Solutions to exercises in chapter 6

1. **Muscles**

   a) The cross-sectional area of a sarcomere is given by the area of the hexagon formed by the six actin strands. With an edge length of 30 nm, we have six equilateral triangles with a sidelength of 30 nm and hence the area is $A_S = 3\sqrt{3} \cdot 30$ nm $\cdot 15$ nm $= 2335$ nm^2 $= 2.3 \cdot 10^{-15}$ m^2. With 225 myosin molecules per sarcomere, we obtain a force per sarcomere of $F_S = N \cdot f = 225 \cdot 4$ pN $= 900 \cdot 10^{-12}$ N $=9 \cdot 10^{-10}$ N. This gives a force per area of $F_S/A_S = \frac{9 \cdot 10^{-10} N}{2.3 \cdot 10^{-15} m^2} = 3.85 \cdot 10^5 N/m^2$. Multiplying this with the cross-sectional area of the muscle of $A_M = 5$ cm^2 $= 5 \cdot 10^{-4}$ m^2 gives the force exerted by the muscle as $F_M = F_S \cdot A_M/A_S = 3.85 \cdot 10^5$ N/m^2 $\cdot 5 \cdot 10^{-4}$m^2 $= 193$ N.

   b) The number of myosin molecules has an uncertainty of $r_N = 10\%$, that of the area of the sarcomere one of $r_A = 13\%$ and the force of a single molecule one of $r_f = 10\%$. Together they give an uncertainty in force of $r_F = \sqrt{r_N^2 + r_A^2 + r_f^2} = \sqrt{369}\% \simeq 19\%$ or $\sigma_F = 38N$. A reasonable estimate therefore would be $F = 2.0(4) \cdot 10^2 N$.

2. **Pendulum on a scale**

   a) Since the frame is not moving, the sum of all forces acting on it must be zero. These forces are the force of gravity on the frame ($\vec{G}_R$), the normal force from the scale ($\vec{N}_{WR}$) as well as the force of the spring ($\vec{F}_{FR}$). Hence $\vec{G}_R + \vec{N}_{WR} + \vec{F}_{FR} = 0$. The force that the scale will show is $\vec{N}_{RW} = -\vec{N}_{WR} = \vec{F}_{FR} + \vec{G}_R$.

   b) Equation of motion of the bob: $m_K \frac{d^2\vec{x}_K(t)}{dt^2} = \vec{G}_K + \vec{F}_{FK}$, where $\vec{F}_{FK} = -k(\vec{x}_K - \vec{x}_{K,0})$. Moreover, we can connect part a) to this equation of motion to see what the scale will show by considering the equation of motion of the spring itself, where we find: $\vec{F}_{RF} + \vec{F}_{KF} + \vec{G}_F = 0$, such that $\vec{F}_{FR} = -\vec{F}_{RF} = \vec{F}_{KF} + \vec{G}_F = -\vec{F}_{FK} + \vec{G}_F$.

   Taking all of this together we find for the force on the scale: $\vec{N}_{RW} = -\vec{F}_{FK} + \vec{G}_F + \vec{G}_R = \vec{G}_K + \vec{G}_F + \vec{G}_R - m_K \frac{d^2\vec{x}_K(t)}{dt^2}$. Since the term with acceleration corresponds to a harmonic oscillation, the indicated weight will show a similar oscillation around the sum of the forces of gravity on the frame, the spring and the bob.

   c) Choosing $z(t)$ such that the zero of z is at the rest position of the bob, the equation of motion in b) becomes: $m_K \frac{d^2 z(t)}{dt^2} = -kz(t)$. As this gives a harmonic oscillation, we obtain for the acceleration: $\frac{d^2 z(t)}{dt^2} = -\omega^2 z_0 \cos(\omega t)$, which we can insert in th eequation of motion: $-m_K \omega^2 z_0 \cos(\omega t) = -kz_0 \cos(\omega t)$. As this has to be valid for all times, we find that $\omega = \sqrt{k/m}$ must be the angular frequency of the oscillation. Similarly, this is the angular frequency with which the indicated weight oscillates. Numerically: $\omega = 10 s^{-1}$.

