Quantum Field Theory
A Diagrammatic Approach

Solutions manual

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Contents

1 Exercises for chapter 1

Solution to exercise 1

1. We write

$$Z(J) = \sum_{n\geq 0} \frac{J^n}{n!} G_n \quad , \quad W(J) = \log Z(J) = \sum_{n\geq 0} \frac{J^n}{n!} C_n$$

with $G_0 = 1$ and $C_0 = 0$. Direct expansion leads to

$$\begin{aligned} C_1 &= G_1 \ , \\ C_2 &= G_2 - G_1^2 \ , \\ C_3 &= G_3 - 3\,G_2G_1 + 2\,G_1^3 \ , \\ C_4 &= G_4 - 4\,G_3G_1 - 3\,G_2^2 + 12\,G_2G_1^2 - 6\,G_1^4 \ , \\ C_5 &= G_5 - 5\,G_4G_1 - 10\,G_3G_2 + 20\,G_3G_1^2 + 30\,G_2^2G_1 - 60\,G_2G_1^3 + 24\,G_1^5 \ . \end{aligned}$$

A check is possible: in each line the coefficients must sum to zero.

2. The equation $Z' = WZ$ can be written out:

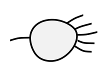
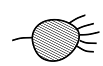
$$\sum_{m\geq 0} \frac{J^m}{m!} G_{m+1} = \sum_{n,k\geq 0} \frac{J^{n+k}}{n!\,k!} C_n G_k$$

and equation powers of J on both sides gives

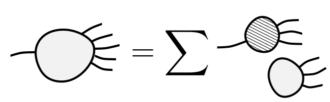
$$G_{m+1} = \sum_{n,k\geq 0} \frac{m!}{n!\,k!} C_n G_k \,\theta(n+k=m)$$

which is the desired result after we shift $m \to m-1$.

3. Denoting connected diagrams by hatching, and unconnected ones by light shading, we have

$G_n =$  , $C_n =$ 

Singling out one external leg, we then have



where the sum runs over all ways to distribute the other $m-1$ legs over the two blobs: this explains the binomial. The split is unambiguous since the left-hand leg must be part of a connected (sub)diagram.

Solution to exercise 2

The action, including the source,

$$S(\varphi) = \mu\varphi^2/2 + \lambda\varphi^4/4! - J\varphi$$

will, as $|\varphi| \to \infty$, go to *positive* infinity for *any* $\lambda > 0$, irrespective of μ and J, and the integral $Z(J)$ always exists. For $\lambda < 0$ the action will always go to *negative* infinity, and $Z(J)$ *never* exists.

Solution to exercise 3
In this exercise, $\hbar$ is introduced as a 'sneak preview'. The action must have the dimensionality of $\hbar$ since $S(\varphi)/\hbar$ occurs in the exponent.

1. The combinations $\mu\varphi^2$ and $\lambda\varphi^4$ both have the dimensionality of $\hbar$. Therefore $g = \hbar\lambda/\mu^2 = \hbar(\lambda\varphi^4)/(\mu\varphi^2)^2$ has dimension $\hbar \cdot \hbar/(\hbar)^2$

2. Under the substitution, φ takes on all real values if ψ runs from 0 to infinity. The action and the integration element take the form

$$S(\varphi) = \frac{3\hbar}{8g}\left(\psi + \frac{1}{\psi} - 2\right) \quad , \quad d\varphi = \sqrt{\frac{3\hbar}{16\mu g}}\left(\psi^{-3/4} + \psi^{-5/4}\right)\, d\psi$$

   By taking $\psi \to \psi^{-1}$ the term with $\psi^{-5/4}$ is brought in the same form as that with $\psi^{-3/4}$, and the result follows.

3. The integral can be written as

$$H = \sqrt{\frac{\hbar x}{\mu}}\, e^x\, K_{1/4}(x) \quad , \quad x = \frac{3}{4g}$$

4. $g \to 0$ corresponds to $x \to \infty$, and $g \to \infty$ corresponds to $x \to 0$.

Solution to exercise 4
Denoting the order of the derivative by superscripts, the SDe can be written as

$$\frac{\hbar^5}{5!}\lambda_6 Z^{(5)} + \frac{\hbar^4}{4!}\lambda_5 Z^{(4)} + \frac{\hbar^3}{3!}\lambda_4 Z^{(3)} + \frac{\hbar^2}{2!}\lambda_3 Z^{(2)} + \hbar\mu Z^{(1)} - JZ = 0$$

Solution to exercise 5
From the definition of $\phi(J)$ we see that

$$\frac{\partial}{\partial J}Z(J) = \phi(J)Z(J) \quad \Rightarrow \quad \frac{\partial^2}{(\partial J)^2}Z(J) = \left(\phi(J)^2 + \phi'(J)\right) Z(J)$$

and so on. The unit function $e(J)$ is introduced to give $\partial/\partial J$ something to work on.

Solution to exercise 6
Denoting $\partial^n\phi(J)/\partial J^n$ by ϕ_n, the SDe reads

$$\begin{aligned}
\phi_0 \;=\; & \frac{J}{\mu} - \frac{\lambda_3}{2\mu}\left(\phi_0^2 + \phi_1\right) - \frac{\lambda_4}{6\mu}\left(\phi_0^3 + 3\phi_0\phi_1 + \phi_2\right) \\
& - \frac{\lambda_5}{24\mu}\left(\phi_0^3 + 6\phi_0^2\phi_1 + 4\phi_0\phi_2 + 3\phi_1^2 + \phi_3\right) \\
& - \frac{\lambda_6}{120\mu}\left(\phi_0^5 + 10\phi_0^3\phi_1 + 10\phi_0^2\phi_2 + 15\phi_0\phi_1^2 + 5\phi_0\phi_3 + 10\phi_1\phi_2 + \phi_4\right)
\end{aligned}$$

Solution to exercise 7
An example of `MAPLE` code that does the trick is

```
phi:=0:  for k from 0 to 6 do
  phi:=convert(expand(series( z*J/mu
    - u*mu/6*(phi^3 + 3*phi*diff(phi,J) + diff(phi,J$2)),
    z=0,k+1)),polynom);
od:  phi:=subs(z=1,phi):
for n from 2 by 2 to 6 do C[n]:=coeff(phi,J,n-1)*(n-1)! od;
```

<u>Solution to exercise</u> 8

1. Denoting the exponential by A, we have

$$\mu\frac{\partial}{\partial J}A = \mu\varphi A \ \ , \ \ \lambda_3\frac{\partial}{\partial\mu}A = -\frac{\lambda_3}{2}\varphi^2 A \ \ , \ \ \lambda_3\frac{\partial}{\partial\lambda_3}A = -\frac{\lambda_4}{6}\varphi^3 A$$

   and the proof is the same as that of the SDe. We find

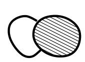
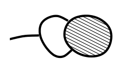
$$\mathfrak{S} = \left(\mu\frac{\partial}{\partial J} - \lambda_3\frac{\partial}{\partial\mu} - \lambda_4\frac{\partial}{\partial\lambda_3} - J\right) Z = 0$$

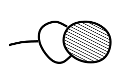
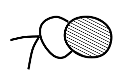
2. The object $\mathfrak{S}$ is srtrictly zero. Taking the combination

$$\left(\frac{\partial}{\partial J} - \phi\right)\mathfrak{S} = 0$$

   and recalling that $\partial Z/\partial J = \phi Z$ gives the desired result.

3. The operation $\partial/\partial J$ adds an eternal line to each diagram. This can be done by attaching the new line to a propagator, or to a three-point vertex:

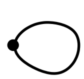
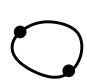
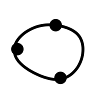
 $\to$  $\sim \frac{1}{\mu} \to -\frac{\lambda_3}{\mu^3}$,

 $\to$  $\sim \lambda_3 \to \frac{\lambda_4}{\mu}$.

   The derivative ensures that all propagators and vertices undergo this operation. The only diagram that is not accessible is the bare propagator, which explains the term $1/\mu$.

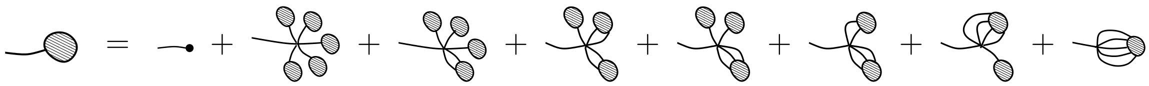
<u>Solution to exercise</u> 9

The easiest solution is to use two-point vertices on a single loop. From the symmetry factors of the diagrams

 $\sim \frac{1}{2}$,  $\sim \frac{1}{4}$,  $\sim \frac{1}{6}$

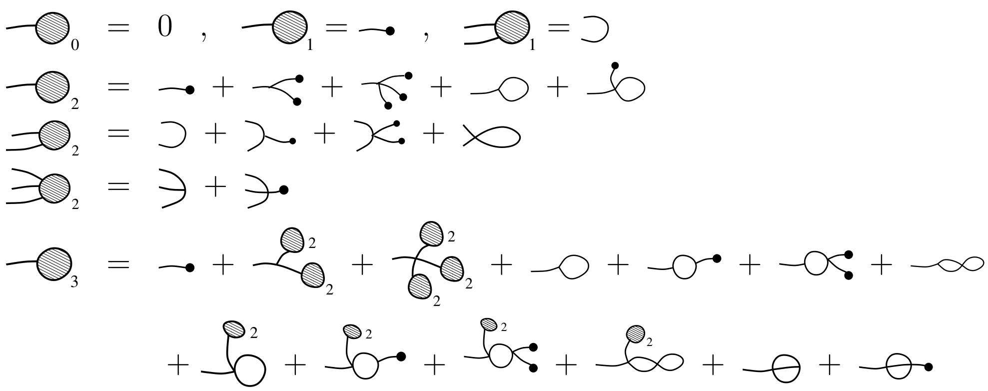
we see that a symmetry factor of $1/42$ belongs to such a diagram with 21 two-point vertices, the *icosikaihenagon*.

Solution to exercise 10



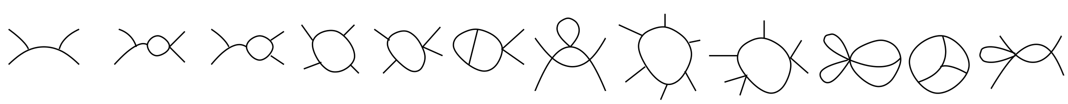
The algebraic form is given in exercise 6.

Solution to exercise 11
Indicating the order of the iteration by a subscript we write



After the third iteration we end up with 81 distinct diagrams, with up to 9 sources.

Solution to exercise 12
Treating the 12 diagram in order:



we have the following multiplicity factor, (inverse) symmetry factor, and number of loops:

	1	2	3	4	5	6	7	8	9	10	11	12
mult.	3	6	6	3	6	1	3	60	90	1	1	6
symm.	1	2	1	1	1	4	2	1	1	48	24	4
loops	0	1	1	1	1	2	2	1	1	4	3	2

Solution to exercise 13
The diagram has 8 loops, its symmetry factor is $(2!)^{-4}(3!)^{-3}/2 = 1/6912$, and its multiplicity is $(14)!/2!/3!/(4!)^3 = 525525$.

Solution to exercise 14

1. The Dyson series can be written as

$$\frac{1}{\mu_0} - \frac{1}{\mu_0}\delta\mu\frac{1}{\mu_0} + \frac{1}{\mu_0}\delta\mu\frac{1}{\mu_0}\delta\mu\frac{1}{\mu_0}\cdots = \frac{1}{\mu_0}\;\frac{1}{1+\delta\mu/\mu_0} = \frac{1}{\mu}$$

2.

$$\begin{aligned}\mathcal{R}(\mu) - \mathcal{R}(\mu_0) &= -\frac{1}{2}\frac{\delta\mu}{\mu_0} + \frac{1}{4}\left(\frac{\delta\mu}{\mu_0}\right)^2 - \frac{1}{6}\left(\frac{\delta\mu}{\mu_0}\right)^3 + \cdots \\ &= -\frac{1}{2}\log\left(1 + \frac{\delta\mu}{\mu_0}\right) = -\frac{1}{2}\log\left(\frac{\mu}{\mu_0}\right)\end{aligned}$$

Dividing by $\delta\mu$ and letting $\delta\mu$ go to zero we find that

$$\mathcal{R}'(\mu) = -\frac{1}{2\mu}$$

and this proves the result.

3. The stepping equation implies the correct tadpole:

$$\frac{\lambda_3}{\mu}\frac{\partial}{\partial\mu}\left(\frac{1}{2}\log\left(\frac{c}{\mu}\right)\right) = -\frac{\lambda_3}{2\mu^2}$$

4. The sum of these diagrams is

$$1 + \sum_{k\geq 1}\frac{1}{k!}\,\mathcal{R}(\mu)^k = \exp(\mathcal{R}(\mu)) = \sqrt{\frac{c}{\mu}} = \sqrt{\frac{2\pi}{\mu}} = \int d\varphi\ \exp\left(-\frac{\mu}{2}\varphi^2\right)$$

2 Exercises for chapter 2

Solution to exercise 15

This is essentially a repetition of exercise 8, with the inclusion of $\hbar$.

1. It suffices to include a factor $\hbar$ with every derivative, and the remember that $\hbar Z' = \phi Z$.

2. In the stepping term $(\hbar\lambda_3/\mu)\frac{\partial}{\partial\mu}$ the λ_3 requires a $1/\hbar$, but we have *two* extra propagators, giving a $\hbar^2$. In the stepping term $(\hbar\lambda_4/\mu)\frac{\partial}{\partial\lambda_3}$ the coupling is simply changed from λ_3 to λ_4, and there is *one* new propagator.

Solution to exercise 16

This is a repetition of exercise 3, with two couplings instead of one. We use that $\mu\varphi^2$, $\lambda_3\varphi^3$ and $\lambda_4\varphi^4$ all have the dimensionality of $\hbar$. Therefore $\hbar(\lambda_4\varphi^4)/(\mu\varphi^2)^2$ and $\hbar(\lambda_3\varphi^3)^2/(\mu\varphi^2)^3$ are dimensionless, as is $\lambda_3^2/(\mu\lambda_4)$.

Solution to exercise 17

1. The significant step is to realize that

$$\frac{dy}{dJ} = \frac{1}{y} \quad \to \quad \frac{d}{dJ} = \frac{1}{y}\frac{d}{dy}$$

Simple substitution then leads to the result.

2. Under the Ansatz we can write

$$\begin{aligned} yw^2 &= \sum_{n\geq 0} \hbar^n \, y^{3-3n} \sum_{m,\ell\geq 0} a_m a_\ell \, \theta(m+\ell=n) \ , \\ \hbar w' &= \sum_{n\geq 1} \hbar^n a_{n-1} \, y^{3-3n} \end{aligned}$$

and the result follows from equating the terms with given powers of y. The fact that they come out so nicely proves the consistency of the Ansatz.

3. By putting $n = 0$ we immediately see that $a_0^2 = 1$. The standard tree-level solution is $w = y$ so that $a_0 = 1$. Furthermore we can write

$$\sum_{m,\ell\geq 0} a_m a_\ell \, \theta(m+\ell=n) = 2a_n + \sum_{\ell=1}^{n-1} a_{n-\ell} a_\ell$$

4. Putting $n = 1$ in the recursion immediately gives $a_1 = -1/2$. For factorially divergent series, we can approximate, for large n,

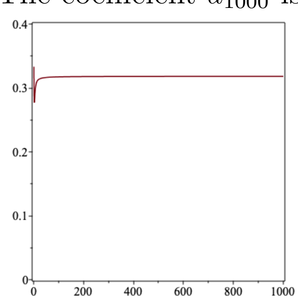
$$\sum_{\ell=1}^{n-1} a_{n-\ell} a_\ell \approx 2a_1 a_{n-1} = -a_{n-1}$$

so that $a_n \approx a_{n-1} \, 3(n-1)/2$.

5. A simple `MAPLE` code to compute the a_n is

```
a[0]:=1:a[1]:=-1/2: nmax:=1000;
for n from 2 to nmax do
  a[n]:=1/2*(a[n-1]*(3*n-4)-sum(a[j]*a[n-j],j=1..n-1));
od:
```

The coefficient a_{1000} is approximately $-1.579910647 \times 10^{2740}$.



This plot shows the ratio of a_n and $-(3/2)^n\Gamma(n)$. The asymptotic ratio of about 0.32 can be read off, or by direct inspection of a_{1000}, which gives a ratio of 0.3182213017.

6. The series is Borel summable since the coefficients do not increase faster than $n!$. The integral is ambiguous because, apart from a_0, all coefficients have the same sign.

<u>Solution to exercise</u> 18

1. We can write

$$-\hbar\frac{\partial}{\partial\varphi}\left(\varphi^m\, e^{-S/\hbar}\right) = \left(-\hbar m\varphi^{m-1} + \mu\varphi^{m+1} + \frac{\lambda}{6}\varphi^{m+3}\right) e^{-S/\hbar}$$

   and this integrates to zero since it is a total derivative. The assumption on H_{-1} is to ensure that $mH_{m-1} = 0$ for $m = 0$.

2. This goes in exactly the same way. The assumption on the endpoint behaviour is necessary to have the integral vanish.

Solution to exercise 19

1. The interaction term reads $\lambda\varphi^{m+1}/(m+1)!$. The classical equatuion follows from $S'(\phi) = J$.

2. Writing x for J/μ we have $\phi = x - (\lambda/\mu m!)\phi^m$. The Lagrange expansion then tells us that

$$\phi = x + \sum_{n\geq 1}\frac{1}{n!}\frac{\partial^{n-1}}{(\partial x)^{n-1}}\left(-\frac{\lambda}{\mu m!}x^m\right)^n \quad , \quad \frac{\partial^{n-1}}{(\partial x)^{n-1}}x^{mn} = \frac{(mn)!}{(mn-n+1)!}x^{mn-n+1}$$

   and the result follows immediately.

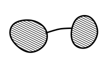
3. For φ^3 theory:

$$C_2 = \frac{\hbar}{\mu}\ ,\ C_3 = -\frac{\hbar^2\lambda}{\mu^3}\ ,\ C_4 = 3\frac{\hbar^3\lambda^2}{\mu^5}\ ,\ C_5 = -15\frac{\hbar^4\lambda^3}{\mu^7}$$

   For φ^4 theory:

$$C_2 = \frac{\hbar}{\mu}\ ,\ C_4 = -\frac{\hbar^3\lambda}{\mu^4}\ ,\ C_6 = 10\frac{\hbar^5\lambda^2}{\mu^7}\ ,\ C_8 = -280\frac{\hbar^7\lambda^3}{\mu^{10}}$$

Solution to exercise 20

A 1P reducible vacuum graph has to look like  and consists of two connected pieces both with a *single* leg, which is impossible in φ^4 theory.

Solution to exercise 21

From $\Gamma'(\phi) = J$ we find, after differentiation to J:

$$\Gamma''(\phi)\phi'(J) = 1\ ,\ \Gamma'''(\phi)\phi'(J)^2 + \Gamma''(\phi)\phi''(J) = 0$$

and therefore $\phi' = 1/\Gamma''$ and $\phi'' = -\Gamma'''/\Gamma''^3$. Inserting and expanding up to $\mathcal{O}(\hbar^3)$ then gives

$$0 = \Gamma_0' - \mu\phi - \frac{\lambda}{6}\phi^3 + \hbar\left(\Gamma_1' - \frac{\lambda\phi}{2\Gamma_0''}\right) + \hbar^2\left(\Gamma_2' + \frac{\lambda\phi\Gamma_1''}{2\Gamma_0''^2} + \frac{\lambda\Gamma_0'''}{6\Gamma_0''^3}\right)$$

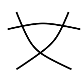
Sucessively asking the coefficients of $\hbar^{0,1,2}$ to vanish, we find

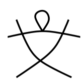
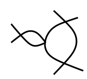
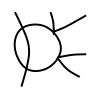
$$\begin{aligned}
\Gamma_0' &= \mu\phi + \frac{\lambda}{6}\phi^3 \ , \\
\Gamma_1' &= \frac{\lambda\phi}{2\mu+\lambda\phi^2} = \frac{\lambda}{2\mu}\phi - \frac{\lambda^2}{4\mu^2}\phi^3 + \frac{\lambda^3}{8\mu^3}\phi^5 + \cdots \ , \\
\Gamma_2' &= \frac{2\lambda^2\phi(\lambda\phi^2-10\mu)}{3(2\mu+\lambda\varphi^2)^4} = -\frac{5\lambda^2}{12\mu^3}\phi + \frac{7\lambda^3}{8\mu^4}\phi^3 - \frac{9\lambda^4}{8\mu^5}\phi^5 + \cdots
\end{aligned}$$

hence

$$\begin{aligned}
\Gamma_0 &= \frac{\mu}{2}\phi^2 + \frac{\lambda}{24}\phi^4 \ , \\
\Gamma_1 &= \frac{\lambda}{4\mu}\phi^2 - \frac{\lambda^2}{16\mu^2}\phi^4 + \frac{\lambda^3}{48\mu^3}\phi^6 \ , \\
\Gamma_2 &= -\frac{5\lambda^2}{24\mu^3}\phi^2 + \frac{7\lambda^3}{32\mu^4}\phi^4 - \frac{9\lambda^4}{48\mu^5}\phi^6
\end{aligned}$$

The 1PI diagrams with 2 and 4 legs are given in appendix 19.5.8-9. The 6-leg diagrams, with their $\mathfrak{sm}$, are

one loop :  $\mathfrak{sm} = 15$

two loops :  $\mathfrak{sm} = \frac{45}{2}$,  $\mathfrak{sm} = \frac{45}{2}$,  $\mathfrak{sm} = \frac{90}{2}$,  $\mathfrak{sm} = 45$

<u>Solution to exercise</u> 22

1. This step is completely analogous to that of exercise 21.

2. Here the significant issue is to realize that $\Gamma''(\phi) = x + A'(x)$ for $x = 1 + \phi$.

3. We can write

$$\begin{aligned}
xA(x) &= \sum_{n\geq 1}\left(\frac{\hbar}{2}\right)^n \kappa_n\, x^{3-3n} \\
A(x)A'(x) &= \sum_{n\geq 2}\left(\frac{\hbar}{2}\right)^n x^{3-3n}\sum_{m,\ell\geq 12}\kappa_m\kappa_\ell(3-2\ell)\theta(m+\ell=n)
\end{aligned}$$

   Inspection of the terms with equal powers of x gives the recursion relation. The values of $\kappa_{1,\ldots,5}$ follow immediately.

4. This can be read off immediately: the only point to keep in mind is that a term with ϕ^t in the effective action picks up a Feynman rule with a factor $t!$. The number $(3L-4+n)!/(3L-3)!$ is equal to $(3L-2)(3L-1)\cdots(3L+n-5)(3L+n-4)$, the product of $n-1$ subsequent numbers. This is guaranteed to contain more than L factors of 2 if $n > L+1$,[1] and hence the total $\mathfrak{sm}$ is integer if n is sufficiently large.

[1]Actually for smaller n since we do not count factors of 4,8,... in this argument.

Solution to exercise 23

1. The domain of φ is $(-\infty, 1/a)$. As $\varphi \to -\infty$, $\exp(-S/\hbar)$ decreases exponentially; for $\varphi \to 1/a$ we have $\exp(-S/\hbar) \approx (1 - a\varphi)^{\mu/a^2\hbar}$.

2. Expanding in powers of φ we have

$$S(\varphi) = \frac{\mu}{2}\varphi^2 + \sum_{n\geq 3} \frac{\mu a^{n-2}}{n}\varphi^n$$

and this gives the Feynman rule $-(n-1)!\mu a^{n-2}/\hbar$ for the n-point vertex.

3. From

$$S'(\varphi) = \frac{\mu\varphi}{1 - a\varphi} = \sum_{n\geq 0} \mu a^n\, \varphi^{n+1}$$

we find the SDe

$$\frac{\mu\hbar\frac{\partial}{\partial J}}{1 - a\hbar\frac{\partial}{\partial J}} Z(J) = \sum_{n\geq 0} \mu a^n \hbar^{n+1} \frac{\partial^{n+1}}{(\partial J)^{n+1}} Z = J Z(J)$$

Multiplying this on both sides by $1 - a\hbar\frac{\partial}{\partial J}$ gives the result.

4. Dividing the equation by $Z(J)$ leads immediately to $\mu\phi = J - a\hbar - aJ\phi$.

5. This is easier than it looks! For this action the stepping equation reads

$$C_{n+1} = \frac{\hbar}{\mu}\left(\lambda_3 \frac{\partial}{\partial\mu} + \sum_{k\geq 3} \lambda_{k+1} \frac{\partial}{\partial\lambda_k}\right) C_n$$

The three two-loop vacuum diagrams (see appendix 19.5.1) evaluate to

$$\frac{5\hbar\lambda_3^2}{24\mu^3} - \frac{h\lambda_4}{8\mu^2}$$

Applying the stepping rule once gives the two-loop *tadpole*:

$$-\frac{5\hbar^2\lambda_3^3}{8\mu^5} + \frac{2\hbar^2\lambda_3\lambda_4}{3\mu^4} - \frac{\hbar^2\lambda_5}{8\mu^3}$$

and this vanishes if we insert the values of the couplings. Therefore the two-loop propagator also vanishes.Alternatively, you might also write out the 7 diagrams for the two-loop tadpole.

6. From $\phi(J)$ in point 4 we can rewrite to find $J(\phi)$:

$$J(\phi) = \frac{\mu\phi + a\hbar}{1 - a\phi} = \Gamma'(\phi)$$

and then integration gives the effective action.

7. Differentiating $J(\phi)$ gives

$$\frac{\partial}{\partial \phi} J(\phi) = \frac{\mu + a^2\hbar}{(1-a\phi)^2} = \Gamma''(\phi)$$

and this is positive on $(-\infty, 1/a)$. Hence the effective action is concave.

8. The action including the source is

$$S(\varphi) = -\frac{\mu}{a^2}\log(1-a\varphi) - \left(\frac{\mu}{a} + J\right)\varphi$$

Therefore the action does not go to positive infinity for $\varphi \to -\infty$ when J becomes too negative, and the path integral becomes undefined.

Solution to exercise 24

This can be done by direct inspection of the vacuum diagrams, see appendix 19.5.1-2. The three-loop result is intimately related to the fact that in the Stirling approximation of the factorial the asymptotic correction terms of $\log(n!)$ contain only odd powers of $1/n$ (see equation 19.405).

Solution to exercise 25

1. The action has dimensionality μ/a^2 which must be the dimensionality of $\hbar$. The expansion

$$S(\varphi) = \frac{\mu}{2}\varphi^2 + \sum_{k\geq 3}\frac{\mu a^{k-2}}{k!}\varphi^k$$

proves the Feynman rules.

2. From

$$S'(\varphi) = \frac{\mu}{a}\left(e^{a\varphi} - 1\right)$$

the SDe for the path integral reads

$$\frac{\mu}{a}\left(\exp\left(a\hbar\frac{\partial}{\partial J}\right) - 1\right) Z(J) = \frac{\mu}{a}\left(Z(J+a\hbar) - Z(J)\right) = JZ(J)$$

3. You do not have to *find* the solution, only to *prove* it! Replacing J by $J + a\hbar$ inside the Gamma function is the crucial step:

$$\Gamma\left(\frac{\mu + a(J+a\hbar)}{a^2\hbar}\right) = \Gamma\left(\frac{\mu + aJ}{a^2\hbar}\right)\frac{\mu + aJ}{a^2\hbar}$$

It is trivially checked that $Z(0) = 1$. You can also compute $Z(J)$ directly as an integral, where the variable transformation $\varphi = \log(a^2\hbar z/\mu)/a$ is useful (with $0 \leq z < \infty$).

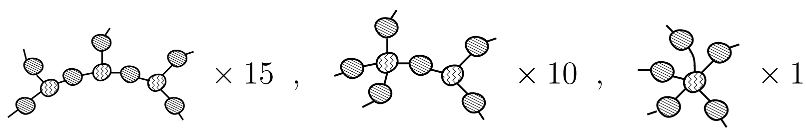
4. Up to an additive constant

$$\log(Z(J)) = \frac{J}{a\hbar}\log\left(\frac{a^2\hbar}{\mu}\right) + \log\Gamma\left(\frac{\mu + aJ}{a^2\hbar}\right)$$

and differentiating to J gives the result.

5. This step involves simply writing out the asymptotic expansion of the ψ function according to appendix 19.20.6
6. This follows directly from the properties of the ψ function.
7. For $\varphi \to +\infty$ the action always goes to positive infinity, but for $\varphi \to -\infty$ only as long as J larger than $-\mu/a$, just like in exercise 23.

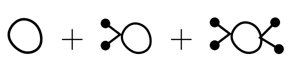
<u>Solution to exercise</u> 26
The 5-point skeleton diagrams are



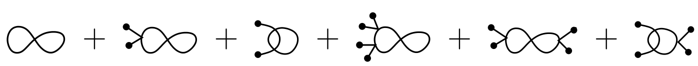
The last diagram has no tree-level contribution.

<u>Solution to exercise</u> 27
For simplicity we adopt $\lambda_{3,4} = 1$. The external fields ϕ_c are represented as source vertices. Notice that the external field lines are *not* external lines and therefore contribute to the symmetry factor!

1. The one-loop diagrams contributing to the effective action are

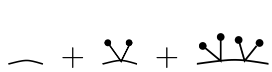

$$= -\frac{1}{2}\log\mu - \frac{\phi_c^2}{4\mu} + \frac{\phi_c^4}{16\mu^2}$$

The two-loop diagrams are

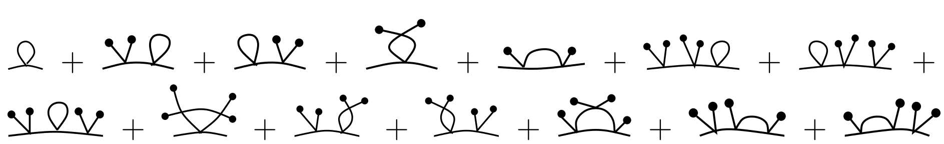

$$= -\frac{\hbar}{8\mu^2} + \frac{5\hbar\phi_c^2}{24\mu^3} - \frac{7\hbar\phi_c^4}{32\mu^4}$$

To get the effective action we must multiply this by $(-\hbar)$. The result is consisten with exercise 21, except that that approach cannot give the ϕ_c^0 terms.

2. The tree-level contributions to the propagator are

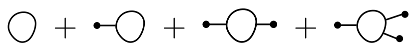

$$= \frac{\hbar}{\mu} - \frac{\hbar\phi_c^2}{2\mu^2} + \frac{\hbar\phi_c^4}{4\mu^3}$$

The one-loop contribution is


$$= -\frac{\hbar^2}{2\mu^3} + \frac{5\hbar^2\phi_c^2}{4\mu^4} - \frac{7\hbar^2\phi_c^4}{4\mu^5}$$

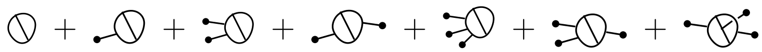
3. It is easy to verify that the propagator of item 2 equals $-2\hbar\frac{\partial}{\partial\mu}$ of the diagrams of item 1.

4. The one-loop diagrams contributing to the effective action are



$$= -\frac{1}{2}\log\mu - \frac{\phi_c}{2\mu} + \frac{\phi_c^2}{4\mu^2} - \frac{\phi_c^3}{6\mu^3}$$

The two-loop diagrams are

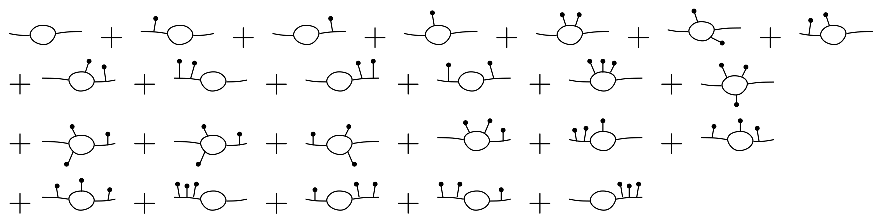


$$= \frac{\hbar}{12\mu^3} - \frac{\hbar\phi_c}{4\mu^4} + \frac{\hbar\phi_c^2}{2\mu^5} - \frac{5\hbar\phi_c^3}{6\mu^6}$$

The tree diagrams contributing to the propagator are

$$= \frac{\hbar}{\mu} - \frac{\hbar\phi_c}{\mu^2} - \frac{\hbar\phi_c^2}{\mu^3} - \frac{\hbar\phi_c^3}{\mu^4}$$

The one-loop diagrams contributing to the 'propagator' are



$$= \frac{\hbar^2}{2\mu^4} - \frac{2\hbar^2\phi_c}{\mu^5} + \frac{5\hbar^2\phi_c^2}{\mu^6} - \frac{10\hbar^2\phi_c^3}{\mu^7}$$

3 Exercises for chapter 3

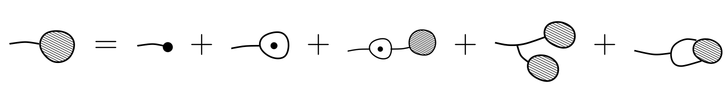
Solution to exercise 28

Consider an arbitrary connected diagram without dotted loops, and single out one external line to be on the left-hand side in the SDe. Then walk into the diagram up to the first vertex. This has a definite place in the SDe. Remove the line and the vertex, and proceed the the (possibly several disconnected) pieces of what is left. Any ambiguities are resolved by the symmetry factors, that are there precisely for this reason. This way we can, for each diagram, write a 'history' of its construction. If we include (a combination of) dotted loops, once a dotted loop is encountered we can determine in which of the 'black boxes' it belongs, and threat these as extra vertices, and proceed as above.

Solution to exercise 29

The only dotted 1PI diagrams possible in φ^3 are

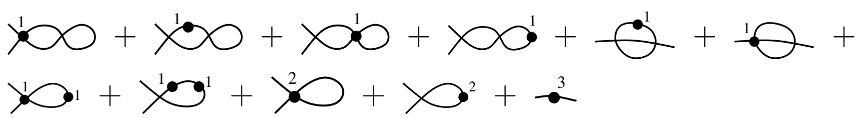
The SDe is a simplified version of Eq.(3.11):



If we include a dot on loops with 3 propagators, it is still not possible to form divergent 1PI diagrams with more than 3 legs, and altough the number of divergent 1PI diagrams becomes infinite, the theory remains renormalizable.

<u>Solution to exercise</u> 30
Any skeleton diagram is built up from (possibly pollywog) vertices and dressed propagators. These can be unambiguously split up into 1PI pieces, but *not* unambigously into non-1PI pieces. Therefore the only acceptable objects to pick up counterterms are the 1PI (sub)diagrams. Note that tadpoles are simply removed by their counterterms.

<u>Solution to exercise</u> 31
We put $\hbar = \mu = \lambda = 1$ for simplicity in this, and the following, exercise. The diagrams without counterterms amount to $\mathfrak{sm} = -5/6$ (see Eq.(19.56)). The countertermed diagrams are

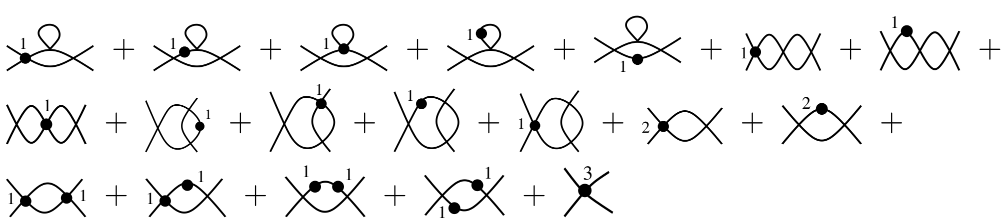


The requirement that the renormalized correction vanishes is therefore

$$-\frac{5}{6} + \frac{5}{6}\eta_1 - \frac{5}{4}\delta_1 - \frac{1}{2}\delta_1^2 + \frac{1}{2}\delta_1\eta_1 + \frac{1}{2}\delta_2 - \frac{1}{2}\eta_2 - \delta_3 = 0$$

and inserting $\delta_{1,2}$ and $\eta_{1,2}$ from Eqs.(3.52-54) gives $\delta_3 = -1/8$.

<u>Solution to exercise</u> 32
The diagrams without counterterms amount to $\mathfrak{sm} = 45/2$ (see Eq.(19.60)). The countertermed diagrams are

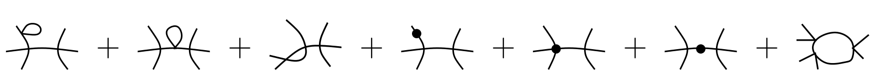


The requirement that the renormalized correction vanishes is therefore

$$\frac{45}{2} - \frac{63}{4}\eta_1 + 21\delta_1 + \frac{9}{2}\delta_1^2 - 6\delta_1\eta_1 + \frac{3}{2}\eta_1^2 - 3\delta_2 + 3\eta_2 - \eta_3 = 0$$

and inserting $\delta_{1,2}$ and $\eta_{1,2}$ from Eqs.(3.52-54) gives $\eta_3 = 11/8$.

<u>Solution to exercise</u> 33
The diagrams are



The one-loop counterterms cancel the one-loop diagrams *except* the last one, which has $\mathfrak{sm} = 15$, and this is consistent with Eq.(3.4).

