a sphere translating in otherwise quiescent fluid at $\mathbf{U^p} = -\mathbf{U}^\infty$

Streamlines for translation

Disturbance streamlines for a translating sphere and full streamlines for a particle fixed in uniform stream





Three features of this Stokes flow field

- Slow decay of disturbance fields away from the translating sphere, as r^{-2} for the pressure and as r^{-1} for the dominant portion of the velocity
- Fore-aft symmetry: instantaneous streamlines of the disturbance velocity field converging above the particle and diverging below
- Disturbance twice as large at a point on the axis of symmetry $\approx 3\mathbf{U^p}a/2r$ as at a point at an equal distance in the transverse direction $\approx 3\mathbf{U^p}a/4r$ (due to the pressure field maintaining the flow as divergent-free)

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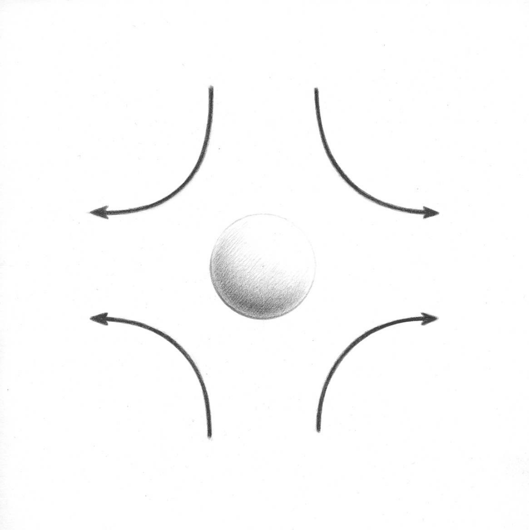
Sphere fixed in a strain field



Disturbance problem for the sphere at the origin in the straining flow $\mathbf{E}^\infty \cdot \mathbf{x}$

Homogeneous Stokes equations for the disturbance fields

$$\begin{aligned} \nabla \cdot \mathbf{u} &= 0 \\ \mu \nabla^2 \mathbf{u} &= \nabla p \end{aligned}$$

Boundary conditions

$$\begin{array}{lcl} \mathbf{u} = -\mathbf{E}^\infty \cdot \mathbf{x} & \text{at} & r = |\mathbf{x}| = a \\ \mathbf{u} \text{ and } p \to 0 & \text{as} & r = |\mathbf{x}| \to \infty \end{array}$$

Solution for the pressure

$p = \sum$ decaying harmonics

$$\frac{1}{r}, \frac{x_i}{r^3}, \frac{\delta_{ij}}{r^3} - \frac{3x_i x_j}{r^5}, \frac{\delta_{ij}x_k + \delta_{ik}x_j + \delta_{kj}x_i}{r^5} - \frac{5x_i x_j x_k}{r^7}, \ldots$$

$p =$ scalar and linear in second-rank tensor $\mathbf{E}^\infty$

$$\begin{aligned} p &\propto \left(\frac{\delta_{ij}}{r^3} - 3\frac{x_i x_j}{r^5}\right) E_{ij}^\infty \\ &= \lambda_1 \frac{x_i E_{ij}^\infty x_j}{r^5} \end{aligned}$$

because $\delta_{ij} E_{ij}^\infty = E_{ii}^\infty = \nabla \cdot \mathbf{u}^\infty = 0$ (continuity)

Solution for the velocity

Particular solution

$$u_i^{(p)} = \frac{p}{2\mu} x_i = \frac{\lambda_1}{2\mu} x_i \frac{x_j E_{jk}^{\infty} x_k}{r^5}$$

Homogeneous solution

- $\mathbf{u}^{(\mathbf{h})} = \sum$ decaying harmonics:

$$\frac{1}{r}, \frac{x_i}{r^3}, \frac{\delta_{ij}}{r^3} - \frac{3x_i x_j}{r^5}, \frac{\delta_{ij}x_k + \delta_{ik}x_j + \delta_{kj}x_i}{r^5} - \frac{5x_i x_j x_k}{r^7}, \ldots$$

