Appendix SA5.3 Left and Right Inverses in Nonsquare Input-Output Systems[1]

A5.3.1 Introduction

A number of "generalized" inverses have been defined for rectangular matrices. Among the most common are left and right inverses. We concentrate on these in this Appendix.

Given $\underset{(m\times n)}{\mathbf{A}}$, where $m > n$, a *left inverse* for **A** can be found as long as the rank of **A** is n [denoted $\rho(\mathbf{A}) = n$; this is the largest possible rank for $\underset{(m\times n)}{\mathbf{A}}$ when $m > n$]. This inverse is developed as follows: if **A** is premultiplied by $\mathbf{A}'$, the resulting matrix $\mathbf{A}'\mathbf{A}$ is square (of order n), and it can be shown that $\rho(\mathbf{A}'\mathbf{A}) = n$ and hence $\mathbf{A}'\mathbf{A}$ is nonsingular. Then

$$(\mathbf{A}'\mathbf{A})^{-1}\mathbf{A}'\mathbf{A} = \mathbf{I}$$

and $(\mathbf{A}'\mathbf{A})^{-1}\mathbf{A}'$ is said to be a left inverse of **A**, from the defining property of an inverse—it generates an identity matrix when multiplying **A** (in this case, on the left);

$$\mathbf{A}_L^{-1} = (\mathbf{A}'\mathbf{A})^{-1}\mathbf{A}' \qquad \text{(A5.3.1)}$$

In a similar fashion, if $m < n$, then a *right inverse* can be found for **A**, now provided that $\rho(\mathbf{A}) = m$. (This is the maximum possible rank for $\underset{(m\times n)}{\mathbf{A}}$ when $m < n$.) In this case $\mathbf{A}\mathbf{A}'$ is square and of order m [and also $\rho(\mathbf{A}\mathbf{A}') = m$] so

$$\mathbf{A}\mathbf{A}'(\mathbf{A}\mathbf{A}')^{-1} = \mathbf{I}$$

and the right-inverse of **A** is

$$\mathbf{A}_R^{-1} = \mathbf{A}'(\mathbf{A}\mathbf{A}')^{-1} \qquad \text{(A5.3.2)}$$

We investigate how these inverses for nonsquare matrices might appear to have appeal for rectangular input-output systems.

A5.3.2 More Commodities than Industries (m > n)

Numerical Illustration

We reproduce the illustrative data from section 5.6 for $m = 3$ and $n = 2$. We will use these figures in what follows.

[1] This appendix contains a good deal of algebraic detail; it will be of interest primarily to readers who are curious about the concepts of "left" and "right" inverses and why they do not offer much help for rectangular input-output models.

Table A5.3.1 A Three-Commodity, Two-Industry Example

		Commodities			Industries		Final Demand	Total Output
		1	2	3	1	2		
Commodities	1				18	12	70	100
	2				20	16	40	76
	3				2	6	7	15
					U		**e**	**q**
Industries	1	90	10	8				108
	2	10	66	7				83
			V					**x**
Value Added					68	49	117	
					v′			
Total Output		100	76	15	108	83		
			q′		**x′**			

In section 5.6 we found that

$$\mathbf{B}=\begin{bmatrix}.1667 & .1446\\ .1852 & .1928\\ .0185 & .0723\end{bmatrix},\ \mathbf{C}=\begin{bmatrix}.8333 & .1205\\ .0926 & .7952\\ .0741 & .0843\end{bmatrix}\ \text{and}\ \mathbf{D}=\begin{bmatrix}.9 & .1316 & .5333\\ .1 & .8684 & .4667\end{bmatrix}$$

Commodity Technology

Approach I. We saw in section 5.5 that commodity technology generates the need for $\mathbf{C}^{-1}$ in both commodity-by-commodity and industry-by-industry direct requirements matrices— $\underset{(c\times c)}{\mathbf{A}_C}=\mathbf{B}\mathbf{C}^{-1}$ and $\underset{(i\times i)}{\mathbf{A}_C}=\mathbf{C}^{-1}\mathbf{B}$, respectively. This need for $\mathbf{C}^{-1}$ goes back to the operation of transforming (5.18) [$\mathbf{C}\mathbf{x}=\mathbf{q}$] into (5.19) [$\mathbf{x}=\mathbf{C}^{-1}\mathbf{q}$] and then substituting into the right-hand side of (5.13) [$\mathbf{q}=\mathbf{B}\mathbf{x}+\mathbf{e}$] to replace **x** with a function of **q**. We explore this transformation in the rectangular case when $m > n$.

