Chapter 1: Real Numbers and Functions

Part B: Real Functions and Graphs

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Definition of a Function

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In the above definition, X is called the **domain** of f and Y is called the **codomain** of f. The notation $f: X \to Y$ is used as shorthand for "f is a function with domain X and codomain Y", and is read as "f is a function from X to Y".

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The subset of Y consisting of the values actually taken by the function is called its **image** or **range**.

Consider $f\colon X \to Y$. Let $a \in X$ and $b \in Y$ such that $f(a) = b$. Then b is called the **image** of the point a under f, while a is called the **pre-image** of b under f.

Examples

Consider $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x + 1$.
The domain and codomain are both $\mathbb{R}$.

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Consider $g\colon \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^2$.
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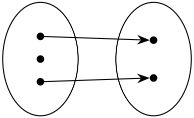
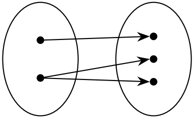
The image of g is $[0, \infty)$.

Examples



(A)



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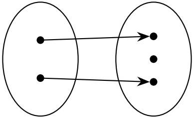

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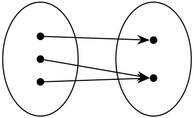

(D)

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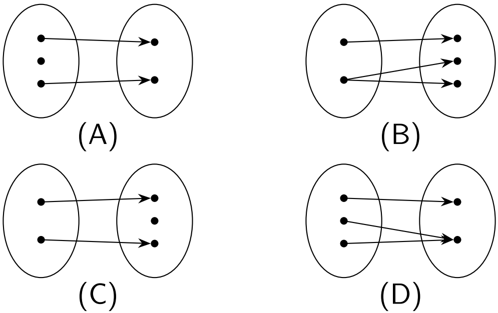

(A) does not represent a function because there is a point in the domain that has no image.

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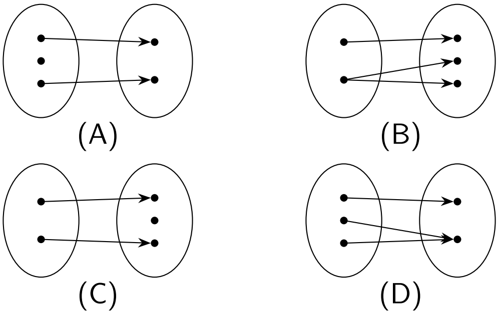

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(B) does not represent a function, since there is a point in the domain that has two images.

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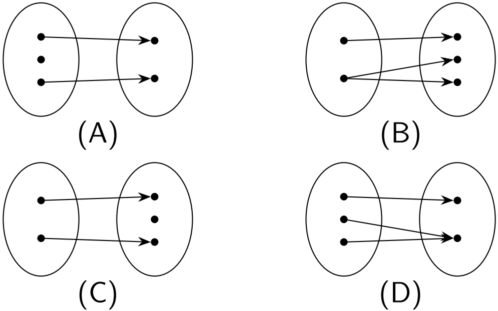

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(C) and (D) represent functions, since it is permitted for points in the *codomain* of a function to have no pre-image as well as to have multiple pre-images.

One-one Functions

$f : X \to Y$ is called **one-one** or **injective** if distinct points in X have distinct images in Y: If $a, b \in X$ and $a \neq b$ then $f(a) \neq f(b)$.

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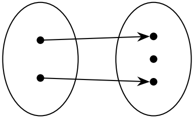

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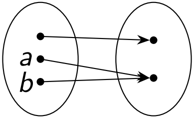

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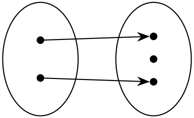

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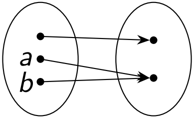

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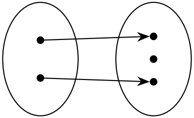

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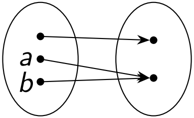

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The function in (A) is one-one because distinct points in the domain are taken to distinct points in the codomain.
The function in (B) is not one-one because the points a and b are taken to the same value.