- $\mathbf{u}^{(\mathbf{h})}$ = vector and linear in $\mathbf{E}^{\infty}$

$$u_i^{(h)} = \lambda_2 E_{ij}^{\infty} \frac{x_j}{r^3} + \lambda_3 E_{jk}^{\infty} \left(\frac{\delta_{ij}x_k + \delta_{ik}x_j}{r^5} - \frac{5x_i x_j x_k}{r^7} \right)$$

Disturbance velocity and pressure

Determination of constants

- Continuity: $\nabla \cdot \mathbf{u} = 0 \Rightarrow \lambda_2 = 0$
- Boundary conditions: $\Rightarrow \lambda_1 = -5\mu a^3 \quad \text{and} \quad \lambda_3 = -a^5/2$

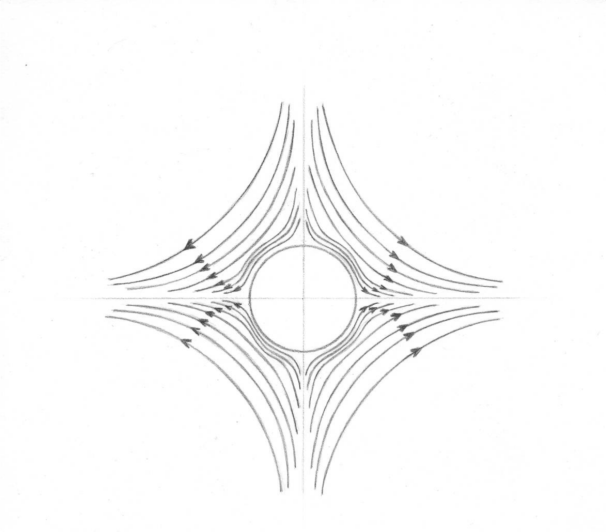
Disturbance velocity: dominant portion decays as r^{-2}

$$u_i = -\frac{5a^3}{2}\frac{x_i(x_j E^{\infty}_{jk} x_k)}{r^5} - \frac{a^5}{2} E^{\infty}_{jk}\left[\frac{\delta_{ij}x_k + \delta_{ik}x_j}{r^5} - \frac{5x_i x_j x_k}{r^7}\right]$$

Disturbance pressure: decays as r^{-3} and 'quadrupolar' form

$$p = -5\mu a^3 \frac{x_i E^{\infty}_{ij} x_j}{r^5}$$

Streamlines for a sphere fixed in a strain field



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Hydrodynamic drag force

Hydrodynamic force on the particle

$$\mathbf{F^h} = \int_{S_p} \boldsymbol{\sigma} \cdot \mathbf{n} \, dS$$

Force balance for inertialess motion

$$\mathbf{F^h} + \mathbf{F^e} = 0$$

$\mathbf{F^e}$ = external force (e.g. due to gravity or an interparticle force)

Stokes drag law (sphere held fixed in a uniform stream $\mathbf{U}^\infty$)

$$\mathbf{F^h} = 6\pi\mu a \mathbf{U}^\infty$$

linear in $\mathbf{U}^\infty$ and in a

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Hydrodynamic torque

Hydrodynamic torque on the particle

$$\mathbf{T^h} = \int_{S_p} \mathbf{x} \times \boldsymbol{\sigma} \cdot \mathbf{n} dS$$