<u>Solution to exercise</u> 34
Rather than giving all diagrams explicitly, it is better to use combinatorics here. The unrenormalized 4-loop diagrams have $\mathfrak{sm} = 115/48$. The three-loop diagrams

have $\mathfrak{sm} = -5/6$, and they all contain 3 vertices and 5 propagataors (not counting the two legs, of course). Adding one-loop counterterms therefore gives for each diagram a factor $3\eta_1 - 5\delta_1$. The two-loop diagrams, with total $\mathfrak{sm} = 5/12$, have 2 vertices and 3 propagators. We can add either two one-loop counterterms or one two-loop counterterm, leading to a factor $\eta_1^2 - 6\eta_1\delta_1 + 6\delta_1^2 + 2\eta_2 - 3\delta_3$. The one-loop diagram ($\mathfrak{sm} = -1/2$) allows for only one vertex counterterm but any number of mass counterterms, so is multiplied by $\eta 1\delta_1^2 - \delta_1^3 - \eta_2\delta_1 - \eta_1\delta_2 + \eta_3 - \delta_3$. The total condition on δ_4 therefore becomes

$$\begin{aligned}\delta_4 &= \frac{115}{48} - \frac{5}{6}(3\eta_1 - 5\delta_1) + \frac{5}{12}(\eta_1^2 - 6\eta_1\delta_1 + 6\delta_1^2 + 2\eta_2 - 3\delta_2) \\ &\quad - \frac{1}{2}(\eta_1\delta_1^2 - \delta_1^3 - \eta_2\delta_1 - \eta_1\delta_2 + 2\delta_1\delta_2 + \eta_3 - \delta_3) = -\frac{9}{16}\end{aligned}$$

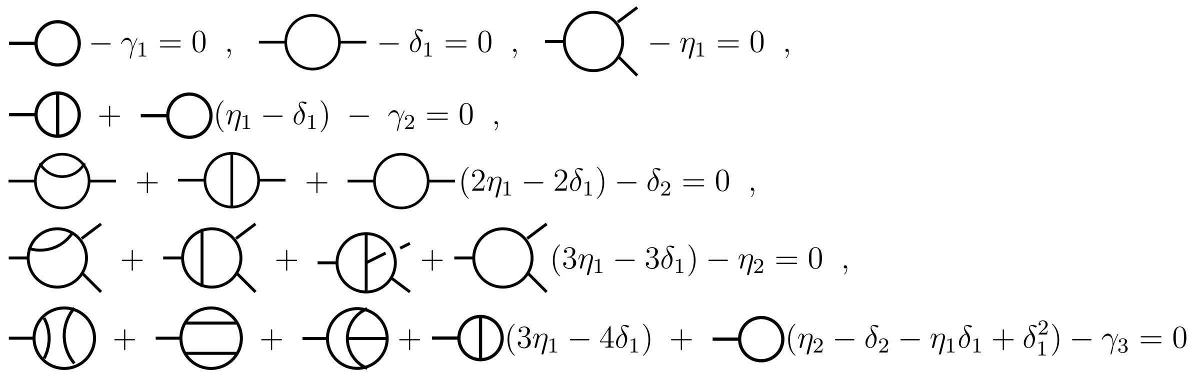
<u>Solution to exercise</u> 35

Below, we give a simple `MAPLE` code to perform renormalization in φ^4 theory. The end result is the comparison of the coefficients in $t_{2n}(u)$ and $\tau_{2n}(\hat{u})$, that gives the improvement factors, for n up to 12.

```
N:=50: Order:=N+1: nmax:=12:
w:=expand(series(log(1+sum(G[2*j]*x^(2*j)/(2*j)!,j=1..Order/2)),x)):
for n from 2 by 2 to N do C[n]:=coeff(w,x,n)*n! od:
for n from 0 by 2 to nmax do
  H[n]:=0:
  for k from 0 to N do
    m:=2*k+n/2;
    H[n]:=H[n]+(-lambda/24/h)^k*(h/mu)^m*(2*m)!/2^m/m!/k!;
  od;
  G[n]:=series(H[n]/H[0],h=0);
od:
for n from 2 by 2 to nmax do
  C[n]:=subs(lambda=u*mu^2/h,convert(series(C[n],h=0),polynom));
  t[n]:=expand(C[n]/(-u)^(n/2-1)*(mu/h)^(n/2));
od:
U:=convert(solve(v=series(u*t[4]/t[2]^2,u=0,N-1),u),polynom):
s:=expand(U/v):
for n from 2 by 2 to nmax do
  tau[n]:=series(s^(n/2-1)*subs(u=U,t[n]/t[2]^(n/2)),v=0,N-2);
od:
for n from 6 by 2 to nmax do  w[n]:=[]:
  for k from 0 to N-2 do
    w[n]:=[op(w[n]),[k,coeff(t[n],u,k)/coeff(tau[n],v,k)]];
 od;  od:
plot([seq(w[2*j],j=3..nmax/2),[[0,exp(15./4)],[N-nmax,exp(15./4)]]],0..N-nmax);
```

<u>Solution to exercise</u> 36

This can be solved in analogy to exercise 34. All relevant diagrams are found in appendix 19.5.5-7. The pertinent diagrammatic equations are



<u>Solution to exercise</u> 37

1. The SDe for the ψ field reads

$$J = m\psi + \frac{g_3}{2}(\psi^2 + \hbar\psi') + \frac{g_4}{6}(\psi^3 + 3\hbar\psi\psi' + \hbar^2\psi'')$$

2. Replacing ψ by $\phi + t$ we can rearrange terms:

$$\begin{aligned} J &= m(\phi + t) + \frac{g_3}{2}(\phi^2 + 2t\phi + t^2 + \hbar\phi') \\ &\quad + \frac{g_4}{6}(\phi^2 + 3t\phi^2 + 3t^2\phi + t^3 + 3\hbar\phi\phi' + 3\hbar t\phi' + \hbar^2\phi'') \\ &= \left(m + tg_3 + \frac{g_4}{2}t^2\right)\phi + \frac{g_3 + g_4 t}{2}(\phi^2 + \hbar\phi') \\ &\quad + \frac{g_4}{6}(\phi^3 + 3\hbar\phi\phi' + \hbar^2\phi'') + \left(mt + \frac{1}{2}g_3t^2 + \frac{1}{6}t^3\right) \\ &= S''(t)\phi + \frac{1}{2}S'''(t)(\phi^2 + \hbar\phi') + \frac{1}{6}S''''(t)(\phi^3 + 3\hbar\phi\phi' + \hbar^2\phi'') + S'(t) \end{aligned}$$

and this proves the form of the action in terms of φ.

<u>Solution to exercise</u> 38

Let the renormalized parameters be μ_r and λ_r, and $u_r = \hbar\lambda_r^2/\mu_r^3$. Exercise 36 then tells us that the *bare* parameters, those occurring in the action, are given by

$$\mu = \mu_r(1 + u_r/2 - u_r^2./2 + \cdots) \quad , \quad \lambda = \lambda_r(1 - u_r - u_r^2/2 + \cdots)$$

Writing $u = \hbar\lambda^2/\mu^3$ and inserting everything into Eq.(3.71), expansion in powers of u_r gives

$$T = \frac{\hbar\lambda_r}{\mu_r}\left(-\frac{1}{2} + \frac{1}{2}u_r + \frac{1}{4}u_r^2 + \cdots\right)$$

which corresponds precisely to $\gamma_1 = -1/2$, $\gamma_2 = 1/2$, $\gamma_3 = 1/4$.

<u>Solution to exercise</u> 39

1. This follows immediately from $G_k = H_k/H_0$.
2. This is necessary since the tadpole in completely cancelled by the counterterms, and an 1PI tadpole graph cannot contain a lower-loop tadpole since then it would not be 1PI.
3. The SDe-like equation is a (linear) recurrence relation, so we can find all G_k by successive calculation. With $x = 2/\lambda$:

$$G_2 = -xT \ , \ G_3 = x^2\mu T + x\hbar \ , \ G_4 = -x^3\mu^2 T + x^2T^2 - x^2\mu\hbar \ ,$$
$$G_5 = x^4\mu^3 T - 2x^3\mu T^2 + x^3\mu^2\hbar - 4x^2T\hbar$$

4. Write $T = \sum_{k\geq 1} t_k\hbar^k$. Then we can successively put the coefficient of $\hbar^k$ in G_{2k+1} to zero, and so we find

$$t_1 = -\frac{1}{\mu x} \ , \ t_2 = -\frac{2}{\mu^4 x^3} \ , \ t_3 = -\frac{20}{\mu^7 x^5} \ , \ t_4 = -\frac{352}{\mu^{10} x^7} \ , \ \ldots$$

Solution to exercise 40

1. This follows directly from the perturbation expansion, where the tadpole term is also expanded. The only delicate points are the realization that the tadpole term must be $\mathcal{O}(\hbar)$, and that odd powers of φ must integrate to zero.
2. This is governed by the fact that in each term the power of $\hbar$ is $p-k = (n+k)/2$. The rest of this exercise is embodied in the following `MAPLE` code:

```
N:=30; Order:=N+1: nmax:=6;
w:=expand(series(log(1+sum(G[j]*x^j/j!,j=1..nmax)),x=0,nmax+1)):
for k from 1 to nmax do C[k]:=coeff(w,x,k)*k! od:
#T:=0;
for n from 0 to nmax do
  H[n]:=0;
  for k from 0 to 2*N-n do
    for r from 0 to 2*N-n-k do
      if (n+3*k+r) mod 2 =0 then
        p := (n+3*k+r)/2;
        H[n]:=H[n]+(-lambda/6/h)^k*(-K)^r*(h/mu)^p*(2*p)!/2^p/p!/k!/r!;
      fi;
    od;
  od;
  G[n]:=expand(series(H[n]/H[0],h=0));
od:
for k from 1 to nmax do C[k]:=convert(expand(series(C[k],h=0)),polynom); od:
K:=sum(d[j+1]*h^j,j=0..N):
C[1]:=series(C[1],h=0):
for k from 1 to N do
  d[k]:=solve(coeff(C[1],h,k)=0,d[k]);
```

```
  C[1]:=expand(C[1]);
od:
expand(algsubs(lambda^2=u/h*mu^3,convert(expand(series(C[1],h=0)),polynom)));
for n from 2 to nmax do
  C[n]:=convert(expand(series(C[n],h=0)),polynom);
  t[n]:=algsubs(lambda^2=u*mu^3/h,expand((-1)^n*C[n]/lambda^(n-2)/h^(n-1)*mu^(2*
od:
us:=solve(v=series(u*t[3]^2/t[2]^3,u=0),u):
for n from 2 to nmax do
  tau[n]:=series(subs(u=us,convert(series(t[n]*t[2]^(n-3)/t[3]^(n-2),u=0),polyno
od:
for n from 4 to nmax do
  w[n]:=[];
  for k from 0 to N-nmax do
    w[n]:=[op(w[n]),[k,coeff(t[n],u,k)/coeff(tau[n],v,k)]];
  od;
od:
plot([seq(w[j],j=4..nmax),[[0,exp(7/3.)],[N-nmax,exp(7/3.)]]]);
```

This is not the fastest possible code, but at least it is transparent. For the purely one-loop tadpole renormalization, it suffices to put

```
 K= lambda/(2*mu^2)
```

and skip the computation of the coefficients `d[k]`.

4 Exercises for chapter 4

Solution to exercise 41

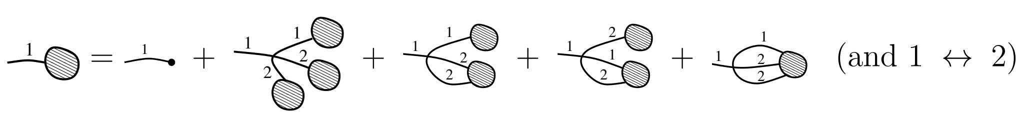
1. The SDe for the path integral:

$$\hbar\mu\frac{\partial}{\partial J_1} + \hbar^3\frac{\lambda}{2}\frac{\partial}{\partial J_1}\frac{\partial^2}{(\partial J_2)^2} = J_1 Z \quad (\text{and } 1 \leftrightarrow 2)$$

2. The SDe for the field functions:

$$\phi_1 = \frac{J_1}{\mu} - \frac{\lambda}{2\mu}\left(\phi_1\phi_2^2 + \hbar\phi_1\frac{\partial}{\partial J_2}\phi_2 + 2\hbar\phi_2\frac{\partial}{\partial J_2}\phi_1 + \hbar^2\frac{\partial^2}{(\partial J_1)^2}\phi_2\right) \quad (\text{and } 1 \leftrightarrow 2)$$

3. Diagrammatically:



Solution to exercise 42

1. Doing the φ_2 integral:

$$\int d\varphi_2 \exp\left(-\frac{\mu\varphi_2^2}{2\hbar}\left(1+\frac{\lambda\varphi_1^2}{2\mu}\right)\right) \sim \left(1+\frac{\lambda\varphi_1^2}{2\mu}\right)^{-1/2} = \exp\left(-\frac{1}{\hbar}\left[\frac{\hbar}{2}\log\left(1+\frac{\lambda\varphi_1^2}{2\mu}\right)\right]\right)$$

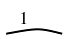
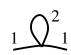
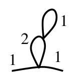
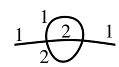
2. After the φ_2 integral:

$$S(\varphi_1) = -J_1\varphi_1 + \frac{1}{2}\left(\mu+\frac{\hbar\lambda}{2\mu}\right)\varphi_1^2 - \frac{\hbar\lambda^2}{16\mu^2}\varphi_1^4 + \frac{\hbar\lambda^3}{48\mu^3}\varphi_1^6 + \cdots$$

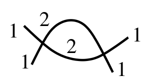
which leads to the Feynman rules

 $= \dfrac{\hbar}{\mu+\frac{\hbar\lambda}{2\mu}}$,  $= \dfrac{3\lambda^2}{2\mu^2}$,  $= -\dfrac{15\lambda^3}{\mu^3}$

3. Before the φ_2 integral:

C_2 =  +  +  +  +  $+ \cdots$

$$= \frac{\hbar}{\mu} - \frac{\hbar^2\lambda}{2\mu^3} + \frac{\hbar^3\lambda^2}{\mu^5}\left(\frac{1}{4}+\frac{1}{4}+\frac{1}{2}\right) + \cdots ,$$

C_4 =  $+ \cdots = \dfrac{3\hbar^4\lambda^2}{2\mu^6} + \cdots$

After integrating out φ_2:

C_2 = +  $= \dfrac{\hbar}{\mu+\frac{\hbar\lambda}{2\mu}} + \dfrac{3\lambda^2}{4\mu^2}\left(\dfrac{\hbar}{\mu+\frac{\hbar\lambda}{2\mu}}\right)^3 + \cdots$,

C_4 =  $+ \cdots = \dfrac{3\lambda^2}{2\mu^2}\left(\dfrac{\hbar}{\mu+\frac{\hbar\lambda}{2\mu}}\right)^4 + \cdots$

If we expand in $\hbar$ the two formulations are identical up to the indicated order.

<u>Solution to exercise</u> 43

1. Concentrate on ϕ_1, and let n be $2,3,\ldots,N$ where we implicitly sum over n. The interaction term is $(\lambda/4!)(\varphi_1^2+\varphi_n^2)^2 = \lambda\varphi_1^4/24 + \lambda\varphi_1^2\varphi_n^2/12 + \cdots$ so that the Feynman rules are $-\lambda/\hbar$ for a φ_1^4 coupling and $-\lambda/3\hbar$ for a $\varphi_1^2\varphi_n^2$ one. The SDe for ϕ_1 is therefore

$$\begin{aligned}
\phi_1 &= \frac{J_1}{\mu} - \frac{\lambda}{\mu}K \ , \\
K &= \frac{1}{6}\phi_1^3 + \frac{1}{2}\frac{1}{3}\phi_2\phi_n^2 + \frac{\hbar}{2}\phi_1\nabla_1\phi_1 + \frac{\hbar}{3}\phi_n\nabla_1\phi_n + \frac{\hbar}{2}\frac{1}{3}\phi_1\nabla_n\phi_n + \frac{\hbar^2}{6}\nabla_1^2\phi_1 + \frac{\hbar}{2}\frac{1}{3}\nabla_n^2\phi_1 \\
&= \frac{1}{6}\phi_1(\phi_1^2+\phi_n^2) + \frac{\hbar}{6}\phi_1(\nabla_1\phi_1+\nabla_n\phi_n) + \frac{\hbar}{6}\nabla_1(\phi_1^2+\phi_n^2) + \frac{\hbar^2}{6}(\nabla_1^2+\nabla_n^2)\phi_1 \\
&= \frac{1}{6}\left(\phi_1|\vec{\phi}|^2 + \hbar\phi_1\vec{\nabla}\cdot\vec{\phi} + \hbar\nabla_1|\vec{\phi}|^2 + \hbar^2\vec{\nabla}^2\phi_1\right)
\end{aligned}$$

2. We use

$$S_1(\vec{\phi}) = \mu\varphi_1 + \frac{\lambda}{6}\varphi_1|\vec{\phi}|^2$$

and follow the (more-field analogue of) the SDe

3. We use

$$|\vec{\phi}|^2 = 2xF(x)^2 \ \ , \ \ \nabla_j\phi_k = \delta_{jk}F(x) + J_jJ_kF'(x) \ \ ,$$
$$\nabla_j\nabla_k\phi_\ell = (J_j\delta_{k\ell} + J_k\delta_{j\ell} + J_\ell\delta_{jk})F'(x) + J_jJ_kJ_\ell F''(x)$$

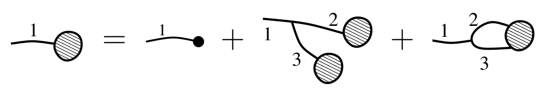
and perform the appropriate sums.

Solution to exercise 44

1. This follows directly from, for instance,

$$\frac{\partial}{\partial\varphi_1}S(\varphi_1,\varphi_2,\varphi_3) = \mu\varphi_1 + g\varphi_2\varphi_3 - J_1$$

2.



3. This holds trivially for the tree propagators. There are two operations on connected diagrams: splitting an external line into two (where for instance $1 \to 2+3$), and this reduces n_1 by one, while increasing $n_{2,3}$ by one; or closing a loop, which reduces one of the n_j by two (the reverse operations are also possible of course). Starting with the tree diagram for $1 \to 2+3$ we thus prove the statement.

4. A two-point diagram for, say, Π_{12} would have $n_1 = n_2 = 1$ and $n_3 = 0$ and is therefore impossible. A tadpole for field 1, say, is necessarily built from a single 123 vertex and a Π_{23}, and hence is impossible.

Solution to exercise 45

1. This is simply the more-field version of the classical SDe, which prescribes that the action including the sources must be at a minimum. Here, we shall first assume that the lines are not oriented.

2. Write the action as

$$S(\vec{\phi}) = \sum_n \frac{\mu_n}{2}\phi_n^2 + V(\phi_n)$$

then

$$W_{ab} = \mu_a\delta_{ab} + R_{ab} \ \ , \ \ R_{ab} = \frac{\partial}{\partial\phi_a}\frac{\partial}{\partial\phi_b}V(\vec{\phi})$$

we can interpret this as a matrix expression . The inverse is (summing over repeated indices)

$$W_{bk}^{-1} = \frac{1}{\mu_b}\left(\delta_{bk} - R_{bk}\frac{1}{\mu_k} + R_{bc}\frac{1}{\mu_c}R_{ck}\frac{1}{\mu_k} - R_{bc}\frac{1}{\mu_c}R_{cd}\frac{1}{\mu_d}R_{dk}\frac{1}{\mu_k} + \cdots\right)$$

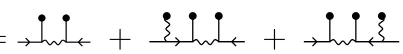
so that in $\hbar W_{ab}^{-1}$ we recognize the Dyson-summed propagator with an effective two-point interaction R_{ab}. The external fields in R do not undergo any additional interactions: they are classical, external fields.

3. For (Bald) QED, we have

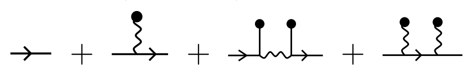
$$W = \begin{pmatrix} 0 & m + eA & e\bar{\psi} \\ eA + m & 0 & e\psi \\ e\psi & e\bar{\psi} & \mu \end{pmatrix}$$

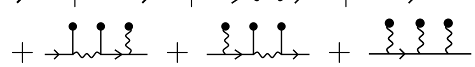
This can be inverted by standard operations, and we find the following distinct propagators up to $\mathcal{O}(e^4)$:

$$\hbar(W^{-1})_{\psi\psi} = \frac{\hbar e^2\psi^2}{m^2\mu} - 2\frac{\hbar e^3\psi^2 A}{m^3\mu} =$$



$$\hbar(W^{-1})_{\psi\bar{\psi}} = \frac{\hbar}{m} - \frac{\hbar e A}{m^2\mu} + \frac{\hbar^2\psi\bar{\psi}}{m^2\mu} + \frac{\hbar e^2 A^2}{m^3} - 2\frac{\hbar\psi\bar{\psi}A}{m^3\mu} - \frac{\hbar e^3 A^3}{m^4}$$

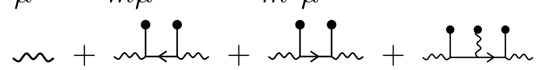
$=$ 



$$\hbar(W^{-1})_{\psi A} = -\frac{\hbar e\psi}{m\mu} + \frac{\hbar e^2\psi A}{m^2\mu} - 2\frac{\hbar e^3\psi^2\bar{\psi}}{m^2\mu^2} - \frac{\hbar e^3\psi\bar{\psi}^2}{m^3\mu}$$

$=$ 

$$\hbar(W^{-1})_{AA} = \frac{\hbar}{\mu} + 2\frac{\hbar e^2\psi\bar{\psi}}{m\mu^2} - 2\frac{\hbar e^3\psi\bar{\psi}A}{m^2\mu^2}$$

$=$ 

The other propagators are obtained by reversing all arrows.

Solution to exercise 46

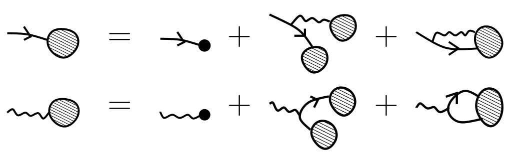
1. This is easiest studied if we assume a single interaction $\lambda\varphi_1^n\varphi_2^k/(n!k!)$. Up to one loop, the SDe for ϕ_1 can then be written as

$$\mu_1\phi_1 = J_1 - \frac{\lambda}{(n-1)!k!}\phi_1^{n-1}\phi_2^k - \hbar\lambda\left(\frac{1}{2}\frac{1}{(n-2)!k!}\phi_1^{n-3}\phi_2^k\frac{\partial}{\partial J_1}\phi_1 + \frac{1}{(n-2)!(k-1)!}\phi_1^{n-2}\phi_2^{k-1}\frac{\partial}{\partial J_2}\phi_1 + \frac{1}{2}\frac{1}{(n-1)!(k-2)!}\phi_1^{n-1}\phi_2^{k-2}\frac{\partial}{\partial J_2}\phi_2\right)$$

which has precisely the indicated form as long as we remember that $\frac{\partial}{\partial J_1}\phi_2 = \frac{\partial}{\partial J_2}\phi_1$.

2. This follows immediately from Eq.(19.470), where we have to use that $\partial\phi_k/\partial J_\ell$ is the inverse of $\partial^2\Gamma/\partial\phi_k\partial\phi_\ell$, and that both this term and the $W_{k\ell}$ have to be taken at the tree level, since the $\hbar$ is already explicitly there. The factor $1/\det(W(\vec{0}))$ is included to make sure that $\Gamma^{(1)}(\vec{0}) = 0$.

Solution to exercise 47



(and with arrows reversed)

Since in the vertex no legs are equivalent, all symmetry factors are unity.

Solution to exercise 48
The 2- and 3-loop connected vacuum diagrams are

The symmetry factors are, respectively, 1/2, 1/2, 1/2, 1, 1/4, 1/4, 1/2. Therefore $\mathfrak{sm} = 1$ for the 2-loop vacuum diagrams, and $\mathfrak{sm} = 5/2$ for the 3-loop diagrams.

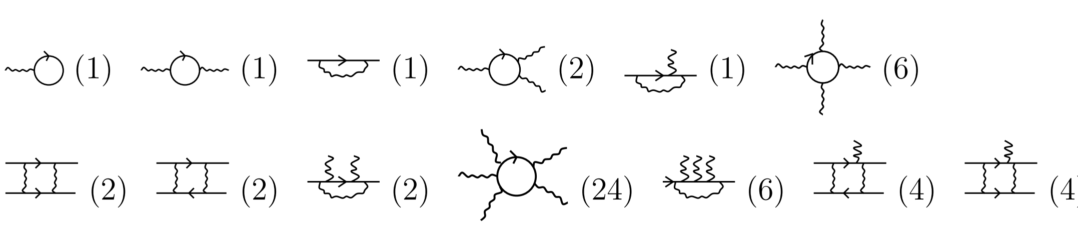
Solution to exercise 49
From exercise we have the form of W, and we can immediately compute

$$\frac{\det\left(W(\psi,\bar{\psi},A)\right)}{\det\left(W(0,0,0)\right)} = \left(1+\frac{e}{m}A\right)\left(1+\frac{e}{m}A - 2\frac{e^2}{m\mu}\psi\bar{\psi}\right)$$

and this gives the quoted form for $\Gamma^{(1)}$. Expanding that to fifth order, we find

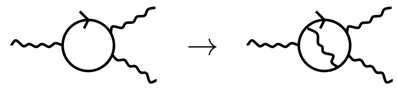
$$\begin{aligned}\Gamma^{(1)} &= e\frac{A}{m} + e^2\left(-\frac{A^2}{2m^2} - \frac{\psi\bar{\psi}}{m\mu}\right) + e^3\left(\frac{A^3}{3m^3} + \frac{A\psi\bar{\psi}}{m^2\mu}\right)\\ &\quad + e^4\left(-\frac{A^4}{4m^4} - \frac{\psi^2\bar{\psi}^2}{m^2\mu^2} - \frac{A^2\psi\bar{\psi}}{m^3\mu}\right) + e^5\left(\frac{A^5}{5m^5} + \frac{A^3\psi\bar{\psi}}{m^4\mu} + \frac{2A\psi^2\bar{\psi}^2}{m^3\mu^2}\right)\end{aligned}$$

The corresponding 1PI diagrams (with their $\mathfrak{sm}$ indicated) are



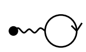
Solution to exercise 50
An 'electron' loop with n 'photon' vertices contains n powers of $(-e/m)$ and has $\mathfrak{sm} = (n-1)!$. The sum of the effective vertices is therefore precisely that of Eq.(4.27).

Solution to exercise 51
Adding another loop retains the even/odd character of the number of vertices on a fermion loop: for instance

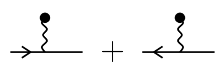


Of course one could have the extra photon line connected to another part of the diagram, but in that case Furry's theorem no longer applies to the modified diagram. Also note that a $\varphi\bar{\varphi}B^2$ vertex would destroy the validity of Furry's theorem in higher orders.

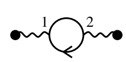
Solution to exercise 52
Here we do not give the complete result, but rather discuss how to obtain it. As before, we can replace the derivative to the 'regulator' by simply a derivative to eiher m or μ since that has the same effect. We shall drop the overall factor $\hbar$.The first example is (the arrow denoting derivatives)

 $= \dfrac{eA}{m} \;\to -\dfrac{eA}{m^2}$

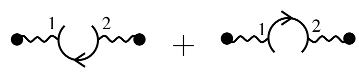
The corresponding diagrams are obtained by opening up every propagator (in this case just one) and entering it on on side or the other: this gives

 $= -2\dfrac{eA}{m^2}$

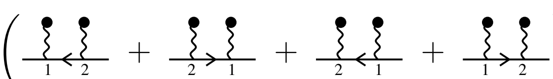
as required (remember the factor 1/2 that one has to add). The second example is

 $= -\dfrac{e^2A^2}{2m^2} \;\to\; \dfrac{e^2A^2}{m^3}$

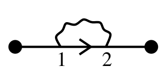
where we have arbitrarily numbered the vertices for clarity. There are now two ways to open propagators:

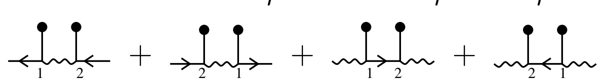


The corresponding propagator diagrams (each carrying the symmetry factor 1/2) are

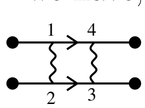
$\dfrac{1}{2}\Big($  $\Big) = \dfrac{2e^2A^2}{m^3}$

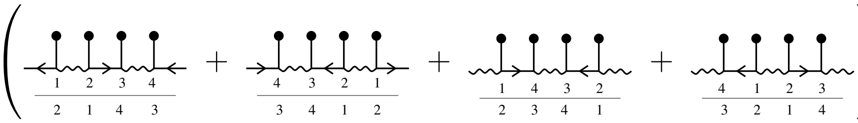
The same works for different internal lines:

 $= -\dfrac{e^2\psi\bar{\psi}}{m\mu} \;\to\; \dfrac{e^2\psi\bar{\psi}}{m^2\mu} + \dfrac{e^2\psi\bar{\psi}}{m\mu^2}$

 $= \dfrac{2e^2\psi\bar{\psi}}{m^2\mu} + \dfrac{2e^2\psi\bar{\psi}}{m\mu^2}$

In order e^4 we have, for example,

 $= -\dfrac{e^4\psi^2\bar{\psi}^2}{2m^2\mu^2} \;\to\; \dfrac{e^4\psi^2\bar{\psi}^2}{m^3\mu^2} + \dfrac{e^4\psi^2\bar{\psi}^2}{m^2\mu^3}$

$\dfrac{1}{2}\Bigg($  $\Bigg)$

$$= \frac{2e^4\psi^2\bar{\psi}^2}{m^3\mu^2} + \frac{2e^4\psi^2\bar{\psi}^2}{m^2\mu^3}$$

5 Exercises for chapter 5

<u>Solution to exercise</u> 53

First we take $n \geq 2$. The recursion relation leads, under the Ansatz, to

$$AB^n = \frac{\gamma}{\mu}\left(AB^{n+1} + AB^{n-1}\right) \quad \rightarrow \quad B = \frac{1}{2\gamma}\left(\mu - \sqrt{\mu^2 - 4\gamma^2}\right)$$

Then, for $n = 0$,

$$A = \frac{\hbar}{\mu} + \frac{2\gamma}{\mu}AB \quad \rightarrow \quad A = \frac{\hbar}{\sqrt{\mu^2 - 4\gamma^2}}$$

<u>Solution to exercise</u> 54

1. We have

$$\varphi(x) = \frac{1}{2\pi}\int dk\, e^{ikx}\varphi(k) \;\rightarrow\; \overline{\varphi(x)} = \frac{1}{2\pi}\int dk\, e^{-ikx}\varphi(k) = \frac{1}{2\pi}\int dk\, e^{ikx}\varphi(-k)$$

   and if φ is real we must therefore have $\varphi(k) = \varphi(-k)$. The same holds for the sources.

2. This is most easily done by first looking at the m^2 term:

$$\begin{aligned}\int dx\, \varphi(x)^2 &= \frac{1}{(2\pi)^2}\int dx\, dk_1\, dk_2\, e^{ix(k_1+k_2)}\, \varphi(k_1)\, \varphi(k_2) \\ &= \frac{1}{2\pi}\int dk_1\, dk_2\, \delta(k_1+k_2)\, \varphi(k_1)\, \varphi(k_2) \\ &= \frac{1}{2\pi}\int dk\, \varphi(k)\, \varphi(-k) = \frac{1}{2\pi}\int dk\, \varphi(k)^2\end{aligned}$$

   Note how important it is that m^2 does not depend on x! In exactly the same way:

$$\begin{aligned}\int dx\, \varphi'(x)^2 &= \frac{1}{2\pi}\int dk\, k^2\, \varphi(k)^2 \ , \\ \int dx\, \varphi(x)^4 &= \frac{1}{(2\pi)^4}\int dk_1 \cdots dk_4\ (2\pi)\delta(k_1 + \cdots + k_4)\, \varphi(k_1)\, \varphi(k_2)\, \varphi(k_3)\, \varphi(k_4) \ , \\ \int \varphi(x)J(x) &= \frac{1}{2\pi}\int dk\varphi(k)\, J(k)\end{aligned}$$

   Here we have used $\int dx\, \exp(ixy) = 2\pi\, \delta(y)$, proven in exercise 61.

3. The Feynman rules in the momentum representation are now precisely those of Eq.(5.45) provided that we include the factor $(2\pi)^{-1}$ in the sum over momenta.

<u>Solution to exercise</u> 55

1. (a) No momentum can flow through the external line because it has nowehere to go.

(b) Here we disregard the external line. The diagram, including the symmetry factor, is then

$$\frac{1}{2\pi}\int dk\,\frac{-\lambda_3}{\hbar}\frac{1}{2}\frac{\hbar}{k^2+m^2} = -\frac{\lambda_3}{4\pi}\int dk\frac{1}{k^2+m^2}$$
$$= -\frac{\lambda_3}{4\pi}\left[\frac{1}{m}\arctan\left(\frac{k}{m}\right)\right]_{k=-\infty}^{k=+\infty} = -\frac{\lambda_3}{4m}$$

(c) With contour integration we can close the integration contour around $k = im$:

$$\int dk\,\frac{1}{k^2+m^2} = \int dk\frac{1}{(k-im)(k+im)}$$
$$= \oint_{k\sim im} dk\frac{1}{(k-im)(k+im)} = 2\pi i\left[\frac{1}{k+im}\right]_{k=im} = \frac{\pi}{m}$$

which of course leads to the same result.

(d) We might also close the contour around $k = -im$, but then it runs clockwise rather than anticlockwise, which gives an additional minus sign.

2. The diagram is given by

$$\frac{\lambda_3^2}{2}\frac{1}{2\pi}\int dk\,\frac{1}{(k^2+m^2)((k-p)^2+m^2)}$$

Respectively taking the residue at $k = im$ and $k = p + im$ gives

$$\frac{\lambda_3^2}{4m}\left(\frac{1}{((p-im)^2+m^2)}+\frac{1}{(p+im)^2+m^2}\right) = \frac{\lambda_3^2}{2m(p^2+4m^2)}$$

The poles below the real axes give the same contribution by the same argument as above. You might also take one pole above, and one below the real axis but this simply makes life more difficult.

3. This two-loop graph is given by

$$\frac{\lambda_4^2}{6}\frac{1}{(2\pi)^2}\int dk_1\,dk_2\,\frac{1}{(k_1^2+m^2)(k_2^2+m^2)((k_1+k_2-p)^2+m^2)}$$

The k_1 integral has residues at $k_1 = im$ and $k_1 = p - k_2 + im$, and this gives

$$\frac{\lambda_4^2}{12\pi m}\int dk_2\,\frac{1}{(k_2^2+m^2)((k_2-p)^2+4m^2)}$$

Note the loss of 'symmetry' in k_2. The residues at $k_2 = im$ and $k_2 = p + 2im$ then give

$$\frac{\lambda_4^2}{24m^2}\left(\frac{2}{(p+im)(p-3im)}+\frac{1}{(p+im)(p+3im)}\right) = \frac{\lambda_4^2}{8m^2(p^2+9m^2)}$$

4. In this case the diagram reads

$$-\frac{\lambda_3^3}{2\pi}\int dk\,\frac{1}{(k^2+m^2)((k+p_1)^2+m^2)((k-p_3)^2+m^2)}$$

Using the three residues at $k = im$, $k = -p_1 + im$ and $k = p_3 + im$, and momentum conservation, yields

$$\begin{aligned}
&-\frac{\lambda_3^2}{2m}\Bigg(\frac{1}{((p_1+im)^2+m^2)((p_3-im)^2+m^2)}\\
&+\frac{1}{((p_1-im)^2+m^2)((p_2+im)^2+m^2)}+\frac{1}{((p_3+im)^2)((p_2-im)^2+m^2)}\Bigg)\\
&=-\frac{\lambda_3^3}{2m}\,\frac{24m^2+p_1^2+p_2^2+p_3^2}{(p_1^2+4m^2)(p_2^2+4m^2)(p_3^2+4m^2)}
\end{aligned}$$

Solution to exercise 56

The point-to-point jumps $\delta\varphi = \varphi(x+\Delta) - \varphi(x)$ are, for any D, of the order of $\sqrt{\gamma/\hbar}$. As discussed in the text, for $D = 1$, $\gamma \sim \Delta^{-1}$ and $\delta\varphi \sim \Delta^{1/2}$. For $D = 2$, $\gamma \sim 1$ and $\delta\varphi \sim 1$: the function is not necessarily continuous. For $D \geq 3$, the typical $\delta\varphi$ diverges as $\Delta \to 0$ and the jumps are only kept in check by the mass term $m^2\varphi^2$. For massless theories this term is absent.

Solution to exercise 57

After angular integration, the propagator becomes

$$\Pi(\vec{x}) \sim \int_0^\infty dk\,\frac{k^{D-1}}{k^2+m^2}\,\frac{e^{ikr}-e^{-ikr}}{kr} = \frac{1}{r^{D-2}}\int_0^\infty d\ell\,\frac{\ell^{D-1}}{\ell^2+(rm)^2}\,\frac{e^{i\ell}-e^{-i\ell}}{\ell}$$

As $r \to 0$ the value of m becomes irrelevant. The object $rm \to 0$ only serves as a regulator to make the propagator defined. The same of course also follows from the properties of the modified Bessel function of the second kind.

Solution to exercise 58

1. In the diagrammatic derivation of the Wetterich equation we have to cut through propagators that are part of loops. In these propagators all momentum modes contribute (weighed by their propagators, of course). We therefore not only have to sum over all propagagors but also over all ther modes, *i.e.* integrate of the momentum.

2. Using the Litim prescription $f_\Lambda(p) = \theta(\Lambda > |\vec{p}|)(\Lambda^2 - |\vec{p}|^2)$ the propagator reads

$$\frac{\hbar}{\Lambda^2+m^2}\;(|\vec{p}| < \Lambda) \quad , \quad \frac{\hbar}{|\vec{p}|^2+m^2}\;(|\vec{p}| > \Lambda)$$

The lower momenta are therefore suppressd, the propagator reaches its maximum for any $|\vec{p}| < \Lambda$. In the Wetterich equation we use

$$\frac{\partial}{\partial\Lambda}f_\Lambda(p) = \delta(\Lambda - |\vec{p}|)(\Lambda^2 - |\vec{p}|^2) + 2\Lambda\theta(|\vec{p}|^2 < \Lambda^2) = 2\Lambda\theta(|\vec{p}|^2 < \Lambda^2)$$

The momentum integral is therefore cut off above $|\vec{p}| = \Lambda$.

6 Exercises for chapter 6

Solution to exercise 59
Under the choice (6.11), all parameters of the action are purely imaginary except η:

$$\left|\exp\left(-\frac{S}{\hbar}\right)\right| = \exp\left(-\frac{\eta}{2\hbar}\int d^Dx\,\varphi(x)^2\right)$$

If *eta* were negative the path integral would not converge. Note that here we use the action *without* the factor $-i$ removed.