Onto Functions

$f : X \to Y$ is called **onto** or **surjective** if its image is all of Y, that is, for each $b \in Y$ there exists $a \in X$ such that $f(a) = b$.

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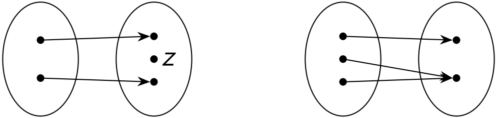

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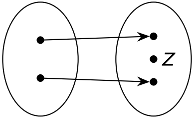

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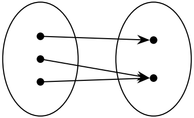
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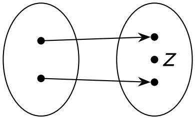

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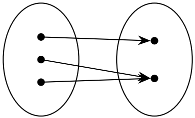

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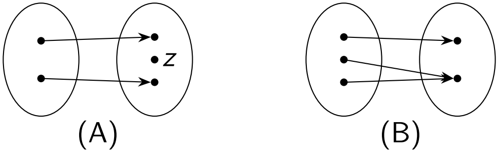

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Task

Find out whether the functions $f, h\colon \mathbb{R} \to \mathbb{R}$ are one-one or onto. If a function is not onto, give its image.

(a) $f(x) = \frac{1}{2}(x + |x|)$. *(b)* $h(x) = \begin{cases} x^2 + x + 1 & \text{if } x \geq 0, \\ x + 1 & \text{if } x < 0. \end{cases}$

Bijections

$f : X \to Y$ is called a **one-one correspondence** or **bijection** if it is both one-one and onto.

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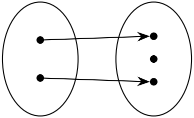
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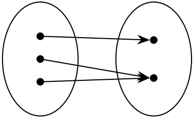
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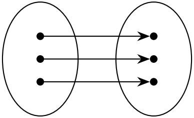
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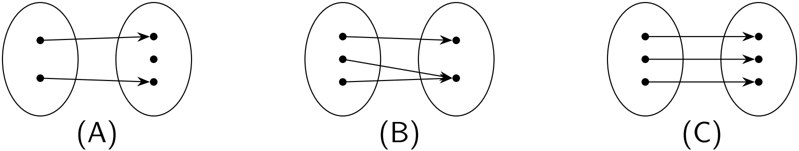

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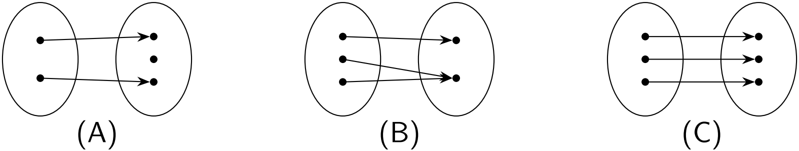

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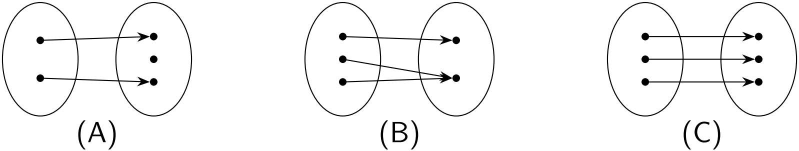

The function in (A) is one-one but not onto.

The function in (B) is onto but not one-one.

The function in (C) is both one-one and onto, hence it is a bijection.

Inverse Functions

Suppose $f\colon X \to Y$ and $g\colon Y \to X$. (Their domain and codomain are switched)

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Suppose f is a bijection.
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We set $g(b) = a$. $\square$

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Example 2: $f : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $f(x) = x^2$. Then

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Example: Consider $f\colon \mathbb{R} \to [0, \infty)$, $f(x) = x^2$ and $g\colon [0, \infty) \to \mathbb{R}$, $g(x) = \sqrt{x}$.
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Task

Show that composition of functions is associative: If $f\colon W \to X$, $g\colon X \to Y$ and $h\colon Y \to Z$ then $h \circ (g \circ f) = (h \circ g) \circ f$.

