Applied Quantum Mechanics

Chapter 1 problems

LAST NAME FIRST NAME

Problem 1.1

A metal ball is buried in an ice cube that is in a bucket of water.

(a) If the ice cube with the metal ball is initially under water, what happens to the water level when the ice melts?

(b) If the ice cube with the metal ball is initially floating in the water, what happens to the water level when the ice melts?

(c) Explain how the Earth's average sea level could have increased by at least 100 m compared to about 20 000 years ago.

(d) Estimate the thickness and weight per unit area of the ice that melted in (c). You may wish to use the fact that the density of ice is 920 kg m^{-3}, today the land surface area of the Earth is about 148 300 000 km^2 and water area is about 361 800 000 km^2.

Problem 1.2

Sketch and find the volume of the largest and smallest convex plug manufactured from a sphere of radius $r = 1$ cm to fit exactly into a circular hole of radius $r = 1$ cm, an isosceles triangle with base 2 cm and a height $h = 1$ cm, and a half circle radius $r = 1$ cm and base 2 cm.

Problem 1.3

An initially stationary particle mass m_1 is on a frictionless table surface and another particle mass m_2 is positioned vertically below the edge of the table. The distance from the particle mass m_1 to the edge of the table is l. The two particles are connected by a taught, light, inextensible string of length $L > l$.

(a) How much time elapses before the particle mass m_1 is launched off the edge of the table?

(b) What is the subsequent motion of the particles?

(c) How is your answer for (a) and (b) modified if the string has spring constant κ?

Problem 1.4

The velocity of waves in shallow water may be approximated as $\upsilon = \sqrt{gh}$ where g is the acceleration due to gravity and h is the depth of the water. Sketch the lowest frequency standing water wave in a 5 m long garden pond that is 0.9 m deep and estimate its frequency.

Problem 1.5

(a) What is the dispersion relation of a wave whose group velocity is half the phase velocity?

(b) What is the dispersion relation of a wave whose group velocity is twice the phase velocity?

(c) What is the dispersion relation when the group velocity is four times the phase velocity?

(d) What is the dispersion relation when the group velocity is the negative of the phase velocity?

Problem 1.6

A stationary ground-based radar uses a continuous electromagnetic wave at 10 GHz frequency to measure the speed of a passing airplane moving at a constant altitude and in a straight line at 1000 km hr^{-1}. What is the maximum beat frequency between the out going and reflected radar beams? Sketch how the beat frequency varies as a function of time. What happens to the beat frequency if the airplane moves in an arc?

Problem 1.7

(a) If classical electromagnetism were described in terms of a single *complex* field $\mathbf{G}$ show that Maxwell's equations in free space and in the absence of free charges may written as the complex equations

$$\nabla \cdot \mathbf{G} = 0$$

and

$$i\frac{\partial \mathbf{G}}{\partial t} = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \nabla \times \mathbf{G}$$

where $\mathbf{G} = \frac{1}{\sqrt{2}}\left(\frac{\mathbf{D}}{\sqrt{\varepsilon_0}} + i\frac{\mathbf{B}}{\sqrt{\mu_0}}\right)$

(b) Show that the energy flux density in the electromagnetic field given by the Poynting vector is

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} = \frac{-i}{\sqrt{\varepsilon_0 \mu_0}}(\mathbf{G}^* \times \mathbf{G})$$

(c) If the field $\mathbf{G}$ is purely real, what is the value of $\mathbf{S}$?

(d) Show that the electromagnetic energy density is $U = |\mathbf{G}|^2$.

Problem 1.8

How would Maxwell's equations be modified if magnetic charge g (magnetic monopoles) were discovered? Derive an expression for conservation of magnetic current and write down a generalized Lorentz force law that includes magnetic charge. Write Maxwell's equations with magnetic charge in terms of a field $\mathbf{G} = \frac{1}{\sqrt{2}}\left(\frac{\mathbf{D}}{\sqrt{\varepsilon}} + i\frac{\mathbf{B}}{\sqrt{\mu}}\right)$.

