Applied Quantum Mechanics

Chapter 10 problems

LAST NAME FIRST NAME

Problem 10.1

Consider a three-dimensional potential $V(x, y, z)$ that is infinite except in a region $0 < x < L$, $0 < y < L$, and $0 < z < L$ where $V(x, y, z) = 0$.

(a) Write down the time-independent Schrödinger equation for a particle mass m confined to motion in the potential and solve for the eigenfunctions.

(b) Show that the eigenenergies are $E^{(0)}_{n_x, n_y, n_z} = \frac{\pi^2\hbar^2}{2mL^2}(n_x^2 + n_y^2 + n_z^2)$, where n_x, n_y, and n_z are non-zero positive integers. What is the degeneracy of the ground state and what is the degeneracy of the first excited state?

(c) The system is perturbed by introducing a potential $\hat{W} = V_0$ in a region for which $0 < x < \frac{L}{2}$, $0 < y < \frac{L}{2}$, and $0 < z < L$. The perturbation $\hat{W} = 0$ elsewhere and V_0 is a constant. Use first-order perturbation theory to find the new ground state energy.

(d) What are the new eigenenergies and eigenfunctions of the first excited state?

Problem 10.2

A particular unperturbed Hamiltonian expressed in matrix form is

$$\boldsymbol{H}^{(0)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

The system is subject to the perturbation

$$\mathbf{W} = \begin{bmatrix} 0 & \Delta & 0 \\ \Delta & 0 & 0 \\ 0 & 0 & \Delta \end{bmatrix}$$

where $\Delta \ll 1$.

(a) Find the exact eigenvalues of $\boldsymbol{H} = \boldsymbol{H}^{(0)} + \mathbf{W}$.

(b) Find the eigenvalues to second-order using time-independent non-degenerate perturbation theory.

(c) Compare the results obtained in (a) and (b).

Problem 10.3

(a) An electron moves in a one-dimensional box of length X. Apply the periodic boundary condition $\phi(x) = \phi(x + X)$ to find the electron eigenfunction and eigenvalues.

(b) Now apply a weak periodic potential $V(x) = V(x + L)$ to the system, where $X = NL$ and N is a large positive integer. Using nondegenerate perturbation theory, find the first order correction to the wave functions and the second order correction to the eigenenergies.

(c) When wave vector k is close to $n\pi/L$, where n is an integer, the result in (b) is no longer valid. Use two-state degenerate perturbation theory to find the corrected energy values for $k = n\pi((1 + \Delta)/L)$, and $k' = n\pi((1 - \Delta)/L)$ where Δ is small compared with π/L.

(d) Use the results of (b) an (c) to draw the electron dispersion relation, $E(k)$.

(e) If we choose the lowest-frequency Fourier component of the perturbative periodic potential in part (b), then $V(x) = V_1 \cos(\pi x/L)$. Repeat (b), (c), and (d) using this potential.

Hint: $V(x) = V_0 + \sum_{n \neq 0} V_n e^{i2\pi nx/L}$, and choose $V_0 = 0$.

Problem 10.4

A semiconductor quantum dot is modeled as a three-dimensional box of side L and infinite barrier energy. An electron in the quantum dot has energy $E = \dfrac{3\pi^2\hbar^2}{m^* L^2}$, where m^* is the effective electron mass.

(a) Calculate the first-order correction to the electron energy when a uniform electric field **E** is applied in the z-direction.

(b) If $L = 20\ \text{nm}$, the effective electron mass is $m^* = 0.07 \times m_0$, and the strength of the applied electric field is $\mathbf{E} = 2 \times 10^4\ \text{V cm}^{-1}$, what is the value of the new electron energy level?

(c) Explain the degeneracy of the system after the perturbation is applied.

Problem 10.5

The first four lowest energy states of a one-dimensional harmonic oscillator with characteristic frequency ω_0 are subject to the perturbation

$$\mathbf{W} = \begin{bmatrix} W_{00} & W_{01} & W_{02} & W_{03} \\ W_{10} & W_{11} & W_{12} & W_{13} \\ W_{20} & W_{21} & W_{22} & W_{23} \\ W_{30} & W_{31} & W_{32} & W_{33} \end{bmatrix} = \Delta\hbar\omega_0 \begin{bmatrix} 1 & 0 & \frac{-1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 \\ \frac{-1}{\sqrt{2}} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

where $\Delta \ll 1$.