Torque balance for inertialess motion

$$\mathbf{T^h} + \mathbf{T^e} = 0$$

$\mathbf{T^e}$ = external torque

Stokes law (sphere held fixed in rotational flow $\boldsymbol{\omega}^\infty \times \mathbf{x}$)

$$\mathbf{T^h} = 8\pi\mu a^3 \boldsymbol{\omega}^\infty$$

linear in $\boldsymbol{\omega}^\infty$ and in a^3

Stresslet

First moment of the surface traction over the surface

$$M_{ij} = \int_{S_p} x_i \sigma_{jk} n_k dS$$

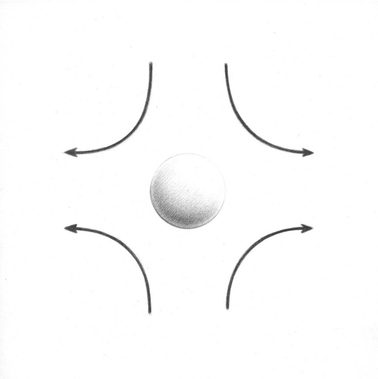
Symmetric portion = stresslet

$$S_{ij} = \frac{1}{2} \int_{S_p} [\sigma_{ik} x_j + \sigma_{jk} x_i] n_k dS$$

Antisymmetric portion = same information as torque

$$A_{ij} = \frac{1}{2} \int_{S_p} (x_i \sigma_{jk} - x_j \sigma_{ik}) n_k dS = -\frac{1}{2} \epsilon_{ijk} T_k$$

Stresslet for a solid sphere in a straining flow $\mathbf{E}^{\infty}\cdot\mathbf{x}$



$$\mathbf{S} = \frac{20\pi}{3}\mu a^3 \mathbf{E}^{\infty}$$

linear in $\mathbf{E}^{\infty}$ and in a^3

Result of the resistance of the rigid particle to the straining motion

Direct computation of the hydrodynamic force from the velocity field due to a sphere fixed in a uniform stream

Total traction on the sphere surface $\equiv$ constant

$$\boldsymbol{\sigma} \cdot \mathbf{n} = \frac{3\mu}{2a}\mathbf{U}^{\infty}$$

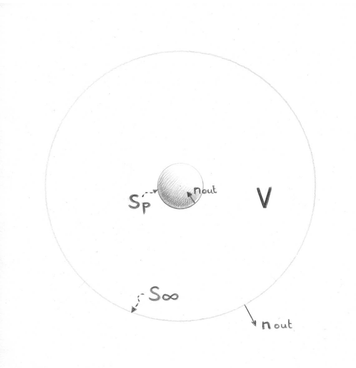
Hydrodynamic force

$$\mathbf{F^h} = \int_{r=a} [\frac{3\mu}{2a}\mathbf{U}^{\infty}]dS = \frac{3\mu}{2a}\mathbf{U}^{\infty} \times 4\pi a^2 = 6\pi\mu a\mathbf{U}^{\infty}$$

Stokes drag force for a particle moving with velocity $\mathbf{U^p}$ in an otherwise static fluid bath

$$\mathbf{F^h} = -6\pi\mu a\mathbf{U^p}$$

Simpler computation of the hydrodynamic force



Computation of dominant stress decaying as R^{-2}

- Divergence theorem applied to the Stokes momentum equations

$$\int_V \frac{\partial \sigma_{ij}}{\partial x_j} dV = \int_{S_p} \sigma_{ij} n_j^{out} dS + \int_{S_\infty} \sigma_{ij} n_j^{out} dS = 0$$

- Drag force

$$\begin{aligned} F_i^h = \int_{S_p} \sigma_{ij} n_j dS &= -\int_{S_p} \sigma_{ij} n_j^{out} dS \\ &= \int_{S_\infty} \sigma_{ij} n_j^{out} dS \\ &= \int_{S_\infty} \sigma_{ij}^{(-2)} n_j^{out} dS \end{aligned}$$