This would seem to be exactly the sort of situation for which a *left inverse* for **C** would be suited. This inverse can be defined, as in (A5.3.1), provided $\rho(\mathbf{C})=n$. Here $\rho(\mathbf{C})=2=n$, and

$$\mathbf{C}\mathbf{x}=\mathbf{q}\Rightarrow \mathbf{C}_L^{-1}\mathbf{C}\mathbf{x}=\mathbf{C}_L^{-1}\mathbf{q}\Rightarrow \mathbf{x}=\mathbf{C}_L^{-1}\mathbf{q}$$

Thus $\mathbf{C}_L^{-1}=(\mathbf{C}'\mathbf{C})^{-1}\mathbf{C}'$ serves as a matrix for transformation of **q** into **x**, as required in (5.19). For this illustration,

$$\mathbf{C}_L^{-1}=\begin{bmatrix}1.2145 & -.1922 & .0771\\ -.1506 & 1.2690 & .1077\end{bmatrix}$$

and the reader can easily check that

$$\mathbf{x}=\begin{bmatrix}108\\83\end{bmatrix}=\mathbf{C}_L^{-1}\mathbf{q}=\begin{bmatrix}1.2145 & -.1922 & .0771\\-.1506 & 1.2690 & .1077\end{bmatrix}\begin{bmatrix}100\\76\\15\end{bmatrix}$$

as required.

Does this "solve" the problem of finding an appropriate $\mathbf{C}^{-1}$ for the $m > n$ case? Unfortunately, it does not. This is because there are infinitely many *other* matrices that could be found to premultiply $\mathbf{q}=\begin{bmatrix}100\\76\\15\end{bmatrix}$ to transform it into $\mathbf{x}=\begin{bmatrix}108\\83\end{bmatrix}$. Here's why. We know from the dimensions of **x** and **q** that we must have a 2×3 matrix $\mathbf{M}=\begin{bmatrix}m_{11} & m_{12} & m_{13}\\m_{21} & m_{22} & m_{23}\end{bmatrix}$ for which $\mathbf{Mq}=\mathbf{x}$. (When **C** is square and nonsingular, $\mathbf{M}=\mathbf{C}^{-1}$.)

For this example, this means finding the six elements m_{ij} that satisfy

$$\begin{bmatrix}m_{11} & m_{12} & m_{13}\\m_{21} & m_{22} & m_{23}\end{bmatrix}\begin{bmatrix}100\\76\\15\end{bmatrix}=\begin{bmatrix}108\\83\end{bmatrix}$$

Written out explicitly,

$$\begin{aligned}100m_{11}+76m_{12}+15m_{13}+\ \ 0m_{21}+\ 0m_{22}+\ 0m_{23}&=108\\0m_{11}+\ 0m_{12}+\ 0m_{13}+100m_{21}+76m_{22}+15m_{23}&=\ 83\end{aligned}\qquad\text{(A5.3.3)}$$

With $\mathbf{Q}=\begin{bmatrix}100 & 76 & 15 & 0 & 0 & 0\\0 & 0 & 0 & 100 & 76 & 15\end{bmatrix}=\left[\begin{array}{c:c}\mathbf{q}' & \mathbf{0}\\\hdashline\mathbf{0} & \mathbf{q}'\end{array}\right]$ and $\mathbf{m}=\begin{bmatrix}m_{11}\\m_{12}\\m_{13}\\m_{21}\\m_{22}\\m_{23}\end{bmatrix}$, (A5.3.3) is:

$$\mathbf{Qm}=\mathbf{x}\qquad\text{(A5.3.4)}$$

This is a set of two linear equations with six unknowns. Such systems are "underdetermined" (there are too many unknowns relative to the number of equations). If the equations are consistent (not contradictory, so that a solution can be found), they have an infinite number of possible solutions. It is easy to see that these equations are consistent.[2] For example, letting $m_{11}=108/100$ and $m_{22}=83/76$ (and all other $m_{ij}=0$) gives one possible solution. This means that

[2] The "formal" test of consistency for a set of linear equations like those in (A5.3.4) is to examine $\rho(\mathbf{Q})$ and $\rho(\mathbf{Q}\ \vdots\ \mathbf{x})$, where the latter matrix is **Q** augmented with **x** as an additional column. If the ranks

$$\mathbf{M}(1)=\begin{bmatrix}1.08 & 0 & 0\\ 0 & 1.0921 & 0\end{bmatrix}$$

accomplishes the task of transforming **q** into **x** and therefore it also performs the role of $\mathbf{C}^{-1}$. This is seen to be vastly different from $\mathbf{C}_L^{-1}$, but its elements are an equally valid "solution" to (A5.3.3) or (A5.3.4).[3]