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Let $f \colon X \to Y$ and $g \colon Y \to X$. Show that g is the inverse function of f if and only if $g \circ f = 1_X$ and $f \circ g = 1_Y$.

Composition and Inverse Functions

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Theorem

Let $f: X \to Y$ and $g: Y \to Z$ be bijections. Then

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&= f^{-1} \circ 1_Y \circ f = f^{-1} \circ f \;=\; 1_X, \\
(g \circ f) \circ (f^{-1} \circ g^{-1}) &= g \circ (f \circ f^{-1}) \circ g^{-1} \\
&= g \circ 1_X \circ g^{-1} = g \circ g^{-1} \;=\; 1_Y. \quad \square
\end{aligned}$$

Table of Contents

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Absolute Value and Unit Step Functions

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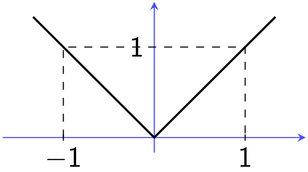
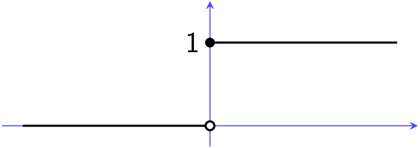
The absolute value $|x|$ defines a real function called the **absolute value** or **modulus function**. Its domain is $\mathbb{R}$ and its graph is:



The **Heaviside** or **unit step function** is defined by

$$H(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0. \end{cases}$$



Signum Function

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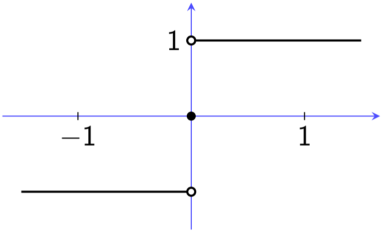
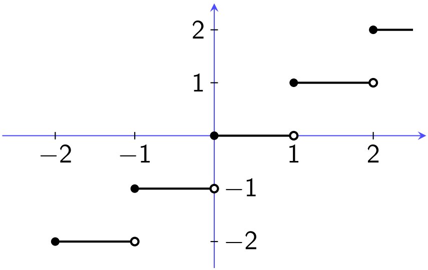
The **sign** or **signum function** is defined by

$$\operatorname{sgn}(x) = \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } 0 < x. \end{cases}$$



Greatest Integer Function

Recall that for every real number x there is a unique integer $[x]$ such that $[x] \leq x < [x] + 1$.
The function which associates $[x]$ to x is called the **greatest integer function**.
Sometimes it is called the **floor function** and is denoted by $\lfloor x \rfloor$.



Vertical Shifts

Given a real function f and a real number c, we define a function called $f + c$ by

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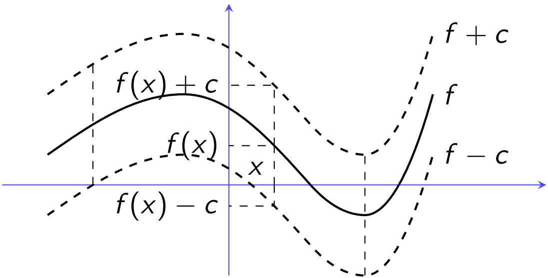

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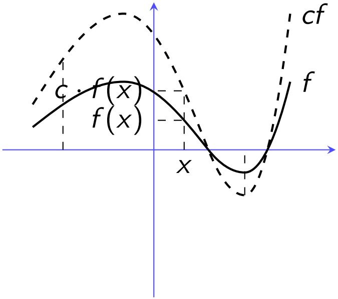