Solution to exercise 60

1. Using the $i\eta$ prescription with infinitesimal η:

$$\int d\varphi\,\exp\left(-i\frac{m^2-i\eta}{2\hbar}\varphi^2\right)\to$$
$$\int d\varphi\,\exp\left(-\frac{\eta+im^2}{2\hbar}\varphi^2\right) = \sqrt{\frac{2\pi\hbar}{\eta+im^2}} \to \sqrt{\frac{2\pi\hbar}{m^2}}\sqrt{-i}$$

2. For the propagator, the φ^4 vertex and the source vertex we have, respectively,

$$\frac{-i\hbar}{m^2-i\eta}\quad,\quad -i\frac{\lambda}{\hbar}\quad,\quad i\frac{J}{\hbar}$$

For the SDe: the path integral

$$Z(J) = N\int d\varphi\,\exp\left(\frac{i}{\hbar}\left(\frac{-(m^2-i\eta)}{2}\varphi^2 - \frac{\lambda}{4!}\varphi^4 + J\varphi\right)\right)$$

obeys the SDe

$$\frac{\lambda}{6}\left(-i\hbar\frac{\partial}{\partial J}\right)^3 Z(J) + (m^2-i\eta)\left(-i\hbar\frac{\partial}{\partial J}\right)Z(J) - mJZ(J) = 0$$

and for the field function $\phi(J) = -i\hbar\frac{\partial}{\partial J}\log Z(J)$ we then have

$$\phi(J) = \frac{J}{m^2-i\eta} - \frac{\lambda}{6(m^2-i\eta)}\left(\phi(J)^3 - 3i\hbar\phi(J)\frac{\partial}{\partial J}\phi(J) - \hbar^2\frac{\partial^2}{(\partial J)^2}\phi(J)\right)$$

Solution to exercise 61

$$I_\epsilon = \int dx\,\exp\left(-\frac{\epsilon}{2}\left(x^2 - 2\frac{ixy}{\epsilon} - \frac{y^2}{\epsilon^2}\right) - \frac{y^2}{2\epsilon}\right) = \sqrt{\frac{2\pi}{\epsilon}}\exp\left(-\frac{y^2}{2\epsilon}\right)$$

As $\epsilon\to 0$, I_ϵ approaches zero for any nonzero y, and infinity for $y=0$. Its integral over y is 2π, and therefore we may write $I_\epsilon \to 2\pi\,\delta(y)$.

Solution to exercise 62

1. For the timelike separation we need to evaluate the integral

$$H_t = \int_1^\infty d\tau\, \sqrt{\tau^2-1}\, \exp(-a\tau) \quad , \quad a = ims$$

We define $y = \tau + \sqrt{\tau^2-1}$, so that

$$\tau = \frac{y^2+1}{2y} \quad , \quad \sqrt{\tau^2-1} = \frac{y^2-1}{2y} \quad , \quad d\tau = \frac{y^2-1}{2y^2} dy$$

Then we can write (using $y \to 1/y$ where appropriate, and partial integration):

$$\begin{aligned}
H_t &= \int_1^\infty dy\, \frac{(y^2-1)^2}{4y^3} E(y) \quad , \quad E(y) = \exp\left(-\frac{a}{2}(y+1/y)\right) = E(1/y) \\
H_t &= \int_1^\infty dy \left(\frac{y}{4} - \frac{1}{2y} + \frac{1}{4y^3}\right) E(y) = \int_0^\infty dy \left(\frac{y}{4} - \frac{1}{4y}\right) E(y) \\
&= \frac{1}{4}\int_0^\infty dyy \left(1 - \frac{1}{y^2}\right) E(y) = -\frac{1}{2a}\int_0^\infty dy\; y\; \frac{d}{dy}E(y) \\
&= \frac{1}{2a}\int_0^\infty E(y) \quad = \quad \frac{1}{a}K_1(a)
\end{aligned}$$

where the definition (19.428) has been used.

2. For the spacelike separation we have to evaluate the integral

$$H_s = -i\int_{-\infty}^{\infty} dk \frac{k}{\sqrt{k^2+m^2}}\; \exp(isk)$$

The deformation of the contour as indicated in figure 6.2 means that we have to replace

$$\int_{-\infty}^{\infty} dk \quad \to \quad \int_{im+\epsilon}^{i\infty+\epsilon} dk - \int_{im-\epsilon}^{i\infty-\epsilon} dk$$

with infinitesimal positive ϵ. We can write $k = i\ell \pm \epsilon$, to find

$$\frac{1}{\sqrt{k^2+m^2}} = \frac{1}{\sqrt{-\ell^2+m^2 \pm i\epsilon}} = \mp\frac{i}{\sqrt{\ell^2-m^2}}$$

so that

$$\begin{aligned}
H_s &= -i\int_m^\infty d(i\ell)\,(i\ell)\,\frac{-2i}{\sqrt{\ell^2+m^2}}\; \exp(-s\ell) \\
&= m\int_1^\infty d\tau\, \frac{\tau}{\sqrt{\tau^2-1}}\; \exp(-ms\tau)
\end{aligned}$$

Now we take

$$
\begin{aligned}
y &= \tau + \sqrt{\tau^2 - 1} \ , \ \ \tau = \frac{y^2+1}{2y} \ \ , \ \ \sqrt{\tau^2-1} = \frac{y^2-1}{2y} \ \ , \ \ d\tau = \frac{y^2-1}{2y^2} dy \\
H_s &= m \int_1^\infty \frac{dy}{y} \, \frac{y^2+1}{2y} E(y) \ \ , \ \ E(y) = \exp\Big(-\frac{ms}{2}(y + 1/y) \Big) \\
H_s &= \frac{m}{2} \int_1^\infty dy \left(1 + \frac{1}{y^2} \right) E(y) \ \ = \ \ \frac{m}{2} \int_0^\infty dy \, E(y) \ \ = \ \ m \, K_1(ms)
\end{aligned}
$$

<u>Solution to exercise</u> 63

With $k = |\vec{k}|$, $r = |\vec{x}|$ and c the cosine of the angle between $\vec{k}$ and $\vec{r}$:

$$
\begin{aligned}
\int d^3\vec{x} \, V(\vec{x}) \, e^{i\vec{k}\cdot\vec{x}} &= \frac{1}{2} \int_0^\infty dr \, r^2 \int_{-1}^{1} dc \, \frac{e^{-mr}}{r} \, e^{ikrc} \\
&= \frac{1}{2ik} \int_0^\infty dr \, \left(e^{-(m-ik)r} - e^{-(m+ik)r} \right) = \frac{1}{2ik} \left(\frac{1}{m-ik} - \frac{1}{m+ik} \right) = \frac{1}{k^2+m^2}
\end{aligned}
$$

<u>Solution to exercise</u> 64

The range of the Yukawa potential $\exp(-mr)/r$ is $1/m$. From Eq.(6.48) we know

$$
\frac{1}{m} = \frac{\hbar}{Mc} = \frac{\hbar c}{Mc^2}
$$

For a particle of mass 90 GeV/c^2, and using Eqs.(1) and (3) we thus find

$$
\frac{1}{m} = \frac{6.58 \times 10^{-25} \, \text{GeV sec} \ \times \ 3 \times \ 10^8 \, \text{meter/sec}}{90 \, \text{GeV}} \sim 2.2 \ \times \ 10^{-18} \, \text{meter}
$$

<u>Solution to exercise</u> 65

$$
\begin{aligned}
\delta(k^2 - m^2) \, \theta(k^0 > 0) &= \delta \left((k^0)^2 - \omega(\vec{k})^2 \right) \, \theta(k^0 > 0) \\
&= \ \delta \left((k^0 - \omega(\vec{k}))(k^0 + \omega(\vec{k}) \right) \ \theta(k^0 > 0) \\
&= \ \left(\frac{\delta(k^0 - \omega(\vec{k}))}{k^0 + \omega(\vec{k})} + \frac{\delta(k^0 + \omega(\vec{k}))}{k^0 - \omega(\vec{k})} \right) \ \theta(k^0 > 0) = \frac{1}{2\omega(\vec{k})} \delta(k^0 - \omega(\vec{k})) \, \theta(k^0)
\end{aligned}
$$

since the second Dirac delta cannot be resolved. Therefore

$$
d^4k \, \delta(k^2 - m^2) \, \theta(k^0 > 0) = dk^0 \, d^3\vec{k} \, \frac{\delta(k^0 - \omega(\vec{k}))}{2\omega(\vec{k})} \, \theta(k^0 > 0) = \frac{d^3\vec{k}}{2\omega(\vec{k})}
$$

<u>Solution to exercise</u> 66

1. The 'field' $\psi(x)$ has no dynamics of its own, it is simply a collection of independent stochastics ψ, one at each point. These can therefore be integrated independently.

2. Dropping the $i\eta$ where possible:

$$\int d\psi\ \exp\left(\frac{i}{\hbar}\left(\frac{\mu^2+i\eta}{2}\psi^2 - \frac{g}{2}\psi\varphi(x)^2\right)\right) =$$
$$\int d\psi\ \exp\left(\frac{i(\mu^2+i\eta)}{2\hbar}\left(\psi + \frac{g}{\mu^2}\varphi(x)^2\right)^2 - \frac{ig}{8\mu^2\hbar}\varphi(x)^4\right) \sim \exp\left(-\frac{ig}{8\mu^2\hbar}\varphi(x)^4\right)$$

3. After integrating out the ψ we are left with an 'effective' φ^4 coupling. The Lagrangian is

$$\mathcal{L}(\varphi) = \frac{1}{2}\partial^\mu\varphi(x)\partial_\mu\varphi(x) - \frac{m^2-i\eta}{2}\varphi(x)^2 - \frac{\lambda_4}{4!}\varphi(x)^4 + J(x)\varphi(x)\ \ ,\ \ \lambda_4 = \frac{3g^2}{\mu^2}$$

4. The propagators for the ϕ and ψ fields are, respectively

$$\Pi_\phi(x) = \frac{i\hbar}{(2\pi)^D}\int d^Dk\,\frac{e^{-ik\cdot x}}{k^2-m^2+i\eta}\ \ ,\ \ \Pi_\psi(x) = \frac{i\hbar}{\mu^2}\delta^D(x)$$

and the SDe's are

$$\begin{aligned}
\phi(x) &= \int d^Dy\ \Pi_\phi(x-y)\left(\frac{iJ(y)}{\hbar} - \frac{ig}{\hbar}\phi(y)\psi(y) - \frac{ig}{\hbar}\left(-i\hbar\frac{\delta}{\delta J(y)}\right)\psi(y)\right)\\
\psi(x) &= \frac{i\hbar}{\mu^2}\frac{-ig}{2\hbar}\left(\phi(x)^2 + \left(-i\hbar\frac{\delta}{\delta J(x)}\right)\phi(x)\right)
\end{aligned}$$

Inserting $\psi(x)$ and using the above definition of λ_4 we obtain

$$\begin{aligned}
\phi(x) &= \int d^Dy\ \Pi_\phi(x-y)\left(\frac{iJ(y)}{\hbar}\right.\\
&\left. -\frac{i\lambda_4}{6}\left(\phi(x)^3 + 3\phi(y)\left(-i\hbar\frac{\delta}{\delta J(y)}\right)\phi(y) + \left(-i\hbar\frac{\delta}{\delta J(y)}\right)^2\phi(y)\right)\right)
\end{aligned}$$

5. The Euler-Lagrange equation for ψ reads

$$\frac{\delta}{\delta\psi(x)}S = \mu\psi(x) - \frac{g}{2}\varphi(x)^2 = 0 \quad\to\quad \psi(x) = \frac{g}{2\mu^2}\varphi(x)^2$$

and inserting this into the Lagrangian gives the same φ^4 theory as above.

6. In Euclidean theory this will not work since (in zero dimensions) we would need the action

$$S(\varphi,\psi) = \frac{m^2}{2}\varphi^2 - \frac{\mu^2}{2}\psi^2 + \frac{g}{2}\psi\varphi^2 - J\varphi$$

so that the integral over ψ is badly divergent.

7 Exercises for chapter 7

Solution to exercise 67

Assume that the particle with momentum p is on-shell, and that its energy p^0 is positive if we count it going upwards (or downwards). In that case the lower (upper) blob describes an on-shell particle going into on-shell particles with positive energies, *i.e.* a *decay* process. Since the incoming particles are stable, such processes cannot occur. Therefore p cannot be on-shell.

Solution to exercise 68

1. The positivity is trivial, and $\int dx(x^2+\alpha^2)^{-1} = (1/\alpha)[\arctan(x/\alpha)]_{-\infty}^{\infty} = \pi/\alpha$.
2. For $x \neq 0$, $\lim_{\alpha\to 0} f_\alpha(x) = \lim_{\alpha\to 0} \alpha/(\pi x) = 0$, and $\lim_{\alpha\to 0} f_\alpha(0) = \lim_{\alpha\to 0} 1/(\pi\alpha) = \infty$.

Solution to exercise 69

If the particle is moving with momentum q, the flux factor is changed from $1/(2m)$ to $1/(2\omega(\vec{q})) = 1/(2q^0)$. The amplitude and the phase space factor are Lorentz invariant. The decay width is therefore decreased, and the lifetime increase, by a factor q^0/m.

Solution to exercise 70

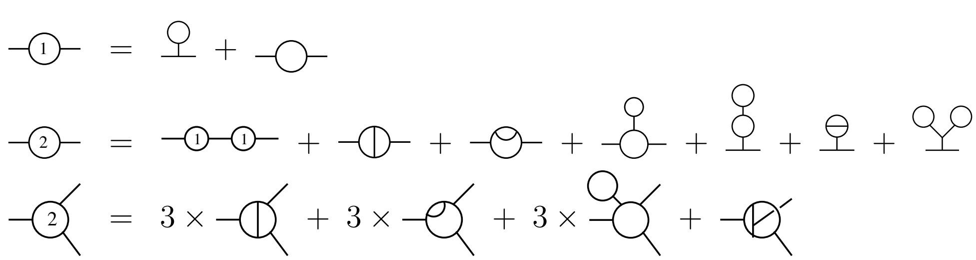
1. This is the Cauchy-Schwartz inequality:
$$\langle c|\ c\rangle = \langle a|\ a\rangle + \langle b|\ b\rangle\,|\,\langle a|\ b\rangle\,|^2 - 2\,\langle b|\ a\rangle\,\langle a|\ b\rangle = 1 - |\,\langle a|\ b\rangle\,|^2 \leq 0$$
2. If $|\,\langle a|\ b\rangle\,| = 0$ then $\langle b|\ a\rangle$ is simply a phase, and $\langle c|\ c\rangle = 0$ so that $|c\rangle$ must vanish.

Solution to exercise 71

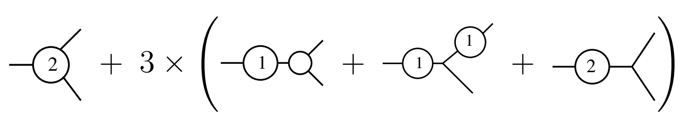
For a φ^5 theory, the $2 \to 3$ amplitude is an energy-independent constant at the tree level, while it ought to decrease with energy if unitarity is to hold.

Solution to exercise 72

We may define



These three lines correspond to $\mathfrak{sm} = -1,\ -25/8,\ -13/2$, respectively the two-loop three-point amplitude is given by



The number of diagrams is therefore $10 + 3 \times (2 + 4 + 10) = 58$. The $\mathfrak{sm}$ is given by

$$\mathfrak{sm} = -\frac{13}{2} + 3\left(-1 - 1 - \frac{25}{8}\right) = -\frac{175}{8}$$

It can also be obtained from the two-loop tadpole by the stepping equation:

$$\left(\frac{\hbar\lambda_3}{\mu}\frac{\partial}{\partial\mu}\right)^2 \left(-\frac{5}{8}\frac{\hbar^2\lambda_3^3}{\mu^5}\right) = -\frac{175}{8}\frac{\hbar^4\lambda_3^5}{\mu^9}$$

<u>Solution to exercise</u> 73
Assuming integration troughout, and using $D = n + 1$ and $p = |\vec{p}_1|$:

$$\begin{aligned} & d^d p_1\, \delta(p_1^2 - m^2)\; d^d p_2\, \delta(p_2^2 - m^2)\; \delta^D(P - p_1 - p_2) \\ &= \frac{1}{2p_1^0}\, d^n\vec{p}_1\; \delta(s - 2p_1^0\sqrt{s}) = \frac{1}{2s}\; d^n\vec{p}_1\, \delta(p_1^0 - \sqrt{s}/2) \\ &= \frac{\pi^{n/2}}{s\,\Gamma(n/2)}\, p^{n-1}\, dp\, \delta(p_1^0 - \sqrt{s}/2) \end{aligned}$$

We can rewrite:

$$\delta(p_1^0 - \sqrt{s}/2) = \sqrt{s}\delta(p^2 + m^2 - s/4) = \frac{1}{\beta}\delta(p - \sqrt{s}\beta/2)$$

so that

$$V = \frac{\pi^{n/2}}{\Gamma(n/2)s\beta}\left(\frac{\sqrt{s}\beta}{2}\right)^{n-1}$$

For $D = 3, 4, 5, 6$ we find, respectively, $V = \pi/(2\sqrt{s})$, $\pi\beta/2$, $\pi^2\sqrt{s}\beta^2/8$ and $\pi^2 s\beta^3/12$.

<u>Solution to exercise</u> 74
We have to be a bit careful here, since $s = m^2$ implies $(p_1 + p_2)^2 = (p_1 - q_2)^2 = m^2$. So we take the limit $s \to m^2$:

$$\begin{aligned} & \frac{1}{2(p_1 \cdot p_2)} - \frac{1}{2(p_1 \cdot q_2)} = \frac{1}{s - m^2}\left(1 - \frac{2s}{s + m^2 + (s - m^2)\cos\theta}\right) \\ &= \frac{\cos\theta - 1}{s + m^2 + (s - m^2)\cos\theta} \quad \to \quad \frac{\cos\theta - 1}{2m^2} \end{aligned}$$

Therefore

$$\mathfrak{M} = \frac{i\hbar\lambda^2}{2}(1 - \cos\theta)$$

This can be considered the sattering of E in a static F field. In the same way, the flux factor and the phase space give a finite product:

$$\Phi_\sigma\, dV(p_1 + p_2; p_2, q_2) = \frac{1}{2(s - m^2)}\;\frac{1}{32\pi^2}\frac{s - m^2}{s}d\Omega \quad \to \quad \frac{1}{64\pi^2 m^2}d\Omega$$

And the cross section is

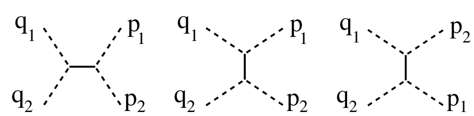
$$\sigma = \int \frac{\hbar^2\lambda^4}{256\pi^2 m^2}(1 - \cos\theta)^2\, d\Omega = \frac{\hbar^2\lambda^4}{48\pi m^2}$$

Solution to exercise 75

1. Remember the symmetry factor F_{symm}!

$$\Gamma = \Gamma(F \to EE) = \frac{1}{2m} \int |-i\hbar^{1/2} m\lambda|^2 \, \frac{d\Omega}{32\pi^2} \, \frac{1}{2} = \frac{\hbar\lambda^2 m}{32\pi}$$

2. Of course we stick to the tree-level diagrams:



3.

$$\mathfrak{M} = -i\hbar\lambda^2 m^2 \left(\frac{1}{s - m^2 + im\Gamma} + \frac{1}{t - m^2} + \frac{1}{u - m^2} \right)$$

where $s = (q_1 + q_2)^2$, $t = (p_1 - q_1)^2$, $u = (p_1 - q_2)^2$. Note that $t, u < 0$ and therefore their propagators do not contain $im\Gamma$: for negative invariant mass squared decays are impossible.

4. The only contribution to $\Im(T)$ comes from the s-channel diagram, and reads

$$\Im(T) = \frac{\hbar\lambda^2 m^3 \Gamma}{(s - m^2)^2 + m^2\Gamma^2} > 0$$

Strictly speaking $\Im(T)$ is only required for forward scattering but this part of the amplitude is angle-independent anyway (at the tree level).

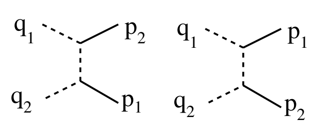
5. For small λ, $\Gamma \ll m^2$. At $s = m^2$ the first diagram dominates:

$$\mathfrak{M} = -i\frac{\hbar\lambda^2 m}{\Gamma} \quad \to \quad \sigma = \frac{1}{2m^2} \left(\frac{\hbar\lambda^2 m}{\Gamma} \right)^2 \frac{1}{32\pi^2} (4\pi) \frac{1}{2} = \frac{\hbar^2\lambda^4}{32\pi\Gamma^2} = \frac{32\pi}{s}$$

This is precisely the unitarity limit as given by Eq.(19.185).

Solution to exercise 76

The process is $E(q_1)\,E(q_2) \;\to\; F(p_1)\,F(p_2)$, described (at the tree level) by two diagrams:



The particle momenta will be described by

$$q_1^\mu = E(1, \vec{e}_q),\; q_2^\mu = E(1, -\vec{e}_q),\; p_1^\mu = (E, q\vec{e}_p),\; q_2^\mu = (E, -q\vec{e}_p)$$

where $\vec{e}_{q,p}$ are unit vectors, and $q = \beta E$, $\beta^2 = 1 - (m/E)^2$. The amplitude is therefore

$$\mathfrak{M} = -i\hbar\lambda^2 m^2 \left(\frac{1}{(q_1 - p_1)^2} + \frac{1}{(q_1 - p_2)^2} \right)$$

We have, with $\vec{e}_q \cdot \vec{e}_p = c = \cos\theta$, θ being the polar scattering angle:

$$(p_1 - q_1)^2 = m^2 - 2E^2 + 2Eqc = -E^2(1+\beta^2 - \beta c) \ \ , \ \ (q_1 - p_2)^2 = -E^2(1+\beta^2+2\beta c)$$

Note that both t and u range from $-E^2(1+\beta)$ to $-E^2(1-\beta)$ so are never zero for $m > 0$. With $s = 4E^2$

$$\mathfrak{M} = i\frac{4\hbar\lambda^2 m^2}{s}\left(\frac{1}{1+\beta^2-2\beta c} + \frac{1}{1+\beta^2+2\beta c}\right)$$

The differential cross section, after integration over the azimuthal angle and including $F_{\text{symm}} = 1/2$, is

$$\frac{d\sigma}{dc} = \frac{\hbar^2\lambda^4 m^4}{4\pi s^3}\left(\frac{1}{1+\beta^2-2\beta c} + \frac{1}{1+\beta^2+2\beta c}\right)^2$$

Performing the final integral:

$$\int\limits_{-1}^{1} dc\ \left(\frac{1}{1+\beta^2-2\beta c} + \frac{1}{1+\beta^2+2\beta c}\right)^2 = \frac{2}{\beta(1+\beta^2)}\log\left(\frac{1+\beta}{1-\beta}\right) + \frac{4}{(1-\beta^2)^2}$$

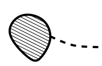
gives the total cross section (recall that $4(1-\beta^2)^{-2} = s^2/(4m^4)$):

$$\sigma = \frac{\hbar^2\lambda^4}{16s}\left(1 + \frac{8m^4}{s^2\beta(1+\beta^2)}\log\left(\frac{1+\beta}{1-\beta}\right)\right)$$

As a possible check, we may take the produced F particles at rest. Then $t = u = -m^2$, $s = 4m^2$ so that $\mathfrak{M} = 2i\hbar\lambda^2$ and $\sigma = \hbar^2\lambda^4/(8\pi m^2)$ which is indeed the limit $\beta \to 0$ of the general expression.

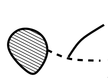
<u>Solution to exercise</u> 77

1. The 'nonradiative' amplitude $\mathfrak{M}_0$ is given by

$$\mathfrak{M}_0 =$$ 

$$= J(p)\,\sqrt{\hbar}$$

2. The 'radiative' amplitude $\mathfrak{M}_1$ is given by

$$\mathfrak{M}_1 =$$ 

$$= J(p+k)\,\frac{i\hbar}{(p+k)^2 - m^2 + i\eta}\,(-i\frac{\lambda m}{\hbar})\,\sqrt{\hbar}^2$$

3. $(p+k)^2 - m^2 + i\eta \ \to \ (m^2 + 2(p\cdot k)) - m^2 = 2(p\cdot k)$.

4. $(p\cdot k) = k^0(p^0 - |\vec{p}|\cos\theta)$, with θ the angle between $\vec{p}$ and $\vec{k}$. Therefore $\mathfrak{M}_1 \sim (k^0)^{-1}$ and $|\mathfrak{M}_1|^2 \sim (k^0)^{-2}$.

5. The radiative phase space integration element contains

$$d^4k\,\delta(k^2) = \frac{1}{2\omega(\vec{k})}d^3\vec{k} = \frac{|\vec{k}|^2}{2\omega(\vec{k})}\,d|\vec{k}|\,d\Omega = \frac{k^0}{2}\,d(k^0)\,d\Omega$$

6. The differential cross section contains $d(k^0)/k^0$. The total cross section therefore involves an integral $\int d(k^0)/k^0$. Since for a massless particle the *minimum* energy is zero, the total cross section diverges logarithmically.

Solution to exercise 78

1. φ^3 theory: here $R(\varphi) = a\varphi$, with $a = \lambda_3/(3m^2)$. Solving Eq.(7.78) gives

$$\tau = \log(x/c) = \int \frac{dg}{g\sqrt{1+ag}} = -\log\left(\frac{\sqrt{1+ag}+1}{\sqrt{1+ag}-1}\right)$$

This implies

$$f(x) = -\frac{4cx}{a(c+x)^2}$$

Requiring $f'(0) = 1$ tells us that $c = -4/a$, so that

$$f(x) = \frac{x}{(1-ax/4)^2} = \frac{x}{(1-x(\lambda_3/12m^2))^2}$$

2. φ^4 theory: now $R(\varphi) = a\varphi^2$ with $a = \lambda_4/(12m^2)$. This leads immediately to

$$\log(x/c) = -\frac{1}{2}\log\left(\frac{\sqrt{1+ag^2}+1}{\sqrt{1+ag^2}-1}\right)$$

which gives us after fixing $f'(x) = 1$:

$$f(x) = \frac{2cx}{\sqrt{s}(c^2-x^2)^2} = \frac{x}{1-xa/4} = \frac{x}{1-x(\lambda_4/48m^2)}$$

3. SB case: we now have $1 + R(\varphi) = (1 + a\varphi)^2$ with $a = \lambda_3/(6m^2)$. This case is actually quite simple:

$$\log(x/c) = \log\left(\frac{g}{1+ag}\right) \quad \to f(x) = \frac{x}{c-ax} = \frac{x}{1-x\lambda_3/(6m^2)}$$

The numbers A_n follow immediately from the series expansions

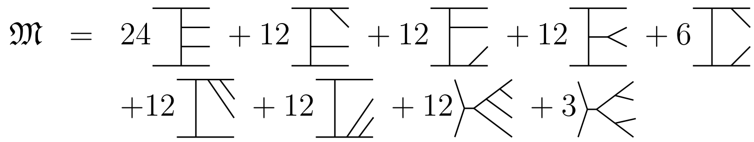
$$\frac{z}{(1-z)^2} = \sum_{k\geq 1} kz^k \quad , \quad \frac{z}{1-z^2} = \sum_{k\geq 0} z^{2k+1} \quad , \quad \frac{z}{1-z} = \sum_{k\geq 1} z^k$$

Solution to exercise 79

1. The particle momentum at rest is $k^\mu = (m, 0)$. In φ^3 theory, the $2 \to 4$ process $p_1 + p_2 \;\to\; k\;k\;k\;k$ has $p_1^\mu = (2m, \sqrt{3}m\vec{e})$ and $p_2^\mu = (2m, -\sqrt{3}m\vec{e})$, so that

$$d_1 = (p_1 - k)^2 - m^2 = -3m^2 \;,\; d_2 = (p_1 - 2k)^2 - m^2 = -4m^2 \;,$$
$$s_2 = (2k)^2 - m^2 = 3m^2 \;,\; s_3 = (3k)^2 - m^2 = 8m^2 \;,\; s_4 = (4k)^2 - m^2 = 15m^2$$

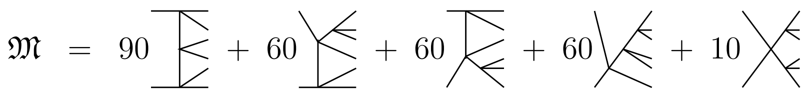
In terms of diagrams,



$$= -i\frac{\hbar^2\lambda_3^4}{m^6}\left(\frac{24}{d_1^2 d_2}+\frac{2\cdot 12}{d_1 d_2 s_2}+\frac{12}{d_1^2 s_2}+\frac{6}{d_1 s_2^2} + \frac{2\cdot 12}{d_1 s_2 s_3}+\frac{12}{s_2 s_3 s_4}+\frac{3}{s_2^2 s_4}\right) = 0$$

2. For $2 \to 6$ in φ^4 theory, we have $p_1^\mu = (3m, \sqrt{8}m\vec{e})$ and therefore

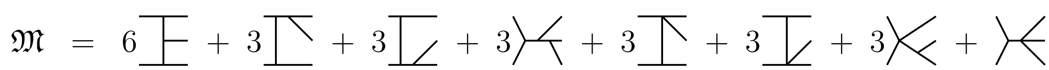
$$d_2 = (p_1 - 2k)^2 - m^2 = -8m^2\ ,\quad s_3 = (3k)^2 - m^2 = 8m^2\ ,\quad s_5 = (5k)^2 - m^2 = 24m^2$$



$$= i\frac{\hbar^3\lambda_4^3}{m^4}\left(\frac{90}{d_2^2}+\frac{2\cdot 60}{d_2 s_3}+\frac{60}{s_3 s_5}+\frac{10}{s_3^2}\right) = 0$$

3. For $2 \to 3$ in general $\varphi^{3/4}$ theory, we have $p_1^\mu = (3m/2, \sqrt{5/4}m\vec{e})$ and

$$d_1 = (p_1 - k)^2 - m^2 = -2m^2\ ,\quad s_2 = (2k)^2 - m^2 = 3m^2\ ,\quad s_3 = (3k)^2 - m^2 = 8m^2$$



$$= -i\frac{\hbar^{1/2}\lambda_3^3}{m^4}\left(\frac{6}{d_1^2}+\frac{2\cdot 3}{d_1 s_2}+\frac{3}{s_2 s_3}\right) - i\frac{\hbar^{1/2}\lambda_3\lambda_4}{m^2}\left(\frac{2\cdot 3}{d_1}+\frac{3}{s_2}+\frac{1}{s_3}\right)$$

$$= -i\frac{15\hbar^{1/2}\lambda_3}{8m^2}\left(\frac{\lambda_3^2}{3m^2}-\lambda_4\right)$$

For the SB case we have precisely $\lambda_4 = \lambda_3^2/(3m^2)$ so that $\mathfrak{M} = 0$.

Solution to exercise 80

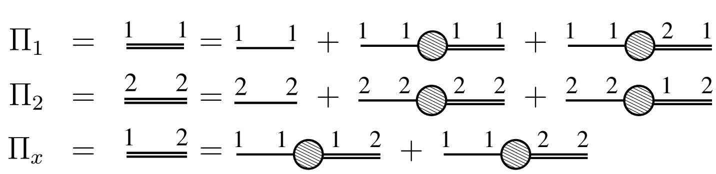
With $\hbar g^2 = 4\pi\alpha$ and $M^2 = 4K/(\alpha N)$, we have $-g^2 M^2 (i\hbar) = -i(16\pi K/N)$. In the total cross section the prefactor becomes $(16\pi/N)^2(1/2s)(1/32\pi^2)(4\pi) = 16\pi/(N^2 s)$. Note that $F_{\text{symm}} = 1$ since the particles and the antiparticles are (assumed to be) distinguishable.

Solution to exercise 81

The (one-loop) diagrams in K_1 contain g_1^2, and those in K_2 contain g_2^2; those in K_x contain $g_1 g_2$. The rest of the diagrams are identical, so $K_1 K_2 = K_x^2$.

Solution to exercise 82

We denote the bare propagators by single, and the dressed propagators by double lines. Then



In the last line 1 and 2 may be interchanged. To prove (7.99) is suffices to insert the results of Eq.(7.100) and do the algebra.

8 Exercises for chapter 8

Solution to exercise 83
Notice that in this exercise we split up the *Euclidean space* and not the set of momenta! Therefore the total invariant mass t is simply the sum of the 'partial' invariant masses t_1 and t_2, and indeed

$$\int_0 dt_1\, dt_2\, t_1^{n-1}\, t_2^{k-1}\, \delta(t_1 + t_2 - t) = \frac{\Gamma(n)\Gamma(k)}{\Gamma(n+k)}\, t^{n+k-1}$$

Solution to exercise 84

1. $$\int_0^\infty dt\, t^{-1-\epsilon} = \left[-\frac{t^{-\epsilon}}{\epsilon}\right]_{t=a}^{t=\infty} + \left[-\frac{t^{-\epsilon}}{\epsilon}\right]_{t=0}^{t=a}$$

   In the first term, we take $\epsilon > 0$; and in the second one we take $\epsilon < 0$. The total vanishes.

2. For general n, we take $\epsilon > n+2$ for $a < t < \infty$ and $\epsilon < n+2$ for $0 < t < a$.

Solution to exercise 85

1. Using the Euler formula:

$$\int_0^1 dx \left(log(m^2) - \sum_{k\geq 1} \frac{1}{k}\left(\frac{s}{m^2}\right)^k x^k(1-x)^k \right) = \log(m^2) - \sum_{k\geq 1}\left(\frac{s}{m^2}\right)^k \frac{\Gamma(k+1)^2}{k\Gamma(2k+2)}$$

2. In fact, this can even be done without the Stirling approximation. We can write

$$T_k = \frac{\Gamma(k+1)^2}{k\Gamma(2k+2)} \quad \to \quad \frac{T_k}{T_{k-1}} = \frac{k-1}{4k+2} \sim \frac{1}{4} \quad (k \to \infty)$$

   and therefore the sum is only convergent if $s/(4m^2) < 1$.

Solution to exercise 86
We can use

$$\int_0^1 dx\, x^{-\epsilon}(1-x)^{-\epsilon} = \frac{\Gamma(1-\epsilon)^2}{\Gamma(2-2\epsilon)} = \frac{(1-\epsilon\gamma_E + \cdots)^2}{(1-2\epsilon)(1-2\epsilon\gamma_E+\cdots)} \sim 1 + 2\epsilon$$

The rest of the ϵ expansion goes in the standard way; the x integral is responsible for the '-2' in Eq.(8.25).

Solution to exercise 87

1. The one-loop effetive potential consists of one-loop diagrams without influx of external momenta. The loop momentum $\vec{p}$ is therefore the same in all propagators. Therefore is suffies to (a) replace every μ by $|\vec{p}|^2 + m^2$, and to include a 2ν-dimensional momemtum integral, whih commutes with the α integral. The α integral is introduced here beause integration $F(\alpha)$ is relatively straightforward.

2.

$$\begin{aligned} F(\alpha) &= \frac{1}{(2\pi)^{2\nu}} \int \frac{d^{2\nu}\vec{p}}{|\vec{p}|^2 + \alpha} = \frac{1}{(4\pi)^\nu \Gamma(\nu)} \int_0^\infty dt \, \frac{t^{\nu-1}}{t+\alpha} = \frac{\alpha^{\nu-1}\Gamma(1-\nu)}{(4\pi)^\nu} \\ V_1(\phi) &= \frac{\hbar\Gamma(1-\nu)}{2(4\pi)^\nu} \int_{m^2}^{m^2+V''(\phi)} d\alpha \, \alpha^{\nu-1} = \frac{\hbar\Gamma(1-\nu)m^{2\nu}}{2\nu(4\pi)^\nu} \left[\left(\frac{\alpha}{m^2} \right)^\nu \right]_{\alpha=m^2}^{\alpha=m^2+V''(\phi)} \end{aligned}$$

3. The only nontrivial step here is to realize that $(1+V''/m^2)^\nu - 1 \;\; \to \;\; \nu \log(1 + V''/m^2)$ as ν approaches zero.

4. We adopt $D = 2k - 2\epsilon$ to do dimensional regularization, so that $\nu = k - \epsilon$. Then $m^{2\nu}$ is to replaced by $m^{2k}(\mu/m)^{2\epsilon}$ if we introduce the engineering dimension. The important step is

$$\begin{aligned} \Gamma(1-\nu) &= \Gamma(\epsilon - (k-1)) = \frac{\Gamma(\epsilon)}{(\epsilon-1)(\epsilon-2)\cdots(\epsilon-(k-1))} \\ &= \frac{(-)^{k-1}\Gamma(\epsilon)}{\Gamma(k)\left(1-\frac{\epsilon}{1}\right)\cdots\left(1-\frac{\epsilon}{k-1}\right)} \sim (-)^{k-1}\frac{\Gamma(\epsilon)}{\Gamma(k)}(1+\epsilon H_k) \end{aligned}$$

   Furthermore,

$$W^\nu - 1 = W^{k-\epsilon} - 1 = W^k(1 - \epsilon\log(W)) - 1 = (W^k - 1) - \epsilon W^k \log(W)$$

   The ϵ expansion is from then on trivially performed.

5. This is almost trivial: if $V''(\phi)$ has ϕ^2 for its highest power of ϕ, then $W^k = W^{D/2}$ has ϕ^D for its highest power (at least for D even). In terms of diagrams, the one-loop contribution fo the φ^{2n} term in the effective action has n propagators. Such a diagram will become divergent if $2n \leq D$. Therefore, for even dimension D the terms in the effective action up to φ^D are divergent, and higher powers have finite coefficients.

<u>**Solution to exercise**</u> **88**

It is enough to look at the one-loop tadpole diagram in φ^3 theory. In Euclidean space this reads (up to factors of 2π and the symmetry factor 1/2)

$$\int d^D\vec{p} \left(-\frac{\lambda_3}{\hbar} \right) \frac{\hbar}{|\vec{p}^2| + m^2} = -\int d^d\vec{p} \frac{\lambda_3}{\|vp^2| + m^2}$$

In Minkowski space, including the Wick rotation:

$$\int d^D p, \left(-i\frac{\lambda_3}{\hbar} \right) \frac{i\hbar}{p^2 - m^2 + i\eta} = \int d^D p \, \frac{\lambda_3}{p^2 - m^2 + i\eta} = -i \int d^D p_E \, \frac{\lambda_3}{|\vec{p}_E|^2 + m^2}$$

The Wick rotation ensures that the minus signs work out identically in the two formulations. The factors $(-)$ in Euclidean, and $(-i)$ in Minkowski space, are precisely the usual prefactors in the Feynman rules and this ensures that the effective potentials have the same form. The minus sign referred to in the exercise is the diference between, say $\lambda_4\varphi^4$ in the Euclidean Lagrangian versus $-\lambda_4\varphi^4$ in the Minkowskian one.