   d) The mass and the spring constant both enter the angular frequency as a square root, i.e. the relative error of ω is given by: $r_\omega^2 = \frac{1}{4} r_k^2 + \frac{1}{4} r_m^2$. Given the relative errors of m, $r_m = 0.05 = r$ and k, $r_k = 0.05 = r$ we obtain: $r_\omega^2 = \frac{1}{2} r^2$ oder $r_\omega = 0.05/\sqrt{2} \simeq 0.03$

   e) The oscillating amplitude of the indicated weight is given by the amplitude of the oscillation and the spring constant, hence $F_0 = kz_0 = 10$ N.

3. **Viscous friction**

   (a) The forces acting on the raindrop are friction and gravity. The equation of motion hence is: $mdv/dt = -mg + 6\pi\eta_{air} rv$. If the drop falls with vonstant speed, $dv/dt = 0$, and we obtain $mg = 6\pi\eta_{Luft} rv$. The mass of the drop depends on its size via the volume, i.e. $m = 4\pi/3 r^3 \rho_{water}$, which gives: $4\pi/3 r^3 \rho_{water} g = 6\pi\eta_{air} rv$. Solving for v gives the speed we are looking for: $v = \frac{2\rho_{water} g r^2}{9\eta_{air}}$.

   (b) Inserting the values in the equation for v above - for r_1 we obtain: $v = \frac{2 10^3 kg/m^3 10 m/s^2 0.45^2 10^{-6} m^2}{9 1.8 10^{-5} Pas} = \frac{0.9 0.45 10^{-2} kg/s^2}{9 1.8 10^{-5} kg/(ms)} = \frac{9 10^{-3} ms/s^2}{4 \cdot 9 10^{-5}} = \frac{10^{-3} m/s}{4 \cdot 10^{-5}} = 25 m/s$. This is very fast and not really realistic, see also part d). For r_2 we can use that $v \propto r^2$, such that $v(r_2) = v(r_1)r_2^2/r_1^2$. $r_2^2/r_1^2 = 1/100$, such that we obtain $v(r_2) = 25m/s * 1/100 = 0.25m/s$, which is more realistic.

Errors: $\partial v/\partial r = 2v/r$, $\partial v/\partial \eta = -v/\eta$. Or for relative errors: $\sigma_v^2/v^2 = \sigma_\eta^2/\eta^2 + 4\sigma_r^2/r^2$. Both for r_1 and r_2 we have: $\sigma_r/r = 0.5/4.5 = 1/9$, $\sigma_\eta/\eta = 1/18$. Numerically: $\sigma_v^2/v^2 = 1/18^2 + 4(1/9)^2 = 17/18^2$ and hence: $\sigma_v/v = \sqrt{17}/18 = 0.22$.

(c) $Re = \rho_{air} \cdot r \cdot v/\eta$, $\rho_{air} = 1$ kg/m^3, $v = 25$ m/s, $r = 0.45$ mm, $\eta_{air} = 1.8 \cdot 10^{-5}$ Pa s. Therefore $Re = \frac{1kg/m^3 \cdot 0.45 \cdot 10^{-3} m \cdot 25 m/s}{1.810^{-5} Pas} = \frac{0.01 \cdot 2.5 \cdot 0.45 kg/(ms)}{1.810^{-5} kg/(ms)} = 0.62510^3 = 625$. This is higher than the critical Reynolds number of $Re_c = 10^2$, which is why we should have used turbulent friction instead.

