Applied Quantum Mechanics

Chapter 11 problems

LAST NAME FIRST NAME

Problem 11.1
Derive the following commutation relations:

(a) $[\hat{L}_z, \hat{x}] = i\hbar\hat{y}$

(b) $[\hat{L}_z, \hat{y}] = -i\hbar\hat{x}$

(c) $[\hat{L}_z, \hat{z}] = 0$

(d) $[\hat{L}^2, \hat{x}] = -2\hbar^2\hat{x} + 2i\hbar(\hat{L}_z\hat{y} - \hat{L}_y\hat{z})$

(e) $[\hat{L}^2, [\hat{L}^2, \hat{\mathbf{r}}]] = 2\hbar^2(\hat{\mathbf{r}}\hat{L}^2 + \hat{L}^2\hat{\mathbf{r}})$

Problem 11.2
The ground state wave function of a hydrogenic atom with nuclear charge Ze is $\psi_1(r) = Ae^{-r/r_1}$, where r is the distance between the electron and the nucleus and r_1 is a characteristic length scale. The electron is subject to a radially-symmetric coulomb potential given by $V(r) = -e^2/4\pi\varepsilon_0\varepsilon_r r$.

(a) Find the value of the normalization constant A.

(b) Find the value of r_1 that minimizes the energy expectation value $\langle E_1 \rangle$.

(c) Use the value of r_1 in (b) to calculate the ground state energy.

(d) Show that $\langle E_{\text{kinetic}} \rangle = -\langle E_{\text{potential}} \rangle / 2$ (which is a result predicted by the virial theorem).

(e) Show that the peak in radial probability occurs at $r = a_B/Z$.

(f) Show that the expectation value $\langle r \rangle = 3a_B/2Z$.

(g) Show that the expectation value of momentum $\langle p \rangle = 0$.

Problem 11.3
To calculate the spontaneous emission rate $A = 1/\tau_{sp}$ for the $2p \to 1s$ ($|n = 2, l = 1, m\rangle \to |n = 1, l = 0, m = 0\rangle$) transition in hydrogen one usually averages over the three possible values of the quantum number m so that

$$A = \frac{e^2\omega^3}{3\hbar c^3\pi\varepsilon_0}\frac{1}{3}\sum_{m=-1}^{m=1}|\langle 2, 1, m|\hat{\mathbf{r}}|1, 0, 0\rangle|^2$$

where $\hbar\omega$ is the energy of the emitted photon. Since $r^2 = x^2 + y^2 + z^2$, this equation can be written as

$$A = \frac{e^2\omega^3}{3\hbar c^3\pi\varepsilon_0}\frac{1}{3}\sum_{m=-1}^{m=1}(|\langle 2, 1, m|\hat{x}|1, 0, 0\rangle|^2 + |\langle 2, 1, m|\hat{y}|1, 0, 0\rangle|^2 + |\langle 2, 1, m|\hat{z}|1, 0, 0\rangle|^2)$$

(a) Show that

$$x = r\sin(\theta)\cos(\phi) = -\sqrt{\frac{2\pi}{3}}r(Y_1^1 - Y_1^{-1})$$

$$y = r\sin(\theta)\sin(\phi) = i\sqrt{\frac{2\pi}{3}}r(Y_1^1 + Y_1^{-1})$$

$$z = r\cos(\theta) = \sqrt{\frac{4\pi}{3}}rY_1^0$$

and rewrite each matrix element appearing in the expression for spontaneous emission in terms of a radial integral and an angular integral.