Problem 1.9

(a) The capacitance of a small metal sphere in air is $C_0 = 1.1 \times 10^{-18}$ F. A thin dielectric film with relative permittivity $\varepsilon_{r_1} = 10$ uniformly coats the sphere and the capacitance increases to 2.2×10^{-18} F. What is the thickness of the dielectric film and what is the single electron charging energy of the dielectric coated metal sphere?

(b) The dielectric coated metal sphere of part (a) is now coated with metal. What is the new value of the single-electron charging energy for the central metal sphere?

(c) Compare the result in (b) to the charging energy of a metal sphere radius $0.5 \text{ nm} < r_0 \leq 10 \text{ nm}$ embedded in a dielectric of relative permittivity $\varepsilon_r = 10$ and surrounded by metal shell of internal radius $r_1 = 2r_0$. Plot single-electron charging energy ΔE as a function of r_0.

Problem 1.10

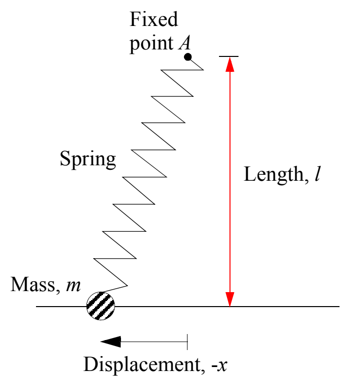
(a) A diatomic molecule has atoms with mass m_1 and m_2. An isotopic form of the molecule has atoms with mass m'_1 and m'_2. Find the ratio of vibration oscillation frequency ω / ω' of the two molecules.

(b) What is the ratio of vibrational frequencies for carbon monoxide isotope 12 ($^{12}C^{16}O$) and carbon monoxide isotope 13 ($^{13}C^{16}O$)?

Problem 1.11

(a) Find the frequency of oscillation of the particle of mass m illustrated in the figure below. The particle is only free to move along a line and is attached to a light spring whose other end is fixed at point A located a distance l perpendicular to the line. A force F_0 is required to extend the spring to length l.

(b) Part (a) describes a new type of child's swing. If the child weighs 20 kg, the length $l = 2.5$ m, and the force $F_0 = 450$ N, what is the period of oscillation?



Problem 1.12

A house in California consumes 10 MW hrs of electricity over a one year period.

(a) What is the minimum vertical distance one would have to lift a 200 000 ton block of concrete to store this amount of energy?

(b) A 25% efficient water driven electric generator is placed at the bottom of a 10 m high waterfall. What is the minimum water flow rate (measured in tons per minute) required to supply the average power consumption of the house?

(c) Iron weights six inches in diameter are attached to a cable and hung in a well of diameter greater than six inches and depth h. The iron weighs $140\ \text{kg m}^{-1}$ and the weight of the cable is insignificant. What is the maximum amount of energy that can be stored using this method if $h = 100\ \text{m}$ and how much more energy can be stored if $h = 1000\ \text{m}$?

Problem 1.13

A one centimeter long linear chain of spherical atoms has nearest neighbor spacing of 0.25 nm.

(a) What is the minimum diameter of a one atom thick disk made of these atoms?

(b) What is the minimum diameter of a sphere made of these atoms?

Problem 1.14

An electromagnetic wave has electric field $\mathbf{E}(\mathbf{r}, \omega) = \mathbf{E}_0(\omega)e^{i(k'(\omega) + ik''(\omega))\tilde{\mathbf{k}} \cdot \mathbf{r}}$ where $k'(\omega)$ and $k''(\omega)$ are the real and imaginary parts, respectively, of the frequency-dependent wave number. The wave propagates in a homogeneous dielectric characterized by $\mu_r = 1$ and complex permittivity function $\varepsilon(\omega) = \varepsilon_0\varepsilon_r(\omega) = \varepsilon_0(\varepsilon_r'(\omega) + i\varepsilon_r''(\omega))$, where $\varepsilon_r'(\omega)$ and $\varepsilon_r''(\omega)$ are the real and imaginary parts, respectively, of the frequency-dependent relative permittivity function.