In fact, another solution for an underdetermined linear system like that in (A5.3.3) can also be found using the *right* inverse of $\underset{(2\times 6)}{\mathbf{Q}}$ as defined in (A5.3.2), provided it exists [that is, provided that $\rho(\mathbf{Q})=2$]. The logic is this: Given a right inverse, so that $\mathbf{Q}\mathbf{Q}_R^{-1}=\mathbf{I}$, then from (A5.3.4)

$$\mathbf{Qm}=\mathbf{x}\Rightarrow\mathbf{Qm}=\mathbf{Q}\mathbf{Q}_R^{-1}\mathbf{x}\Rightarrow\mathbf{m}=\mathbf{Q}_R^{-1}\mathbf{x}$$

Using this approach, we find

$$\mathbf{M}(2)=\begin{bmatrix}.6750 & .5130 & .1012\\ .5187 & .3942 & .0778\end{bmatrix}$$

and this is seen to be entirely different from either $\mathbf{C}_L^{-1}$ or $\mathbf{M}(1)$.[4]

Each of these proxies for $\mathbf{C}^{-1}$ can be combined with **B** to generate an associated direct requirements matrix—$\underset{(c\times c)}{\mathbf{A}_C}=\mathbf{B}\mathbf{C}^{-1}$ or $\underset{(i\times i)}{\mathbf{A}_C}=\mathbf{C}^{-1}\mathbf{B}$. In general, the commodity-by-commodity matrices, $\underset{(c\times c)}{\mathbf{A}_C}$, produced in this way will all be different, as will the industry-by-industry matrices, $\underset{(i\times i)}{\mathbf{A}_C}$. Here are three examples of the $\underset{(c\times c)}{\mathbf{A}_C}$ that result from using $\mathbf{C}_L^{-1}$, $\mathbf{M}(1)$ or $\mathbf{M}(2)$, respectively:

are equal, the system is consistent.

[3] **M**(1) is an example of a "basic" solution to the system in (A5.3.3). Such solutions are found in this case by setting four variables equal to zero and solving the remaining set of two equations in two unknowns (provided that the coefficients matrix of the reduced system is nonsingular). In this case there are up to $C_4^6=15$ possible basic solutions but six of those have singular coefficients matrices. Solutions can also be found when the four selected variables are set equal to *any* four constants, not just zero.

[4] Recall that for a square and nonsingular **C**, all column sums of both **C** and $\mathbf{C}^{-1}$ were 1. This is not true for $\mathbf{C}_L^{-1}$, **M**(1) or **M**(2). However, we could add three linear equations to (A5.3.3), requiring that each column sum in **M** be 1. This will generate a consistent system in which the ranks of the coefficient and augmented matrix are four (one equation is redundant). Removing the last equation (for example) gives a consistent system whose coefficients matrix has maximum rank and so again a right inverse could be used. This will lead to another entirely different **M**.

$$\mathbf{BC}_L^{-1} = \begin{bmatrix} .1806 & .1514 & .0284 \\ .1959 & .2090 & .0350 \\ .0116 & .0882 & .0092 \end{bmatrix} \quad \mathbf{BM}(1) = \begin{bmatrix} .1800 & .1579 & 0 \\ .2000 & .2105 & 0 \\ .0200 & .0789 & 0 \end{bmatrix}$$

$$\mathbf{BM}(2) = \begin{bmatrix} .1875 & .1425 & .0281 \\ .2250 & .1710 & .0337 \\ .0500 & .0380 & .0075 \end{bmatrix}$$

Clearly, these are very different "representations" of the economy whose input-output accounts are given in Table A5.3.1. [Also, the column of zeros in $\mathbf{BM}(1)$ makes little sense.]

Finally, each $\underset{(c\times c)}{\mathbf{A}_C} = \mathbf{BC}^{-1}$ and $\underset{(i\times i)}{\mathbf{A}_C} = \mathbf{C}^{-1}\mathbf{B}$ has a pair of associated total requirements matrices—$(\mathbf{I}-\mathbf{BC}^{-1})^{-1}$ or $\mathbf{C}^{-1}(\mathbf{I}-\mathbf{BC}^{-1})^{-1}$ in the first case (as in Table 5.4) and $(\mathbf{I}-\mathbf{C}^{-1}\mathbf{B})^{-1}$ or $\mathbf{C}(\mathbf{I}-\mathbf{C}^{-1}\mathbf{B})^{-1}$ in the second case (as in Table 5.5). And any one of these matrices will of course be affected by the choice of the stand-in for $\mathbf{C}^{-1}$. Here are the three $(\mathbf{I}-\underset{(c\times c)}{\mathbf{A}_C})^{-1} = (\mathbf{I}-\mathbf{BC}^{-1})^{-1}$ examples that result from using $\mathbf{C}_L^{-1}$, $\mathbf{M}(1)$ and $\mathbf{M}(2)$, respectively:

$$(\mathbf{I}-\mathbf{BC}_L^{-1})^{-1} = \begin{bmatrix} 1.2810 & .2503 & .0456 \\ .3192 & 1.3316 & .0563 \\ .0434 & .1214 & 1.0148 \end{bmatrix} \quad [\mathbf{I}-\mathbf{BM}(1)]^{-1} = \begin{bmatrix} 1.2821 & .2564 & 0 \\ .3248 & 1.3316 & 0 \\ .0513 & .1103 & 1 \end{bmatrix}$$

$$[\mathbf{I}-\mathbf{BM}(2)]^{-1} = \begin{bmatrix} 1.2957 & .2247 & .0443 \\ .3549 & 1.2697 & .0532 \\ .0789 & .0599 & 1.0118 \end{bmatrix}$$

As would be expected, the variations in direct requirements matrices are transmitted to total requirements matrices, and there is no way of choosing a "best" solution from among these or other alternatives.

Approach II. As an alternative indication of the kind of indeterminacy that haunts rectangular input-output systems with commodity technology, we look briefly at a different approach that generates even more possible substitutes for $\mathbf{C}^{-1}$. We go back to $\mathbf{Cx} = \mathbf{q}$ [(5.18)] and ask whether there are other matrices, call them **T**, that could serve (in addition to $\mathbf{C}_L^{-1}$) as a left inverse to **C**. If so, then (5.18) could be transformed in the same way as with $\mathbf{C}_L^{-1}$, namely:

$$\mathbf{Cx} = \mathbf{q} \Rightarrow \mathbf{TCx} = \mathbf{Tq} \Rightarrow \mathbf{x} = \mathbf{Tq}$$

(When **C** is square and nonsingular, $\mathbf{T} = \mathbf{C}^{-1}$.) This means that we are looking for a 2×3 matrix $\mathbf{T} = \begin{bmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \end{bmatrix}$ that is defined by the relationship $\mathbf{TC} = \mathbf{I}$. (This is exactly the role played by $\mathbf{C}_L^{-1}$; the point here is to show that there are other matrices that accomplish exactly the same thing). Explicitly,

$$\mathbf{TC} = \begin{bmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \end{bmatrix} \begin{bmatrix} .8333 & .1205 \\ .0926 & .7952 \\ .0741 & .0843 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$$

or, written out completely,

$$\begin{aligned} .8333t_{11} + .0926t_{12} + .0741t_{13} + \quad 0t_{21} + \quad 0t_{22} + \quad 0t_{23} &= 1 \\ .1205t_{11} + .7952t_{12} + .0843t_{13} + \quad 0t_{21} + \quad 0t_{22} + \quad 0t_{23} &= 0 \\ 0t_{11} + \quad 0t_{12} + \quad 0t_{13} + .8333t_{21} + .0926t_{22} + .0741t_{23} &= 0 \\ 0t_{11} + \quad 0t_{12} + \quad 0t_{13} + .1205t_{21} + .7952t_{22} + .0843t_{23} &= 1 \end{aligned} \qquad \text{(A5.3.5)}$$

Again, we face an underdetermined system [fewer equations (four) than unknowns (six)], the equations can be shown to be consistent, and there are multiple solutions.

The conclusion is that in the case of $m > n$ under commodity technology, there are as many potential direct requirements matrices as there are solutions to an underdetermined linear equation system. And there is more than one relevant set of equations. That is, there are (infinitely) many possible candidates, and there is no way of choosing a "best" solution—even if ridiculous options, such as those with negative elements or zero columns, are discarded—so commodity technology models are impossible when $m > n$.

Industry Technology

We saw in section 5.6.2 that the $m > n$ case presented no problems in deriving unambiguous direct requirements matrices, $\underset{(c\times c)}{\mathbf{A}_I}$ and $\underset{(i\times i)}{\mathbf{A}_I}$. The only instance where $\mathbf{D}^{-1}$ is used is in the total requirements matrix that is of least interest, namely the commodity-by-industry matrix $\mathbf{D}^{-1}(\mathbf{I} - \mathbf{DB})^{-1}$ (Table 5.5).