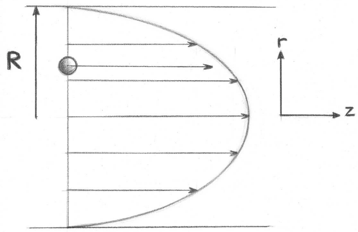
Faxén laws

Hydrodynamic force and force moments for a sphere in a general ambient flow field

$$
\begin{aligned}
\mathbf{F} &= 6\pi\mu a\left[\left(1+\frac{a^2}{6}\nabla^2\right)\mathbf{u}^\infty(\mathbf{x}=0)-\mathbf{U}^\mathbf{p}\right] \\
\mathbf{T} &= 8\pi\mu a^3\left[\boldsymbol{\omega}^\infty(\mathbf{x}=0)-\boldsymbol{\omega}^p\right] \\
\mathbf{S} &= \frac{20}{3}\pi\mu a^3\left(1+\frac{a^2}{10}\nabla^2\right)\mathbf{E}^\infty(\mathbf{x}=0)
\end{aligned}
$$

Additional pieces owing to the curvature of the flow $\nabla^2\mathbf{u}^\infty$ (evaluated at the position $\mathbf{x}=0$ occupied by the center of the particle) for the force and stresslet but not for the torque!

Lag of a sphere in a Poiseuille flow



$$u_z^\infty = U_{\max}\left[1 - \left(\frac{r}{R}\right)^2\right]$$

- Force-free suspended sphere at r_s

$$\begin{aligned} U_s &= [1 + \frac{a^2}{6}\nabla^2]u_z^\infty(r_s) \\ &= u_z^\infty(r_s) - \frac{2}{3}\frac{a^2}{R^2}U_{\max} \end{aligned}$$

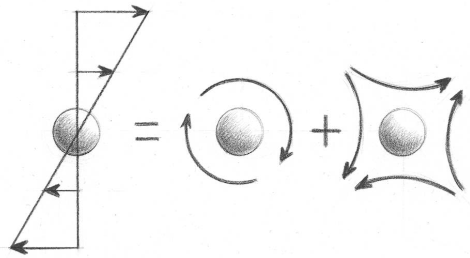
- Torque-free sphere in this situation has rotation vector of $\boldsymbol{\omega}^p = \boldsymbol{\omega}^\infty$

$$\omega_\theta^p = rU_{max}/R^2$$

(along the azimuthal coordinate θ)

Decomposition of a sphere in a shear

sphere in a rotation + sphere in strain



Freely mobile sphere in simple shear flow

Homogeneous Stokes equations

$$\begin{aligned} \nabla \cdot \mathbf{u} &= 0 \\ \mu \nabla^2 \mathbf{u} &= \nabla p \end{aligned}$$

Boundary conditions

$$\begin{aligned} \mathbf{u} &= \boldsymbol{\omega}^p \times \mathbf{x} && \text{at} && r = |\mathbf{x}| = a \\ \mathbf{u} - \mathbf{u}^\infty &\to 0 && \text{as} && r = |\mathbf{x}| \to \infty \end{aligned}$$

Faxen laws

- Force-free: $\Rightarrow \mathbf{U^p} = \mathbf{u}^\infty(\mathbf{x} = 0)$
- Torque-free: $\Rightarrow \boldsymbol{\omega}^p = \boldsymbol{\omega}^\infty(\mathbf{x} = 0)$

Flow caused by a sphere held fixed in linearly varying ambient flow field $\mathbf{u}^\infty = (\dot\gamma y, 0, 0) = (\mathbf{E}^\infty + \boldsymbol{\Omega}^\infty)\cdot\mathbf{x}$

Total velocity field

$$u_i = u_i^\infty - \frac{5a^3}{2}\frac{x_i(x_j E_{jk}^\infty x_k)}{r^5} - \frac{a^5}{2}E_{jk}^\infty\left[\frac{\delta_{ij}x_k + \delta_{ik}x_j}{r^5} - \frac{5x_i x_j x_k}{r^7}\right]$$

- Disturbance flow generated by the sphere only due to its resistance to the straining component of the shearing flow because no disturbance created by a freely-rotating sphere embedded in a solid-body rotation
- Rotationally dominated motion in the vicinity of the rotating sphere $\Rightarrow$ closed streamlines

Streamlines around freely mobile sphere in simple shear