Solutions to exercises in chapter 3

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1. **Kinematics**

   (a) Given $x(t) = x_0 \cdot \cos(\omega t)$, the is given by $v = dx/dt = -x_0 \cdot \omega \cdot \sin(\omega t)$. The maximum velocity is obtained when the sine is equal to -1, i.e. $v_{max} = x_0 \cdot \omega$.

   (b) Given the velocity from a) $v = -x_0 \cdot \omega \cdot \sin(\omega t)$, the acceleration is given by $a = dv/dt = -x_0 \cdot \omega^2 \cdot \cos(\omega t)$. The maximum acceleration is obtained when the cosine is equal to -1, i.e. $a_{max} = x_0 \cdot \omega^2$. Because $x(t) = x_0 \cdot \cos(\omega t)$, one can also write $a(t) = d^2/dt^2 x(t) = -\omega^2 \cdot x(t)$.

   (c) From $\cos^2(\alpha) + \sin^2(\alpha) = 1$, we obtain that $\sin(\alpha) = \sqrt{1 - \cos^2(\alpha)}$. If we insert this in the solution for (a), i.e. $v = -x_0 \cdot \omega \cdot \sin(\omega t)$, we obtain $v = -x_0 \cdot \omega \cdot \sqrt{1 - \cos^2(\omega t)} = -\omega \cdot \sqrt{x_0^2 - (x_0 \cdot \cos(\omega t))^2} = -\omega \cdot \sqrt{x_0^2 - x(t)^2}$.

2. **Circular motion**

   (a) For circular motion, the radius $|\vec{r}|$ is constant, i.e. does not depend on time, thus we know that $\frac{d}{dt}|\vec{r}| = 0$. Therefore also $|\vec{r}|^2$ does not depend on time, which means that we can write $\frac{d}{dt}|\vec{r}|^2 = 0$. Using the product rule to determine $\frac{d}{dt}|\vec{r}|^2 = \frac{d}{dt}(\vec{r} \cdot \vec{r})$, we obtain $\frac{d}{dt}|\vec{r}|^2 = \vec{r} \cdot \frac{d}{dt}(\vec{r}) + \frac{d}{dt}(\vec{r}) \cdot \vec{r} = 2\vec{r} \cdot \frac{d}{dt}(\vec{r}) = 2\vec{v} \cdot \vec{r}$. Since we know that this must be zero, i.e. $\frac{d}{dt}|\vec{r}|^2 = 0$, we also know that $\vec{v} \cdot \vec{r} = 0$. This means that the velocity is always perpendicular to the position vector.

   (b) We know that the speed $|\vec{v}|$ is constant, i.e. $\frac{d}{dt}|\vec{v}| = 0$. As above, we thus know that similarly $|\vec{v}|^2$ is constant, i.e. $\frac{d}{dt}|\vec{v}|^2 = 0$. Again taking the derivative $\frac{d}{dt}|\vec{v}|^2 = \frac{d}{dt}(\vec{v} \cdot \vec{v})$ by using the product rule, we obtain $\frac{d}{dt}|\vec{v}|^2 = \vec{v} \cdot \frac{d}{dt}(\vec{v}) + \frac{d}{dt}(\vec{v}) \cdot \vec{v} = 2\vec{v} \cdot \frac{d}{dt}(\vec{v}) = 2\vec{a} \cdot \vec{v}$. Since $\frac{d}{dt}|\vec{v}|^2 = 0$, we know that $\vec{a} \cdot \vec{v} = 0$, which means that the acceleration is perpendicular to the velocity.

3. **Damped oscillation**

   The acceleration of a pendulum is given by $ma_T = -mg\sin(\phi) - 6\pi\eta r v$, where the second term is Stokes friction as determined in Exercise 2.6

   For a pendulum we have: $s = \ell\phi$, thus $v = \dot{s} = \ell\dot{\phi}$ and $a_T = \dot{v} = \ell\ddot{\phi}$

   Inserting this above, we obtain $m\ell\ddot{\phi} = -mgsin(\phi) - 6\pi\eta r\ell\dot{\phi}$. For small angles, we can approximate the sine by the angle and have a simpler equation given by: $m\ell\ddot{\phi} = -mg\phi - 6\pi\eta r\ell\dot{\phi}$. Dividing by $m\ell$ gives a generic equation for an oscillation: $\ddot{\phi} = -\omega_0^2\phi - 1/\tau_0\dot{\phi}$ where $\omega_0 = \sqrt{g/\ell}$ and $1/\tau_0 = 6\pi\eta r/m$

   a) The pendulum will oscillate with a frequency $\omega = \sqrt{\omega_0^2 - 1/(4\tau_0^2)}$. With the above values we obtain numerically $\omega_0 = \sqrt{10}s^{-1} = \pi s^{-1}$ and $1/\tau_0 = 6\pi \cdot 0.2 \cdot 10^{-2}/0.1\frac{kg \cdot m}{m \cdot s \cdot kg} = 0.12\pi s^{-1}$. This gives an angular frequency of $\omega = \sqrt{\omega_0^2 - 1/(2\tau_0)^2} = \sqrt{1^2 - 0.06^2}\pi s^{-1} \simeq \pi s^{-1}$ or in other words a period of 2 s.

   b) From a), we see that the damping hardly influences the frequency of the pendulum in this case. Therefore, we can use the simpler form of $\omega \simeq \sqrt{g/\ell}$ for estimating the error of ω. Using relative errors, this means that $r_\omega = r_\ell/2 = 0.025$. Therefore the uncertainty of the period will be $\sigma_T = 0.05$ s.

   c) The decay-time of the amplitude is given by $\tau = 2\tau_0$, which means that the envelope of the amplitude is given by $\phi_0 \exp(-t/\tau)$. The pendulums energy is proportional to the amplitude squared, such that the energy will depend on time as $E_0(\exp(-t/\tau))^2 = E_0 \exp(-2t/\tau) = E_0 \exp(-t/\tau_0)$. If after a time $T_{1/2}$, half the energy has been dissipated, we can write $E(T_{1/2} = E_0/2 = E_0 \exp(-T_{1/2}/\tau_0)$. Solving this yields $\ln(2) = T_{1/2}/\tau_0$ or $T_{1/2} = \ln(2)\tau_0$. With $\tau_0 = \frac{m}{6\pi\eta r}$, we obtain $T_{1/2} = \frac{\ln(2)m}{6\pi\eta r}$ or numerically: $T_{1/2} = \frac{0.7 \cdot 0.1kgms}{6\pi 0.2kg 0.01m} = 7/(1.2\pi)s \simeq 1.9s$

   d) Since $T_{1/2} = \frac{\ln(2)m}{6\pi\eta r}$, we see that $r_T^2 = r_m^2 + r_\eta^2 + r_r^2 = 3 \cdot 0.1^2$. Therefore the error in the half-time is about 17% or 0.3 s.