<u>Solution to exercise</u> 89

$$
\begin{aligned}
\mathcal{P}\int_a^b dx\,\frac{f(x)}{x} &= \lim_{\epsilon\to 0}\left(\int_a^{-\epsilon}+\int_\epsilon^b\right)\frac{f_0}{x}+f_1+f_2x+f_x^2+\cdots \\
&= \int_a^b dx\,\left(f_1+f_2x+f_3x^2+\cdots\right)+\lim_{\epsilon\to 0}\left(-\int_\epsilon^{|a|}+\int_\epsilon^b\right)dx\frac{f_0}{x} \\
&= f_0\log\left(\frac{b}{|a|}\right)+\left[f_1x+\frac{1}{2}f_2x^2+\frac{1}{3}f_3x^3+\cdots\right]_{x=a}^{x=b}
\end{aligned}
$$

<u>Solution to exercise</u> 90

1. This is faurly trivial: we may take $\nu = D/2$, $\epsilon = 0$, and impose the upper limit Λ^2 on t after using the t-shell formula.

2. Define $B(\phi) = m^2 + \lambda\phi^2/2$. Then

$$
\begin{aligned}
V_1'(\phi) &= \frac{\hbar}{2}B'(\phi)F(B(\phi)) \\
V_1''(\phi) &= \frac{\hbar}{2}\left(B''(\phi)F(B(\phi))+B'(\phi)^2F'(B(\phi))\right) \\
&= \frac{\hbar\lambda}{2(4\pi)^\nu\Gamma(\nu)}\int_0^{\Lambda^2} dt\,t^{\nu-1}\left(\frac{1}{t+B(\phi)}-\frac{\lambda\phi^2}{(t+B(\phi))^2}\right)
\end{aligned}
$$

3. (**a**) For $\Lambda^2 = 0$ the t integral vanishes; (**b**) if $m^2 > \lambda\phi^2/2$ the numerator of the integrand is always positive; (**c**) if $m^2 < \lambda\phi^2/2$ then the numerator is negative for $0 < t < (\lambda\phi^2/2 - m^2)$, so one-loop concavity is certainly lost for $\Lambda^2 < (\lambda\phi^2/2 - m^2)$. For sufficiently large Λ^2 the larger t values will eventually win out. For instance, if $\lambda\phi^2/2 = 2m^2$, one-loop concavity is restored in four dimensions ($\nu = 2$) for $\Lambda^2 > (1.617\cdots)m^2$: in two dimensions ($\nu = 1$) for $\Lambda^2 > (2.498\cdots)m^2$.

   Eq.(2.32) is the zero-dimensional analogue:

$$
\hbar\Gamma_1''(\phi) = \frac{\hbar}{2}\frac{\mu-\lambda\phi^2/2}{(\mu+\lambda\phi^2/2)^2}
$$

   In this case there is no t integral to save the day, and one-loop concavity is lost for $\lambda\phi^2/2 > \mu$.

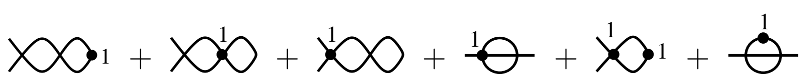
<u>Solution to exercise</u> 91
Pauli-Villars regularization is not a modification of the propagator but rather the addition of extra (unphysical) fields. Any variation in the extra PV propagator can therefore only depend on the extra field, and not on the 'original' propagator. Therefore the Wetterich equation cannot possibly hold in this scheme.

9 Exercises for chapter 9

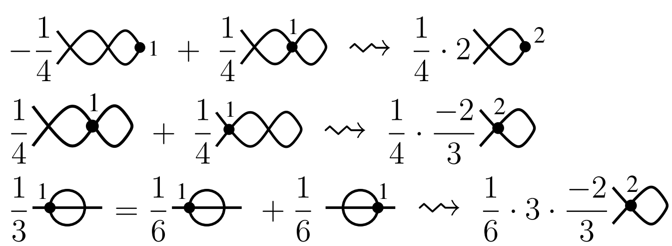
<u>Solution to exercise</u> 92
For φ^6 theory, there are no tadpoles at al ($E = 1$), and the prpoagator ($E = 2$) has no one-loop contribution.

<u>Solution to exercise</u> 93
For φ^3 theory, $\mathcal{Q}$ of Eq.(9.2) is negative for $D \leq 5$, and the theory is then super-renormalizable. Applicatino of Eq.(9.2) gives the following results. The one-point IP1 diagrams are divergent as follows: at one loop for $D = 2, 3, 4, 5$, at two loops for $D = 4, 5$ at 3 and 4 loops for $D = 5$. The two-point IP1 diagrams are divergent as follows: at one loop for $D = 4, 5$, and at 2 loops for $D = 5$. No other IP1 diagrams are divergent. The one-point diagrams are given in Eqs.(19.44-47), the two-point ones in Eq.(19.48-49).

<u>Solution to exercise</u> 94
The $\mathfrak{sm}$ factors of the 3-loop 2-point diagrams are given in Eq.(19.56). After all one-loop contractions have been performed using the rules (9.8) and (9.9), the daughters are

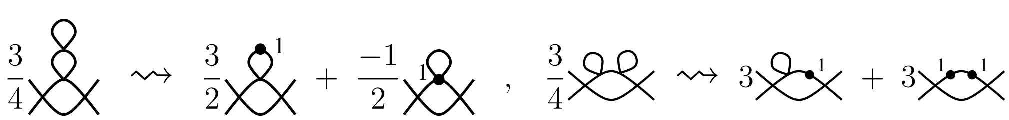


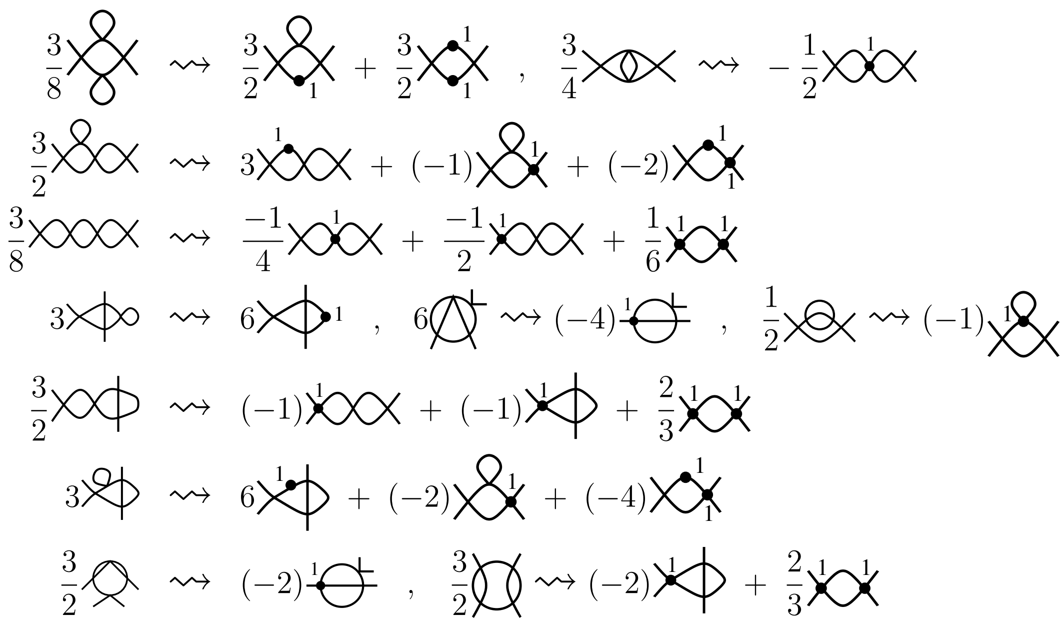
and they all have the correct $\mathfrak{sm}$ after we collect them in Eq.(9.14). The two-loop contractions are, by rule (9.11) and (9.13),



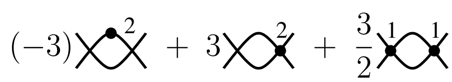
The granddaugters also have the right $\mathfrak{sm}$, and by rule (9.16) so does the 3-loop divergence.

<u>Solution to exercise</u> 95
The uncontracted three-loop four-point diagrams are given in Eq.(19.60) The full one-loop contraction yields





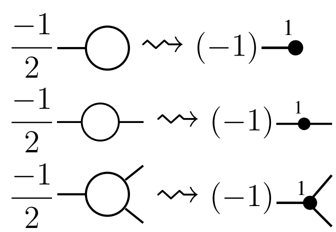
After collecting diagrams, all daughters have the correct $\mathfrak{sm}$. After applying the two-loop contraction rules (9.11) and (9.13), the (effectively) two-loop granddaughters are



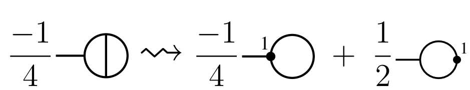
and the last two of these diagrams allow for the three-loop contraction. The uncontracted four-loop two-point diagrams are found in Eq.(19.57); the procedure goes the same way.

<u>Solution to exercise</u> 96

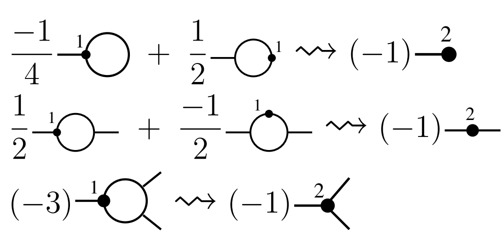
The one-loop contraction rules turn out to be



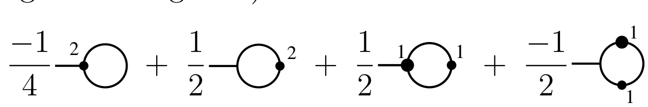
Under these rules,



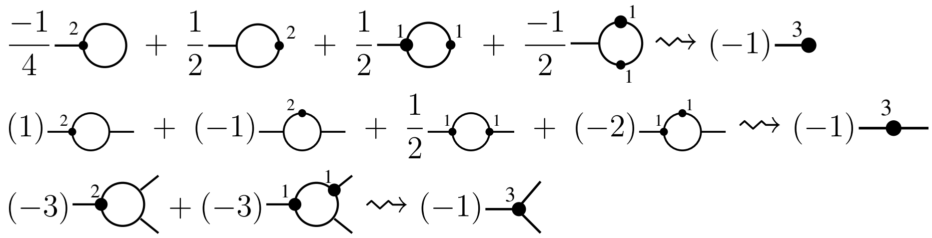
The first diagram has $\mathfrak{sm}$ -1/4, rather than -1/2 as it ought to. The two-loop contraction rules:



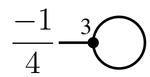
likewise give (among other diagrams) the contractable combination



which again has the first diagram off by a factor 2. The three-loop contraction rules are



again gives rise to (among many other ones) a diagram



too small by a factor of 2. But these diagrams can *only* occur in the tadpole counterterm, and that itself cannot contribute to any other 1PI diagram. This mismatch is therefore not dangerous.

<u>Solution to exercise</u> 97
The self-energy diagram has two internal propagators. As the loop momentum goes to infinity, the loop integral in D dimensions is therefore proportional to $\int d^Dp\ (p^2)^{-2} \sim \mathfrak{m}^{D-4}$ and is therefore logarithmically divergent in 4, and quadratically divergent in 6 dimensions.

<u>Solution to exercise</u> 98
We can simply replace the one-loop 1PI blob in Eq.(9.19) by higher-loop 1PI or pollywog ones. Note that here it is really important to order the loop corrections are collectedinto 1PI/pollywog sets!

<u>Solution to exercise</u> 99
Every derivative to s effectively introduces an addition propagator of dimension $\mathfrak{m}^{-2}$. Since $\Sigma'(s)$ is logarithmically divergent, the higher derivatives are finite.

<u>Solution to exercise</u> 100
The tadpole integral goes, at high loop momentum, as $\int d^Dp(p^2)^{-1} = \mathfrak{m}^{D-2}$, and is therefore quartically divergent in $D = 6$.

<u>Solution to exercise</u> 101
The vertex loop diagram is logarithmically divergent; therefore any derivative introduces at least a factor $\mathfrak{m}^{-1}$ which makes the integral finite, just as in exercise 99.

<u>Solution to exercise</u> 102

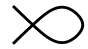
1. The action contains a term $\int d^4x\,(\partial_\mu\varphi)(\partial^\mu\varphi)$ which implies

$$\mathbf{dim}\big[L^2\varphi^2\big] = \mathbf{dim}[\hbar] \quad\to\quad \mathbf{dim}\big[\varphi^4\big] = \mathbf{dim}\big[\hbar^2/L^4\big]$$

   where L is a length scale. Therefore the φ^4 interaction term gives

$$\mathbf{dim}\left[\int d^4x\,\lambda\,\varphi^4\right] = \mathbf{dim}\big[\hbar^2\lambda\big] = \mathbf{dim}[\hbar] \quad\to\quad \mathbf{dim}[\hbar\lambda] = 1$$

2. The one-loop self-energy insertion (including the symmetry factor!) is the diagram



and is given by

$$iK = \left(-i\frac{\lambda_0}{\hbar}\right)\frac{1}{2(2\pi)^4}\int d^4p\,\frac{i\hbar}{p^2 - m_0^2 + i\eta}$$

Adopting dimensional regularization, doing the Wick rotation, and using the t-shell formula we can write

$$\begin{aligned} iK &= \frac{\lambda_0\,\mu^{2\epsilon}}{2(2\pi)^{4-2\epsilon}}\int d^{4-2\epsilon}p\,\frac{1}{p^2 - m_0^2 + i\eta} \\ &= -i\frac{\lambda_0\,\mu^{2\epsilon}\,\pi^{2-\epsilon}}{2(2\pi)^{4-2\epsilon}\Gamma(2-\epsilon)}\int_0^\infty dt\,\frac{t^{1-\epsilon}}{t+m_0^2} \\ &= -i\frac{\lambda_0\,(4\pi\mu^2)^\epsilon}{2(4\pi)^{2-\epsilon}}(m_0^2)^{1-\epsilon}\,\Gamma(\epsilon-1) \end{aligned}$$

Using $\Gamma(\epsilon-1) = -\Gamma(\epsilon)/(1-\epsilon)$ and expanding in ϵ then yields

$$iK = i\frac{\lambda_0 m_0^2}{32\pi^2}\left(\frac{1}{\epsilon} - \gamma_E + \log(4\pi) + \log\left(\frac{\mu^2}{m_0^2}\right) + 1\right)$$

3. The Dyson sum for the propagator reads

$$\begin{aligned} &\frac{i\hbar}{p^2 - m_0^2 + i\eta} + \frac{i\hbar}{p^2 - m_0^2 + i\eta}(iK)\frac{i\hbar}{p^2 - m_0^2 + i\eta} + \cdots \\ &= \frac{i\hbar}{p^2 - m_0^2 + i\eta}\left(1 - \frac{K\hbar}{p^2 - m_0^2 + i\eta} + \cdots\right) = \frac{i\hbar}{p^2 - m_0^2 + K\hbar + i\eta} \end{aligned}$$

and therefore $m^2 = m_0^2 - K$, or

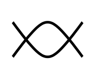
$$m^2 = m^2\left(1 - \frac{\lambda_0\hbar m^2}{32\pi^2}(R_\epsilon - \log(m_0^2) + 1)\right)$$

4. At one loop, the beta function is purely formal since the one-loop self energy has no momentum flowing into it. Nevertheless, we can write

$$\beta(M^2) = M^2\frac{\partial}{\partial M^2}m^2 = -\mu^2\frac{\partial}{\partial\mu^2}m^2 = \frac{\lambda\hbar}{32\pi^2}m^2$$

5. At one loop, the wave function is not renormalized since the one-loop self-energy has no momentum dependence.

6. The one-loop 1PI contributions to the interaction vertex afre three diagrams of the form



It is simplest to start in the situation where all external momenta vanish. Then the one-loop diagrams are all equal, and their sum is given by

$$\begin{aligned} iV &= \frac{3\lambda_0^2\,\mu^{2-\epsilon}}{2(2\pi)^{4-2\epsilon}}\int d^{4-2\epsilon}p\,\frac{1}{(p^2-m_0^2+i\eta)^2} \\ &= i\frac{3\lambda_0^2(4\pi\mu^2)^\epsilon}{32\pi^2\Gamma(2-\epsilon)}\int\limits_0^\infty dt\,\frac{t^{1-\epsilon}}{(t+m_0^2)^2} \\ &= i\frac{3\lambda_0^2(4\pi\mu^2)^\epsilon\,(m_0^2)^{-\epsilon}\,\Gamma(\epsilon)}{32\pi^2}\approx i\frac{3\lambda_0^2}{32\pi^2}(R_\epsilon-\log(m_0^2)) \end{aligned}$$

Introducing external momenta can only introduce finite terms, see exercises 99. and 101.

7. At one loop,

$$-i\frac{\lambda}{\hbar}=-i\frac{\lambda_0}{\hbar}+i\frac{3\lambda_0^2}{32\pi^2}(R_\epsilon+\cdots)\quad\to\quad\lambda=\lambda_0\left(1-\frac{3\hbar\lambda_0}{32\pi^2}(R_\epsilon+\cdots)\right)$$

For the β function we may as well take the derivative to $-R_\epsilon$, and so find $\beta(\lambda)=3\lambda^2/(32\pi^2)$.

8. Write $\rho=1/\lambda$, then with $\beta_0=3/(32\pi^2)$ we have

$$\frac{\partial}{\partial\log M^2}\rho(M^2)=-\beta_0\quad\to\quad\rho(M^2)=\rho(s_0)-\beta_0\log(M^2/s_0)$$

and this leads to the desired result. For any positive value of $\lambda(s_0)$ the Landau pole *will* be reached. Assuming that negative λ are not admissible, the only way to avoid the Landau pole is to have $\lambda(s_0)=0$ which means that $\lambda(M^2)=0$ for all M^2.

10 Exercises for chapter 10

Solution to exercise 103

$$\begin{aligned} \gamma^\mu\not{a}\gamma_\mu &= 2\gamma^\mu\,a_\mu-\gamma^\mu\gamma_\mu\not{a}=2\not{a}-4\not{a}=-2\not{a} \\ \gamma^\mu\not{a}\not{b}\gamma_\mu &= 2\gamma^\mu\not{a}b_\mu-\gamma^\mu\not{a}\gamma_\mu\not{b}=2\not{b}\not{a}+2\not{a}\not{b}=4(a\cdot b) \\ \gamma^\mu\not{a}\not{b}\not{c}\gamma_\mu &= 2\gamma^\mu\not{a}\not{b}c_\mu-\gamma^\mu\not{a}\not{b}\gamma_\mu\not{c}=2\not{c}\not{a}\not{b}-4\not{c}(a\cdot b)=2\not{c}\not{a}\not{b}-2\not{c}(\not{a}\not{b}+\not{b}\not{a})=-2\not{c}\not{b}\not{a} \end{aligned}$$

Solution to exercise 104

By construction, γ^4 anticommutes with γ^μ $(\mu=0,1,2,3)$ since γ^5 does. Also $\bar{\gamma}^5=-\gamma^5$ so $\bar{\gamma}^4=\gamma^4$. Finally $2\gamma^4\gamma^4=-2(\gamma^5)^2=-2$.

Solution to exercise 105

From $\gamma^5\gamma^0\gamma^j=i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^j=-i\gamma^1\gamma^2\gamma^3\gamma^j$ we have

$$\gamma^5\gamma^0=i\gamma^2\gamma^3=\sigma^{23}\ ,\ \gamma^5\gamma^0\gamma^2=-i\gamma^1\gamma^3=\sigma^{31}\ ,\ \gamma^5\gamma^0\gamma^3=i\gamma^1\gamma^2=\sigma^{12}$$

and from there

$$\gamma^5\gamma^j\gamma^k=-i\gamma^5\sigma^{jk}=-i\gamma^5\gamma^5\gamma^0\gamma^n=-i\gamma^0\gamma^n=\sigma^{n0}$$

Solution to exercise 106
It is easy to see that $\gamma^5\sigma^\alpha\beta$ only has tensor components, and therefore

$$\gamma^5\sigma^{\alpha\beta} = \frac{\sigma_{\mu\nu}}{8}\mathrm{Tr}\left(\gamma^5\sigma^{\alpha\beta}\sigma^{\mu\nu}\right)$$

The trace is only nonzero if all the indices are different, and we find

$$\frac{1}{8}\mathrm{Tr}\left(\gamma^5\sigma^{\alpha\beta}\sigma^{\mu\nu}\right) = -\frac{1}{8}\mathrm{Tr}\left(\gamma^5\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu\right) = -\frac{i}{2}\epsilon^{\alpha\beta\mu\nu}$$

Solution to exercise 107
For $m = 0, 1$:

$$\gamma_\nu\Gamma_0\gamma^\nu = \gamma_n u\gamma^\nu\Gamma_0 = D\Gamma_0 \quad , \quad \gamma_\nu\Gamma_1\gamma^\nu = (2-D)\Gamma_1$$

Then, using the induction step:

$$\begin{aligned}\gamma_\nu\gamma^\alpha\gamma^\beta\Gamma_{m-2}\gamma^\nu &= 2\gamma^\beta\Gamma_{m-2}\gamma^\alpha - 2\gamma^\alpha\Gamma_{m-2}\gamma^\beta + \gamma^\alpha\gamma^\beta\gamma_\nu\Gamma_{m-2}\gamma^\nu \\ &= 2(-)^{m-1}\gamma^\alpha\gamma^\beta\Gamma_{m-2} + 2(-)^{m-2}\gamma^\alpha\gamma^\beta\Gamma_{-2} + (-)^{m-2}(D-2(m-2))\gamma^\alpha\gamma^\beta\Gamma_{m-2} + \cdots \\ &= (-)^m(D-2m)\gamma^\alpha\gamma^\beta\Gamma_{m-2} + \cdots\end{aligned}$$

where the ellipsis denotes terms with fewer Dirac matrices.

Solution to exercise 108

1. Multiplying by $\langle a|$ we have $0 = \langle a|\left\{|a\rangle\langle a|b\rangle - \lambda|b\rangle\right\} = \langle a|b\rangle(1-\lambda)$.
2. If $\lambda = 1$ we multiply by $\langle b|$ to find $\langle b|a\rangle\langle a|b\rangle = 1$. If $\langle a|b\rangle = 0$ then $|a\rangle\langle a|b\rangle = 0 = \lambda|b\rangle$.

Solution to exercise 109

1. It suffices to take $N = 8$ in Eq.(10.79). With $q^2 = 0$ we then have $\det(1 - z\mathfrak{W}) = (1 - 2zq^0)^2$.
2. $\mathfrak{W}$ cannot be a dyad since it has two nonzero eigenvalues. But since it is a Hermitian matrix it allows for the $N = 8$-dimensional spae to be split up into two 4-dimensional ones, each containing one of the nonzero eigenvalues. In each of these subspaces we have a vector $\xi_{1,2}$ to serve as the basis for a dyad. Also, we have $\xi_1^d ag\xi_2 = 0$ since they have nonzero components in different subspaces.
3. This follows from the previous discussion.

Solution to exercise 110
Expanding the logarithm:

$$\begin{aligned}\log(1 - z\not{p}) &= -\sum_{n\geq 1}\frac{z^n}{n}\not{p}^n = -\sum_{n\geq 1}\frac{z^2}{2n}\not{p}^{2n} - \sum_{n\geq 0}\frac{z^{2n+1}}{2n+1}\not{p}^{2n+1} \\ &= \frac{1}{2}\sum_{n\geq 1}\frac{(z^2v^2)^n}{n} - \frac{\not{p}}{2v}\sum_{n\geq 1}\frac{(zv)^n}{n}\left(1 - (-1)^n\right)\end{aligned}$$

and this gives the result. Taking the trace and using $\mathrm{Tr}\,(1) = 4$, $\mathrm{Tr}\,(\not{p}) = 0$ gives

$$\mathrm{Tr}\,(\log(1 - z\not{p})) = \log\det\,(1 - z\not{p}) = 2\log(1 - z^2v^2) = \log\left((1 - zv)^2(1 + zv)^2\right)$$

so there are two eigenvalues v and two eigenvalues $-v$.

Solution to exercise 111
All this follows immediately from the fact that $\not{p} \pm m)$ and $(1 + \gamma^5\not{s})$ commute, and

$$(\not{p}\pm m)^2 = \pm 2m(\not{p}\pm m)\ ,\ (\not{p}+m)(\not{p}-m) = 0\ ,\ (1\pm\gamma^5\not{s})^2 = 2(1\pm\gamma^5\not{s})\ ,\ (1+\gamma^5\not{s})(1-\gamma^5\not{s}) = 0$$

Solution to exercise 112
By the Casimir trick and the trace identities:

$$\begin{aligned}\bar\eta\gamma^\mu\eta &= \mathrm{Tr}\,(\eta\bar\eta\gamma^\mu) = 4w\,k^\mu\\ \bar\eta\gamma^5\gamma^\mu\eta &= \mathrm{Tr}\,\left(\eta\bar\eta\gamma^5\gamma^\mu\right) = -4w\,s^\mu\\ \bar\eta\omega_+\eta &= \mathrm{Tr}\,\left(\eta\bar\eta(1+\gamma^5)\right) = 2w(\cos(\theta) + i\sin(\theta))\end{aligned}$$

It is enough to have the spinor η as a aset of four complex nunbers to work out k,s,w and θ.

Solution to exercise 113
This is simply a matter of using the Casimir trick. For instance

$$\overline{v}(p,s)\gamma^5\gamma^\mu v(p,s) = \frac{1}{2}\mathrm{Tr}\,\left((\not{p} - m)(1 + \gamma^5\not{s})\gamma^5\gamma^\mu\right) = \frac{1}{2}\mathrm{Tr}\,\left(-m\gamma^5\not{s}\gamma^5\gamma^\mu\right) = 2ms^\mu$$

Solution to exercise 114
Again using the Casimir trick, and contracting the repeated indices:

$$\begin{aligned}p^0 &= \overline{u}\gamma^0u/2 = u^\dagger u/2 = \Sigma_{a=1}^4|u^a|^2/2 > 0\\ p\cdot p &= \frac{1}{4}\overline{u}\gamma^\mu u\overline{u}\gamma_\mu u = \frac{1}{16}\mathrm{Tr}\,\left((\not{p}+m)(1+\gamma^5\not{s})\gamma^\mu(\not{p}+m)(1+\gamma^5\not{s})\gamma_\mu\right)\\ &= \frac{1}{16}\mathrm{Tr}\,\left((\not{p}+m)(1+\gamma^5\not{s})(-2\not{p}+4m+2m\gamma^5\not{s})\right)\\ &= \frac{m}{4}\mathrm{Tr}\,\left((\not{p}+m)(1+\gamma^5\not{s})\right) = m^2\\ p\cdot s &\sim \mathrm{Tr}\,\left((\not{p}+m)(1+\gamma^5\not{s})\gamma^\mu(\not{p}+m)(1+\gamma^5\not{s})\gamma^5\gamma_\mu\right)\\ &= \mathrm{Tr}\,\left((\not{p}+m)(1+\gamma^5\not{s})(2\not{p}-4m-2m\gamma^5\not{s})\gamma^5\right) = 0\\ s\cdot s &= \frac{1}{16m^2}\mathrm{Tr}\,\left((\not{p}+m)(1+\gamma^5\not{s})\gamma^5(2\not{p}-4m-2m\gamma^5\not{s})\gamma^5\right)\\ &= -\frac{1}{4m}\mathrm{Tr}\,\left((\not{p}+m)(1+\gamma^5\not{s})\right) = -1\end{aligned}$$

Solution to exercise 115
It is an eigenspinor of both $(\not{p}+m)$ and $(1+\gamma^5\not{s})$ QED.

Solution to exercise 116
Since the four spinors $u(p,\pm s)$ and $v(p,\pm s)$ form a basis, we always have

$$u(p,s') = a_+u(p,s) + a_-u(p,-s) + b_+v(p,s) + b_-v(p,s)$$

Multiplying from the left by $(\not{p} + m)$ shows that $b_+ = b_- = 0$. Furthermore, $\overline{u}(p,s)u(p,s') = a_+\overline{u}(p,s)u(p,s) + a_-\overline{u}(p,s)u(p,-s) = 2ma_+$, and similarly $\overline{u}(p,-s)u(p,s') = 2mb$.

Solution to exercise 117

The numerator of the left-hand side amounts to

$$\hbar^2\left(\gamma^\mu((p+q)^2-m^2)-2(p+q)^\mu(\not{p}+\not{q}+m)\right)$$

and that on the right-hand side is

$$-\hbar^2(\not{p}+\not{q}+m)\gamma^\mu(\not{p}+\not{q}+m)$$

By anticommutation these are the same.

Solution to exercise 118

$$\Sigma_z(\theta)=\sqrt{\frac{x^2}{(\tilde{x}+x)^2}}\left(1+\frac{\tilde{\not{x}}\not{x}}{x^2}\right)=\frac{1}{2\cos(\theta/2)}(1+\cos(\theta)+\sin(\theta)\not{x}\not{y})$$

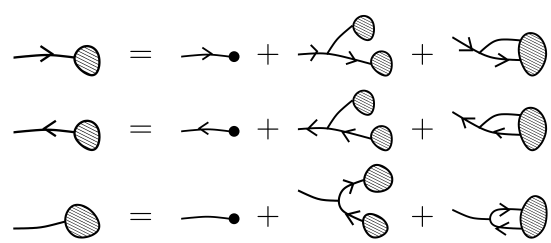
and we use $1+\cos(\theta)=2\cos(\theta/2)^2$, $\sin(\theta)=2\cos(\theta/2)\sin(\theta/2)$.

Solution to exercise 119

Using $\cos(\pi/6)=\sqrt{3}/2$ and $\sin(\pi/6)=1/2$ we have, for a rotation over $\pi/3$, $\Sigma=(1+\sqrt{3}\not{x}\not{y})/2$. This leads to $\Sigma^2=(1+2\sqrt{3}\not{x}\not{y}+3\not{x}\not{y}\not{x}\not{y})/4=(-1+\sqrt{3}\not{x}\not{y})/2$ and so finally to $\Sigma^3=-1$.

Solution to exercise 120

1.



2. The single diagram for $M\;\to\;E(p_1)\,\bar{E}(p_2)$ has the value

$$\mathfrak{M}=i\hbar^{1/2}\lambda\,\overline{u}(p_1)\,v(p_2)$$

where (as is usual) we do not indicate the spins if they are to be summed over. Note that at the threshold $M=2m$, $p_1=p_2$ so that $\mathfrak{M}=0$. This is a check.

$$\begin{aligned}\left\langle|\mathfrak{M}|^2\right\rangle &= \hbar\lambda^2\,\mathrm{Tr}\left((\not{p}_1+m)(\not{p}_2-m)\right)=2\hbar\lambda^2\left(M^2-4m^2\right)\\ \Gamma(F\to E\bar{E}) &= \frac{1}{2M}(2\hbar\lambda^2)\left(M^2-4m^2\right)\frac{1}{32\pi^2}\left(1-\frac{4m^2}{M^2}\right)^{1/2}(4\pi)\\ &= \frac{\hbar\lambda^2}{8\pi}M\left(1-\frac{4m^2}{M^2}\right)^{3/2}\end{aligned}$$

3. We shall use $s=(p_1+p_2)^2=(q_1+q_2)^2$ and $t=(p_1-q_1)^2=(p_2-q_2)^2$.

(a)

(b)

$$
\begin{aligned}
\mathfrak{M} &= -i\hbar\lambda^2\left(\frac{\mathcal{A}_1}{s-M^2}-\frac{\mathcal{A}_2}{t-M^2}\right)\\
\mathcal{A}_1 &= \overline{v}(p_2)u(p_1)\,\overline{u}(q_1)v(q_2) \quad , \quad \mathcal{A}_2 \;=\; \overline{u}(q_1)u(p_1)\,\overline{v}(p_2)v(q_2)\\
\left\langle|\mathcal{A}_1|^2\right\rangle &= \frac{1}{4}\mathrm{Tr}\left((\not{p}_2-m)(\not{p}_1+m)\right)\mathrm{Tr}\left((\not{q}_1+m)(\not{q}_2-m)\right) = (s-4m^2)^2\\
\left\langle|\mathcal{A}_2|^2\right\rangle &= \frac{1}{4}\mathrm{Tr}\left((\not{q}_1+m)(\not{p}_1+m)\right)\mathrm{Tr}\left((\not{p}_2-m)(\not{q}_2-m)\right) = (t-4m^2)^2\\
\langle\mathcal{A}_1\mathcal{A}_2^*\rangle &= \frac{1}{4}\mathrm{Tr}\left((\not{p}_2-m)(\not{p}_1+m)(\not{q}_1+m)(\not{q}_2-m)\right)\\
&= -\frac{1}{2}st-2m^2(s+t)+8m^4\\
\left\langle|\mathfrak{M}|^2\right\rangle &= \hbar^2\lambda^4\left(\frac{\langle|\mathcal{A}_1|^2\rangle}{(s-M^2)^2}+\frac{\langle|\mathcal{A}_2|^2\rangle}{(t-M^2)^2}-\frac{2\,\langle\mathcal{A}_1\mathcal{A}_2^*\rangle}{(s-M^2)(t-M^2)}\right)\\
&= \hbar^2\lambda^4\left(B_0+B_1\frac{1}{t-M^2}+B_2\frac{1}{(t-M^2)^2}\right)\\
B_0 &= 1+\frac{s+4m^2}{s-M^2}+\frac{(s-4m^2)^2}{(s-M^2)^2}\\
B_1 &= \left((s+4m^2)(3M^2-4m^2)-2M^4\right)/(s-M^2)\\
B_2 &= (M^2-4m^2)^2
\end{aligned}
$$

(c) The scattering angle θ between $\vec{p}_1$ and $\vec{q}_1$ and the variable t are related by

$$t=\frac{4m^2-s}{2}(1-\cos\theta)$$

and therefor the *average* over $\cos\theta$ is given by the integral

$$\frac{1}{2}\int_{-1}^{1}d\cos\theta=\frac{1}{s-4m^2}\int_{4m^2-s}^{0}dt$$

Thus we find

$$\sigma=\frac{\hbar^2\lambda^4}{16\pi s}\left(B_0-\frac{B_1}{s-4m^2}\log\left(1+\frac{s-4m^2}{M^2}\right)+\frac{B_2}{M^2(M^2+s-4m^2)}\right)$$

4. For $m=M=0$, $\langle|\mathfrak{M}|^2\rangle=3$.

5. (a)

(b)

$$\begin{aligned}
\mathfrak{M} &= -i\hbar\lambda^2 \left(\frac{\mathcal{A}_1}{2(p_1k_1)} - \frac{\mathcal{A}_2}{2(p_1k_2)} \right) \\
\mathcal{A}_1 &= \overline{u}(p_2))(\not{p}_1 + \not{k}_1 + m)u(p_1) \\
\mathcal{A}_2 &= \overline{u}(p_2))(\not{p}_1 - \not{k}_2 + m)u(p_1) \\
\left\langle |\mathcal{A}_1|^2 \right\rangle &= 8m^2(s+m^2) + 2(s-m^2)(p_1k_2) \\
\left\langle |\mathcal{A}_2|^2 \right\rangle &= 16m^4 + 2(s-9m^2)(p_1k_2) \\
\langle \mathcal{A}_1\mathcal{A}_2^* \rangle &= 2(s+3m^2)(2m^2-(p_1k_2)) \\
\left\langle |\mathfrak{M}|^2 \right\rangle &= \hbar^2\lambda^4 \left(\frac{4m^4}{(p_1k_2)^2} + \frac{s^2-18m^2s+15m^4}{2(s-m^2)(p_1k_2)} \right. \\
&\qquad \left. + \frac{2(s^2+6m^2s+m^4)}{(s-m^2)^2} + \frac{2(p_1k_2)}{s-m^2} \right)
\end{aligned}$$

where we used $(p_1k_1) = (s-m^2)/2$. In the CM frame, with θ the angle between $\vec{p}_1$ and $\vec{p}_2$,

$$(p_1k_2) = \frac{s-m^2}{4s} \left(s + m^2 + (s-m^2)\cos\theta \right)$$

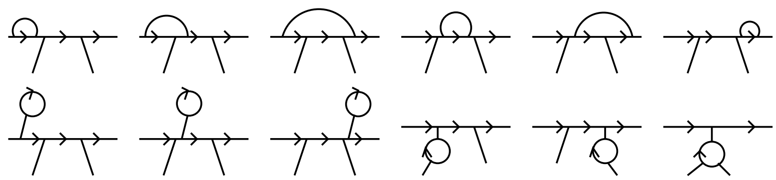
and

$$\sigma = \frac{\hbar^2\lambda^4}{16\pi s} \left(\frac{s(s^2-18m^2s-15m^4)}{(s-m^2)^3} \log\left(\frac{s}{m^2}\right) + \frac{5s^3+55s^2m^2+3sm^4+m^6}{2s(s-m^2)^2} \right)$$

(c) The limit $s \to m^2$ is regular:

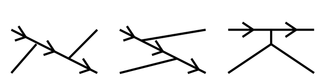
$$\lim_{s\to m^2} \sigma = \frac{\hbar^2\lambda^4}{12\pi m^2}$$

6. (a) Twelve of the diagrams are

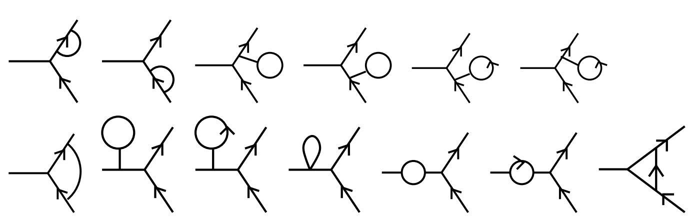


The other ones are obtained by interchanging the two external F lines.

(b) i.



ii.



iii. A:10, B:74, C:42, D:2424.