The trouble arises in trying to move from $\mathbf{Dq} = \mathbf{x}$ [in (5.16)] to $\mathbf{q} = \mathbf{D}^{-1}\mathbf{x}$ [in (5.17)]. From Table 5.9, we know $\mathbf{D} = \begin{bmatrix} .9 & .1316 & .5333 \\ .1 & .8684 & .4667 \end{bmatrix}$, and the problem is to find a transformation from $\mathbf{x} = \begin{bmatrix} 108 \\ 83 \end{bmatrix}$ to $\mathbf{q} = \begin{bmatrix} 100 \\ 76 \\ 15 \end{bmatrix}$. Since $\mathbf{D}$ is a 2×3 matrix, we cannot use a left inverse. We need to find a 3×2 matrix, $\mathbf{R}$, for which $\mathbf{q} = \mathbf{Rx}$. (If $\mathbf{D}$ were square and nonsingular, $\mathbf{R}$ would be $\mathbf{D}^{-1}$.) The requirement is

$$\begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \\ r_{31} & r_{32} \end{bmatrix} \begin{bmatrix} 108 \\ 83 \end{bmatrix} = \begin{bmatrix} 100 \\ 76 \\ 15 \end{bmatrix}$$

So we are faced with three linear equations containing six unknowns:

$$\begin{aligned}
108r_{11} + 83r_{12} + 0r_{21} + 0r_{22} + 0r_{31} + 0r_{32} &= 100 \\
0r_{11} + 0r_{12} + 108r_{21} + 83r_{22} + 0r_{31} + 0r_{32} &= 76 \\
0r_{11} + 0r_{12} + 0r_{21} + 0r_{22} + 108r_{31} + 83r_{32} &= 15
\end{aligned}$$

The issues here are exactly the same as we have already explored—multiple solutions to an underdetermined system of linear equations with no valid criteria for preferring any one of the possible solutions.[5] Hence with an $m > n$ rectangular system, a satisfactory commodity-by-industry total requirements matrix, $\mathbf{D}^{-1}(\mathbf{I}-\mathbf{DB})^{-1}$, cannot be found. As already noted, this is the least interesting or useful of the total requirements matrices under industry technology, so it is not a matter of great concern.

A5.3.3 Fewer Commodities than Industries (m < n)

Numerical Illustration

In some cases, real world input-output *accounts* might be presented with more industries than commodities [e.g., a "scrap" or a "second-hand/used goods" row (an industry) in a make matrix, where the total amount of scrap or second-hand goods in the economy is counted, without an accompanying commodity]. In general, however, the accounts will be aggregated in such a way that $m \geq n$ before using the data in an input-output *model*. However, some writers have considered the implications of a model with $m < n$, and we do that briefly here, primarily for completeness. Table A5.3.2 contains data that we will use in what follows.

Table A5.3.2 A Two-Commodity, Three-Industry Example

		Commodities		Industries			Final Demand	Total Output
		1	2	1	2	3		
Commodities	1			18	12	2	70	102
	2			20	16	6	40	82
					U		**e**	**q**
Industries	1	90	10					100
	2	8	60					68
	3	4	12					16
		V						**x**
Value Added				62	40	8	110	
					v′			
Total Output		102	82	100	68	16		
		q′			**x′**			

[5] Again, one could also include the requirement that columns in **R** sum to 1. That adds two more linear equations, but the system is still underdetermined, with an infinite number of solutions.

As was true in the $m > n$ case, there is no problem with respect to dimensions of the usual matrices in the commodity-industry model. Here these will be:

$$\underset{(2\times3)}{\mathbf{B}} = \underset{(2\times3)}{\mathbf{U}}\,\underset{(3\times3)}{\hat{\mathbf{x}}^{-1}},\quad \underset{(2\times3)}{\mathbf{C}} = \underset{(2\times3)}{\mathbf{V}'}\,\underset{(3\times3)}{\hat{\mathbf{x}}^{-1}},\quad \underset{(3\times2)}{\mathbf{D}} = \underset{(3\times2)}{\mathbf{V}}\,\underset{(2\times2)}{\hat{\mathbf{q}}^{-1}}$$

For this example, it is easily found that

$$\mathbf{B} = \begin{bmatrix} .18 & .1765 & .125 \\ .20 & .2353 & .375 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} .9 & .1176 & .25 \\ .1 & .8824 & .75 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} .8824 & .1220 \\ .0784 & .7317 \\ .0392 & .1463 \end{bmatrix}$$