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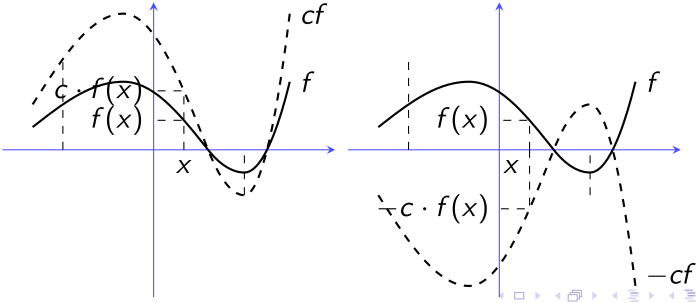

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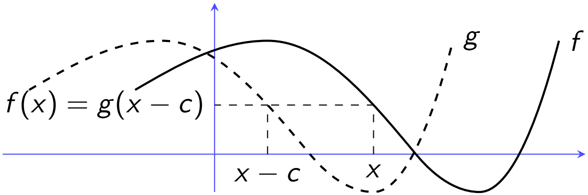
Thus, the graph of g is a horizontal shift to the left of the graph of f.



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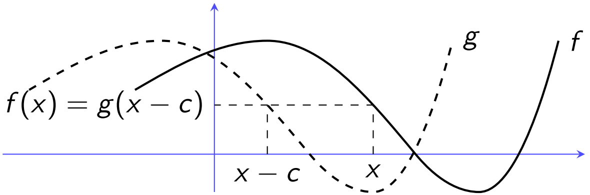

Task

Describe the graph of $g(x) = f(x + c)$ *when* $c < 0$.

Horizontal Scaling

Consider $h(x) = f(cx)$ with $c > 0$.

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Reasoning as before (try!) we conclude that the value taken by f at x is taken by h at x/c.

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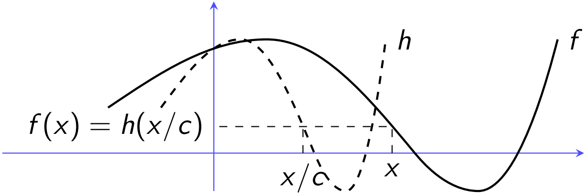

Note that the graph will contract if $c > 1$ and will stretch if $c < 1$.

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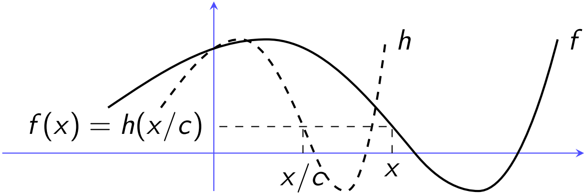

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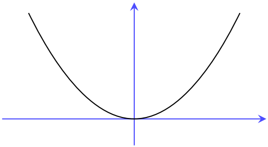
Task

Describe the graph of $h(x) = f(cx)$ when $c < 0$.

Exercise

Recall that the graph of $f(x) = x^2$ is an upward opening parabola.



Use your understanding of shifts and scalings to plot the graphs of the following on the same xy-plane.

1. $g(x) = (x-2)^2 + 1$.
2. $h(x) = 4x^2 + 12x + 5$.

Arithmetic of Functions

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Let f, g be real functions. We use them to define new functions:

$$(f+g)(x) = f(x) + g(x), \qquad (f-g)(x) = f(x) - g(x),$$
$$(fg)(x) = f(x)g(x), \qquad \frac{f}{g}(x) = \frac{f(x)}{g(x)}.$$

Task

Let f, g be real functions with domains A, B respectively. Describe the domains of the following functions: $f+g$, $f-g$, fg, f/g.

Graph of Inverse Function

Suppose I, J are subsets of $\mathbb{R}$ and $f\colon I \to J$ is a bijection. It has an inverse function $f^{-1}\colon J \to I$.

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Now (y,x) is the reflection of (x,y) in the line $y = x$.

Therefore the graph of f^{-1} can be obtained by reflecting the graph of f in the line $y = x$.