Solution to exercise 121

1. For the process $E(p) \to D(q)F(k)$:

$$\begin{aligned}
\mathfrak{M} &= i\hbar^{1/2}\lambda\,\overline{u}(q)u(p)\\
\left\langle|\mathfrak{M}|^2\right\rangle &= \frac{\hbar\lambda^2}{2}\mathrm{Tr}\left((\not{q}+m_D)(\not{p}+m_E)\right) = \hbar\lambda^2(m_E+m_D)^2\\
\Gamma(E\to DF) &= \frac{\hbar\lambda^2}{16\pi}\frac{(m_E+m_D)^2(m_E^2-m_D^2)}{m_E^3}
\end{aligned}$$

2. For the process $F(k_1)F(k_2) \to E(p_1)\bar{E}(p_2)$, with $m_D = 0$:

$$\begin{aligned}
\mathfrak{M} &= -i\hbar\lambda^2\left(\frac{\mathcal{A}_1}{D_1}+\frac{\mathcal{A}_2}{D_2}\right)\quad,\quad D_{1,2}=(p_1-k_{1,2})^2\\
\mathcal{A}_1 &= \overline{u}(p_1)(\not{p}_1-\not{k}_1)v(p_2)\quad,\quad \mathcal{A}_2 \;=\; \overline{u}(p_1)(\not{p}_1-\not{k}_2)v(p_2)\\
\left\langle|\mathcal{A}_1|^2\right\rangle &= 8(p_1k_1)(p_2k_1)+8m_E^2(p_1k_1)-8m_E^2(p_2k_1)+4m_E^2(p_1p_2)-4m_E^4\\
\left\langle|\mathcal{A}_2|^2\right\rangle &= 8(p_1k_2)(p_2k_2)+8m_E^2(p_1k_2)-8m_E^2(p_2k_2)+4m_E^2(p_1p_2)-4m_E^4\\
\left\langle\mathcal{A}_1\mathcal{A}_2^*\right\rangle &= 4(p_1k_1)(p_2K_2)+4(p_1k_2)(p_2k_1)-4(k_1k_2)(p_1p2)\\
&\quad +4m_E^2(p_1k_1)+4m_E^2(p_1k_2)-4m_E^2(p_2k_1)-4m_E^2(p_2k_2)\\
&\quad +4m_E^2(p_1p_2)-4m_E^2(k_1k_2)-4m_E^4\\
\sigma &= \frac{\hbar^2\lambda^4}{16\pi s}\left(\frac{16E^4-10m_E^4}{EP(E^2+P^2)}\log\left(\frac{E+P}{E-P}\right)-12\right)\sqrt{1-\frac{m_E^2}{E^2}}
\end{aligned}$$

with $s = 4E^2$, $P^2 = E^2 - m_E^2$.

Solution to exercise 122

If the derivative operator cuts a fermion line, this line may be part of a closed fermion loop. That loop has a minus sign that the propagator does not have, so this must be compensated. Alternatively, we may cut through e ferrmion line that extends throughout the diagram (and thus has external fields at both ends. Cutting through such a line causes these external fields not to be connected to one another any more, but to the endpoints of the propagator, a rearrangment of external fermions that also calles for a minus sign.

11 Exercises for chapter 11

Solution to exercise 123

For simplicity, let $\vec{p}$ point along the z axis. Then $[\not{p},\gamma^0] = |\vec{p}|\gamma^0\gamma^3$ and by exercise 105 we see that the second term becomes simply $\sin(\theta/2)\gamma^1\gamma^2$, *i.e.* a rotation around the z axis. For massless momenta we have $|\vec{p}| = p^0$ and use $[\not{p},\gamma^0]u_\lambda(p) = \not{p}\gamma^0u_\lambda(p) = 2p^0u_\lambda(p)$ so that $\Sigma_p u_l a(p) = (\cos(\theta/2) + i\sin(\theta/2)\gamma^5)u_\lambda(p) = (\cos(\theta/2) + i\lambda\sin(\theta/2))u_\lambda(p)$.

Solution to exercise 124

$$
\begin{aligned}
|s_+(p_1,p_2)|^2 &= |p_1^y + ip_1^z|^2\frac{p_2^0-p_2^x}{p_1^0-p_1^x} + |p_2^y+ip_2^z|^2\frac{p_1^0-p_1^x}{p_2^0-p_2^x} \\
&\quad - \Big((p_1^y+ip_1^z)(p_2^y-ip_2^z) + (\text{c.c})\Big) \\
&= (p_1^0+p_1^x)(p_2^0-p_2^x) + (p_1^0-p_1^x)(p_2^0+p_2^x) - 2(p_1^yp_2^y + p_1^zp_2^z) \\
&= 2\left(p_1^0p_2^0 - p_1^xp_2^x - p_1^yp_2^y - p_1^zp_2^z\right)
\end{aligned}
$$

Solution to exercise 125

$$
\begin{aligned}
A &= \overline{u}_+(p_1)\omega_-\not{p}_2\omega_+\not{p}_3\omega_-\not{p}_4\omega_+\not{p}_5 u_-(p_6) \\
&= \overline{u}_+(p_1)u_-(p_2)\overline{u}_-(p_2)u_+(p_3)\overline{u}_+(p_3)u_-(p_4)\overline{u}_-(p_4)u_+(p_5)\overline{u}_+(p_5)u_-(p_6) \\
&= s_+(p_1,p_2)s_-(p_2,p_3)s_+(p_3,p_4)s_-(p_4,p_5)s_+(p_5,p_6) \\
|A|^2 &= 32(p_1p_2)(p_2p_3)(p_3p_4)(p_4p_5)(p_5p_6)
\end{aligned}
$$

Solution to exercise 126

A_1 can only be nonzero if the helicities of $u(p_1)$ and $u(p_7)$ are *opposite*; A_2 is nonzero if they are *identical*. So either $A_1 = 0$ or $A_2 = 0$.

Solution to exercise 127

We start by realizing that there must be vector q such that

$$
u_+(p_2)\overline{u}(p_3) = \omega_+u_+(p_2)\overline{u}_+(p_3)\omega_- = \omega_+\not{q}
$$

So that, by the Casimir trick,

$$
q^\rho = \frac{1}{2}\overline{u}_+(p_3)\gamma^\rho u_+(p_2)
$$

Therefore

$$
\begin{aligned}
&i\epsilon^{\mu\nu\alpha\beta}\overline{u}_+(p_1)\gamma_\alpha u_+(p_2)\overline{u}_+(p_3)\gamma_\beta u_+(p_4) \\
&\quad = \frac{i}{2}\epsilon^{\mu\nu\alpha\beta}\overline{u}_+(p_1)\gamma_\alpha\gamma_\rho\gamma_\beta u_+(p_4)\overline{u}_+(p_3)\gamma^\rho u_+(p_2)
\end{aligned}
$$

The Pauli identity: $\gamma_\alpha\gamma_\rho\gamma_\beta = -i\epsilon_{\sigma\alpha\rho\beta}\gamma^5\gamma^\rho$ plus terms symmetric in α and β, then tells us that

$$
\overline{u}_+(p_1)\gamma_\alpha\gamma_\rho\gamma_\beta u_+(p_4) = i\epsilon_{\sigma\alpha\rho\beta}\overline{u}_+(p_1)\gamma^\sigma u_+(p_4)
$$

And finally

$$
\frac{i}{2}\epsilon^{\mu\nu\alpha\beta}\, i\epsilon_{\sigma\alpha\rho\beta} = \left\{ \begin{array}{cc} \mu & \nu \\ \rho & \sigma \end{array} \right\}
$$

Solution to exercise 128

We use $s = (p_1+p_2)^2 = (q_1+q_2)^2$, $t = (p_1-q_1)^2 = (p_2-q_2)^2$, $u = (p_1-q_2)^2 = (p_{;2}-q_1)^2$. This also implies $s+t+u = 0$ for massless particles.

1. Of the sixteen helicity cases, six are nonzero. Using the Chisholm identity (and reversal where necessary):

$$\begin{aligned}
\mathfrak{M}(+,+,+,+) &= 2i\hbar Q_{\mathrm{e}}{}^2 s_+(p_1,q_2)s_-(p_2,q_1)\left(\frac{1}{s}+\frac{1}{t}\right) \sim 2\hbar Q_{\mathrm{e}}{}^2\,\frac{u^2}{st} \\
\mathfrak{M}(+,+,-,-) &= 2i\hbar Q_{\mathrm{e}}{}^2 \frac{s_+(p_1,q_1)s_-(p_2,q_2)}{s} \sim 2\hbar Q_{\mathrm{e}}{}^2\frac{t}{s} \\
\mathfrak{M}(+,-,+,-) &= -2i\hbar Q_{\mathrm{e}}{}^2 \frac{s_+(p_1,p_2)s_-(q_1,q_2)}{t} \sim 2\hbar Q_{\mathrm{e}}{}^2\frac{s}{t}
\end{aligned}$$

plus interchange of $+$ and $-$. Note how nicely the Fermi sign makes $\mathfrak{M}(+,+,+,+)$ come out simple!

2. In terms of the Mandelstam variables s, t, and u:

$$\left\langle|\mathfrak{M}|^2\right\rangle = 2\hbar^2 Q_{\mathrm{e}}{}^4\,\frac{s^4+t^4+u^4}{s^2t^2}$$

In the centre-of-mass frame we have

$$s = 4E^2 \ \ , \ \ t = -2E^2(1 = \cos\theta) \ \ , \ \ u = -2E^2(1+\cos\theta)^2$$

This gives

$$\left\langle|\mathfrak{M}|^2\right\rangle = \hbar^2 Q_{\mathrm{e}}{}^4\,\frac{(3+(\cos\theta)^2)^2}{(1-\cos\theta)^2}$$

3. It is kinematically possible for q_1^μ to equal p_1^μ since the particles have the same energy and mass. In that case $t = (p_1 - q_1)^2 = 0$ strictly.

Solution to exercise 129

1. We can do this by looking at Eq.(11.40). We now have 4 helicity states rtather than just one. If the two neutrinos have opposite helicity, we can account for this by integrchanging k_2 and q. Therefore

$$\left\langle|\mathfrak{M}|^2\right\rangle = 128\hbar^2 G^2\left((pk_2)(qk_1)+(pq)(k_1k_2)\right)$$

2. $\langle|\mathfrak{M}|^2\rangle = 64\hbar^2 G^2 m_\mu^3\left(k_2^0(1-2k_2^0/m_\mu)+q^0(1-2q^0/m_\mu)\right)$

3. The distribution of $y = (2q^0)/m_\mu$ is

$$\frac{1}{\Gamma}\,\frac{d\Gamma}{dy} = y^2(9-8y)$$

4. The difference between $y^2(9-8y)$ and $y^2(6-4y)$ is quite clear, especially for large y values since the first spectrum has a maximum at $y = 3/4$ and the second one is monotonic. Also the averages are different: in the first case, $\langle y\rangle = 0.65$ and in the second case $\langle y\rangle = 0.70$.

Solution to exercise 130

1. The relation between m and M is $m = (Mc^2)/(\hbar c)$. Using $\hbar c = 3.16 \times 10^{-26}$ kg m^3/sec^2, we find $m_\mu = 3.34 \times 10^{24}$/m and $m_\pi = 4.42 \times 10^{24}$/m. These numbers are unwieldy: that is why in practice we often settle for Fundamental HEP units.

2. It is easy to verify that $f_\pi\sqrt{\hbar c^2}$ has dimension GeV. From that point on we can use Fundamental HEP units. The lifetime τ_π is converted into GeV by using $\hbar = 6.58 \times 10^{-25}$ GeV sec, which gives $\Gamma_\pi = 2.53\ 10^{-17}$ GeV. A little algebra using Eq.(11.65) then gives $f_\pi = 0.09$ GeV.

Solution to exercise 131
This is most easily seen by taking the massless limit. The right-handed antineutrino is described by $v(q_2) = v_-(q_2)$, in agreement with the factor $(1+\gamma^5)$. This also asks for $\overline{u}(q_1) = \overline{u}_-(q_1)$, corresponding to a left-handed charged lepton. And this is just what nature asks.

12 Exercises for chapter 12

Solution to exercise 132
From Lorentz covariance we must have

$$X^{\mu\nu} = \frac{1}{4\pi}\int d\Omega\ \epsilon^\mu\,\epsilon^\nu \quad = \quad Ag^{\mu\nu} + Bp^\mu p^\nu$$

From $\epsilon \cdot p = 0$ we have $X^{\mu\nu}p_\mu = 0$ so that $A = -B\,p^2 = -B\,m^2$, and from $\epsilon\cdot\epsilon = -1$ we find $X^\mu_\mu = -1$. We can then solve to find $A = -1/3$ and $B = 1/(3m^2)$.

Solution to exercise 133
Taking a trace with γ^α:

$$\mathrm{Tr}\left(\Lambda(p;q)^\mu_\nu\gamma_\mu\gamma^\alpha\right) = 4\Lambda(p;q)^\alpha_\nu = \mathrm{Tr}\left(\Sigma\gamma_\nu\overline{\Sigma}\gamma^\alpha\right)$$

and the Chisholm identity then gives

$$4\Lambda(p;q)^\alpha_\nu\gamma^\nu = \mathrm{Tr}\left(\Sigma\gamma_\nu\overline{\Sigma}\gamma^\alpha\right)\gamma^\nu = 2\left(\overline{\Sigma}\gamma^\alpha\Sigma + \Sigma^R(\gamma^\alpha)^R\overline{\Sigma}^R\right) = 4\overline{\Sigma}\gamma^\alpha\Sigma$$

since $\Sigma^R = \overline{\Sigma}$.

Solution to exercise 134
Under a Lorentz transformation, and using $\overline{\Sigma}\gamma^5 = \gamma^5\overline{\Sigma}$:

$$\begin{aligned}
J_S &\rightarrow \overline{u}(p)\overline{\Sigma}\Sigma u(q) = J_S \quad , \quad J_P \rightarrow \overline{u}(p)\overline{\Sigma}\gamma^5\Sigma u(q) = J_P \quad , \\
J_V^\mu &\rightarrow \overline{u}(p)\overline{\Sigma}\gamma^\mu\Sigma u(q) = \Lambda^\mu_\nu\overline{u}(p)\gamma^\nu u(q) = \Lambda^\mu_\nu J_V^\nu \quad , \\
J_A^\mu &\rightarrow \overline{u}(p)\overline{\Sigma}\gamma^5\gamma^\mu\Sigma u(q) = \Lambda^\mu_\nu\overline{u}(p)\gamma^5\gamma^\nu u(q) = \Lambda^\mu_\nu J_A^\nu \quad , \\
J_T^{\mu\nu} &\rightarrow \overline{u}(p)\overline{\Sigma}\gamma^\mu\gamma^\nu\Sigma u(q) = \overline{u}(p)\overline{\Sigma}\gamma^\mu\Sigma\overline{\Sigma}\gamma^\nu\Sigma u(q) = \Lambda^\mu_\alpha\Lambda^\nu_\beta J_T^{\alpha\beta}
\end{aligned}$$

The minus signs under parity arise from the fact that $\gamma^0\gamma^k\gamma^0 = -\gamma^k$ for $k = 1, 2, 3, 5$.

Solution to exercise 135

$$
\begin{aligned}
b^\mu &= a^\mu - \frac{2}{(p+q)^2}(p+q)^\mu(p+q\cdot a) + \frac{2}{p^2}q^\mu(pa)\\
(bb) &= (aa) + \frac{4}{(p+q)^4}(p+q)^2(p+q\cdot a)^2 + \frac{4}{p^4}q^2(pa)^2 - \frac{4}{(p+q)^2}(p+q\cdot a)^2\\
&\quad + \frac{4}{p^2}(qa)(pa) - \frac{8}{p^2(p+q)^2}(p+q\cdot q)(p+q\cdot a)(pa)\\
&\quad (aa) + \frac{4}{(p+q)^2}(p+q\cdot a)^2 + \frac{4}{p^2}(pa)^2 - \frac{4}{(p+q)^2}(p+q\cdot a)^2\\
&\quad + \frac{4}{p^2}(qa)(pa) - \frac{4}{p^2}(p+q\cdot a)(pa) \quad = (aa)
\end{aligned}
$$

Solution to exercise 136

We use the fact that any four-vector is a linear combination of t, x, y, and z, and that

$$t\cdot t = -x\cdot x = -y\cdot y = -z\cdot z = 1$$

Solution to exercise 137

Consider a vector u^μ with $(uk_{1,2}) = 0$. Then $u^2 < 0$ so we may take $u^2 = -1$. For $v^\mu = \epsilon^\mu(u, k_1, k_2)/(k_1k_2)$ we have $(vu) = (vk_{1,2}) = 0$ and $v^2 = -1$. Moreover, $\epsilon^\mu(v, k_1, k_2)/(k_1k_2) = -u^\mu$. Therefore we have

$$\Lambda_{\mu\nu}k^\nu_{1,2} = k^\mu_{1,2} \;\;,\;\; \Lambda_{\mu\nu}u^\mu = \cos\theta u^\mu - \sin\theta v^\mu \;\;,\;\; \Lambda_{\mu\nu}v^\mu = \cos\theta v^\mu + \sin\theta u^\mu$$

so this is a minimal Lorentz transform and in the frame with $\vec{k}_{1,2}$ antiparallel it is a rotation in the u, v plane.

Solution to exercise 138

The generator of rotations aounrd the z axis is, from Eq.(12.16):

$$(T_z)^\alpha_\beta = i\hbar\,(x^\alpha y_\beta - y^\alpha x_\beta)$$

Therefore

$$
\begin{aligned}
(T_z\epsilon_\pm)^\mu &= \frac{i\hbar}{\sqrt{2}}\,(x^\mu y_\nu - y^\mu x_\nu)\,(x^\nu \pm iy^\nu) = \frac{i\hbar}{\sqrt{2}}\,(\mp\, ix^\mu + y^\mu) = \pm\,\hbar\epsilon^\mu_+\\
(T_z\epsilon_0)^\mu &= i\hbar\,(x^\mu y_\nu - y^\mu x_\nu)\,z^\nu = 0
\end{aligned}
$$

Solution to exercise 139

1. This is essentially the same as exercise 123, except that we no longer require $p^2 = 0$.

2. It is a Lorentz-covariiant expression.

3. Since $[\gamma^0, \not{p}]$ depends only on $\vec{p}$ we can write

$$\frac{1}{|\vec{p}|}\gamma^5[\gamma^0, \not{p}] = \frac{1}{k_1^0}\gamma^5[\gamma^0 \not{k}_1] = -\frac{1}{k_2^0}\gamma^5[\gamma^0 \not{k}_2]$$

and then use

$$\gamma^5[\gamma^0 \not{k}_1]u_+(k_1) = -\gamma^5 \not{k}_1 \gamma^0 u_+(k_1) = -2k_1^0 \gamma^5 u_+(k_1) = -2k_1^0 u_+(k_1)$$

and similar for k_2 (up to the additional minus sign). This leads immediately to the results.

4.

$$\begin{aligned} \overline{u}_+(k_1)\gamma^\mu u_+(k_2) &\rightarrow \overline{(\Sigma u_+(k_1))}\gamma^\mu \Sigma u_+(k_2) \\ &= \overline{(e^{-i\alpha/2}u_+(k_1))}\gamma^\mu \left(e^{i\alpha/2}u_+(k_2)\right) = e^{i\alpha}\overline{u}_+(k_1)\gamma^\mu u_+(k_2) \end{aligned}$$

Solution to exercise 140

1. The Chisholm identity gives

$$\not{\epsilon}_+ = \frac{\sqrt{2}}{m}\left(u_+(k_2)\overline{u}_+(k_1) + u_-(k_1)\overline{u}_-(k_2)\right) \quad , \quad \not{\epsilon}_- = \frac{\sqrt{2}}{m}\left(u_-(k_2)\overline{u}_-(k_1) + u_+(k_1)\overline{u}_+(k_2)\right)$$

Hence

$$\begin{aligned} \not{\epsilon}_-\not{\epsilon}_+\not{p} &= \frac{2}{m^2}\left(s_-(k_1,k_2)s_+(k_1,k_2)\omega_-\not{k}_2 + s_+(k_2,k_1)s_-(k_2,k_1)\omega_+\not{k}_1\right) \\ &= -2\left(\omega_-\not{k}_2 + \omega_+\not{k}_1\right) \end{aligned}$$

and this proves the result.

2. The Pauli identity gives

$$\not{p}\not{\epsilon}_+\not{\epsilon}_- - \not{\epsilon}_-\not{\epsilon}_+\not{p} = -2i\gamma^5\gamma_\mu\epsilon^\mu(p,\epsilon_+,\epsilon_-)$$

and therefore $\epsilon_0^\mu \sim \epsilon^\mu(p,\epsilon_+,\epsilon_-)$. This is the main point; the other orthogonalities are trivial.

Solution to exercise 141

Let us take $r = q$. Then

$$\epsilon_\lambda^\mu(k) = \frac{\overline{u}_\lambda(k)\gamma^\mu u_\lambda(q)}{\lambda\sqrt{2}\, s_{-\lambda}(k,q)} \quad \rightarrow \quad \epsilon_\lambda \cdot q = 0 \quad , \quad \epsilon_\lambda \cdot p = \frac{s_\lambda(k,p)s_{-\lambda}(p,q)}{\lambda\sqrt{2}\, s_{-\lambda}(k,q)}$$

so that

$$\frac{\epsilon_\lambda \cdot p}{2(kp)} = \frac{\epsilon_\lambda \cdot p}{s_\lambda(k,p)s_{-\lambda}(p,k)} = \frac{-\lambda s_{-\lambda}(p,q)}{\sqrt{2}\, s_{-\lambda}(k,p)s_{-\lambda}(k,q)}$$

13 Exercises for chapter 13

Solution to exercise 142

1. Applying the handlebar gives

$$\not{k}\not{\epsilon} \;\to\; \not{k}\not{k} = k^2 = 0$$

2. The amplitude for a $1 \to 2$ process must have dimensionality $\mathfrak{m}$, see Eq.(7.23). Since $\not{k} \sim \mathfrak{m}$ and $u, \overline{u} \sim \mathfrak{m}^{1/2}$ we must have $\mathbf{dim}[g] = \mathfrak{m}^{-1}\hbar^{-1/2} = L/\hbar^{1/2}$.

3. Since we may use $\sum\, \epsilon^\alpha\, \bar{\epsilon}^\beta = -g^{\alpha\beta}$, and with $\omega = \cos\theta + i\sin\theta\gamma^5 = \bar{\omega}$:

$$\begin{aligned}\left\langle|\mathfrak{M}|^2\right\rangle &= -\frac{\hbar g^2}{2}\mathrm{Tr}\left((\not{q}+m_e)\omega\not{k}\gamma^\alpha(\not{p}+m_\mu)\gamma_\alpha\not{k}\omega\right) = -\frac{\hbar g^2}{2}\mathrm{Tr}\left(\not{q}\omega\not{k}\gamma^\alpha\not{p}\gamma_\alpha\not{k}\omega\right)\\ &= \hbar g^2\mathrm{Tr}\left(\not{q}\omega\not{k}\not{p}\not{k}\omega\right) = 2(pk)\hbar g^2\mathrm{Tr}\left(\not{q}\omega\not{k}\omega\right) = 8\hbar g^2\,(pk)(qk)\\ &= 2\hbar g^2\,(m_\mu^2 - m_e^2)^2 = 8\pi\alpha\,\frac{(m_\mu^2 - m_e^2)^2}{\Lambda^2}\end{aligned}$$

4.

$$\Gamma(\mu\to e\gamma) \;=\; \frac{1}{2m_\mu}\left\langle|\mathfrak{M}|^2\right\rangle\frac{1}{32\pi^2}\left(1-\frac{m_e^2}{m_\mu^2}\right)(4\pi) = \frac{\alpha(m_\mu^2-m_e^2)^3}{2m_\mu^3\Lambda^2}$$

5. Using the rsuul for muon decay of chapter 11, we have

$$B = \frac{\Gamma(\mu\to e\gamma)}{\Gamma(\mu\to e\bar{\nu}_e\nu_\mu)} = \frac{96\alpha\pi^3(m_\mu^2-m_e^2)^3}{m_\mu^8\Lambda^2G_F^2} \approx \frac{1.43\;10^{13}\,\mathrm{GeV}^2}{\Lambda^2}$$

   Therefore $K \approx 3.8\;10^6$ GeV, and $\Lambda > 10^{12}$ GeV.

Solution to exercise 143
The (sum of the) two diagrams is

$$\mathfrak{M} = -ie^2\hbar\,\overline{u}(p_2)\left[\not{\epsilon}_2\,\frac{\not{p}_1+\not{k}_1+m}{2(p_1k_1)}\not{\epsilon}_1 - \not{\epsilon}_1\,\frac{\not{p}_2-\not{k}_1+m}{2(p_2k_1)}\not{\epsilon}_2\right]u(p_1)$$

The handlebar on photon 1 leads to

$$\begin{aligned}\frac{\not{p}_1+\not{k}_1+m}{2(p_1k_1)}\not{\epsilon}_1u(p_1) &\;\to\; \frac{\not{p}_1+m}{2(p_1k_1)}\not{k}_1u(p_1) = u(p_1)\\ \overline{u}(p_2)\not{\epsilon}_1\,\frac{\not{p}_2-\not{k}_1+m}{2(p_2k_1)} &\;\to\; \overline{u}(p_2)\not{k}_1\,\frac{\not{p}_2+m}{2(p_2k_1)} = \overline{u}(p_2)\end{aligned}$$

so that $\mathfrak{M}$ vanishes.

Solution to exercise 144
The identity $4\pi\alpha = Q^2\hbar$, when translated according to Eq.(13.34), reads $4\pi\alpha = q^2/(\hbar c)$, which is correct.

Solution to exercise 145

It is simplest to examine the *differences* with the 'original' Furry theorem, that gave $C = -1$ for vector particles. With one scalar vertex, only the *odd* powers of m survive, leading to $C = +1$ for scalar particles. Replacing γ^μ by $\sigma^{\mu\nu}$ again only the odd powers of m survive, but also $(\sigma^{\mu\nu})^R = -\sigma^{\mu\nu}$ so that again $C = -1$. Replacing γ^μ by γ^5 has the same effect as replacing it by 1 since $(\gamma^5)^R = \gamma^5$, so gives $C = +1$. Finally, $\gamma^5\gamma^\mu$ keeps only the even powers of m but $(\gamma^5\gamma^\mu)^R = -\gamma^5\gamma^\mu$ so that $C = +1$. Note that here the C-character of the particle is completely determined by its *coupling* to fermions

Solution to exercise 146

1. The minus sign is (what else?) the Fermi sign, that is unavoidable whenever $v(p, s)$ and $\overline{v}(p, s)$ occur together.

2. By direct computation using the Casimir trick we have

$$\begin{aligned} \Gamma = 1 &\;:\; U = 2m \;\;,\;\; V = 2m \\ \Gamma = \gamma^\mu &\;:\; U = 2p^\mu \;\;,\;\; V = -2p^\mu \\ \Gamma = \sigma^{\mu\nu} &\;:\; U = 2\epsilon^{\mu\nu}(p, s) \;\;,\;\; V = -2\epsilon^{\mu\nu}(p, s) \\ \Gamma = \gamma^5\gamma^\mu &\;:\; U = -2ms^\mu \;\;,\;\; V = -2ms^\mu \end{aligned}$$

   For $\Gamma = \gamma^5$ we find $U = V = 0$, with no conclusion possible.

Solution to exercise 147

We tacitly assume to be working in the centre-of-mass frame, where the process is

$$e^+(p_1)\, e^-(p_2) \;\to\; \mu^-(q_1)\, \mu^+(q_2)$$

so that $\vec{q}_2 = -\vec{q}_1$.

1. By the assumption of C conservation, we only have to look at the C character of the intermediate state: -1 for one photon, +1 for two photons.

2. In what follows, the incoming e^+e^- pair is simply a machine that produces a superposition of one- and two-photon states. At the tree level, one one-photon states are produced: the initial state is then $|\gamma\rangle$, with $C\,|\gamma\rangle = -\,|\gamma\rangle$. The final state $|f\rangle$ is $|\mu^-(q_1)\mu^+(q_2)\rangle$ and $C\,|f\rangle = |\mu^-(q_2)\mu^+(q_1)\rangle$. The differential cross section is essentially the transition rate $|\mathfrak{M}(q_1, q_2)|^2$:

$$\begin{aligned} |\mathfrak{M}(q_1, q_2)|^2 &= \langle i|\, f\rangle \langle f|\, i\rangle = \left\langle \gamma \left|C^\dagger C\right| f\right\rangle \left\langle f \left|C^\dagger C\right| i\right\rangle \\ &= \langle \gamma\, |C|\, f\rangle \left\langle f \left|C^\dagger\right| \gamma\right\rangle = |\mathfrak{M}(q_2, q_1)|^2 \end{aligned}$$

   The interchange $q_1 \leftrightarrow q_2$ is equivalent to replacing the scattering angle cosine $\cos\theta$ by $-\cos\theta$. The differntial cross section is therefore symmetric in $\cos\theta$ at the tree level.

3. For more complicated observables A we define

$$\Omega = \sum_f |f\rangle f \quad , \quad A = \langle i \,|\Omega|\, i\rangle$$

where the sum Σ_f is defined by the experiment. We define C-even and C-odd parts by

$$C\Omega_\pm C^\dagger = \pm\,\Omega_\pm$$

The full initial state is now

$$|i\rangle = |\gamma\rangle + |\gamma\gamma\rangle \quad , \quad C\,|i\rangle = -\,|\gamma\rangle + |\gamma\gamma\rangle$$

where we ignore possible coefficients in the superposition. By the same reasoning as above we find

$$\begin{aligned} A_\pm &= (\langle\gamma| + \langle\gamma\gamma|)\,\Omega_\pm\,(|\gamma\rangle + |\gamma\gamma\rangle) \\ &= \pm\,(\langle\gamma\,|\Omega_\pm|\,\gamma\rangle + \langle\gamma\gamma\,|\Omega_\pm|\,\gamma\gamma\rangle) \mp (\langle\gamma\,|\Omega_\pm|\,\gamma\gamma\rangle + \langle\gamma\,|\Omega_\pm|\,\gamma\gamma\rangle) \end{aligned}$$

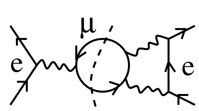
therefore

$$\begin{aligned} A_+ &= \langle\gamma\,|\Omega_+|\,\gamma\rangle + \langle\gamma\gamma\,|\Omega_+|\,\gamma\gamma\rangle \\ A_- &= \langle\gamma\,|\Omega_-|\,\gamma\gamma\rangle + \langle\gamma\,|\Omega_-|\,\gamma\gamma\rangle \end{aligned}$$

A_+ is any experimental observable that is *symmetric* in $q_1 \leftrightarrow q_2$.

4. An *antisymmetric* observable A_- gets contributions only from the single- and double-photon channel. The second diagram interfering with the third one is a *two*-loop contribution.

As an illustration, an excellent proof for the *total* cross section follows from the fact that, directly by Furry's theorem,



$= 0$

<u>Solution to exercise</u> 148

1. This is because under integrhange of particle and antiparticle $u\overline{u}$ and $d\bar{d}$ are invariant as long as they are not distinguished by any other variables. Since π^0 is a spin-zero particle (an s-wave bound state) this holds.

2. As fas as we know, there is no *'anti'*-light. If photons and antiphotons were different, then for instance black-body radiation would look different sice for every wavenumber there would not be two but four photon states.

3. If the photons had $C = +1$ then $\pi^0 \to \gamma\gamma\gamma$ would be far less severely suppressed (only by a factor α or so). Therefore $C = -1$, and the pion then has $C = (-1)^2 = +1$.

Solution to exercise 149

1. (a) The photons can be attached in $n!$ permutations; there are no vertices with 2 or more photons.

   (b) As Eq.(13.75) shows, in this case always the photon closest to either the electron or to the positron has that fermion's momentum as a gauge vector, and the amplitude vanishes.

   (c) The diagrams are nonzero for which the 'exceptional' photon is closest to either the electron or to the positron. Then there are $(n-1)!$ ways to attach the other photons.

2. We use p_2 for all three gauge vectors, so that

$$\omega_- \not\epsilon_1 \doteq \frac{\sqrt{2}\, u_-(k_1)\overline{u}_-(p_2)}{s_-(k_1,p_2)}\ ,\ \omega_- \not\epsilon_2 \doteq \frac{\sqrt{2}\, u_-(k_2)\overline{u}_-(p_2)}{s_-(k_2,p_2)}\ ,\ \omega_- \not\epsilon_3 \doteq \frac{\sqrt{2}\, u_-(p_2)\overline{u}_-(k_3)}{s_+(k_3,p_2)}$$

Then, up to an overall factor $\sqrt{8e^6\hbar^3}$,

$$\begin{aligned}
\mathfrak{M} &\doteq \frac{\overline{u}_+(p_1)\not\epsilon_1(\not k_1 - \not p_1)\not\epsilon_2(\not p_2 - \not k_3)\not\epsilon_3 u_+(p_3)}{2(p_1k_1)\,2(p_2k_2)\,\sqrt{8}} + (1 \leftrightarrow 2) \\
&\doteq \frac{s_+(p_1,k_1)\overline{u}_-(p_2)(\not k_1 - \not p_1)u_-(k_2)\overline{u}_-(p_2)\not k_3 u_-(p_2)\overline{u}_-(k_3)u_+(p_2)}{2(p_1k_1)\,2(p_2k_3)\,s_-(k_1,p_2)s_-(k_2,p_2)s_+(k_3,p_2)} + (1 \leftrightarrow 2) \\
&\doteq \frac{\overline{u}_-(p_2)\not k_3 u_-(k_2)s_-(k_3,p_2)}{s_-(k_1,p_1)s_-(k_1,p_2)s_-(k_2,p_2)s_+(k_3,p_2)} + (1 \leftrightarrow 2) \\
&\doteq \frac{\overline{u}_-(p_2)\not k_3\not k_2 u_+(p_1)s_-(k_3,p_2) + (1 \leftrightarrow 2)}{s_-(k_1,p_1)s_-(k_1,p_2)s_-(k_2,p_1)s_-(k_2,p_2)s_+(k_3,p_2)} \\
&= \frac{\overline{u}_-(p_2)\not k_3\not p_2 u_+(p_1)s_-(k_3,p_2)}{s_-(k_1,p_1)s_-(k_1,p_2)s_-(k_2,p_1)s_-(k_2,p_2)s_+(k_3,p_2)} \\
&\doteq \frac{s_-(p_1,p_2)s_-(k_3,p_2)^2}{s_-(k_1,p_1)s_-(k_1,p_2)s_-(k_2,p_1)s_-(k_2,p_2)}
\end{aligned}$$

3. For this helicity configuration, we have

$$|\mathfrak{M}|^2 = 2e^6\hbar^3\,\frac{s\,(p_2k_3)^2}{(p_1k_1)(p_2k_1)(p_1k_2)(p_2k_2)} = 2e^6\hbar^3\,\frac{s\,(p_2k_3)^3(p_1k_3)}{\prod_{j=1}^{3}(p_1k_j)(p_2k_j)}$$

The other contributions come from (a) taking the fermions' helicity - and changing the gauge vector from $p2$ to p_1, (b) flipping *all* helicities, (c) taking also k_1 and k_2 to be the 'exceptional' photon. And let us not forget the factor 1/4 in the average!

Solution to exercise 150

1. We use the fact that $p_1 + p_2 - q_1 - q_2 = 0$ and therefore

$$\begin{aligned}0 &= (p_1 + p_2 - q_1 - q_2)^2 \\ &= 2m^2 + 2M^2 + 2(p_1p_2) + 2(q_1q_2) - 2(p_1q_1) - 2(p_1q_2) - 2(p_2q_1) - 2(p_2q_2) \\ &= 2m^2 + 2M^2 + (s - 2m^2) + (s - 2M^2) + 2(t - m^2 - M^2) + 2(u - m^2 - M^2) \\ &= 2(s + t + u) - 4m^2 - 4M^2\end{aligned}$$

2. This goes precisely the same way if we take the square of $0 = p_1+p_2-q_1-q_2-k$.

Solution to exercise 151
Here exercise 128 is extremely useful. We have seen there that the only amplitudes where both diagrams contribute are

$$\mathfrak{M}(+,+,+,+) \hat{=} \mathfrak{M}(-,-,-,-) \hat{=} 2\hbar {Q_e}^2\, u\, \frac{s+t}{st}$$

If we forget the Fermi sign, the numerator becomes $(s - t)$. We will therefore be off by a factor (with $c = \cos\theta$):

$$\frac{(s-t)^2u^2 + s^4 + t^4}{u^4 + s^4 + t^4} = \frac{c^4 - 4c^3 + 2c^2 + 4c + 13}{(c^2 + 3)^2}$$

This is equal to 1 for $c = \pm 1$ and reaches a maximum of 3/2 for $c = \sqrt{5} - 2 \sim 0.236$ or 76.3 degrees.