Commodity Technology

Approach I. Again, the problem emerges in trying to convert $\underset{(2\times3)}{\mathbf{C}}\,\underset{(3\times1)}{\mathbf{x}} = \underset{(2\times1)}{\mathbf{q}}$ into $\underset{(3\times1)}{\mathbf{x}} = \underset{(3\times2)}{\mathbf{C}^{-1}}\,\underset{(2\times1)}{\mathbf{q}}$. A left inverse for $\underset{(m\times n)}{\mathbf{C}}$ is impossible since $m < n$. Letting $\mathbf{M} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \\ m_{31} & m_{32} \end{bmatrix}$, we want to find the elements in **M** so that

$$\mathbf{x} = \begin{bmatrix} 100 \\ 68 \\ 16 \end{bmatrix} = \mathbf{Mq} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \\ m_{31} & m_{32} \end{bmatrix} \begin{bmatrix} 102 \\ 82 \end{bmatrix}$$

which means

$$\begin{aligned} 102m_{11} + 82m_{12} + 0m_{21} + 0m_{22} + 0m_{31} + 0m_{32} &= 100 \\ 0m_{11} + 0m_{12} + 102m_{21} + 82m_{22} + 0m_{31} + 0m_{32} &= 68 \\ 0m_{11} + 0m_{12} + 0m_{21} + 0m_{22} + 102m_{31} + 82m_{32} &= 16 \end{aligned}$$

The issues here are exactly the same as what we faced in looking for $\mathbf{D}^{-1}$ in the $m > n$ case immediately above. We have an underdetermined system of three linear equations and six unknowns, meaning that basic solutions and infinitely many others can be found. For example, by inspection $m_{11} = 100/102$, $m_{22} = 68/82$ and $m_{31} = 16/102$ is one solution that satisfies the equations, generating $\underset{(c\times c)}{\mathbf{A}_C} = \mathbf{BM} = \begin{bmatrix} .1961 & .1463 \\ .2549 & .1951 \end{bmatrix}$ and $(\mathbf{I} - \mathbf{BM})^{-1} = \begin{bmatrix} 1.3200 & .2400 \\ .4180 & 1.3184 \end{bmatrix}$. Again, requirements that either or both column sums in **M** be 1 can also be added but the system remains underdetermined.

Consequently, as in the $m > n$ case, there are no unique direct requirements matrices with the commodity technology assumption (either commodity-by-commodity or industry-by-industry), and thus there are also no unique total requirements matrices. And there is no way to choose from among the alternatives.

Approach II. As with $\mathbf{C}^{-1}$ in the $m > n$ case, we could also look for a matrix **T** such that $\underset{(3\times2)}{\mathbf{T}}\,\underset{(2\times3)}{\mathbf{C}} = \underset{(3\times3)}{\mathbf{I}}$. As can easily be established, and in contrast to the $m > n$ case, this generates nine linear equations and six unknowns (the elements in **T**). However, the system is inconsistent (the rank of the coefficients matrix is six and the rank of the augmented matrix is seven) and there are no solutions.

Approach III. As a matter primarily of historical curiosity, we note that some writers (e.g., Cressy, 1976) have tried to salvage the commodity technology model in the $m < n$ case, using a *right* inverse to **C**. The argument goes like this. Start with the fundamental commodity technology assumption, $\underset{(c\times i)}{\mathbf{B}} = \underset{(c\times c)}{\mathbf{A}_C}\,\underset{(c\times i)}{\mathbf{C}}$. For our illustration with $m = 2$ and $n = 3$, if $\rho(\mathbf{C}) = 2$, we can find $\mathbf{C}_R^{-1}$ and multiply through on the right, so that

$$\mathbf{B} = \underset{(c\times c)}{\mathbf{A}_C}\,\mathbf{C} \Rightarrow \mathbf{B}\mathbf{C}_R^{-1} = \underset{(c\times c)}{\mathbf{A}_C}\,\mathbf{C}\mathbf{C}_R^{-1} = \underset{(c\times c)}{\mathbf{A}_C}$$

Using **C** from Table A5.3.2, we find $\rho(\mathbf{C}) = 2$, so

$$\mathbf{C}_R^{-1} = \mathbf{C}'(\mathbf{C}\mathbf{C}')^{-1} = \begin{bmatrix} 1.1195 & -.2419 \\ -.1687 & .7007 \\ .0492 & .5412 \end{bmatrix}$$

and

$$\underset{(c\times c)}{\mathbf{A}_C} = \mathbf{B}\mathbf{C}_R^{-1} = \begin{bmatrix} .1779 & .1478 \\ .2027 & .3194 \end{bmatrix} \text{ and } (\mathbf{I} - \mathbf{B}\mathbf{C}_R^{-1}) = \begin{bmatrix} 1.2852 & .2790 \\ .3827 & 1.5525 \end{bmatrix}$$