(a) Derive the expression for refractive index $n_r(\omega) = \sqrt{\frac{1}{2}(\varepsilon_r'(\omega) + \sqrt{\varepsilon_r'^2(\omega) + \varepsilon_r''^2(\omega)})}$.

(b) Introduce absorption coefficient $\alpha(\omega) = 2k''(\omega)$ and show that $\alpha(\omega) = \frac{\omega}{c}\frac{\varepsilon_r''(\omega)}{n_r(\omega)}$.

Problem 1.15

A particle moves between two points A and B in a vertical plane as illustrated in the figure below. If acceleration due to gravity is g and velocity is initially zero, find the shape of the frictionless surface on which the particle must move to give a trajectory that takes the shortest time.

Problem 1.16

Materials with negative relative permeability and negative relative permittivity can display negative refractive index. In this situation group velocity is the negative of phase velocity. Suppose a point source of electromagnetic radiation in air is placed at a distance z_1 normal to the surface of a slab of negative refractive index material of thickness $z_2 > z_1$. The value of the negative refractive index material is $n_r = -1$. Use ray tracing to find the positions at which electromagnetic radiation from the point source is focused to a point. Comment on the statement that this slab of negative index material makes a "perfect lens".

Problem 1.17

An electromagnetic field of wavelength $\lambda_0 = 1500$ nm in free space propagates around the inside circumference of a silica dielectric disk resonator of density $\rho = 2.2$ g cm^{-3} and refractive index $n_r = 1.5$. The disk has radius $R = \frac{0.2}{2\pi}$ mm and thickness $d = 1$ μm. Electromagnetic field loss in the disk is dominated by surface roughness with average value $\alpha = 0.016$ cm^{-1}.

(a) Calculate the lowest natural radial mechanical oscillation frequency of the disk in the absence of light. Young's modulus for the dielectric material is 73 GPa and the Poisson ratio is 0.17.

(b) Calculate the maximum resonant enhancement in electric field and electric field intensity and the full-width half-maximum (FWHM) in the field intensity frequency spectrum.

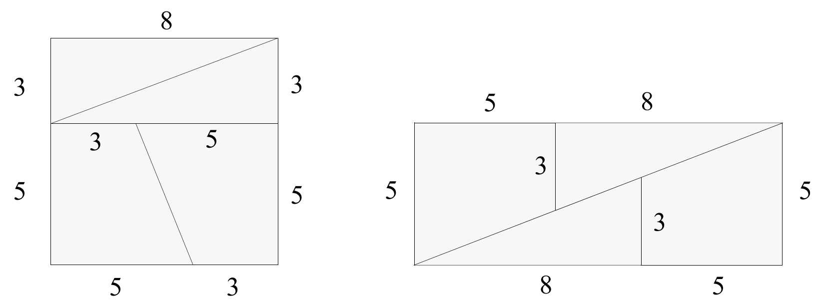
(c) Repeat the calculation in (b) but for $R = \frac{0.02}{2\pi}$ mm.

(d) Compare and explain the results obtained in (b) and (c).

(e) 10 μW of optical power at $\lambda_0 = 1500$ nm wavelength is coupled into the dielectric disk in (b). Estimate the force exerted on the disk due to radiation pressure and estimate the change in disk radius. Compare the resulting shift in resonant frequency to the optical FWHM. What optical modulation depth might be achievable in the system?

(f) What is the FWHM of the frequency intensity spectrum resonance if the electromagnetic field propagating around the inside circumference of the dielectric disk resonator experiences a negative refractive index $n_r' = -1.5$ for exactly one half of each round trip? How does this change the results you obtained in part (d)?