Solution to exercise 152
With $s = (p_1+p_2)^2 = (q_1+q_2)^2$, $t = (p_1-q_1)^2 = (p_2-q_2)^2$, $u = (p_1-q_2)^2 = (p_2-q_1)^2$, and $s + t + u = 4m^2$:

$$\begin{aligned}\mathfrak{M} &= i\hbar {Q_e}^2 \left(\frac{\mathcal{A}_s}{s} - \frac{\mathcal{A}_t}{t}\right) \\ \mathcal{A}_s &= \overline{v}(p_1)\gamma_\alpha u(p_2)\, \overline{u}(q_2)\gamma^\alpha v(q_2) \\ \mathcal{A}_t &= \overline{v}(p_1)\gamma_\alpha v(q_1)\, \overline{u}(q_2)\gamma^\alpha u(p_2) \\ \left\langle|\mathcal{A}_s|^2\right\rangle &= \frac{1}{4}\mathrm{Tr}\left((\not{p}_1 - m)\gamma_\alpha(\not{p}_2 + m)\gamma_\beta\right) \mathrm{Tr}\left((\not{q}_2 + m)\gamma^\alpha(\not{q}_1 - m)\gamma^\beta\right) \\ &= 2(t^2 + u^2) + 16m^2s - 16m^4 \\ \left\langle|\mathcal{A}_t|^2\right\rangle &= \frac{1}{4}\mathrm{Tr}\left((\not{p}_1 - m)\gamma_\alpha(\not{q}_1 - m)\gamma_\beta\right) \mathrm{Tr}\left((\not{q}_2 + m)\gamma^\alpha(\not{p}_2 + m)\gamma^\beta\right) \\ &= 2(s^2 + u^2) + 16m^2t - 16m^4 \\ \langle\mathcal{A}_1\mathcal{A}_2^*\rangle &= \frac{1}{4}\mathrm{Tr}\left((\not{p}_1 - m)\gamma_\alpha(\not{p}_2 + m)\gamma_\beta(\not{q}_2 + m)\gamma^\alpha(\not{q}_1 - m)\gamma^\beta\right) \\ &= -2u^2 + 16m^2u - 24m^4 \\ \left\langle|\mathfrak{M}|^2\right\rangle &= \frac{\hbar^2 {Q_e}^4}{s^2t^2}\Big(2s^4 + 2t^4 + 2u^2(s+t)^2 + 16m^4(t^2 + st + s^2) \\ &\qquad + 8m^2(-t^2u - t^3 - 3stu + 2st^2 - s^2u + 2s^2t - s^3\Big)\end{aligned}$$

<u>Solution to exercise</u> 153

For $e^+(p_1)e^-(p_2) \;\rightarrow\; \mu^+(q_1)\mu^-(q_2)\gamma(k)$:

$$\left\langle|\mathfrak{M}|^2\right\rangle = \hbar^3 Q_{\rm e}^{\,6}\;\frac{t^2+t'^2+u^2+u'^2}{ss'}K$$

with

$$\begin{aligned} K \;&=\; \frac{2(p_1p_2)}{(p_1k)(p_2k)} + \frac{2(q_1q_2)}{(q_1k)(q_2k)} \\ &+ \frac{2(p_1q_1)}{(p_1k)(q_1k)} + \frac{2(p_2q_2)}{(p_2k)(q_2k)} - \frac{2(p_1q_2)}{(p_1k)(q_2k)} - \frac{2(p_2q_1)}{(p_2k)(q_1k)} \\ &- \frac{m^2}{(p_1k)^2} - \frac{m^2}{(p_2k)^2} - \frac{m^2}{(q_1k)^2} - \frac{m^2}{(q_2k)^2} \end{aligned}$$

For $e^+(p_1)e^-(p_2) \;\rightarrow\; e^+(q_1)e^-(q_2)\gamma(k)$:

$$\left\langle|\mathfrak{M}|^2\right\rangle = \hbar^3 Q_{\rm e}^{\,6}\;\frac{ss'(s^2+s'^2)+tt'(t^2+t'^2)+uu'(u^2+u'^2)}{ss'tt'}K$$

<u>Solution to exercise</u> 154

1. The transition rate for final-state radiation can be found from exercise 152:

$$\left\langle|\mathfrak{M}|^2\right\rangle = \hbar^3 Q_{\rm e}^{\,6}\,\frac{t^2+t'^2+u^2+u'^2}{s(q_1k)(q_2k)}$$

With θ_j the angle between $\vec{q_j}$ and $\vec{p_1}$, we have

$$\begin{aligned} t &= -sx_1(1-\cos\theta_1)/2\;,\;\; t' = -sx_2(1+\cos\theta_2)/2 \\ u &= -sx_2(1-\cos\theta_2)/2\;,\;\; u' = -sx_1(1+\cos\theta_1)/2 \end{aligned}$$

Averaging over the direction of $\vec{p_1}$, we therefore have

$$\left\langle t^2+t'^2+u^2+u'^2\right\rangle = \frac{2s^2}{3}\left(x_1^2+x_2^2\right)$$

Using momentum conservation,

$$(q_1k) = s(1-x_2)/2\;,\;\;(q_2k) = s(1-x_1)/2$$

The angle-averaged rate is therefore, using $\hbar Q_{\rm e}^{\,2} = 4\pi\alpha$:

$$\left\langle\left\langle\|\mathfrak{M}|^2\right\rangle\right\rangle = \frac{512\pi^3}{3s}\;\frac{x_1^2+x_2^2}{(1-x_1)(1-x_2)}$$

The phase space factor, after angular integration but before the energy integration, is

$$d(x_1,x_2) = \int dV(p_1{+}p_2;q_1,q_2,k) = \frac{1}{(2\pi)^5}\int\frac{1}{8}dq_1^0\,dq_2^0\,d\Omega_1\,d\phi_2 = \frac{s}{128\pi^3}dx_1\,dx_2$$

For the cross section, therefore

$$d\sigma = \frac{1}{2s}\;\left\langle\left\langle|\mathfrak{M}|^2\right\rangle\right\rangle\;d(x_1,x_2) = \frac{2\alpha^3}{3s}\;\frac{x_1^2+x_2^2}{(1-x_1)(1-x_2)}dx_1\,dx_2$$

2. The prefactor $\hbar^3 {Q_e}^6$ is proportional to $\pi^3\alpha^3$; the phase space contains two angles, implying a factor π^2; and the phase space also contains $(2\pi)^{-5}$. Therefore all factors of π cancel here.

3. For $x_1 \to 1$, $(q_2 k)$ vanishes so that either $k^0 \to 0$ or $\vec{q}_2 \parallel \vec{k}$, and similar for x_2. If both x_1 and x_2 approach 1 this means that $k^0 \to 0$. Therefore $x_1 = 1, x_2 < 1$ and $x_2 = 1, x_1 < 1$ are collinear singularities, and $x_1 = x_2 = 1$ is an infrared singularity.

Solution to exercise 155

1. According to Eq.(13.115) the distribuition of the abgle θ between the electron and photon momenta (in the overall centre-of-mass system) is proportional, for small angles, to

$$\sim (\theta^2 + (m/E)^2)^{-1}$$

and this is invoked to state that the 'typical' angle is around $m/E \approx 0.0005/100 \approx 5\ 10^{-6}$.

2. From

$$r(\alpha) = \int_{\cos(\alpha)}^{1} d\cos\theta\, \frac{1}{1+\delta-\cos\theta} = \log\left(\frac{1+\delta-\cos(\alpha)}{\delta}\right)$$

with $\delta = m^2/(2E^2)$, we find that α must obey

$$r(\alpha) = f\, r(\pi) \quad \to \quad \cos(\alpha) = 1 + \delta - \delta\left(\frac{2}{\delta}\right)^f$$

3. If α is small(ish) then $\alpha \approx \sqrt{2\delta}(2/\delta)^{f/2}$ and this is about $3\ 10^{-3}$ for this δ and $f = 0.5$.

Solution to exercise 156

1. This is simply

$$S = \sum_{\lambda=\pm}\left|\frac{p_1\cdot\epsilon_\lambda}{p_1\cdot k} - \frac{p_2\cdot\epsilon_\lambda}{p2\cdot k}\right|^2$$

2. We use $p_1^\mu = (E,\vec{p})$, $p_2^\mu = (E,-\vec{p})$, $k^\mu = k(1,\vec{e})$ and $p = |\vec{p}|$, and insert that into S.

3. Simplifying S we obtain

$$S = \frac{4E^2p^2(1-c^2)}{k^2(E^2-p^2c^2)^2}$$

and this vanishes for $c = \pm 1$. Another way to understand this is to remark that the photon polarization can be chosen such that $\vec{\epsilon}_\lambda \perp \vec{k}$ and $\epsilon^0_\lambda = 0$ (by taking the gauge vector to point opposite to $\vec{k}$). Then, for $c = \pm 1$, also $\vec{\epsilon}_\lambda \perp \vec{p}$ so that $p_{1,2}\cdot\epsilon_\lambda = 0$.

Solution to exercise 157

1. We use $t = (p_1 - q_1)^2 = m^2 - 2(p_1 q_1)$, $u = (p_2 - q_1)^2 = m^2 = 2(p_2 q_1)$.

$$\begin{aligned}\mathfrak{M} &= \frac{ie^2\hbar}{s}\overline{v}(p_1)(\not{q}_1 - \not{q}_2)u(p_2) = \frac{2ie^2\hbar}{s}\overline{v}(p_1)\not{q}_1 u(p_2) \\ \left\langle|\mathfrak{M}|^2\right\rangle &= \frac{e^4\hbar^2 \mathrm{Tr}\left(\not{p}_1\not{q}_1\not{p}_2\not{q}_1\right)}{s^2} = \frac{e^4\hbar^2\left(8(p_1q_1)(p_2q_2) - 4m^2(p_1p_2)\right)}{s^2} = \frac{2e^4\hbar^2(tu - m^4)}{s^2}\end{aligned}$$

2. With $E = p^0_{1,2} = q^0_{1,2}$, $\beta = |\vec{q}_1|/E$ and θ the angle between $\vec{p}_1$ and $\vec{q}_1$:

$$\begin{aligned}&8(p_1q_1)(p_2q_1) - 4m^2(p_1p_2) = 8E^4\beta^2(1-\cos\theta^2) \\ &\frac{d\sigma}{d\Omega} = \frac{1}{2s}(4\pi\alpha)^2\frac{\beta^2(1-\cos\theta^2)}{2}\frac{1}{32\pi^2}\beta \\ &\sigma = \frac{\alpha^2\pi}{3s}\beta^3 = \frac{\alpha^2\pi}{3s}\left(1 - \frac{4m^2}{s}\right)^{3/2}\end{aligned}$$

3. As $m^2/s \to 0$,

$$\sigma(e^+e^- \to \mu^+\mu^-) = \frac{4\alpha^2\pi}{3s} \quad , \quad \sigma(e^+e^- \to w\bar{w}) = \frac{\alpha^2\pi}{3s}$$

For the muons there are 2 contributing helicity configurations. Moreover, the Gordon decomposition shows that the $\mu^-\gamma\mu^+$ interaction contains the scalar (convection) part as well as a 'magnetic' part. At high energy these contribute equally.

Solution to exercise 158

It is enough to compute $\mathfrak{M}$. For $w(p_1)\bar{w}(p_2) \;\to\; w'(q_1)\bar{w}'(q_2)$:

$$\mathfrak{M} = ie^2\hbar\,\frac{(p_1 - p_2 \cdot q_1 - q_2)}{s} \simeq e^2\hbar\,\frac{u-t}{s}$$

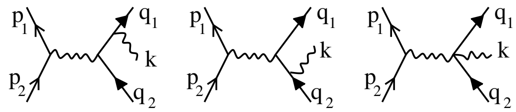
For $w(p_1)\bar{w}(p_2) \;\to\; w(q_1)\bar{w}(q_2)$:

$$\begin{aligned}\mathfrak{M} &= ie^2\hbar\left(\frac{(p_1 - p_2 \cdot q_1 - q_2)}{s} + \frac{(q_1 + p_1 \cdot -p_2 - q_2)}{t}\right) \\ &\simeq e^2\hbar\left(\frac{u-t}{s} + \frac{u-s}{t}\right)\end{aligned}$$

Note that crossing symmetry is preserved since the amplitude is invariant under $p_2 \;\leftrightarrow\; -q_1$.

Solution to exercise 159

1. The process is $e^+(p_1)e^-(p_2) \to w(q_1)\bar{w}(q_2)\gamma(k)$. The diagrams are



The amplitude, with photon polarization $\rho = \pm$:

$$\begin{aligned}
\mathfrak{M}(\lambda,\rho) &= -i\frac{e^3\hbar^{3/2}}{s}\,\overline{v}_\lambda(p_1)\gamma_\mu u_\lambda(p_2)\,\left(J_1^\mu + J_2^\mu + J_3^\mu\right)\\
J_1^\mu &= \frac{(q_1+q_1+k\cdot\epsilon)(q_1+k-q_2)^\mu}{2(q_1k)} \quad\to\quad -2q_2^\mu\frac{q_1\cdot\epsilon}{q_1\cdot k}\\
J_2^\mu &= \frac{(-q_2-q_2-k\cdot\epsilon)(q_1-k-q_2)^\mu}{2(q_2k)} \quad\to\quad -2q_1^\mu\frac{q_2\cdot\epsilon}{q_2\cdot k}\\
J_3^\mu &= -2\epsilon^\mu
\end{aligned}$$

Here we used $k\cdot\epsilon = 0$ and $\overline{v}(p_1)(\not{q}_1+\not{q}_2+\not{k})u(p_2) = 0$.

2. We start with $\lambda = \rho = +$, and pick p_2 as the gauge vector of the photon. Then

$$\begin{aligned}
\epsilon^\mu &= \frac{\overline{u}_+(k)\gamma^\mu u_+(p_2)}{\sqrt{2}\;s_-(k,p_2)} \quad,\quad \overline{u}_+(p_1)\not{\epsilon}_+u_+(p_2) = 0\\
\mathfrak{M} &= i\frac{e^3\hbar^{3/2}\sqrt{8}}{s\;s_-(k,p_2)}B\\
B &= \frac{s_+(k,q_1)s_-(q_1,p_2)s_+(p_1,q_2)s_-(q_2,p_2)}{2(q_1k)} + \frac{s_+(k,q_2)s_-(q_2,p_2)s_+(p_1,q_1)s_-(q_1,p_2)}{2(q_2k)}\\
&= s_-(q_1,p_2)s_-(q_2,p_2)\left(\frac{s_+(p_1,q_2)}{s_-(q_1,k)} + \frac{s_+(p_1,q_1)}{s_-(q_2,k)}\right)\\
&= \frac{s_-(q_1,p_2)s_-(q_2,p_2)\,\overline{u}_+(p_1)(\not{q}_1+\not{q}_2)u_+(k)}{s_-(k,q_1)s_-(k,q_2)}\\
&= \frac{s_-(q_1,p_2)s_-(q_2,p_2)s_+(p_1,p_2)s_-(p_2,k)}{s_-(k,q_1)s_-(k,q_2)}\\
|\mathfrak{M}|^2 &= 2e^6\hbar^3\,\frac{t'u'}{s(q_1k)(q_2k)} \quad,\quad t' = (p_2-q_2)^2\;,\; u' = (p_2-q_1)^2
\end{aligned}$$

The case $\lambda = -, \rho = +$ is obtained by $p_1 \leftrightarrow p_2$, and we can also flip both helicities. This gives the quoted result.

<u>Solution to exercise</u> 160

1. It is easily checked that $\epsilon^{(1)}\cdot k_{1,2} = \epsilon^{(2)}\cdot k_{1,2} = 0$ and $\epsilon^{(1)}\cdot\epsilon^{(2)} = 0$. The amplitude is

$$\mathfrak{M}(\epsilon_1,\epsilon_2) = -2ie^2\hbar\left(\frac{(p_1\epsilon_1)(p_2\epsilon_2)}{(p_1k_1)} + \frac{(p_2\epsilon_1)(p_1\epsilon_2)}{(p_2k_1)} - (\epsilon_1\epsilon_2)\right)$$

We also have

$$(p_{1,2}\epsilon^{(1)}) = 0\;\;,\;\;(p_1\epsilon^{(2)}) = -(p_2\epsilon^{(2)}) = -E^2\sin\theta\;\;,\;\;(p_{1,2}k_1) = E^2(1\pm\cos\theta)$$

Thus we find

$$
\begin{aligned}
\mathfrak{M}(\epsilon^{(1)},\epsilon^{(2)}) &= -2ie^2\hbar \\
\mathfrak{M}(\epsilon^{(1)},\epsilon^{(2)}) &= \mathfrak{M}(\epsilon^{(2)},\epsilon^{(1)}) = 0 \\
\mathfrak{M}(\epsilon^{(2)},\epsilon^{(2)}) &= -2ie^2\hbar\left(-\frac{\sin\theta^2}{1-\cos\theta} - \frac{\sin\theta^2}{1\cos\theta} + 1\right) \\
&= -2ie^2\hbar(-(1+\cos\theta) - (1-\cos\theta) + 1) = +2ie^2\hbar \\
\left\langle|\mathfrak{M}|^2\right\rangle &= 8e^4\hbar^2
\end{aligned}
$$

2. For this choice, we have $(p\epsilon_1) = (p\epsilon_2) = 0$ so only the seagull survives. We have

$$
\left(\epsilon_1^+\right)^\mu = \frac{\overline{u}_+(k_1)\gamma^\mu u_+(p_1)}{\sqrt{2}\, s_-(k_1,p_1)} \quad , \quad \left(\epsilon_2^-\right)^\mu = \frac{\overline{u}_-(k_2)\gamma^\mu u_-(p_1)}{-\sqrt{2}\, s_+(k_2,p_1)} = \frac{\overline{u}_+(p_1)\gamma^\mu u_+(k_2)}{\sqrt{2}\, s_+(p_1,k_2)}
$$

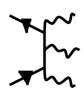
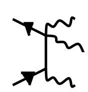
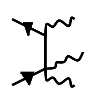
$$
\epsilon_1^+\cdot\epsilon_2^- = \frac{\overline{u}_+(k_1)\gamma^\mu \not{p}_1\gamma_\mu u_+(k_2)}{2s_-(k_1,p_1)s_+(p_1,k_2)} = -\frac{s_+(k_1,p_1)s_-(p_1,k_2)}{s_-(k_1,p_1)s_+(p_1,k_2)} \doteq 1
$$

3. We can take p_1, say, as gauge vector for all polarizations. This if photon j and i have the same helicity, $\epsilon_j\cdot\epsilon_i = 0$. Therefore diagrams with seagulls do not contribute. In a given diagram, if photon n is the first one to be emitted by the incoming p_1, there will be a vertex factor $(p_1 + p_1 - k_n\cdot\epsilon_n) = 2(p_1\epsilon_n) = 0$ and that diagram vanishes. Thus $\mathfrak{M} = 0$.

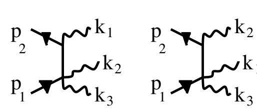
4. (a) For $w\bar{w}$ to n photons, the number of diagrams a_n can be shown to be

$$
a_n = \frac{n!}{\sqrt{3}}\left(\left((1+\sqrt{3})/2\right)^{n+1} - \left((1-\sqrt{3})/2\right)^{n+1}\right)
$$

so that $a_3 = 12$. The diagrams are

$6\times$  $+\ 3\times$  $+\ 3\times$ 

(b) By the reasoning of item 3, the only possibly nonzero diagrams are



(c)

$$
\begin{aligned}
\mathfrak{M} &= -4ie^3\hbar^{3/2}\,\frac{(p_2\epsilon_1^+)(\epsilon_2^+\epsilon_3^-)}{2(p_2k_1)} + (1\leftrightarrow 2) \\
\frac{(p_2\epsilon_1^+)}{(p_2k_1)} &= \frac{\overline{u}_+(k_1)\not{p}_2 u_+(p_1)}{2(p_2k_1)\sqrt{2}\, s_-(k_1,p_1)} = -\frac{s_-(p_2,p_1)}{\sqrt{2}s_-(k_1,p_2)s_-(k_1,p_1)} \\
(\epsilon_2^+\epsilon_3^-) &= -\frac{\overline{u}_+(k_2)\gamma^\alpha\not{p}_1\gamma_\alpha u_+(k_3)}{2s_-(k_2,p_1)s_+(k_3,p_1)} = \frac{s_+(p_1,k_2)s_-(k_3,p_1)}{s_-(k_2,p_1)s_+(k_3,p_1)} \\
\mathfrak{M} &= ie^3\hbar^{3/2}\sqrt{8}\left(\frac{s_-(p_2,p_1)s_+(p_1,k_2)s_-(k_3,p_1)}{s_-(k_1,p_2)s_-(k_1,p_1)s_-(k_2,p_1)s_+(k_3,p_1)} + (1\leftrightarrow 2)\right)
\end{aligned}
$$

$$
\begin{aligned}
&\doteq e^3\hbar^{3/2}\sqrt{8}\,\frac{s_-(p_2,p_1)s_-(k_3,p_1)}{s_-(k_1,p_1)s_-(k_2,p_1)s_+(k_3,p_1)}\left(\frac{s_+(p_1,k_2)}{s_-(k_1,p_2)}+\frac{s_+(p_1,k_1)}{s_-(k_2,p_2)}\right)\\
&= e^3\hbar^{3/2}\sqrt{8}\,\frac{s_-(p_2,p_1)s_-(k_3,p_1)\;\overline{u}_+(p_1)(\not{k}_1+\not{k}_2)u_+(p_2)}{s_-(k_1,p_1)s_-(k_1,p_2)s_-(k_2,p_1)s_-(k_2,p_2)s_+(k_3,p_1)}\\
&\doteq e^3\hbar^{3/2}\sqrt{8}\,\frac{s_-(p_2,p_1)s_-(k_3,p_1)s_-(k_3,p_2)}{s_-(k_1,p_1)s_-(k_1,p_2)s_-(k_2,p_1)s_-(k_2,p_2)}
\end{aligned}
$$

(d) Each of the other photons may also be the 'exceptional one', and we can flip all helicities as well. This gives the quoted result. If we let $k_3^0 \to 0$ then

$$
\left\langle|\mathfrak{M}|^2\right\rangle \to 8e^6\hbar^3\,\frac{(p_1p_2)}{(k_3p_1)(k_3p_2)}\left(\frac{(k_1p_1)^2(k_1p_2)^2+(k_2p_1)^2(k_2p_2)^2}{(k_1p_1)(k_1p_2)(k_2p_1)(k_2p_2)}\right)
$$

and since for $k_3 = 0$ we have $(p_1k_1) = (p_2k_2)$ and $(p_1k_2) = (p_2k_1)$, the last factor in brackets is simply equal to 2.

Solution to exercise 161

1. There is just one diagram, with a photon with momentum k exchanged. The only new ingredient is the factor Z_1Z_2.

2. This is, in fact, the original definition of the flux factor, see Eq.(7.17).

3. Taking $M \to \infty$ where possible, and working the q_1 rest frame, we have

$$
\begin{aligned}
&d^4q_2\,\delta(q_2^2-M^2)\,\delta^4(q+1+k-q_2) = \delta((q_1+k)^2-M^2)\\
&= \delta(2(q_1k)+k^2) = \delta(2Mk^0+k^2) = \frac{1}{2M}\delta(k^0+k^2/M) \approx \frac{1}{2M}\delta(p_1^0-p_2^0)\\
&= \frac{p_1^0}{M}\delta((p_1^0)^2-(p_2^0)^2) = \frac{p_1^0}{M}\delta(|\vec{p}_1|^2-|\vec{p}_2|^2) = \frac{p_1^0}{2M|\vec{p}_1|}\delta(|\vec{p}_1|-|\vec{p}_2|)
\end{aligned}
$$

The result for dV follows immediately. Additionally,

$$
k^2 = -|\vec{k}|^2 = -|\vec{p}_1-\vec{p}_2|^2 = -2|\vec{p}_1|^2(1-\cos\theta) = -4|\vec{p}_1|^2(\sin\theta/2)^2
$$

4. In the $M \to \infty$ limit, we have $(p_1+p_2\cdot q_1+q_2) = 4Mp_1^0$ so that

$$
\begin{aligned}
|\mathfrak{M}|^2 &= 16\pi^2(\alpha Z_1Z_2)^2\frac{M^2(p_1^0)^2}{|\vec{p}_1|^2(\sin\theta/2)^4}\\
\frac{d\sigma}{d\Omega} &= \frac{1}{4M|\vec{p}_1|}16\pi^2(\alpha Z_1Z_2)^2\frac{M^2(p_1^0)^2}{|\vec{p}_1|^4(\sin\theta/2)^4}\frac{|\vec{p}_1|}{16\pi^2M}
\end{aligned}
$$

5. Denoting by p_c^μ the classical (non-quantum) momentum, we have in the non-relativistic limit

$$
\frac{p_1^0}{|\vec{p}_1|^2} = \frac{p_c^0}{\hbar}\frac{\hbar^2}{|\vec{p}_c|^2} \approx \frac{\hbar m_c c}{|\vec{p}_c|^2} = \frac{\hbar c}{2E_K}
$$

and inserting this leads immediately to the Rutherford formula.

<u>Solution to exercise</u> 162

Replacing scalars by Dirac particles means replacing

$$(p_1 + p_2 \cdot q_1 + q_2) \approx 4Mp_1^0 \quad \to \quad \overline{u}(p_2)\gamma_\mu u(p_1)\,\overline{u}(q_2)\gamma^\mu u(q_1)$$

In the $M \to \infty$ limit we can replace q_2 by q_1, and then we have, looking at the spins:

$$\overline{u}(q_1,s)\gamma^\mu u(q,s) = 2Mg^{0\mu} \;\;,\;\; \overline{u}(q_1,s)\gamma^\mu u(q_1,-s) = 0$$

We must therefore replace

$$\begin{aligned} 16M^2(p_1^0)^2 &= (p_1 + p_2 \cdot q_1 + q_2)^2 \quad \to \\ &\frac{1}{4}\,8M^2\,\mathrm{Tr}\left((\not{p}_2 + m)\gamma^0(\not{p}_1 + m)\gamma^0\right) = 8M^2(2p_1^0p_2^0 - (p1p_2) + m^2) \\ &= 8M^2((p_1)^2 + |\vec{p}_1|^2\cos\theta + m^2) = 16M^2\left((p_1^0)^2 - |\vec{p}|^2(\sin\theta/2)^2\right) \end{aligned}$$

<u>Solution to exercise</u> 163

This is simple verification. If, in Eq.(13.144), we leave out the $\sigma^{\mu\nu}$ term then we reproduce exactly Eq.(13.141).

<u>Solution to exercise</u> 164

For muon pair production we can write the transition rate in various ways ways, for example:

$$F(p_1,p_2,q_1,q_2) = 2e^4\hbar^2\,\frac{(p_2q_2)^2 + (p_2q_1)^2}{(q_1q_2)^2} = 2e^4\hbar^2\,\frac{(p_1q_2)^2 + (p_1q_1)^2}{(q_1q_2)^2}$$

This makes it particularly easy to implement Eq.(13.163). For the choice $p_{1,2}^+$ we take the first form, and for the choice $p_{1,2}^-$ we take the second one. This nicely cancels the factors $(s' + 2(p_{1,2}k))^2$:

$$\begin{aligned} (s' + 2(p_1k))^2F(p_1^+,p_2^+,q1,q2) &= 2e^4\hbar^2\,(t'^2 + u'^2) \\ (s' + 2(p_2k))^2F(p_1^-,p_2^-,q1,q2) &= 2e^4\hbar^2\,(t^2 + u^2) \end{aligned}$$

and the radiative transition rate indeed beomes

$$\left\langle|\mathfrak{M}^{\mathrm{r}}|^2\right\rangle = e^6\hbar^3\,\frac{t^2 + t'^2 + u^2 + u'^2}{ss'}\,\frac{s}{(p_1k)(p_2k)}$$

<u>Solution to exercise</u> 165

When the photon is soft but not collinear, the last term in brackets is not IR divergent and can be neglected, so the soft-photon fator remains. When the photon is collinear with q_1, say, the double-pole term is automatically correct since it is always so. The coefficient of $e^2\hbar/(q_1k)$ can then be written as

$$\frac{2(q_1q_2)}{(kq_2)} + \frac{(kq_2)}{(q_1 + k \cdot q_2)} = \frac{2(q_1q_2)^2 + 2(q_1q_2)(kq_2) + (kq_2)^2}{(kq_2)(q_1 + k \cdot q_2)} = \frac{(q_1q_2)^2 + (q_1 + k \cdot q_2)^2}{(kq_2)(q_1 + k \cdot q_2)}$$

which gives Eq.(13.171) for the choice $r_0 = q_2$.

Solution to exercise 166

We use the explicit representationwith gauge vector r, and choose positive helicity for the photon.

$$\begin{aligned}
\eta^\mu &= \epsilon_+^\mu - \frac{(\epsilon_+ p)}{(qp)} q^\mu = \frac{A}{2(pq)\sqrt{2}\, s_-(q,r)} \\
A &= -2q^\mu \overline{u}_+(q) \not{p} u_+(r) + 2(pq)\overline{u}_+(q)\gamma^\mu u_+(r) \\
&= -\overline{u}_+(q)\gamma^\mu u_+(q)\overline{u}_+(q)\not{p} u_+(r) + 2(pq)\overline{u}_+(q)\gamma^\mu u_+(r) \\
&= -\overline{u}_+(q)\gamma^\mu \not{q}\not{p} u_+(r) + 2(pq)\overline{u}_+(q)\gamma^\mu u_+(r) \\
&= \overline{u}_+(q)\gamma^\mu \not{p}\not{q} u_+(r) = \overline{u}_+(q)\gamma^\mu u_+(p) s_+(p,q) s_-(q,r) \\
\to \; \eta^\mu &= \frac{\overline{u}_+(q)\gamma^\mu u_+(p)\; s_+(p,q)}{2(pq)\sqrt{2}} = \frac{\overline{u}_+(q)\gamma^\mu u_+(p)}{\sqrt{2}\, s_-(q,p)}
\end{aligned}$$

For negative helicity the proof is completely analogous.

Solution to exercise 167

1. The amplitude reads

$$\begin{aligned}
\mathfrak{M}(p_1,s_1,p_2,s_2) &= ie^2\hbar\, \overline{v}(p_1,s_1)\left(\not{\epsilon}_2 \frac{\not{p}_2 - \not{k}_1 + m}{2(p_1k_1)}\not{\epsilon}_1 + \not{\epsilon}_1 \frac{\not{p}_2 - \not{k}_2 + m}{2(p_1k_2)}\not{\epsilon}_2\right) u(p_2,s_2) \\
&= i\frac{e^2\hbar}{4}\overline{v}(p_1,s_1)\left(\not{\epsilon}_2 \frac{(\not{p}_2 + m) - (\not{p}_1 - m) - (\not{k}_1 - \not{k}2)}{(p_1k_1)}\not{\epsilon}_1 \right. \\
&\qquad \left. + \not{\epsilon}_1 \frac{(\not{p}_2 + m) - (\not{p}_1 - m) + (\not{k}_1 - \not{k}2)}{(p_1k_2)}\not{\epsilon}_2\right) u(p_2,s_2) \qquad (1)
\end{aligned}$$

2. We have $(\epsilon_{1,2} \cdot k_1 + k_2) = 0$, and $k_1 + k_2$ is at rest in the centre-of-mass frame.

3. In the static limit, we have $(p_1k_2) = (p_1k_2) = mk_{1,2}^0 = m^2$. Also, $\not{p}_{1,2}$ anticommute with $\not{\epsilon}_{1,2}$ in that limit. Therefore

$$\begin{aligned}
\mathfrak{M}(p_1,s_1,p_1,s_2) &= -\frac{ie^2\hbar}{4m^2}\overline{v}(p_1,s_1)\left(\not{\epsilon}_2(\not{k}_1 - \not{k}_2)\not{\epsilon}_1 - \not{\epsilon}_1(\not{k}_1 - \not{k}_2)\not{\epsilon}_2\right) u(p_1,s_2) \\
&= -\frac{e^2\hbar}{2m^2}\,\epsilon^\mu(\epsilon_1, k_1 - k_2, \epsilon_2)\; \overline{v}(p_1,s_1)\gamma^5\gamma^\mu u(p_1,s_2)
\end{aligned}$$

4. Since all three vectors ϵ_1, ϵ_2 and $k_1 - k_2$ lack a timelike component, their combination $\epsilon^\mu(\epsilon_1, k_1 - k_2, \epsilon_2)$ can *only* have a timelike component, so it must be proportional to p_1. In fact,

$$\epsilon^\mu(\epsilon_1, k_1 - k_2, \epsilon_2) = \frac{1}{m^2}\epsilon(p_1, \epsilon_1, k_1 - k_2, \epsilon_2)\, p_1^\mu$$

Therefore

$$\mathfrak{M}(p_1,s_1,p_1,s_2) = -\frac{e^2\hbar}{2m^3}\epsilon(p_1, \epsilon_1, k_1 - k_2, \epsilon_2)\; \overline{v}(p_1,s_1)\gamma^5 u(p_1,s_2)$$

(It may be interesting to note that both $\epsilon(p_1,\epsilon_1,k_1-k_2,\epsilon_2)$ and $\overline{v}(p_1,s_1)\gamma^5 u(p_1,s_2)$ are pseudoscalar objects, so the amplitude is still a Lorentz scalar.) From

$$\gamma^5 u(p,s)\overline{(\gamma^5 u(p,s))} = -\gamma^5(\not{p}+m)(1+\gamma^5\not{s})\gamma^5 = v(p,-s)\overline{v}(p,-s)$$

we see that

$$\overline{v}(p_1,s_1)\gamma^5 u(p_1,s_2) \sim \overline{v}(p_1,s_1)v(p_1,-s_2)$$

and therefore

$$\begin{aligned}\mathfrak{M}(p_1,s_1,p_1,s_1) &= 0\\ \mathfrak{M}(p_1,s_1,p_1,-s_1) &\doteq \frac{e^2\hbar}{m^2}\,\epsilon(p_1,\epsilon_1,k_1-k_2,\epsilon_2) \doteq 2e^2\hbar\,\sin\phi\end{aligned}$$

where ϕ is the angle between $\vec{\epsilon}_1$ and $\vec{\epsilon}_2$.

14 Exercises for chapter 14

Solution to exercise 168
If we take the initial state to be Dirac particles (and using $Q = Q = e$ for simplcity) we have

$$\mathfrak{M}_a = \frac{ie^2\hbar}{s}J_\mu\,\overline{u}(q_1)\gamma^\mu v(q_2)\quad,\quad J_\mu = \overline{v}(p_1,s_1)\gamma_\mu u(p_2,s_2)$$

where we indicate the spins of the initial fermions as well. Summing only over the final-state spins and performing the phase-space integration gives, with $P = p_1+p_2 = q_1+q_2$,

$$\begin{aligned}\int dV\,\Sigma\,|\mathfrak{M}_a|^2 &= \int d\Omega\,\frac{e^4\hbar^2}{s^2}J_\mu J_\nu^*\,\mathrm{Tr}\left((\not{q}_1+m)\gamma^\mu(\not{q}_2-m)\gamma^\nu\right)\frac{1}{32\pi^2}\beta\\ &= \frac{e^4\hbar^2}{12\pi s^2}J_\mu J_\nu^*\left(P^\mu P^\nu - s g^{\mu\nu}\right)\end{aligned}$$

The forward scattering amplitude is

$$\begin{aligned}\mathfrak{M}_f &= \frac{e^2\hbar^2}{s^2}\,\overline{v}(p_1,s_1)\gamma_\mu u(p_2,s_2)\,\mathcal{F}^{\mu\nu}(m^2,P)\,\overline{u}(p_2,s_2)\gamma_\nu v(p_1,s_1)\\ &= \frac{e^2\hbar^2}{s^2}\,J_\mu J_\nu^*\left(P^\mu P^\nu - s g^{\mu\nu}\right)\mathcal{K}(m^2,s)\end{aligned}$$

with the same conclusion $\Re\mathcal{K}(m^2,s) = -\alpha\beta(3-\beta^2)/(6\hbar)\theta(s>4m^2)$. Summing over the initial spins does not lead to an amplitude but to a *sum* of amplitudes

Solution to exercise 169
The change from x to y in the integral implies

$$\int_0^1 dx \;\to\; \frac{1}{2}\int_{-1}^1 dy \;=\; \int_0^1 dy$$

for any integrand symmetric in $x \leftrightarrow 1-x$. Then, assuming $\gamma = zi\beta$ with $z = \pm$,

$$\begin{aligned}
B &= \left[\left(\frac{3y}{2} - \frac{y^3}{2}\right)\log\left(\frac{s}{4}(\gamma^2+y^2)\right)\right]_{y=0}^{y=1} - \int_0^1 dy\,\frac{3y^2-y^4}{\gamma^2+y^2} \\
&= \log\left(\frac{s(1+\gamma^2)}{4}\right) - \int_0^1 dy\,\left(3+\gamma^2-y^2-\frac{\gamma^2(3+\gamma^2)}{\gamma^2+y^2}\right) \\
&= \log(m^2) - \frac{8}{3} - \gamma^2 + \gamma(3+\gamma^2)\arctan(1/\gamma) \\
&= \log(m^2) - \frac{8}{3} + \beta^2 + \frac{z\beta(3-\beta^2)}{2}\log\left(\frac{\beta+z}{\beta-z}\right)
\end{aligned}$$

and we see that the sign z drops out.

<u>Solution to exercise</u> 170
For $m = 0$, we have $\log(-sx(1-x)-i\eta) = \log(s)+\log(x(1-x)) - i\pi\theta(s>0)$. Then,

$$\int_0^1 dx\;6x(1-x) = 1 \quad , \quad \int_0^1 dx\;6x(1-x)\log(x(1-x)) = 12\int_0^1 dx(x-x^2)\log(x) = -\frac{5}{3}$$

For $s \to \infty$ we have $\beta \to 1$ so

$$\frac{\beta(3-\beta^2)}{2}\log\left(\frac{1+\beta}{1-\beta}\right) \approx \log\left(\frac{(1+\beta)^2}{1-\beta^2}\right) \approx \log\left(\frac{4}{1-\beta^2}\right) = \log\left(\frac{s}{m^2}\right)$$

<u>Solution to exercise</u> 171

1. The tricky bit is the integration along the cut. To the right, we have, with $z = |q| > 0$, $q = iz + \eta$ so $-q^2 = z^2 - i\eta$, and to the left we have $q = iz - eta$ hence $-q^2 = z^2 + i\eta$; that part of the integral is therefore

$$\frac{\alpha}{12i\pi^3 r}\int_{2m}^{\infty} d(iz)\,\frac{e^{-zr}}{iz}\left(F_{\Pi}(z^2+i\eta) - F_{\Pi}(z^2-i\eta)\right)$$

2. Expanding to the next order, the right-hand side of Eq.(14.38) becomes

$$\frac{3}{4}\frac{y^{1/2}}{m^{3/2}} - \frac{29}{32}\frac{y^{3/2}}{m^{5/2}}$$

and the Uehling correction then reads

$$\frac{1}{4\pi r} \quad\to\quad \frac{1}{4\pi r}\left(1 + e^{-2mr}\,\frac{\alpha}{4\sqrt{\pi}}\left(\frac{1}{(mr)^{3/2}} - \frac{29}{16(mr)^{5/2}}\right)\right)$$

3. The series expansion in y in Eq.(14.38) is convergent up to the point where $q = 0$ is reached for $y = -2m$. It therefore has a finite radius of convergence of $2m$. The y *integral*, however, extends to infinity. Compare this to the discussion in appendix 19.1.

4. If m_e is the classical electron mass (in kg) the Bohr radius is $r_B = \hbar/(\alpha m_e c)$ so in our convention it is $r_B = 1/(\alpha m)$. The exponent $\exp(-2mr_B)$ is then about $\exp(-274)$. To have a correction of 10^{-4} we would need $mr \approx 0.4$ hence a distance $r \approx 0.003\, r_B$. If we increase m the distances for a given correction become proportionally smaller; or, at a given r, their effect becomes exponentially suppressed.

Solution to exercise 172

1. For very small Γ we may approximate (see Eq.(7.10)):

$$\sigma_{\text{resonance}}(s) \approx 12\pi^2 \frac{\Gamma_\ell}{M}\, \delta(s - M^2)$$

so that

$$[\Re\Pi_h(s) - \Re\Pi_h(0)]_{\text{resonance}} = -\frac{s}{4\pi^2\alpha}\mathcal{P}\int_{4m_\pi^2}^{\infty} dz\, \frac{\sigma_{\text{resonance}}(z)}{z - s} \approx \frac{3}{\alpha}\,\frac{\Gamma_\ell}{M}\,\frac{s}{s - M^2}$$

If $s - M^2$ is of the order of $M\Gamma$ then the δ-approximation is no longer valid, and we get correction terms. To leading order in Γ:

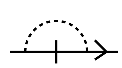
$$[\Re\Pi_h(s) - \Re\Pi_h(0)]_{\text{resonance}} \approx \frac{3}{\alpha}\,\frac{\Gamma_\ell}{M}\,\frac{s}{(s - M^2)^2 + M^2\Gamma^2}\left((s - M^2)\left(1 - \frac{\Gamma}{\pi M}\right) + \frac{M\Gamma}{\pi}\log\left(\frac{s}{M^2}\right)\right)$$

2.

$$[\Re\Pi_h(s) - \Re\Pi_h(0)]_{\text{plateau}} = -\frac{s}{4\pi^2\alpha}\mathcal{P}\int_{4m_\pi^2}^{\infty} dz\, \frac{\sigma_{\text{plateau}}(z)}{z - s}$$

$$= -\frac{\alpha}{3\pi}q^2 N_c \mathcal{P}\int_{4M^2}^{\infty} dz\, \frac{s}{z(z - s)} = \frac{\alpha}{3\pi}q^2 N_c \log\left|\frac{s - 4M^2}{4M^2}\right|$$

Solution to exercise 173
This is done by applying Eq.(13.11-15). It is important to realize that, for the $k^\mu n^\nu$ term in the photon propagator, the piece of diagram



is just a complicated way of writing $(-ie^2/\hbar)\not{n}$ so that it does not matter on which side the handlebar was actually performed.