At first glance, this seems to be a potentially useful approach, but in fact it is a failure. If we put $\underset{(2\times2)}{\mathbf{A}_C} = \mathbf{B}\mathbf{C}_R^{-1} = \begin{bmatrix} .1779 & .1478 \\ .2027 & .3194 \end{bmatrix}$ back into the defining equation—$\underset{(2\times3)}{\mathbf{B}} = \underset{(2\times2)}{\mathbf{A}_C}\,\underset{(2\times3)}{\mathbf{C}}$—we find that it doesn't satisfy the equation at all.

$$\mathbf{B} = \begin{bmatrix} .18 & .1765 & .125 \\ .20 & .2353 & .375 \end{bmatrix} \neq \mathbf{A}_C\mathbf{C} = \begin{bmatrix} .1749 & .1513 & .1553 \\ .2143 & .3057 & .2902 \end{bmatrix}$$

To simplify notation let the unknown matrix $\mathbf{A}_C$ be denoted by $\mathbf{Q} = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix}$. Then the elements q_{ij} must satisfy $\underset{(2\times3)}{\mathbf{B}} = \underset{(2\times2)}{\mathbf{Q}}\,\underset{(2\times3)}{\mathbf{C}}$. This is the same as

$$\underset{(3\times2)}{\mathbf{B}'} = (\underset{(2\times2)}{\mathbf{Q}}\,\underset{(2\times3)}{\mathbf{C}})' = \underset{(3\times2)}{\mathbf{C}'}\,\underset{(2\times2)}{\mathbf{Q}'}$$

Using $(\mathbf{Q}')_1$ for the first column of $\mathbf{Q}'$ (the first row of **Q**) and $(\mathbf{B}')_1$ for the first column of $\mathbf{B}'$ (the first row of **B**) and similarly for $(\mathbf{Q}')_2$ and $(\mathbf{B}')_2$, we have $\mathbf{C}'\left[\mathbf{Q}_1' \mid \mathbf{Q}_2'\right] = \left[\mathbf{B}_1' \mid \mathbf{B}_2'\right]$ or

$$\underset{(3\times2)}{\mathbf{C}'}\underset{(2\times1)}{(\mathbf{Q}')_1} = \underset{(3\times1)}{(\mathbf{B}')_1} \text{ and } \underset{(3\times2)}{\mathbf{C}'}\underset{(2\times1)}{(\mathbf{Q}')_2} = \underset{(3\times1)}{(\mathbf{B}')_2}$$

Each of these is three linear equations in two unknowns; q_{11} and q_{12} in the first equation, q_{21} and q_{22} in the second.

It is sufficient to look closely at the first set only (exactly similar reasoning applies to the second set):

$$\begin{aligned} .9q_{11} + \quad .1q_{12} &= .18 \\ .1176q_{11} + .8824q_{12} &= .1765 \\ .25q_{11} + \quad .75q_{12} &= .125 \end{aligned}$$

These equations are inconsistent, since $\rho(\mathbf{C}') = 2$ and $\rho\left(\mathbf{C}' \mid (\mathbf{B}')_1\right) = 3$. At the same time, $\rho(\mathbf{C}') = 2$ and a left inverse can be found for $\mathbf{C}'$,

$$(\mathbf{C}')_L^{-1} = (\mathbf{C}\mathbf{C}')^{-1}\mathbf{C} = \begin{bmatrix} 1.1195 & -.1687 & .0492 \\ -.2419 & .7007 & .5412 \end{bmatrix}$$

and so

$$(\mathbf{C}')_L^{-1}\mathbf{C}'(\mathbf{Q}')_1 = (\mathbf{C}')_L^{-1}(\mathbf{B}')_1 \Rightarrow (\mathbf{Q}')_1 = (\mathbf{C}')_L^{-1}(\mathbf{B}')_1$$

This yields $(\mathbf{Q}')_1 = (\mathbf{C}')_L^{-1}(\mathbf{B}')_1 = \begin{bmatrix} .1779 \\ .1478 \end{bmatrix}$. Similarly, $(\mathbf{Q}')_2 = (\mathbf{C}')_L^{-1}(\mathbf{B}')_2 = \begin{bmatrix} .2027 \\ .3194 \end{bmatrix}$

What kinds of "solutions" to inconsistent equations are these? It turns out that for an inconsistent linear equation system with more equations than unknowns, a left inverse is known to provide a "solution" that does not satisfy the equations exactly (it can't; they are inconsistent) but it is "least wrong" in a specific sense.[6] Nonetheless, from the point of view of dealing with the problem of finding an acceptable $\underset{(c\times c)}{\mathbf{A}_C}$ for the $m < n$ case, it is of no use.