(d) With the friction force of $F = 2 \cdot \pi \cdot \rho_{air} \cdot r^2 \cdot v^2$ the equation of motion changes to $mdv/dt = -mg + 2 \cdot \pi \cdot \rho_{air} \cdot r^2 \cdot v^2$. Again we are looking for the case where $dv/dt = 0$ and hence $mg = 2 \cdot \pi \cdot \rho_{air} \cdot r^2 \cdot v^2$. With the volume of the drop we get: $4\pi/3r^3\rho_{water}g = 2 \cdot \pi \cdot \rho_{air} \cdot r^2 \cdot v^2$. Solving this for v gives: $v = \sqrt{\frac{2r\rho_{water}g}{3\rho_{air}}}$. Numerically: $v = \sqrt{\frac{20.4510^{-3}m10m/s^2 10^3 kg/m^3}{3kg/m^3}} = \sqrt{9/3}m/s = \sqrt{3}m/s = 1.7m/s$. This makes sense with everyday experience.

4. **Conservation of energy**

   Conservation of energy dictates that the potential energy $E_{pot} = mgh$ in the beginning is equal to the kinetic energy $E_{kin} = mv^2/2$ at the end. Hence $mgh = mv^2/2$. Solving this for v gives $v = \sqrt{2gh}$. This speed is reached after $T = v/g = \sqrt{2h/g}$ given a constant acceleration g.

5. **Conservation of energy 2**

   a) Die total energy is $E = E_{pot} + E_{kin} = (kx^2 + mv^2)/2$. For a harmonic oscillation, $x = x_0 \cos(\omega t)$ and therefore $v = -\omega x_0 sin(\omega t)$, which we can insert into the energy to obtain

   $E = (m \cdot \omega^2 \cdot x_0^2 sin^2(\omega t) + k \cdot x_0^2 \cdot cos^2(\omega t))/2$.

   b) Using $sin^2\alpha + cos^2\alpha = 1$ we obtain: $E = (m \cdot \omega^2 \cdot x_0^2 + (k - m\omega^2) \cdot x_0^2 \cdot cos^2(\omega t))/2 = m \cdot \omega^2 \cdot x_0^2 + (k - m\omega^2) \cdot x(t)^2)/2$. The term proportional to $x(t)^2$ depends on time. In order for conservation of energy to hold, the prefactor of this term needs to be zero, i.e. $(k - m\omega^2) = 0$ or $\omega = \sqrt{k/m}$.

6. **Hydro power plant**

   a) The maximum energy is the total potential energy of all of the water, i.e. $E_{pot} = mgh = \rho V gh$. Numerically: $E = 1000kg/m^3 10 \cdot 10^6 m^3 10m/s^2 200m = 20 \cdot 10^{12} J = 20TJ$. The speed of the water in th eturbine we obtain from the kinetic energy: $mv^2/2 = mgh$ hence $v = \sqrt{2gh} = \sqrt{2 \cdot 10m/s^2 200m} = \sqrt{4000}m/s = 20\pi m/s \simeq 60m/s$

   b) Height and volume both have a relative error of $r = 1\%$. The energy thus has an error of $r_E^2 = r_V^2 + r_h^2 = 2r^2$ or 1.4%.

   For the speed, we only have the height with an error, which enters as a square root. Therefore the relative error of v is: $r_v = r_h/2 = 0.5\%$.

   c) The total energy is given by $mv^2/2$. If v diminishes by 1%, the energy correspondingly diminishes by 2%. For a total energy of 20 TJ these are losses of 0.4 TJ. With a flow of $\Phi = 400m^3/s$ it takes $V/\Phi = \frac{10^7 m^3}{400m^3/s} = 2.5 \cdot 10^4 s$ until the storage lake is empty, i.e. until the losses of 0.4 TJ have been incurred. The power dissipated therefore is $Q = \frac{4 \cdot 10^{11} J}{2.5 \cdot 10^4 s} = 1.6 \cdot 10^7 W$

   d) If the dissipated power corresponds to losses of 2%, the generated power is the 50fold of this power, i.e. $P = 50 \cdot 1.6 \cdot 10^7 W = 800MW$. This power is turned into electrical power with an efficiency of 90 %, which is a power generated of 720 MW the total generated energy is 18 TJ (see a), which is 5 million kWh (1kWh = 3.6 MJ).