Solution to exercise 174
The SDe for the fermion propagator can be written as

$$i\hbar\Pi(p) = \frac{i\hbar}{\not{p} - m} - i\hbar\frac{a(s)\not{p} + m_0 b(s)}{\not{p} - m}\Pi(p)$$

Multiplying with $\not{p} - m$ immediately gives the solution

$$\Pi(p) = \frac{1}{\not{p}(1 + a(s)) - m_0(1 - b(s))} = \frac{\not{p}(1 + a(s)) + m_0(1 - b(s))}{s(1 + a(s))^2 - m_0^2(1 - b(s))^2}$$

Solution to exercise 175
The denominator of Eq.(14.51), $D(s)$, vanishes (by construction!) at $s = m^2$. Its leading term is therefore $(s - m^2)D'(m^2)$, with

$$\begin{aligned} D'(m^2) &= \left[(1+a(s))^2 + 2s(1+a(s))a'(s) + 2m^2 b'(s)(1-b(s))\frac{(1-a(m^2))^2}{(1-b(m^2))^2}\right]_{s=m^2} \\ &= (1+a(m^2))\left((1+a(m^2)) + 2m^2 a'(m^2) + 2m^2 b'(m^2)\frac{(1+a(m^2))}{1-b(m^2)}\right) \end{aligned}$$

The numerator, evaluated at $s = m^2$, cancels the overall factor $(1 + a(m^2))$. The subtlety here is that of course $\not{p}$ *has* to change if we vary s; but such effects are necessarily proportional to $s - m^2$ and hence contribute only to the non-pole terms.

Solution to exercise 176
The denominator is

$$\mathcal{N} = \not{p}(1 + a(s)) - m_0(1 - b(s))$$

Formally putting $\not{p} = m$, *even though this is actually impossible*,[2] and requiring $\mathcal{N}$ to vanish, gives Eq.(14.50) correctly. We can take the derivative unambiguously if we adhere to $\partial/\partial\not{p} = 2\not{p}\partial/\partial s$:

$$\frac{\partial}{\partial\not{p}}\mathcal{N} = 1 + a(s) + 2sa'(s) + 2m_0\not{p}b'(s) = 1 + a(s) + 2sa's(s) + 2\not{p}m\frac{(1+a(s)}{1-b(s)}$$

so that we can write

$$\mathcal{N} \sim (\not{p} - m)\left(1 + a(m^2) + 2m^2 a'(m^2) + 2\not{p}mb'(m^2)\frac{1+a(m^2)}{1-b(m^2)}\right)$$

If, in the coefficient of $b'(m^2)$, we simply put $\not{p} \to m$ then we get Eq.(14.52), but if we first allow ourselves to multiply with the factor $(\not{p} - m)$ then we obtain the wrong sign.

Solution to exercise 177
In D dimensions, we must use $\gamma^\alpha\gamma_\alpha = D = 4 - 2\epsilon$. Hence

$$\begin{aligned} \gamma^\alpha\not{p}\gamma_\alpha &= 2\gamma^\alpha p_\alpha - \gamma^\alpha\gamma_\alpha\not{p} = (-2+2\epsilon)\not{p} \\ \gamma^\alpha\not{p}\not{q}\gamma_\alpha &= 2\not{q}\not{p} - (-2+2\epsilon)\not{p}\not{q} = 4(pq) - 2\epsilon\not{p}\not{q} \\ \gamma^\alpha\not{p}\not{q}\not{k}\gamma_\alpha &= 2\not{k}\not{p}\not{q} - (2(pq) - 2\epsilon\not{p}\not{q})\not{k} = 2\not{k}\not{p}\not{q} + 2\epsilon\not{p}\not{q}\not{k} - \not{k}(2(pq)) \\ &= 2\not{k}\not{p}\not{q} + 2\epsilon\not{p}\not{q}\not{k} - \not{k}(2\not{p}\not{q} + 2\not{q}\not{p}) = -2\not{k}\not{q}\not{p} + 2\epsilon\not{p}\not{q}\not{k} \end{aligned}$$

Solution to exercise 178
If we put $m = s = \lambda = 0$, the k integral reduces to $\int d^{4-2\epsilon}k(k^2)^{-2}$ which is strictly zero; no finite terms can survive. If we keep $\lambda \neq 0$ then

$$a(0) = \frac{\alpha}{4\pi}(4\pi\mu^2)^\epsilon\Gamma(\epsilon)(2-2\epsilon)\int_0^1 dx\ (1-x)\ \left(\lambda^2(1-x)\right)^{-\epsilon}$$

[2] Since $\mathrm{Tr}\,(\not{p}) = 0$ and $\mathrm{Tr}\,(m) = 4m$.

$$= \frac{\alpha}{4\pi}\left(\frac{4\pi\mu^2}{\lambda^2}\right)^{\epsilon}\Gamma(\epsilon)(2-2\epsilon)\int_0^1 dx\ (1-x)^{1-\epsilon}$$
$$= \frac{\alpha}{4\pi}\left(\frac{4\pi\mu^2}{\lambda^2}\right)^{\epsilon}\Gamma(\epsilon)\frac{2-2\epsilon}{2-\epsilon} \sim \frac{\alpha}{4\pi}\left(R_\epsilon - \log(\lambda^2) - \frac{1}{2}\right)$$

Solution to exercise 179

The phase space formula in its full form reads, including the correct compensating factors of μ:

$$V_{1\to 2} = \frac{(2\pi)^{4-2\epsilon}}{\left((2\pi)^{3-2\epsilon}\right)^2}\mu^{2\epsilon+2\epsilon-2\epsilon}\int d^{4-2\epsilon}q_1\,\delta(q_1^2-m_1^2)\,d^{4-2\epsilon}q_2\,\delta(q_2^2-m_2^2)\,\delta^{4-2\epsilon}(P-q_1-q_2)$$
$$= \frac{\mu^{2\epsilon}}{(4\pi^2)^{1-\epsilon}}\int d^{3-2\epsilon}\vec{q}_1\,\frac{1}{2q_1^0}\delta((P-q_1)^2-m_2^2)$$

With $t = |\vec{q}_1|^2$ the δ function can be rewritten as

$$\delta((P-q_1)^2-m_2^2) = \frac{1}{2\sqrt{s}}\delta\left(q_1^0 - \frac{s+m_1^2-m_2^2}{2\sqrt{s}}\right) = \frac{q_1^0}{\sqrt{s}}\delta\left(t - \frac{s\beta^2}{4}\right)$$

so that with the t-shell formula we have

$$V_{1\to 2} = \frac{\mu^{2\epsilon}\pi^{3/2-\epsilon}}{(4\pi^2)^{1-\epsilon}\Gamma(3/2-\epsilon)}\frac{1}{2\sqrt{s}}\int dt\ t^{1/2-\epsilon}\,\delta(t-s\beta^2/4)$$
$$= \frac{1}{8\pi}\left(\frac{4\pi\mu^2}{s}\right)^{\epsilon}\beta^{1-2\epsilon}\,\frac{\pi^{1/2}4^\epsilon}{2\Gamma(3/2-\epsilon)}$$

and finally, to first order in ϵ:

$$\frac{\pi^{1/2}4^\epsilon}{2\Gamma(3/2-\epsilon)} = \frac{\pi^{1/2}4^\epsilon}{2(1/2-\epsilon)\Gamma(1/2-\epsilon)} \approx \frac{4^\epsilon}{(1-2\epsilon)(1-\epsilon\psi(1/2))} \approx 1 - \epsilon\gamma_E + 2\epsilon$$

so that

$$V_{1\to 2} = \frac{s^{-\epsilon}\beta^{1-2\epsilon}}{8\pi}\left(1-\epsilon\gamma_E+\epsilon\log(4\pi\mu^2)+2\epsilon\right)$$

Solution to exercise 180

We have two relations:

$$v = \frac{k^0+k}{k^0-k}\ \ ,\ \ \lambda^2 = (k^0+k)(k^0-k)$$

so that

$$k^0 = \frac{\lambda}{2}(v^{1/2}+v^{-1/2})\ \ ,\ \ k = \frac{\lambda}{2}(v^{1/2}-v^{-1/2})$$

and we find

$$\frac{dk^0}{dv} = \frac{k}{2v}\ \ ,\ \ \frac{k}{(k^0)^2}\frac{dk^0}{dv} = \frac{(v-1)^2}{v(v+1)^2} = \frac{1}{v} - \frac{4}{(v+1)^2}$$

Solution to exercise 181

1. The boost does not affect the 2nd and 3rd components, and

$$x^0 \;\to\; \gamma x^0 + \beta\gamma x^1 = \gamma(1+\beta\cos\xi) \quad , \quad x^1 \;\to\; \beta\gamma x^0 + \gamma x^1 = \gamma(\beta+\cos\xi)$$

2. Note that $\cos\theta = \pm 1$ for $\cos\xi = \pm 1$, so $\theta \leftrightarrow \xi$ is indeed a bijection.

$$\gamma(1+\beta\cos\xi) = \gamma\left(1 + \frac{\beta\cos\theta - \beta^2}{1-\beta\cos\theta}\right) = \frac{\gamma(1-\beta^2)}{1-\beta\cos\theta} = \frac{1}{r(\theta)}$$
$$\gamma(\beta+\cos\xi) = \gamma\left(\beta + \frac{\cos\theta - \beta}{1-\beta\cos\theta}\right) = \frac{\gamma(1-\beta^2)\cos\theta}{1-\beta\cos\theta} = \frac{\cos\theta}{r(\theta)}$$

Solution to exercise 182
It is easiest to take the derivatives, and see that

$$\frac{d}{d\beta}\log\left(\frac{1+\beta}{1-\beta}\right) = \frac{2}{1-\beta^2}$$
$$\frac{d}{d\beta}\left(\frac{1}{4}\log\left(\frac{1+\beta}{1-\beta}\right)^2 + \mathrm{Li}_2\left(-\frac{2\beta}{1-\beta}\right)\right) = \frac{-1}{\beta(1-\beta^2)}\log\left(\frac{1+\beta}{1-\beta}\right)$$

Solution to exercise 183
The first step is

$$\frac{1}{s_1 s_{12}} + \frac{1}{s_2 s_{12}} = \frac{s_1+s_2}{s_1 s_2 s_{12}} = \frac{s_{12}}{s_1 s_2 s_{12}} = \frac{1}{s_1 s_2}$$

and the induction step relies on the fact that $s_1 s_2 \cdots s_{j-1} s_{j+1} \cdots s_n = (s_1 s_2 \cdots s_n)/s_j$ so that

$$\sum_{\mathcal{P}} \frac{1}{s_{\ell_1} s_{\ell_1\ell_2} \cdots s_{\ell_1\cdots\ell_n}} = \sum_{j=1}^{n} \frac{s_{\ell_j}}{s_{\ell_1} s_{\ell_2} \cdots s_{\ell_n} s_{\ell_1\cdots\ell_n}} = \frac{1}{s_{\ell_1}\cdots s_{\ell_n}}$$

Solution to exercise 184
We can write

$$\frac{1}{B} = \frac{2}{s\beta}\left(\frac{1}{z-\beta} - \frac{1}{z+\beta}\right)$$

leading to

$$\int_0^1 dz \frac{1}{B} = \frac{2}{s\beta}\left(\log\left(\frac{1-\beta}{-\beta}\right) - \log\left(\frac{1+\beta}{\beta}\right)\right)$$

For $s > 4m^2$, $\beta = |\beta| + i\eta$ so that $\log(-\beta) - \log(\beta) = -i\pi$. The other integral is most easily done as follows:

$$\begin{aligned}
\frac{\beta s}{2}\int_0^1 dz \frac{\log B}{B} &= \int_0^1 dz \left(\frac{1}{z-\beta} - \frac{1}{z+\beta}\right)\left(\log\left(\frac{s}{4}\right) + \log(z-\beta) + \log(z+\beta)\right) \\
&= \log(s/4)(-L+i\pi) + a_1 + a_2 + a_3 + a_4 \\
a_1 &= \int_0^1 dz \frac{\log(z-\beta)}{z-\beta} = \frac{1}{2}\left(\log(1-\beta)^2 - \log(-\beta)^2\right)
\end{aligned}$$

$$
\begin{aligned}
a_2 &= \int_0^1 dz \frac{-\log(z+\beta)}{z+\beta} = \frac{-1}{2}\left(\log(1+\beta)^2 - \log(\beta)^2\right) \\
a_3 &= \int_0^1 dz \frac{\log(z+\beta)}{z-\beta} = \int_{-\beta}^{1-\beta} \frac{dy}{y}\left(\log(2\beta) + \log\left(1+\frac{y}{2\beta}\right)\right) \\
&= \log(2\beta)\log\left(\frac{1-\beta}{-\beta}\right) - \mathrm{Li}_2\left(\frac{\beta-1}{2\beta}\right) + \mathrm{Li}_2(1/2) \\
a_4 &= \int_0^1 dz \frac{-\log(z-\beta)}{z+\beta} = -\log(1+\beta)\log(1-\beta) + \log(\beta)\log(-\beta) + a_3
\end{aligned}
$$

and then we use $\log(-\beta) = \log(\beta) - i\pi$ and

$$
\mathrm{Li}_2\left(\frac{\beta-1}{2\beta}\right) = -\frac{1}{2}\log\left(\frac{1+\beta}{2\beta}\right)^2 - \mathrm{Li}_2\left(\frac{1-\beta}{1+\beta}\right)
$$

which is the last identity of Eq.(19.441) with $z = (1+\beta)/(2\beta)$.

<u>Solution to exercise</u> 185

1. We take q the incoming fermion, and p the outgoing fermion. The diagram to be computed is the expression $ieV^\mu/\hbar$ where

$$
\begin{aligned}
V^\mu &= -i\frac{e^2\hbar}{(2\pi)^4}\int d^4k \frac{T^\mu}{D} \\
D &= (k^2+i\eta)((k+p)^2+i\eta)((k+q)^2+i\eta) \\
T^\mu &= \gamma^\alpha(\not{p}+\not{k})\gamma^\mu(\not{q}+\not{k})\gamma_\alpha
\end{aligned} \tag{2}
$$

   and T^μ is sandwidched between $\overline{u}(q)$ and $u(p)$.

2. The Feynman trick:

$$
\frac{1}{D} = 2\int \delta_{xy}\ (k^2 + 2x(qk) + 2y(pk) + i\eta)^{-3} = 2\int \delta_{xy}\ (\ell^2 - sxy + i\eta)^{-3}
$$

   where $s = 2(pq)$ and we must shift $k \to \ell = k - xq - yp$.

3. In $4-2\epsilon$ dimensions, shifting the integration momentum, and using the Dirac equations on both sides:

$$
\begin{aligned}
T^\mu &= -2((\not{q}+\not{k})\gamma^\mu(\not{p}+\not{k}) + 2\epsilon(\not{p}+\not{k})\gamma^\mu(\not{q}+\not{k}) \\
&= -2(\not{\ell}+(1-x)\not{q})\gamma^\mu(\not{\ell}+(1-y)\not{p}) + 2\epsilon(\not{\ell}-x\not{q})\gamma^\mu(\not{\ell}-y\not{p}) \\
&\sim (-2+2\epsilon)\not{\ell}\gamma^\mu\not{\ell} - 2\not{q}\gamma^\mu\not{p}((1-x)(1-y)+\epsilon xy)
\end{aligned}
$$

   since the terms linear in ℓ integrate to zero. Further,

$$
\not{\ell}\gamma^\mu\not{\ell} \sim \frac{-2+2\epsilon}{4-2\epsilon}\ell^2\gamma^\mu \quad , \quad \not{q}\gamma^\mu\not{p} \sim -s\gamma^\mu
$$

4. We now have

$$V^\mu = -i\frac{4\pi\alpha\,\mu^{2\epsilon}}{(4\pi^2)^{2-\epsilon}}2\int\delta_{xy}\int d^{4-2\epsilon}\ell\,\frac{\gamma^\mu}{(\ell^2-sxy+i\eta)^3}\gamma^\mu \times\left(\frac{(2-2\epsilon)^2}{4-2\epsilon}\ell^2+s(1-x-y+(1-\epsilon)xy)\right)$$

After the Wick rotation and using the t-shell formula:

$$V^\mu = \frac{\alpha(4\pi\mu^2)^\epsilon}{2\pi\Gamma(2-\epsilon)}\int\delta_{xy}\int\limits_0^\infty\frac{dt\;t^{1-\epsilon}}{(t+sxy-i\eta)^3} \times\left(\frac{(2-2\epsilon)^2}{4-2\epsilon}t-s(1-x-y+(1-\epsilon)xy)\right)\gamma^\mu$$

$$\int\limits_0^\infty dt\,\frac{t^{2-\epsilon}}{(t+sxy-i\eta)^3} = (sxy-i\eta)^{-\epsilon}\frac{2-\epsilon}{2}\Gamma(\epsilon)\Gamma(2-\epsilon)$$

$$\int\limits_0^\infty dt\,\frac{t^{1-\epsilon}}{(t+sxy-i\eta)^3} = (sxy-i\eta)^{-1-\epsilon}\Gamma(1+\epsilon)\Gamma(2-\epsilon)/2$$

5. We can therefore write

$$V^\mu = \frac{\alpha}{2\pi}\left(\frac{4\pi\mu^2}{s}\right)^\epsilon\gamma^\mu\left((1-\epsilon)^2\Gamma(\epsilon)b_0-\Gamma(1+\epsilon)(b_1-2b_2+(1-\epsilon)b_0)\right)$$

$$b_0 = \int\delta_{xy}\,(xy)^{-\epsilon} = \int\limits_0^1 dx\,\frac{x^{-\epsilon}(1-x)^{1-\epsilon}}{1-\epsilon} = \frac{\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{(1-\epsilon)\Gamma(3-2\epsilon)} = \frac{\Gamma(1-\epsilon)^2}{\Gamma(2-2\epsilon)}\left(\frac{1}{2(1-\epsilon)}\right)$$

$$b_1 = \int\delta_{xy}\,(xy)^{-1-\epsilon} = \int\limits_0^1 dx\,\frac{x^{-1-\epsilon}(1-x)^{-\epsilon}}{-\epsilon} = \frac{\Gamma(-\epsilon)\Gamma(1-\epsilon)}{-\epsilon\Gamma(1-2\epsilon)} = \frac{\Gamma(1-\epsilon)^2}{\Gamma(2-2\epsilon)}\left(\frac{1-2\epsilon}{\epsilon^2}\right)$$

$$b_2 = \int\delta_{xy}x(xy)^{-1-\epsilon} = \int\limits_0^1 dx\,\frac{x^{-\epsilon}(1-x)^{-\epsilon}}{-\epsilon} = \frac{\Gamma(1-\epsilon)^2}{\Gamma(2-2\epsilon)}\left(\frac{-1}{\epsilon}\right)$$

and inserting all this gives Eq.(14.128).

Solution to exercise 186

As in the previous exercise we replace the outgoing antifermion with momentum q by an incoming ferimon with momentum q. This amounts to replacing q by $-q$ in our treatment of section 14.4, and we must also realize that now $B=(xp+(1-x)q)^2$.

After we have split off the $[\gamma^\mu, \not{p} - \not{q}]$ term in T_1^μ, we can let $q \to p$ for the static limit, as long as we work to first order in $p - q$. That means that $B = m^2$ so that the integral over x (or z) becomes trivial, and we only have to integrate over y. We therefore find

$$
\begin{aligned}
V^\mu &= \frac{\alpha}{4\pi}\left(V_0\gamma^\mu + V_1\gamma^\mu + V_2\gamma^\mu + [\gamma^\mu, \not{p} - \not{q}]V_3\right) \\
V_0 &= R_\epsilon \int_0^1 dy\; 2y(1-\epsilon)^2(y^2m^2)^{-\epsilon} = R_\epsilon - \log(m^2) - 1 \\
(V_1)_0 &= R_\epsilon \epsilon \int_0^1 dy\; 2y(-2m^2)(y^2m^2)^{-1-\epsilon} = 2(R_\epsilon - \log(m^2)) \\
(V_1)_1 &= \int_0^1 dy\; 2y \frac{-2m^2}{m^2y^2 + \lambda^2} = -2\log(m^2/\lambda^2) \\
V_2 &= \int_0^1 dy\; 2y(4m^2y + m^2y^2)({}^2m^2)^{-1} = 5 \\
V_3 &= \int_0^1 dy\; 2y\, \frac{m}{2}(y^2 - y)(m^2y^2)^{-1} = -\frac{1}{2m}
\end{aligned}
$$

Note that, compared to Eq.(13.6.3), p and q have been interchanged! This explains the sign of the coefficient in Eq.(14.130).

<u>Solution to exercise</u> 187

1. Eq.(14.156) can be written as the simple linear relation

$$d\left(\frac{1}{\alpha}\right) = -\frac{n_f}{3\pi} d\log(M^2)$$

   and this can be integrated trivially.

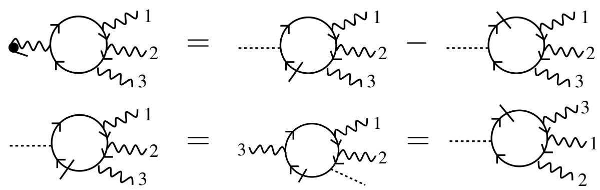
2. $3 \cdot (-1)^2 \;+\; 3 \cdot 3 \cdot (2/3)^2 \;+\; 3 \cdot 3 \cdot (-1/3)^2 = 8$

3. At the Landau pole we have $1/\alpha(M^2) = 0$ and therefore

$$M = M_0\, \exp\left(\frac{3\pi}{2n_f\alpha(M_0^2)}\right)$$

   For the numbers given this evaluates to about 2.5×10^{33} GeV, very far above the Planck scale. QED is certainly not valid at such high scales, and possibly QFT itself has to be replaced by something else, *at least* involving quantum gravity.

Solution to exercise 188
Numbering the photons to keep them apart, and concentrating on one diagram:



(3)

So if we include all permutations of $(1,2,3)$ everything cancels

Solution to exercise 189
Contracting with one and two metric tensors gives

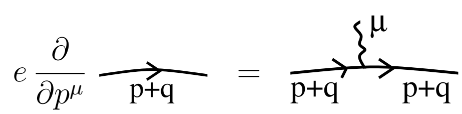
$$g^{\mu\nu} g_{\mu\nu} = g^{\mu}_{\mu} = 4$$
$$\mathfrak{X}\, g_{\mu\nu} g_{P\rho\sigma} = g^{\mu}_{\mu}\, g^{\rho}_{\rho} + g^{\mu\rho} g_{\mu\rho} + g^{\mu}_{\rho} g^{\rho}_{\mu} = 4 \cdot 4 + 4 + 4 = 24$$

Solution to exercise 190
Compared with exercise 189 we have to replace

$$4 \;\rightarrow\; (4-2\epsilon)$$
$$24 \;\rightarrow\; (4-2\epsilon)^2 + 2(4-2\epsilon) = (16 - 16\epsilon + 4\epsilon^2) + (8 - 4\epsilon) = 24\left(1 - \frac{5\epsilon}{6}\right) + \mathcal{O}\left(\epsilon^2\right)$$

Solution to exercise 191

1. Diagrammatically, the Ward derivative of exercise 117 can be written as



   Therefore, attaching a zero-momentum external photon to a fermion loop with loop momentum p amounts to taking the derivative of the loop integrand to the loop momentum.

2. The loop integral over loop momentum p has the form

$$\int d^4p \, \frac{\partial}{\partial p_{\mu}} F(p^0, p^1, \ldots, dp^D) =$$
$$\int dp^0\, dp^1 \cdots dp^{\mu-1}\, dp^{\mu+1} \cdots dp^D \left[F(p^0, \ldots, p^{\mu-1}, X, p^{\mu+1}, \ldots, p^D)\right]_{X=-\infty}^{X=+\infty}$$

   since for a nice theory like QED the loop integrand vanishes at infinity, this gives zero.

15 Exercises for chapter 15

Solution to exercise 192
These things are most easily done by concentrating on the coefficients of $g_{\mu\nu}$, say, and requesting a zero result for aribtrary momenta. If one γ^5 is present we find not $a_1 = a_3 = a_5 = -a_2 = -a_4 = -a_6$ but $a_1 = a_2 = a_3 = a_4 = a_5 = a_6$, and then the result follows from momentum conservation $q_1 + q_2 + q_3 = 0$.

Solution to exercise 193
Algebraically, the diagram of Eq.(15.32) reads

$$
\begin{aligned}
&\left(-q^2 g^{\mu\alpha} + q^\mu q^\alpha\right) \frac{i\hbar}{q^2} \left(-g^{\alpha\nu} + \frac{q^\alpha n^\nu + n^\alpha q^\nu}{(qn}\right) \delta^{jk} \\
&= \frac{i\hbar\delta^{jk}}{q^2} \left(-q^2 g^{\mu\alpha} + q^\mu q^\alpha\right) \left(-g^{\alpha\nu} + \frac{n^\alpha q^\nu}{(qn)}\right) \\
&= \frac{i\hbar\delta^{jk}}{q^2} \left(q^2 g^{\mu\nu} - q^\mu q^\nu - q^2 \frac{n^\mu q^\nu}{(qn)} + \frac{q^\mu (qn) q^\nu}{(qn)}\right) \\
&= i\hbar \left(g^{\mu\nu} - \frac{n^\mu q^\nu}{(qn)}\right) \delta^{jk}
\end{aligned}
$$

Solution to exercise 194
For the $(+,-)$ case we have

$$
\begin{aligned}
\omega_- \not{\epsilon}_1(+) &= \frac{\sqrt{2}}{s_-(q_1,p_2)} u_-(q_1)\overline{u}_-(p_2) \quad , \quad \omega_- \not{\epsilon}_2(-) = \frac{\sqrt{2}}{s_+(q_2,p_1)} u_-(p_1)\overline{u}_-(q_2) \\
\epsilon_1(+)\cdot\epsilon_2(-) &= \frac{s_+(q_1,p_1)s_-(q_2,p_2)}{s_-(q_1,p_2)s_+(q_2,p_1)} \\
A_1(+,-) &= 2\frac{\overline{u}_+(p_1)u_-(q_1)\overline{u}_-(p_2)(\not{q}_1 - \not{p}_1)u_-(p_1)\overline{u}_-(q_2)u_+(p_2)}{s_-(q_1,p_2)s_+(q_2,p_1)} \\
&= 2\frac{s_+(p_1,q_1)s_-(p_2,q_1)s_+(q_1,p_1)s_-(q_2,p_2)}{s_-(q_1,p_2)s_+(q_2,p_1)} \\
&= 2s_+(p_1,q_1)^2 \frac{s_-(q_2,p_2)}{s_+(q_2,p_1)} \quad = \quad -2s_+(p_1,q_1)s_-(q_1,p_2)(\epsilon_1(+)\cdot\epsilon_2(-)) \\
&= -\overline{u}_+(p_1)(\not{q}_1 - \not{q}_2)u_+(p_2)(\epsilon_1(+)\cdot\epsilon_2(-)) = -A_3(+,-)
\end{aligned}
$$

The case $(-,+)$ is obtained by interchanging the two gluons so that $A_2(-,+) = A_1(+,-)\rfloor_{q_1\leftrightarrow q_2}$ and A_3 picks up a relative minus sign, leading to $A_3(-,+) = A_2(-,+)$.

Solution to exercise 195

$$
\begin{aligned}
\mathrm{Tr}\left(T^j T^k T^k T^j\right) &= \mathrm{Tr}\left(T^j T^k T^k T^j 1\right) = \frac{1}{2}\left(\mathrm{Tr}\left(T^k T^k\right)\mathrm{Tr}\left(1\right) - \frac{1}{N}\mathrm{Tr}\left(T^k T^k 1\right)\right) \\
&= \frac{1}{2}\left(N - \frac{1}{N}\right)\mathrm{Tr}\left(T^k T^k\right) = \frac{N^2-1}{2N}\frac{N^2-1}{2} \\
\mathrm{Tr}\left(T^j T^k T^j T^k\right) &= \frac{1}{2}\left(\mathrm{Tr}\left(T^k\right)^2 - \frac{1}{N}\mathrm{Tr}\left(T^k T^k\right)\right) \\
&= \frac{-1}{2N}\mathrm{Tr}\left(T^k T^k\right) = -\frac{1}{2N}\frac{(N^2-1)}{2}
\end{aligned}
$$

Solution to exercise 196
This frelies on the cyclic property of traces.

$$\begin{aligned}
\mathrm{Tr}\left([A,B]C\right) &= \mathrm{Tr}\left(ABC - BAC\right) = \mathrm{Tr}\left(ABC - ACB\right) = \mathrm{Tr}\left(A[B,C]\right)\\
[jkmn] &= h^{jk\ell}h^{mn\ell} = 4\mathrm{Tr}\left(T^\ell[T^j,T^k]\right)\mathrm{Tr}\left(T^\ell[T^m,T^n]\right)\\
&= 2\mathrm{Tr}\left([T^j,T^k][T^m,T^n]\right) = 2\mathrm{Tr}\left(T^j[T^k,[T^m,T^n]]\right)
\end{aligned}$$

Here we have used $B = T^k$ and $C = [T^m, T^n]$.

Solution to exercise 197
This goes the same way as exercise 196, now with $C = [T^\ell, [T^m, T^n]]$:

$$\begin{aligned}
[jk\ell mn] &= h^{jkp}[p\ell mn] = 4\mathrm{Tr}\left(T^p[T^j,T^k]\right)\mathrm{Tr}\left(T^p[T^\ell,[T^m,T^n]]\right)\\
&= 2\mathrm{Tr}\left([T^j,T^k][T^\ell,[T^m,T^n]]\right) = 2\mathrm{Tr}\left(T^j[T^k,[T^\ell,[T^m,T^n]]]\right)
\end{aligned}$$

Solution to exercise 198
The most straightforward way to prove this (preferably using computer algebra) is simply to use the fully expanded form

$$\begin{aligned}
[jklmn] &= r(j,k,l,m,n) - r(j,k,l,n,m) - r(j,k,m,n,l) + r(j,k,n,m,l)\\
&\quad - r(j,l,m,n,k) + r(j,l,n,m,k) + r(j,m,n,l,k) - r(j,n,m,l,k)
\end{aligned}$$

where $r(a,b,c,d,e) = \mathrm{Tr}\left(T^aT^bT^cT^dT^e\right)$, and write out all terms; then, to cyclically shift the arguments of the traces so that T^j, say, occurs in the first position. All terms will cancel precisely.

Solution to exercise 199

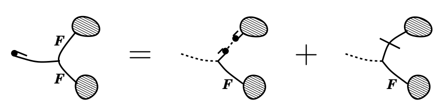
1. The single-gluon case gives zero because that is simply the transversality of the gluon polarization. By Eq.(15.40), for the two-gluon case the single diagram diagram is proportional to

$$\begin{aligned}
&Y(q_1,\epsilon_1;q_2,\epsilon_2;-q_1-q_2,-q_1-q_2) =\\
&\quad (q_1\epsilon_1)(q_1\epsilon_2) - (q_2\epsilon_1)(q_2\epsilon_2) - q_1^2(\epsilon_1\epsilon_2) + q_2^2(\epsilon_1\epsilon_2) = 0
\end{aligned}$$

   because of transversality and masslesness, see Eq.(15.39).

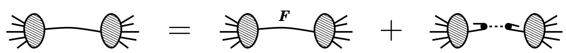
2. This is precisely the argument of section 15.4.2. The qq parts of the Δ's drop out because of the previous result.

3. By Eq.(15.42) we can write for tree amplitudes:



   We have to sum over all possible ways to assign the outgoing gluons to the blobs. By the induction assumption, the first diagram on the right-hand side vanishes. The rest of the argument is precisely that leading to Eq.(15.96) except that the very last diagram is now absent.

4. For tree diagrams, an internal propagator is specified *completely* by the gluon momenta occurring on one of its endpoints, leading to a split-up of the set of all diagrams that contain that propagator. We can therefore write



and the last diagram vanishes. Since we can pick any internal propagator the argument holds generally.

5. As Eq.(15.96) shows, the inclusion of quarks is totally straightforward.

Solution to exercise 200

If $N = 4$ then 'protons', 'neutrons', and generally 'baryons' would consist of 4 quarks rather than 3.[3] They would therefore be bosons rather than fermions, with no Pauli exclusion principle. Nuclear matter would tend to consist of arbitrarily large numbers of nucleons all in the ground state, that is big enormous nuclei of many stellar masses. These would tend to collapse into black holes, leaving the observable universe empty except for leptons.

Solution to exercise 201

1. Since the gluons are massless, E is the *only* energy scale. Therefore amplitudes either go as E^{4-n} or the theory is unacceptable.

2. The maximum number of propagators occurs in those diagrams that contain *only* 3-point vertices. The diagrammatic sum rules for such connected tree diagrams read

$$n + 2I = 3V_3 \quad , \quad V_3 = I + 1 \quad \Rightarrow \quad I = n - 3$$

3. For $n - 3$ propagators we have a total denominator $(E^2)^{n-3}$; contracting two 3-point vertices into a 4-point one can only lower this power.

4. We have to contract all Lorentz indices, so that $(p \cdot p)$, $(p \cdot \epsilon)$, and $(\epsilon \cdot \epsilon)$ are the only possible scalar factors in any numerator.

5. Every diagram must be $\sim E^{4-n}$, coming from powers of p in the numerator and powers of E in the denominators. Since the denominators go at most as E^{2n-6} there can be at most $n - 2$ powers of p in the numerator. Note the important fact that, since E is the only energy scale, it is not possible for contributions $\mathcal{O}\left(E^{2-n}\right)$ to cancel 'partially' down to $\mathcal{O}\left(E^{4-n}\right)$.

6. If both gluons have positive helicity:

$$(\epsilon_j \cdot \epsilon_k) \sim \overline{u}_+(q_j)\gamma_\mu u_+(r_j)\,\overline{u}_+(q_k)\gamma^\mu u_+(r_k) \sim s_+(q_j, q_k)s_-(r_k, r_j)$$

If gluon j has positive helicity and gluon k has negative helicity:

$$(\epsilon_j \cdot \epsilon_k) \sim \overline{u}_+(q_j)\gamma_\mu u_+(r_j)\,\overline{u}_-(q_k)\gamma^\mu u_(r_k) \sim s_+(q_j, r_k)s_-(q_k, r_j)$$

[3] As section 17.7 shows, the 'up' quarks would have charge 3/4, and the 'down' quarks charge -1/4, leading to 'nucleons' of charge 3,2,1,0, or -1.

7. If all gluons have the same helicity, we choose $r_j = r_1$ for all j, so that $(\epsilon_j \cdot \epsilon_k)$ always vanishes.

8. If gluon 1 has helicity opposite from gluon $2, 3, \ldots, n$ we choose $r_1 = q_2$ and $r_{2,3,\ldots,n} = q_1$. Then, again all $(\epsilon \cdot \epsilon) = 0$.

9. If gluon 1 and 2 have helicity opposite from gluons $3, 4, \ldots, n$ then we can ensure $(\epsilon_1 \cdot \epsilon_2) = 0$ by giving them the same $r_1 = r_2$, and also $(\epsilon_k \cdot \epsilon_k) = 0$ by having $r_3 = \cdots = r_n$. Then we can choose, for instance, $r_3 = q_1$ to have $(\epsilon_1 \cdot \epsilon_{3,\ldots,n}) = 0$ but $(\epsilon_2 \cdot \epsilon_{3,\ldots,n})$ will no longer vanish: at most we can choose $r_{1,2} = q_3$ or so to make also $(\epsilon_2 \cdot \epsilon_3) = 0$.

Solution to exercise 202

1. These two forms are the only ones possible if we want to end up with a single Lorentz index μ.

2. We are allowed to use the Feynman gauge for the gluon propagators. For $n = 2$ we have one 3-point vertex and one propagator, giving p^{-1}. We can add an additional gluon by either attaching it with a new 3-point vertex, giving an extra p in the numerator and an extra propagator p^{-2}, or turning a 3-gluon vertex into a 4-gluon one, thereby removing one p from the numerator.

3. The same argument as above, where the most propagators come from diagrams that contain only 3-point vertices.

4. The two relations follow from power counting of the p's and the ϵ's, respectively. We therefore have $\alpha = 2\gamma - n \leq 2(n-1) - n = n - 2$, and $2\beta = n - \alpha \geq n - (n-2) = 2$.

5. As in exercise 201, we can give all gluons the same gauge vector.

6. Again, power counting. We now have $\alpha \leq n-1$ and $2\beta \geq (n-1)-(n-1) = 0$. Obviously, $\beta \geq 0$ anyway since there is no way to arrive at *negative* powers of the polarization vectors! But we see that $\beta = 0$ is *possible*.

7. If $\beta \geq 1$ then we have at least one factor $(\epsilon \cdot \epsilon)$ which can be made to vanish. Therefore we must have $\beta = 0$, $\alpha = \gamma = n - 1$.

8. The only diagrams left consist of a $q\bar{q}$ line with $\not{\epsilon}$ insertions, just like QED, possibly multiplied by their scalar factors $(\epsilon \cdot p)^\alpha (p \cdot p)^{-\gamma}$. We can therefore always choose the gluons to all have the q or the $\bar{q}$ momentum as their gauge vector, depending on the helicity case, and the amplitude vanishes just as in the QED case $e^+e^- \to \gamma\gamma\cdots\gamma$.