We see that $\mathbf{Q} = \left[\mathbf{Q}_1' \mid \mathbf{Q}_2'\right]' = \begin{bmatrix} .1779 & .1478 \\ .2027 & .3194 \end{bmatrix}$ is precisely $\underset{(c\times c)}{\mathbf{A}_C}$ that was obtained by using $\mathbf{C}_R^{-1}$ on $\mathbf{B} = \underset{(c\times c)}{\mathbf{A}_C}\mathbf{C}$. Here's why. Starting with $\mathbf{C}'\mathbf{Q}' = \mathbf{B}'$ and given $(\mathbf{C}')_L^{-1}$,[7]

$$\mathbf{C}'\mathbf{Q}' = \mathbf{B}' \Rightarrow (\mathbf{C}')_L^{-1}\mathbf{C}'\mathbf{Q}' = (\mathbf{C}')_L^{-1}\mathbf{B}' \Rightarrow \mathbf{Q}' = (\mathbf{C}')_L^{-1}\mathbf{B}' \Rightarrow \mathbf{Q} = [(\mathbf{C}')_L^{-1}\mathbf{B}']' = \mathbf{B}[(\mathbf{C}')_L^{-1}]' = \mathbf{B}\mathbf{C}_R^{-1}$$

[6] Specifically, the sum of the squares of the (straight line) distances from the solution to each of the equations is minimum. For more detail, see, for example, Miller (2000, Chapter 2).

[7] The last step is a result of the matrix algebra fact that for an $m \times n$ matrix **A**, where **A** is of full rank, it can be shown, using (A5.3.1) and (A5.3.2), that $(\mathbf{A}_L^{-1})' = (\mathbf{A}')_R^{-1}$ when $m > n$ and $(\mathbf{A}_R^{-1})' = (\mathbf{A}')_L^{-1}$ when $m < n$.

So using $\mathbf{C}_R^{-1}$ to find $\underset{(c\times c)}{\mathbf{A}_C} = \mathbf{B}\mathbf{C}_R^{-1}$ is equivalent to accepting an invalid "solution" to a set of inconsistent equations, and in dealing with the problem of finding an acceptable $\underset{(c\times c)}{\mathbf{A}_C}$ for the $m < n$ case, neither $\mathbf{C}_L^{-1}$ nor $\mathbf{C}_R^{-1}$ is of any use.

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As with the $m > n$ case, there are no problems here. We can easily find that

$$\underset{(2\times 2)}{\mathbf{A}_I} = \underset{(2\times 3)}{\mathbf{B}}\,\underset{(3\times 2)}{\mathbf{D}} = \begin{bmatrix} .1776 & .1694 \\ .2096 & .2514 \end{bmatrix} \text{ and } \underset{(3\times 3)}{\mathbf{A}_I} = \underset{(3\times 2)}{\mathbf{D}}\,\underset{(2\times 3)}{\mathbf{B}} = \begin{bmatrix} .1832 & .1844 & .1560 \\ .1605 & .1860 & .2842 \\ .0363 & .0414 & .0598 \end{bmatrix}$$

When it comes to total requirements matrices, everything is fine with $(\mathbf{I}-\mathbf{BD})^{-1}$ or $\mathbf{D}(\mathbf{I}-\mathbf{BD})^{-1}$ (Table 5.4) or with $(\mathbf{I}-\mathbf{DB})^{-1}$ (Table 5.5).

As is to be expected, the only problem is with $\mathbf{D}^{-1}(\mathbf{I}-\mathbf{DB})^{-1}$, because of $\mathbf{D}^{-1}$. As usual, the issue arises with the transformation from $\mathbf{Dq} = \mathbf{x}$ [in (5.16)] to $\mathbf{q} = \mathbf{D}^{-1}\mathbf{x}$ [in (5.17)]. Here, however, we could find a *left* inverse for **D** [since $m > n$ and $\rho(\mathbf{D}) = 2$], but it is by no means the only acceptable matrix for transforming **x** into **q**. Following the usual procedure, we want a matrix $\mathbf{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \end{bmatrix}$ for which

$$\mathbf{q} = \begin{bmatrix} 102 \\ 82 \end{bmatrix} = \mathbf{Rx} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \end{bmatrix}\begin{bmatrix} 100 \\ 68 \\ 16 \end{bmatrix}$$

This equation system is underdetermined (two linear equations and six unknowns), and there is no unique transformation from **x** to **q** that would correspond to the $m = n$ case for which $\mathbf{R} = \mathbf{D}^{-1}$ (as long as **D** is nonsingular). And so, again, we have no satisfactory total requirements matrix for this least interesting commodity-by-industry case.

References

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