7. **Energy and friction**

   In the beginning, the block has the potential energy of $E = mgh = mgL/2$. Due to friction, at every pass $E = F \cdot L = \mu_G \cdot N \cdot L = \mu_G mgL = mgL/5$ is dissipated. This means that after 2.5 passes, the initial energy has been converted into heat and the block remains at rest in the middle of the plane.

8. **Elevator**

a) Equation of motion: $ma = -mg + F$. When the elevator is decelerated from its maximum speed to 0 (with an acceleration a), this takes a time of $T = v_0/a$. During this time, the elevator has travelled a distance of $L = -aT^2/2 + v_0T$. Inserting T gives: $L = v_0^2/(2a)$. Hence the deceleration needs to be: $a = v_0^2/(2L)$, which we can insert into the equation of motion to obtain: $mv_0^2/(2L) = -mg + F$. Hence the force is: $F = mg + mv_0^2/(2L)$.

b) The kinetic energy before deceleration $mv^2/2$ plus the potential energy gained during deceleration needs to correspond to the Force on the rope times the deceleration distance. Therefore $mv^2/2 + mgL = FL$. This again gives the force of: $F = \frac{mv^2}{2L} + mg = \frac{1500kg2^2m^2/s^2}{22m} + 1500kg10m/^2 = 16500N$.

9. **Bungee Jump**

a) (1) $E_{pot} = mgh$; $E_{kin} = 0$; $E_{Seil} = 0$

(2) $E_{pot} = mg(h - L)$; $E_{kin} = mv^2/2$; $E_{Seil} = 0$

(3) $E_{pot} = mg(h - (L + x))$; $E_{kin} = 0$; $E_{Seil} = kx^2/2$

b) Conservation of energy says that the energies in (1) and (2) must be equal, i.e. $mgh = mg(h - L) + mv^2/2$. In other words: $mgL = mv^2/2$, such that we obtain for the speed: $v = \sqrt{2gL} = \sqrt{500}m/s = \sqrt{5} \cdot 10m/s \simeq 22m/s$

c) The only value with an error is the length of the rope L with a relative error of 4%. Thus the speed, which goes like the square root of the rope length has half this relative error or $r_v = 2\%$.

d) Here, we have to consider the energies in case (3) and (1) above, where we find: $mgh = mg(h - (L + x)) + kx^2/2$ or $mg(L + x) = kx^2/2$. This quadratic equation in x can be solved to give: $x = \frac{mg+\sqrt{(mg)^2+2mgLk}}{k} = \frac{1000N+\sqrt{10^6N^2+2.5\cdot 10^6N^2}}{50N/m} = \frac{1000(1+\sqrt{3.5})}{50}m = 20(1 + \sqrt{3.5})m \simeq 57m$. The height of the fall therefore is $H = x + L = 82m$ or you end up at the closest $s = h - H = 18m$ above the water surface.

e) $s = h - (L + x) = h - (L + \frac{mg+\sqrt{(mg)^2+2mgLk}}{k})$, where m, k, and L carry uncertainties. We thus need to know:

$\frac{\partial s}{\partial m} = \frac{g}{k} + \frac{mg^2+kgL}{k\sqrt{(mg)^2+2mgLk}}$

Numerically: $\sigma_m \frac{\partial s}{\partial m} = 2.2m$

$\frac{\partial s}{\partial k} = -\frac{mg(mg+kL+\sqrt{(mg)^2+2mgLk})}{k^2\sqrt{(mg)^2+2mgLk}}$

Numerically: $\sigma_k \frac{\partial s}{\partial k} = 0.9m$

$\frac{\partial s}{\partial L} = 1 + \frac{mg}{\sqrt{(mg)^2+2mgLk}}$

Numerically: $\sigma_L \frac{\partial s}{\partial L} = 1.5m$

$\frac{\partial s}{\partial h} = 1$

Numerically: $\sigma_h \frac{\partial s}{\partial h} = 2m$

And finally: $\sigma_s = \sqrt{2.2^2 + 0.9^2 + 1.5^2 + 2^2}m = 3.5m$