(b) Use the standard integral $\int_0^\infty x^n e^{-\mu x} dx = n!\mu^{-n-1}$ to show that the radial integral

$$\int_0^\infty r^3 R_{21}^*(r) R_{10}(r) dr = \frac{2^8}{3^4\sqrt{6}} a_B$$

where $R_{10}(r) = 2\left(\frac{1}{a_B}\right)^{3/2} e^{-r/a_B}$ and $R_{21}(r) = \frac{2}{\sqrt{3}}\left(\frac{1}{2a_B}\right)^{3/2} \frac{r}{2a_B} e^{-r/2a_B}$

(c) Show that the angular integrals in (a) are

$$-\sqrt{\frac{2\pi}{3}} \int_{\theta=0}^{\theta=\pi} \int_{\phi=0}^{\phi=2\pi} (Y_1^m)^* (Y_1^1 - Y_1^{-1}) Y_0^0 \sin(\theta) d\theta d\phi = \frac{-1}{\sqrt{4\pi}}\sqrt{\frac{2\pi}{3}}(\delta_{m,1} - \delta_{m,-1})$$

$$i\sqrt{\frac{2\pi}{3}} \int_{\theta=0}^{\theta=\pi} \int_{\phi=0}^{\phi=2\pi} (Y_1^m)^* (Y_1^1 + Y_1^{-1}) Y_0^0 \sin(\theta) d\theta d\phi = \frac{i}{\sqrt{4\pi}}\sqrt{\frac{2\pi}{3}}(\delta_{m,1} + \delta_{m,-1})$$

$$\sqrt{\frac{4\pi}{3}} \int_{\theta=0}^{\theta=\pi} \int_{\phi=0}^{\phi=2\pi} (Y_1^m)^* Y_1^0 Y_0^0 \sin(\theta) d\theta d\phi = \frac{1}{\sqrt{4\pi}}\sqrt{\frac{4\pi}{3}}(\delta_{m,0})$$

(d) Combine the results of (b) and (c) to show that

$$\sum_{m=-1}^{m=1} |\langle 21m|\hat{\mathbf{r}}|100\rangle|^2 = 96\left(\frac{2}{3}\right)^{10} a_B^2 = \left(\frac{32}{27}\right)^3 a_B^2$$

and that the spontaneous emission time for the $2p \to 1s$ transition in hydrogen is $\tau_{sp} = 1.6\ \text{ns}$.

Problem 11.4

(a) Show using first-order perturbation theory that the correction to the $1s$ ground-state energy of a hydrogen atom subject to a uniform electric field $\mathbf{E}$ in the z direction is zero.

(b) Show that the first-order correction to the ground-state wave function is

$$|\psi_0^{(1)}\rangle = |1,0,0\rangle - e|\mathbf{E}| \sum_{n \neq 1, l, m} \frac{\langle n, l, m|\hat{z}|1,0,0\rangle}{E_1 - E_n} |n, l, m\rangle$$

(c) Show to first-order in $\mathbf{E}$ that the susceptibility for the $1s$ state is

$$\chi_{1s} = \frac{\langle \psi_0^{(1)}|e\hat{z}|\psi_0^{(1)}\rangle}{|\mathbf{E}|} = -2e^2 \sum_{n \neq 1, l, m} \frac{|\langle n, l, m|\hat{z}|1,0,0\rangle|^2}{E_1 - E_n}$$

Problem 11.5

(a) An electron with zero orbital angular momentum ($l = 0$) moves in a radial potential $V(r) = 0$ for $r < a$ and $V(r) = \infty$ for $r \geq a$, where a is the radius of a spherical quantum dot. Use the radial Schrödinger equation to find the eigenenergies and normalized eigenstates of the electron.

(b) Find the eigenenergies and normalized eigenstates of an electron with zero orbital angular momentum ($l = 0$) moving in a radial shell potential with $V(r) = 0$ for $a < r < b$ and $V(r) = \infty$ elsewhere.

Problem 11.6

Show that the radial expectation value $\langle r \rangle$ for a hydrogen atom in state ψ_{nlm} is $\langle r \rangle = \frac{a_B}{2}(3n^2 - l(l+1))$ and that $\langle r^2 \rangle = \frac{n^2 a_B^2}{2}(5n^2 + 1 - 3l(l+1))$, where a_B is the Bohr radius.