Problem 1.18
Explain the discrepancy in the following:



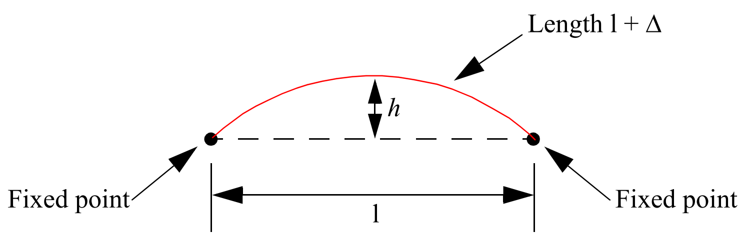
64 square units

65 square units

Problem 1.19
The endpoints of a thin horizontal micro-beam length l = 2 μm is fixed. The beam increases in length by Δ causing the beam to describe the arc of a circle between the fixed points.

(a) Calculate and plot the height of the midpoint h as a function of Δ for 0 nm < Δ < 10 nm.

(b) Calculate and plot the mechanical gain $g = h / \Delta$ for 0 nm < Δ < 10 nm.



Problem 1.20
Example Exercise 1.8 shows how to find the analytic solution of a damped harmonic oscillator of natural frequency ω_0 with motion constrained to the x-direction and subject to a small external oscillatory force, $F(t) = F_0 e^{i\omega t}$. Assume the force is applied for time $t \geq 0$.

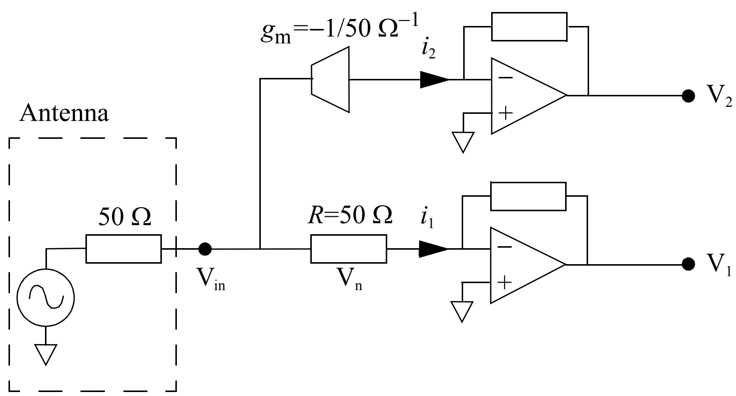
(a) Solve the problem numerically by directly integrating the response of the system in time. Plot $x(t)$ to show the transient and steady-state behavior. Plot and explain the behavior of $\frac{\mathrm{d}}{\mathrm{d}t}x(x) \equiv \dot{x}(x)$ when the system has reached steady-state. What is the time evolution of the total energy of the system when ω is detuned from ω_0? To ensure accuracy use the fourth-order Runge-Kutta method and implement your code in MATLAB.

(b) Add a non-linear term such that the restoring force is of the form $F = -\kappa_1 x - \kappa_3 x^3$. Plot $x(t)$, $\dot{x}(x)$, and the potential $V(x)$ for both positive and negative values of κ_1. Explain the different types of motion visualized by $x(t)$ and $\dot{x}(x)$ that you are able to achieve by varying the potential and driving force.

Problem 1.21
Johnson (*Phys. Rev.* **32**, 97 (1928)) and Nyquist (*Phys. Rev.* **32**, 110 (1928)) showed that thermal fluctuations create RMS voltage noise

$$V_{\mathrm{RMS}} = \sqrt{4Rk_{\mathrm{B}}T\Delta f}$$

in a macroscopic resistor of value R (ohms) at absolute temperature T (kelvin) measured over a frequency bandwidth Δf, so long as the frequencies considered $f \ll \frac{k_{\mathrm{B}}T}{2\pi\hbar}$. This noise can limit sensitivity of a RF receiver. Bruccoleri et al. (*IEEE J. Solid-State Circuits*, **39**, 275 (2004)) showed how the following circuit, in which the current-source transconductance amplifier (g_{m} cell) is an inverter, could be used to cancel thermal noise generated by the input load resistor R.



Explain how this noise cancellation works by evaluating the current i_1 and i_2 for a voltage signal V_{in} at the input and voltage noise V_{n} generated in the resistor R. What physical principals and conservation laws do you exploit to analyze the circuit? What limits the performance of the noise cancellation circuit?