(a) Find the new eigenenergies to first-order in time-independent perturbation theory.

(b) Find the new eigenenergies to second-order in time-independent perturbation theory.

Problem 10.6

An electron is confined in a one-dimensional rectangular potential well of width $2L$ such that $V(x) = 0$ for $0 < x < 2L$ and $V(x) = \infty$ elsewhere. The system is subject to a constant uniform electric field $\mathbf{E}$ in the x direction.

(a) Write down an analytic expression for the new eigenfunctions and energy eigenvalues evaluated to first-order in time-independent perturbation theory.

(b) For an electron with effective electron mass $m_e^* = 0.07 \times m_0$, where m_0 is the bare electron mass, well width $2L = 10\ \text{nm}$, and electric field of $2 \times 10^5\ \text{V cm}^{-1}$, use your result in (a) to find the new eigenfunctions and energy eigenvalues.

(c) Sketch and explain how, according to first-order time-independent perturbation theory, the unperturbed ground-state wave function is modified under the influence of the perturbation. Under what circumstances do you expect the results of first-order time-independent perturbation theory to be invalid?

Problem 10.7

An electron mass m confined to motion in a one-dimensional harmonic potential with characteristic frequency ω is subject to a perturbing potential $\hat{W} = \xi x^3 \hbar\omega(m_0\omega/\hbar)^{3/2}$.

(a) Write down the Hamiltonian for the system.

(b) Calculate to second-order the eigenenergies of the perturbed system.

(c) Calculate to first-order the eigenstates for the perturbed system.

Problem 10.8

The unperturbed Hamiltonian for a particle of mass m with kinetic energy $T = \frac{\hat{p}_x^2 + \hat{p}_y^2}{2m}$ in a two-dimensional harmonic potential $V(x, y) = \frac{\kappa}{2}(\hat{x}^2 + \hat{y}^2)$ is $\hat{H}^{(0)} = \hat{T} + \hat{V}$ where κ is the spring constant. The eigenstates associated with $\hat{H}^{(0)}\phi_{nm} = E_{nm}\phi_{nm}$ are of the form $\phi_{nm} = \phi_n(x)\phi_m(y) = |nm\rangle$ with $n, m = 0, 1, 2, \ldots$. The eigenstates are $(n + m + 1)$-fold degenerate with eigenenergies $E_{nm} = \hbar\omega(n + m + 1)$.

(a) Find the position of minimum potential and the amount by which any perturbing potential $\hat{W} = \frac{\kappa'}{2}\hat{x}$ or $\hat{W} = \frac{\kappa'}{2}(\hat{x} + \hat{y})$ shifts eigenenergy values and show that the perturbation does not break the degeneracy of the states.

(b) Create a contour plot of the potential $V(x, y)$ in the range $-2\text{ nm} < x < 2\text{ nm}$ and $-2\text{ nm} < y < 2\text{ nm}$ for $\kappa = 2\text{ eV nm}^{-2}$ and overlay a contour plot of $\hat{V}(x, y) + \hat{W}(x) = \frac{\kappa}{2}(\hat{x}^2 + \hat{y}^2) + \frac{\kappa'}{2}\hat{x}$ for $\kappa' = 1\text{ eV nm}^{-1}$. Use the contour plots to explain why the perturbing potential fails to break the degeneracy of the states.

(c) Create a contour plot of the potential $V(x, y)$ in the range $-2\text{ nm} < x < 2\text{ nm}$ and $-2\text{ nm} < y < 2\text{ nm}$ for $\kappa = 2\text{ eV nm}^{-2}$ and overlay a contour plot of $\hat{V}(x, y) + \hat{W}(x, y) = \frac{\kappa}{2}(\hat{x}^2 + \hat{y}^2) + \frac{\kappa'}{2}\hat{x}\hat{y}$ for $\kappa' = 1\text{ eV nm}^{-2}$. Use the contour plots to explain why in this case the perturbing potential breaks the degeneracy of the states.