16 Exercises for chapter 16

Solution to exercise 203
For the handlebar on gluon 1 we have

$$Y(-p, \nu; q_1, q_1; q_2, \beta) = \Delta(q_2)^{\beta\nu} - \Delta(p)^{\beta\nu}$$

The second term, multiplied with (the numerator of) the axial-gauge propagator, gives

$$\left(-g^\mu_\nu + \frac{p^\mu n_\nu}{(pn} + \frac{n^\mu p_\nu}{(pn)}\right) \Delta^{\nu\beta}(p) = -p^\mu p^\beta + g^{\mu\nu} p^2 + p^\mu \, \frac{n_\nu \Delta^{\nu\beta}(p)}{(pn)}$$

and every term containing p^μ drops out when contracted with the fermion line. Therefore the Feynman-'t Hooft and axial gauge give the same result (in this case!).

Solution to exercise 204
Performing partial integration twice, and dropping the total derivatives:

$$\begin{aligned}(\partial_\mu V_\nu)(\partial^\nu V^\mu) &= \partial_\mu(V_\nu \partial^\nu V^\mu) - V_\nu(\partial_\mu \partial^\nu V^\mu) \\ &= -\partial^\nu(V_\nu \partial_\mu V^\mu) + (\partial^\nu V_\nu)(\partial_\mu V^\mu) = (\partial \cdot V)^2\end{aligned}$$

In terms of explicit Feynman rules, such partial-integration results are the effect of momentum conservation, since $(k_1 \cdot V_1)(k_2 \cdot V_2) = (k_1 \cdot V_2)(k_2 \cdot V_1)$ if $k_1 + k_2 = 0$.

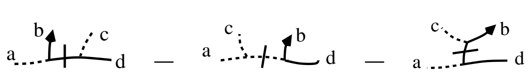
Solution to exercise 205

$$\frac{n^\mu n_\alpha}{\lambda} R^{\alpha\nu} = -\frac{n^\mu q^\nu}{(qn)} \quad , \quad -q^\mu q_\alpha R^{\alpha\nu} = -\frac{q^\mu n^\nu}{(qn)} + \frac{(n^2 + \lambda q^2)}{(qn)^2} q^\mu q^\nu$$

$$q^2 R^{\mu\nu} = -g^{\mu\nu} + \frac{q^\mu n^\nu + n^\mu q^\nu}{(qn)} - \frac{(n^2 + \lambda q^2)}{(qn)^2} q^\mu q^\nu$$

Solution to exercise 206
This is straightforward. The loss of the factor 1/2 in the third line comes about since Eq.(16.28) tells us that the loop has to be manipulated both in the clockwise and in the anticlockwise direction.

Solution to exercise 207

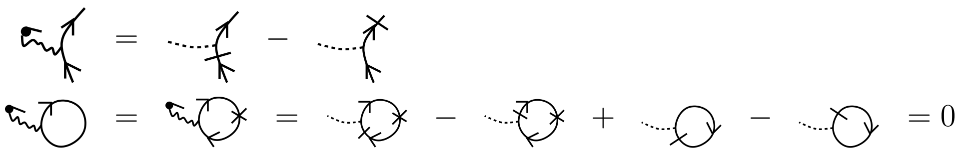


$$\begin{aligned} &\sim \; h^{abn}h^{cnd} - h^{anc}h^{nbd} - h^{cbn}h^{and} \\ &= \; -h^{nba}h^{cnd} - h^{nac}h^{bnd} - h^{ncb}h^{and} = 0 \end{aligned}$$

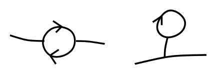
since now we have a cyclic permutation of (c, b, a).

<u>Solution to exercise</u> 208

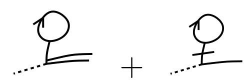
The vanising of the the handlebarred fermionic tadpole followos immediately from Eq.(13.11). If we let a cross vertex stand for the inverse fermion propagator:



Alternatively you may argue that this handlebar amounts to multiplication by zero, but that spoils the diagrammatic fun. The gluonic self-energy due to quarks has two diagrams;



Under the handlebar the first one vanishes *exactly* as in QED, and we are left with



So the gluon self-energy is transverse if these tadpoles vanish (they do).

<u>Solution to exercise</u> 209

$$
\begin{aligned}
h^{jkm}h^{kmn} &= h^{kmj}h^{kmn} = 4\,\mathrm{Tr}\left(T^k[T^m,T^j]\right)\mathrm{Tr}\left(T^k[T^m,T^n]\right)\\
&= 2\left(\mathrm{Tr}\left([T^m,T^j][T^m,T^n]\right) - \frac{1}{N}\mathrm{Tr}\left([T^m,T^j]\right)\mathrm{Tr}\left([T^m,T^n]\right)\right)\\
&= 4\,\mathrm{Tr}\left(T^mT^jT^mT^n - T^mT^jT^nT^m1\right)\\
&= 2\left(\left(\mathrm{Tr}\left(T^j\right)\mathrm{Tr}\left(T^n\right) - \frac{1}{N}\mathrm{Tr}\left(T^jT^n\right)\right)\right.\\
&\quad \left. - \left(\mathrm{Tr}\left(T^jT^n\right)\mathrm{Tr}\left(1\right) - \frac{1}{N}\mathrm{Tr}\left(T^jT^n\right)\right)\right)\\
&= -2N\,\mathrm{Tr}\left(T^jT^n\right) = -N\,\delta^{jn}
\end{aligned}
$$

<u>Solution to exercise</u> 210

Since we are aiming for a zero result we can drop overall factors along the way. With $p^2 = s$:

$$
\begin{aligned}
&\int \frac{d^Dq}{q^2+i\eta} = \int d^Dq\,\frac{(q-p)^2}{(q^2+i\eta)((q-p)^2+i\eta)}\\
&= \int_0^1 dx \int d^Dq \frac{(q-p)^2}{((q-xp)^2+x(1-x)s+i\eta)^2}\\
&\sim \int_0^1 dx \int_0^\infty dt\, t^{1-\epsilon}\,\frac{-t+s(1-x)^2}{(t-x(1-x)s-i\eta)^2}
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^1 dx \Big(-\Gamma(\epsilon-1)\Gamma(3-\epsilon)(-sx(1-x))^{1-\epsilon} \\
&\qquad + \Gamma(\epsilon)\Gamma(2-\epsilon)s(1-x)^2(-sx(1-x))^{-\epsilon}\Big) \\
&\sim \int_0^1 dx \Big((2-\epsilon)x^{1-\epsilon}(1-x)^{1-\epsilon} + (\epsilon-1)x^{-\epsilon}(1-x)^{2-\epsilon}\Big) \\
&= (2-\epsilon)\frac{\Gamma(2-\epsilon)^2}{\Gamma(4-2\epsilon)} + (\epsilon-1)\frac{\Gamma(1-\epsilon)\Gamma(3-\epsilon)}{\Gamma(4-2\epsilon)} \\
&= \frac{\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\Gamma(4-2\epsilon)}\Big((2-\epsilon)(1-\epsilon) + (\epsilon-1)(2-\epsilon)\Big) \quad = 0
\end{aligned}
$$

Solution to exercise 211

For the gluonic self energy, only the first two diagrams occur on the right-hand side of Eq.(16.50). Let the total gluon momentum be q^μ. Iterating the first diagram with the SDe for the ghost, obtain two powers of q, and together with the second diagram this gives precisely $B_\nu\,\Delta^{\nu\mu}(q)$, where B_ν is whatever the result of the loop integrals in the blob is.[4]In a covariant gauge, we must have $B_\nu \propto q_\nu$ since it is the only available vector, and therefore $B_\nu \Delta^{\mu\nu}(q) = 0$.

Solution to exercise 212

We can use Eq.(14.60), the mass renormalization of QED. There we found, to one loop,

$$m = m_0(1-a-b) \sim m_0\left(1 + \frac{3\alpha}{4\pi}(R_\epsilon - \log(M^2)\right)$$

where we introduced the renormalization scale M. Thus

$$M^2\frac{\partial}{\partial M^2} m \sim -\frac{3\alpha}{4\pi} m$$

The only difference for QCD is in the colour factor. Keeping the quark colours explicitly, the colour factor reads

$$(T^jT^j)^a_b = (T^j)^a_c (T^j)^c_b = \frac{1}{2}\left(\delta^a_b\,\delta^c_c - \frac{1}{N}\delta^a_c\delta^c_b\right) = \frac{N^2-1}{2N}\delta^a_b$$

Using $N = 3$, the β function is therefore multiplied by a factor 4/3 when we go from QED to QCD, and α is replaced by α_s.

Solution to exercise 213

1. The Fierz identity tells us that $T^\ell T^j T^\ell$ must be a conbination of T^j and Tr (T^j), and this last trace vanishes.

[4]See also Eq.(7.15) in reference [2].

2.

$$T^\ell T^t T^\ell T^j = \frac{1}{2}\left(\mathrm{Tr}\left(T^j\right)\mathrm{Tr}\left(T^j\right) - \frac{1}{N}\mathrm{Tr}\left(T^jT^j\right)\right) = -\frac{1}{2N}\mathrm{Tr}\left(T^jT^j\right)$$

3. The same argument as in item 1 applies.

4.

$$\begin{aligned} h^{j\ell n}\mathrm{Tr}\left(T^jT^\ell T^n\right) &= 2\,\mathrm{Tr}\left(T^jT^\ell T^n\right)\mathrm{Tr}\left(T^jT^\ell T^n\right) \\ &\quad -2\,\mathrm{Tr}\left(T^\ell T^j T^n\right)\mathrm{Tr}\left(T^jT^\ell T^n\right) \\ &= \left(\mathrm{Tr}\left(T^jT^\ell T^jT^\ell\right) - \frac{1}{N}\mathrm{Tr}\left(T^jT^\ell\right)\mathrm{Tr}\left(T^jT^\ell\right)\right) \\ &\quad -\left(\mathrm{Tr}\left(T^\ell T^jT^jT^\ell 1\right) - \frac{1}{N}\mathrm{Tr}\left(T^\ell T^j\right)\mathrm{Tr}\left(T^jT^\ell\right)\right) \\ &= \frac{1}{2}\mathrm{Tr}\left(T^j\right)\mathrm{Tr}\left(T^j\right) - \frac{1}{2N}\mathrm{Tr}\left(T^jT^j\right) \\ &\quad -\frac{1}{2}\mathrm{Tr}\left(T^jT^j\right)\mathrm{Tr}\left(1\right) + \frac{1}{2N}\mathrm{Tr}\left(T^jT^j1\right) \quad = \quad -\frac{N}{2}\mathrm{Tr}\left(T^jT^j\right) \end{aligned}$$

Solution to exercise 214

The colour current has been proven to be conserved to all orders in physical amplitudes (in section 16.4). The divergent parts of these amplitudes must therefore also conserve the current, and therefore so must the counterterms.

17 Exercises for chapter 17

Solution to exercise 215

1. The tree-level amplitude for the process $W^-(p) \to e^-(q_1)\bar{\nu}_e(q_2)$ is

$$\mathfrak{M} = -ig_{\mathrm{W}}\hbar^{1/2}\,\overline{u}(q_1)(1+\gamma^5)\not{\epsilon}v(q_2) = -2ig_{\mathrm{W}}\hbar^{1/2}\overline{u}_-(q_1)\not{\epsilon}u_-(q_2)$$

We may take for the polarization:

$$\epsilon^\mu_\pm = \frac{1}{m_{\mathrm{W}}\sqrt{2}}\overline{u}_\pm(q_1)\gamma^\mu u_\pm(q_2) \quad , \quad \epsilon^\mu_0 = \frac{1}{m_{\mathrm{W}}}(q_1^\mu - q_2^\mu)$$

The polarizations ϵ_0 and ϵ_- give no contribution, and

$$\overline{u}_-(q_1)\not{\epsilon}_+u_-(q_2) = \frac{\sqrt{2}}{m_{\mathrm{W}}}s_-(q_1,q_2)s_+(q_1,q_2) \sim \sqrt{2}\,m_{\mathrm{W}}$$

so that

$$\left\langle|\mathfrak{M}|^2\right\rangle = \frac{8}{3}\hbar{g_{\mathrm{W}}}^2\,{m_{\mathrm{W}}}^2$$

The same results, of course, from

$$\left\langle|\mathfrak{M}|^2\right\rangle = \frac{\hbar{g_{\mathrm{W}}}^2}{3}\,\mathrm{Tr}\left(\not{q}_1(1+\gamma^5)\gamma^\mu\not{q}_2(1+\gamma^5)\gamma^\nu\right)\left(-g_{\mu\nu} + \frac{p_\mu p_\nu}{{m_{\mathrm{W}}}^2}\right)$$

The width is

$$\Gamma(W^- \to e^-\bar{\nu}_e) = \int \frac{1}{2m_{\rm W}} \left\langle|\mathfrak{M}|^2\right\rangle \frac{d\Omega}{32\pi^2} = \frac{\hbar {g_{\rm W}}^2\, m_{\rm W}}{6\pi} = \frac{\hbar G_{\rm F}\, {m_{\rm W}}^3}{6\sqrt{2}\pi}$$

2. The W^- can decay into $e^-\bar{\nu}_e$, $\mu^-\bar{\nu}_\mu$, $\tau^-\bar{\nu}_\tau$, $d\bar{u}$ and $s\bar{c}$; the decay into $b\bar{t}$ is kinematically out or reach. The total width is therefore

$$\Gamma_W = \frac{\hbar G_{\rm F}\, {m_{\rm W}}^3}{6\sqrt{2}\pi}\big(1+1+1+3\cdot(1+1)\big) = \frac{3\hbar G_{\rm F}\, {m_{\rm W}}^3}{\pi\sqrt{8}}$$

This evaluates to about 2.04 GeV/$\hbar c$ for $m_{\rm W} = 80.4$ GeV/c^2.

3. Let us write $\Gamma_W = k\Gamma(W^- \to e^-\bar{\nu}_e)$, so that in the Standard Model $k = 9$. At total invariant mass $\sqrt{s}$ equal to $m_{\rm W}$, the cross section of Eq.(17.25) evaluates to

$$\sigma(\mu^-\bar{\nu}_\mu \to e^-\bar{\nu}_e)\big]_{s={m_{\rm W}}^2} = \frac{2\hbar^2 {g_{\rm W}}^2}{3\pi\,\Gamma_W^2} = \frac{24\pi}{{m_{\rm W}}^2 k^2} = \frac{3}{2}\,\frac{16\pi}{s}\,\frac{1}{k^2}$$

and this respects the unitarity limit for all $k \geq 1$. Note that $J = 1$ and there are 2 helicity states for the incoming muon, but only 1 for the incoming anti-muon neutrino.

Solution to exercise 216

Without Cabibbo mixing, and writing $\Gamma_h = \hbar {g_{\rm W}}^2 m_{\rm W}/(\pi\sqrt{8})$, we have two hadronic decay modes:

$$\Gamma(W \to q\bar{q}) = \Gamma(W^- \to d\bar{u}) + \Gamma(W^- \to s\bar{c}) = 2\Gamma_h$$

With Cabibbo mixing, we have 4 decay modes:

$$\begin{aligned}\Gamma(W^- \to q\bar{q}) &= \Gamma(W^- \to d\bar{u}) + \Gamma(W^- \to d\bar{c}) + \Gamma(W^- \to s\bar{u}) + \Gamma(W^- \to s\bar{c}) \\ &= \Gamma_h\left((\cos\theta_c)^2 + (\sin\theta_c)^2 + (\sin\theta_c)^2 + (\cos\theta_c)^2\right) \;=\; 2\Gamma_h\end{aligned}$$

Solution to exercise 217

1. The process is $t(p) \to b(q)\, W^+(k)$, so the tree amplitude is

$$\mathfrak{M} = -i\hbar^{1/2} g_{\rm W}\, \overline{u}(q)(1+\gamma^5)\not{\epsilon}u(p)$$

The transition rate, assuming the b quark to be massless, is[5]

$$\begin{aligned}\left\langle|\mathfrak{M}|^2\right\rangle &= \frac{\hbar {g_{\rm W}}^2}{2}\mathrm{Tr}\left(\not{q}(1+\gamma^5)\gamma_\mu(\not{p}+m_{\rm t})(1+\gamma^5)\gamma_\nu\right)\left(-g^{\mu\nu} + \frac{1}{{m_{\rm W}}^2}k^\mu k^\nu\right) \\ &= \hbar {g_{\rm W}}^2 \mathrm{Tr}\left(\not{q}\gamma_\mu\not{p}\gamma_\nu\right)\left(-g^{\mu\nu} + \frac{1}{{m_{\rm W}}^2}k^\mu k^\nu\right) \\ &= \hbar {g_{\rm W}}^2 \mathrm{Tr}\left(2\not{q}\not{p} + \frac{1}{{m_{\rm W}}^2}\not{q}\not{k}\not{p}\not{k}\right) = \hbar {g_{\rm W}}^2 \mathrm{Tr}\left(2\not{q}\not{p} + \frac{1}{{m_{\rm W}}^2}\not{q}\not{p}\not{p}\not{p}\right) \\ &= 2\hbar {g_{\rm W}}^2\left({m_{\rm t}}^2 - {m_{\rm W}}^2\right)\left(2 + \frac{{m_{\rm t}}^2}{{m_{\rm W}}^2}\right)\end{aligned}$$

[5]The averaging over the top colours is exactly compensated by the sum over the bottom colours.

The phase space integration element is

$$\frac{1}{32\pi^2}\left(1-\frac{{m_{\rm W}}^2}{{m_{\rm t}}^2}\right)\,d\Omega$$

and the decay width is

$$\begin{aligned}\Gamma(t\to bW) &= \frac{\hbar {g_{\rm W}}^2}{8\pi}\,\frac{{m_{\rm t}}^3}{{m_{\rm W}}^2}\left(1+\frac{2{m_{\rm W}}^2}{{m_{\rm t}}^2}\right)\left(1-\frac{{m_{\rm W}}^2}{{m_{\rm t}}^2}\right)^2\\ &= \frac{\hbar G_{\rm F}}{8\pi\sqrt{2}}\,{m_{\rm t}}^3\left(1+\frac{2{m_{\rm W}}^2}{{m_{\rm t}}^2}\right)\left(1-\frac{{m_{\rm W}}^2}{{m_{\rm t}}^2}\right)^2\end{aligned}$$

For fixed $m_{\rm W}$ this goes indeed as ${m_{\rm t}}^3$. With $m_{\rm W} = 80.4$ GeV and $m_{\rm t} = 172.4$ GeV, we have $\Gamma(t\to bW)\approx 1.47$ GeV, hence a lifetime of about 4.48×10^{-25} seconds. QCD corrections increase the lifetime by about 10 per cent.

2. This follows from the W decay width, which is 9 times the electronic decay width.

3. In the decay process $t(p)\to b(q)\nu_e(k_1)e^+(k_2)$ the amplitude is

$$\mathfrak{M}\sim \overline{u}_-(q)\gamma_\mu u(p)\,\overline{u}_-(k_1)\gamma^\mu v_-(k_2)\sim s_-(q,k_1)\overline{u}_+(k_2)u(p)$$

If E_{e^+} has its maxumal value $m_{\rm t}/2$, then $\vec{q}$ and $\vec{k}_1$ must be parallel, so that $s_-(q,k_1)=0$.

4. If we introduce the CKM matrix, then we also have available the decays $t-\to sW$ and $t\to dW$, leading to a factor $|V_{tb}|^2+|V_{ts}|^2+|V_{td}|^2$ which equals 1 by the unitarity requirement.

<u>Solution to exercise</u> 218
Eq.(17.54) can be written out:

$$g_1^2+g_2^2+2g_1g_2\gamma^5 = 2{g_{\rm W}}^2+Q_{\rm U}Q_{\rm W}+2{g_{\rm W}}^2\gamma^5$$

this matrix equation implies two separate relations:

$$g_1^2+g_2^2 = 2{g_{\rm W}}^2+Q_{\rm U}Q_{\rm W}\quad,\quad g_1g_2={g_{\rm W}}^2$$

and subtracting these we find $(g_1-g_2)^2=Q_{\rm U}Q_{\rm W}$, which is negative.

<u>Solution to exercise</u> 219
For the process $D\bar{D}\to W^+W^-$ the only difference is in the fermion exchange diagram, in which the W^+ and W^- occur in the opposite order:

$$\mathfrak{M}_1 = -2i\hbar {g_{\rm W}}^2\,\overline{v}(p_1)(1+\gamma^5)\not{\epsilon}_+\frac{\not{q}_+-\not{p}_1}{(p_1-q_+)^2}\not{\epsilon}_-u(p_2)$$

The handlebar therefore results in an opposite sign:

$$\mathfrak{M}_1\rfloor_{\epsilon_+\to q_+} = -2i\hbar {g_{\rm W}}^2\overline{v}(p_1)(1+\gamma^5)\not{\epsilon}_-u(p_2)$$

so that $2{g_{\rm W}}^2$ is replaced by $-2{g_{\rm W}}^2$ to arrive at Eq.(17.58). For the process

$$\bar{D}(p_1)\,U(p_2)\;\to\;W^+(q_+,\epsilon_+)\,Z(q_0,\epsilon_0)$$

there are again three diagrams, two involving fermion exchange. Again neglecting the masses we have

$$\begin{aligned}
\mathfrak{M}_1 &= -i\hbar g_{\rm W}\,\overline{v}(p_1)(1+\gamma^5)\not{\epsilon}_+\frac{\not{q}_+-\not{p}_1}{(q_+-p_1)^2}(v_U+a_U\gamma^5)\not{\epsilon}_0 u(p_2)\\
\mathfrak{M}_2 &= -i\hbar g_{\rm W}\,\overline{v}(p_1)(v_D+a_D\gamma^5)\not{\epsilon}_0\frac{\not{p}_2-\not{q}_+}{(q_+-p_2)^2}(1+\gamma^5)\not{\epsilon}_+ u(p_2)\\
\mathfrak{M}_3 &= i\hbar\frac{g_{\rm W}g_{\rm WWZ}}{s}\,\overline{v}(p_1)(1+\gamma^5)\gamma_\alpha u(p_2)Y(q_+,\epsilon_+;-q_+-q_0,\alpha;q_0,\epsilon_0)
\end{aligned}$$

Upon implementing the handlebar $\epsilon_+\to q_+$ we have

$$\begin{aligned}
\mathfrak{M}_1 &\to -i\hbar g_{\rm W}(v_U+a_U)\,\overline{v}(p_1)(1+\gamma^5)\not{\epsilon}_0 u(p_2)\\
\mathfrak{M}_2 &\to +i\hbar g_{\rm W}(v_D+a_D)\,\overline{v}(p_1)(1+\gamma^5)\not{\epsilon}_0 u(p_2)\\
\mathfrak{M}_3 &\to -i\hbar g_{\rm W}g_{\rm WWZ}\,\overline{v}(p_1)(1+\gamma^5)\not{\epsilon}_0 u(p_2)
\end{aligned}$$

and this leads immediately to Eq.(17.59). Note that from this process only *one* condition is found, not two.

<u>Solution to exercise</u> 220

The decay process $Z(p,\epsilon)\to f(q_1)\bar{f}(p_2)$ has the tree-level amplitude

$$\mathfrak{M}=i\hbar^{1/2}\,\overline{u}(q_1)(v_f+a_f\gamma^5)\not{\epsilon}v(p_2)$$

so that, assuming a single colour for the fermions:

$$\begin{aligned}
\left\langle|\mathfrak{M}|^2\right\rangle &= \frac{\hbar}{3}\mathrm{Tr}\left(\not{q}_1(v_f+a_f\gamma^5)\gamma_\mu\not{q}_2(v_f+a_f\gamma^5)\gamma_\nu\right)\left(-g^{\mu\nu}+\frac{1}{{m_{\rm Z}}^2}p^\mu p^\nu\right)\\
&= \frac{4\hbar}{3}(v_f^2+a_f^2){m_{\rm Z}}^2
\end{aligned}$$

(the $p^\mu p^\nu$ term gives no contribution). We find the partial decay width to be

$$\Gamma(Z-\to f\bar{f})=\frac{1}{2m_{\rm Z}}\left\langle|\mathfrak{M}|^2\right\rangle\frac{4\pi}{32\pi^2}=\frac{\hbar(v_f^2+a_f^2)m_{\rm Z}}{12\pi}$$

Using Eq.(17.65) we can write this as

$$\Gamma(Z\to f\bar{f})=\frac{\alpha\,m_{\rm Z}}{48{c_{\rm W}}^2{s_{\rm W}}^2}\left(1+\left(1-4{s_{\rm W}}^2|q_f|\right)^2\right)=\gamma(q)$$

where q_f is the fractional charge of the fermion. The kinematically available fermion-antifermion pairs are

$$\begin{aligned}
q=0 &\;:\; \nu_e\bar{\nu}_e\;,\;\nu_\mu\bar{\nu}_\mu\;,\;\nu_\tau\bar{\nu}_\tau\\
q=-1 &\;:\; e^+e^-\;,\;\mu^+\mu^-\;,\;\tau^+\tau^-\\
q=+2/3 &\;:\; u\bar{u}\;,\;c\bar{c}\\
q=-1/3 &\;;\; d\bar{d}\;,\;s\bar{s}\;,\;b\bar{b}
\end{aligned}$$

The total decay width, keeping in mind that the quarks come in 3 colours, is therefore

$$\Gamma_Z = 3\gamma(0) + 3\gamma(-1) + 6\gamma(2/3) + 9\gamma(-1/3)$$

Using $\alpha = 1/137$, ${s_\mathrm{W}}^2 = 0.23$, and $m_\mathrm{Z} = 91.19$ GeV/c^2 this amounts to $\Gamma_Z = 2.29$ GeV/$c\hbar$. There are two important corrections: the value of α *at the Z mass* is rather 1/128, and the quark decays have a nonnegligible QCD correction factor of about $1 + \alpha_s/\pi$, with $\alpha_s \sim 0.1$. These change the total width to $\Gamma_Z \sim 2.5$ GeV/$c\hbar$. The measured value is about 2.49.

Solution to exercise 221

1. We consider the process

$$e^+(p_1, \lambda)\, e^-(p_2, \lambda) \;\to\; q(q_1, \rho)\, \bar{q}(q_2, \rho)$$

where we indicate the momenta and the helicities. All fermions are massless, so we use helicity methods. The amplitude is, with $\Sigma = s - {m_\mathrm{Z}}^2 + i m_\mathrm{Z}\Gamma_\mathrm{Z}$:

$$\begin{aligned}
\mathfrak{M}(\lambda, \rho) &= \frac{i\hbar}{\Sigma}\overline{u}_\lambda(p_1)(v_e + a_e\gamma^5)\gamma_\mu u_\lambda(p_2)\; \overline{u}_\rho(q_1)(v_q + a_q)\gamma^\mu u_\rho(q_2) \\
\mathfrak{M}(+,+) &\simeq 2\hbar\, u\,(v_e - a_e)(v_q - a_q)/\Sigma \\
\mathfrak{M}(+,-) &\simeq 2\hbar\, t\,(v_e - a_e)(v_q + a_q)/\Sigma \\
\mathfrak{M}(-,+) &\simeq 2\hbar\, t\,(v_e + a_e)(v_q - a_q)/\Sigma \\
\mathfrak{M}(+,+) &\simeq 2\hbar\, u\,(v_e + a_e)(v_q + a_q)/\Sigma
\end{aligned}$$

The angular *averages* are

$$\langle t^2 \rangle_\Omega = \langle u^2 \rangle_\Omega = s^2/3$$

so that

$$\langle\langle|\mathfrak{M}|^2\rangle\rangle_\Omega = \frac{4\hbar^2 s^2}{3}\frac{(v_e^2 + a_e^2)(v_q^2 + a_q^2)}{(s - {m_\mathrm{Z}}^2)^2 + {m_\mathrm{Z}}^2{\Gamma_\mathrm{Z}}^2} = 192\pi^2\frac{s^2\Gamma(Z \to ee)\Gamma(Z \to qq)}{{m_\mathrm{Z}}^2\left((s - {m_\mathrm{Z}}^2)^2 + {m_\mathrm{Z}}^2{\Gamma_\mathrm{Z}}^2\right)}$$

and the total cross section is

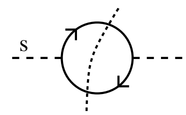
$$\sigma(s) = \frac{1}{2s}\langle\langle|\mathfrak{M}|^2\rangle\rangle_\Omega \frac{4\pi}{32\pi^2} = 12\pi\Gamma(Z \to ee)\Gamma(Z \to qq)\frac{s/{m_\mathrm{Z}}^2}{(s - {m_\mathrm{Z}}^2)^2 + {m_\mathrm{Z}}^2{\Gamma_\mathrm{Z}}^2}$$

2. Taking the numerator $s/{m_\mathrm{Z}}^2$ into account, we have

$$\frac{\partial}{\partial s}\sigma(s) \;\propto\; s^2 - {m_\mathrm{Z}}^4 - mz^2{\Gamma_\mathrm{Z}}^2$$

so that the maximum resides at $s = m_\mathrm{Z}\sqrt{{m_\mathrm{Z}}^2 + {\Gamma_\mathrm{Z}}^2}$, or $\sqrt{s} \sim m_\mathrm{Z} + {\Gamma_\mathrm{Z}}^2/(4m_\mathrm{Z})$.

3. The term $im_Z\Gamma_Z$ comes from the cut fermionic self-energy of the Z:



This cut diagram contains 4 spinors, and therefore scales as s. Therefore we have to replace $m_Z\Gamma_Z$ by $s\Gamma_Z/m_Z$ since it must coincide with $m_Z\Gamma_Z$ at $s = {m_Z}^2$. With this modification we have

$$\frac{\partial}{\partial s}\sigma(s) \;\propto\; s^2({m_Z}^2 + {\Gamma_Z}^2) - {m_Z}^6$$

so the maximum is now shifted to $s = {m_Z}^3/\sqrt{{m_Z}^2 + {\Gamma_Z}^2}$, or $\sqrt{s} = m_Z - {\Gamma_Z}^2/(4m_Z)$. Numerically, ${\Gamma_Z}^2/(4m_Z)$ is about 16 MeV$/c\hbar$.

Solution to exercise 222

Recall the reasoning of exerise 117, employed in exercise 191:

$$\begin{aligned}
&\frac{(\not{q} + m)}{q^2 - m^2}\gamma^\alpha\frac{(\not{q} + m)}{q^2 - m^2} \\
&= \quad 2q^\alpha\frac{\not{q} + m}{(q^2 - m^2)^2} - \frac{\gamma^\alpha}{q^2 - m^2} = -\frac{\partial}{\partial q_\alpha}\left(\frac{\not{q} + m}{q^2 - m^2}\right)
\end{aligned}$$

We similarly have

$$\begin{aligned}
&\frac{-g^{\mu\lambda} + q^\mu q^\lambda/{m_W}^2}{q^2 - {m_W}^2}Y(q,\lambda;-q,\beta;0,\alpha)\frac{-g^{\beta\nu} + q^\beta q^\nu/{m_W}^2}{q^2 - {m_W}^2} \\
&= \frac{1}{(q^2 - {m_W}^2)^2}\left(Y(q,\mu,-q,\nu;0,\alpha) - \frac{q^\mu}{{m_W}^2}Y(q,q;-q,\nu;0,\alpha) + Y(q,\mu;-q,-q;0,\alpha)\frac{q^\nu}{{m_W}^2}\right) \\
&= \frac{1}{(q^2 - {m_W}^2)^2}\left(2q^\alpha g^{\mu\nu} - q^\mu g^{\nu\alpha} - q^\nu g^{\mu\alpha} - \frac{q^\mu}{{m_W}^2}\left(q^\nu q^\alpha - q^2 g^{\nu\alpha}\right) + \left(-q^\mu q^\alpha + q^2 g^{\mu\alpha}\right)\frac{q^\nu}{{m_W}^2}\right) \\
&= \frac{1}{(q^2 - {m_W}^2)^2}\left(2q^\alpha\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{{m_W}^2}\right) + \frac{q^\mu g^{\nu\alpha} + q^\nu g^{\mu\alpha}}{{m_W}^2}(q^2 - {m_W}^2)\right) \\
&= -\frac{\partial}{\partial q_\alpha}\left(\frac{-g^{\mu\nu} + q^\mu q^\nu/{m_W}^2}{q^2 - {m_W}^2}\right)
\end{aligned}$$

and also

$$\begin{aligned}
&\frac{\partial}{\partial q_\alpha}Y(q,\mu;-q-k,\nu;k,\beta) \\
&= \quad \frac{\partial}{\partial q_\alpha}\left((2q+k)^\beta g^{\mu\nu} - (q+2k)^\mu g^{\nu\alpha} + (k-q)^\nu g^{\mu\alpha}\right) = X^{\mu\nu\alpha\beta}
\end{aligned}$$

Solution to exercise 223

1. Using the propagator in the unitary gauge:

$$\Pi(q)\mu\nu = \frac{i\hbar}{q^2 - {m_W}^2}\left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{{m_W}^2}\right)$$

the semi-connected diagram(s) can be written as

$$\Pi(q_1)^{\mu}_{\alpha}\,\frac{iQ_{\rm W}}{\hbar}Y(q_!,\alpha;q_2,\beta;-q_1-q_2,\epsilon)\;\Pi(q_2)^{\nu}_{\beta}\,X_{\mu\nu}$$

where X stands for the rest of the diagrams (the blob). Under the handlebar this gives

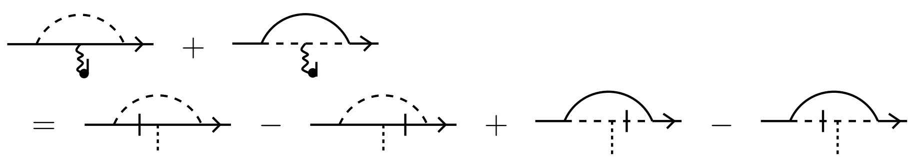
$$\begin{aligned}
&\Pi(q_1)^{\mu}_{\alpha}\,\frac{iQ_{\rm W}}{\hbar}Y(q_!,\alpha;q_2,\beta;-q_1-q_2,-q_1-q_2)\;\Pi(q_2)^{\nu}_{\beta}\,X_{\mu\nu}\\
&=\;\Pi(q_1)^{\mu}_{\alpha}\,\frac{iQ_{\rm W}}{\hbar}\left(q_1^{\alpha}q_1^{\beta}-q_1^2g^{\alpha\beta}-q_2^{\alpha}q_2^{\beta}+q_2^2g^{\alpha\beta}\right)\;\Pi(q_2)^{\nu}_{\beta}\,X_{\mu\nu}\\
&=\;\Pi(q_1)^{\mu}_{\alpha}\,\frac{iQ_{\rm W}}{\hbar}\left(q_1^{\alpha}q_1^{\beta}-(q_1^2-m_{\rm W}{}^2)g^{\alpha\beta}-q_2^{\alpha}q_2^{\beta}+(q_2^2-m_{\rm W}{}^2)g^{\alpha\beta}\right)\;\Pi(q_2)^{\nu}_{\beta}\,X_{\mu\nu} \qquad (4)
\end{aligned}$$

Using the following identity twice, once for q_1 and once for q_2,

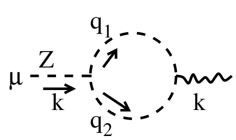
$$\Pi(q)^{\mu}_{\alpha}\left(q^{\alpha}q^{\beta}-(q^2-m_{\rm W}{}^2)g^{\alpha\beta}\right)=i\hbar g^{\mu\beta}$$

then leads to the handlebar rule.

2. For the first two diagrams, the handlebars give



The first and fourth diagram cancel one another, as do the second and third ones. The fermionic contribution to γ, Z mixing vanishes under the handlebar exactly as in QED. The W two-point loop integrand can be written as



$$\frac{ig_{\rm WWZ}}{\hbar}Y(q_1,\lambda;q_2,\rho;-k,\mu)\Pi(q_1)^{\lambda\alpha}\Pi(q_2)^{\rho\beta}\frac{iQ_{\rm W}}{\hbar}Y(-q_2,\beta;-q_1,\alpha;k,\epsilon)$$

Under the handlebar this is transformed into

$$-i\frac{g_{\rm WWZ}Q_{\rm W}}{\hbar}Y(q_1,\alpha;q_2,\beta;-k,\mu)\left(\Pi(q_1)^{\alpha\beta}-\Pi(q_2)^{\alpha\beta}\right)$$

The loop momentum can be chosen either to be q_1 or q_2, and $q_1+q_2=k$. Now,

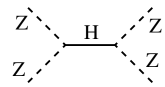
$$\begin{aligned}
Y(q_1,\alpha;q_2,\beta;-k,\mu)&=(q_1-q_2)^{\mu}g^{\alpha\beta}+(q_2+k)^{\alpha}g^{\beta\mu}-(k+q_1)^{\beta}g^{\alpha\mu}\\
&=\;\left(-k^{\mu}g^{\alpha\beta}+2k^{\alpha}g^{\beta\mu}-k^{\beta}g^{\alpha\mu}\right)+\left(2q_1^{\mu}g^{\alpha\beta}-q_1^{\alpha}g^{\beta\mu}-q_1^{\beta}g^{\alpha\mu}\right)\\
&=\;\left(k^{\mu}g^{\alpha\beta}+k^{\alpha}g^{\beta\mu}-2k^{\beta}g^{\alpha\mu}\right)+\left(-2q_2^{\mu}g^{\alpha\beta}+q_2^{\alpha}g^{\beta\mu}+q_2^{\beta}g^{\alpha\mu}\right)
\end{aligned}$$

The terms linear in either q_1 or q_2 drop out as usual, and the integrand is therefore

$$i\frac{g_{\rm WWZ}Q_{\rm W}}{\hbar}\left(2k^{\mu}g^{\alpha\beta}-k^{\alpha}g^{\beta\mu}-k^{\beta}g^{\alpha\mu}\right)\Pi(q)^{\alpha\beta}=i\frac{Q_{\rm W}{}^2c_{\rm W}}{\hbar s_{\rm W}}X^{\alpha\beta\mu\nu}\Pi(q)^{\alpha\beta}k_{\nu}$$

and this is precisely the opposite of the W tadpole diagram (the last one) under the handlebar.

Solution to exercise 224
The s-channel diagram is



The other two diagrams are obtained by crossing $s \leftrightarrow t$ and $s \leftrightarrow u$

1. The s-channel diagram for energy $E \gg m_{\rm Z}$ reads

$$\mathfrak{M}_s = -i\hbar {g_{\rm ZZH}}^2 \left(\frac{s}{2}\right) \frac{1}{s - {m_{\rm H}}^2} \left(\frac{s}{2}\right) \sim -i\frac{\hbar {g_{\rm ZZH}}^2}{4} s$$

   The last lemma obtains when we discard terms of order $\mathcal{O}\left({m_{\rm H}}^2/s\right)$ as well as $\mathcal{O}\left({m_{\rm Z}}^2/s\right)$. This is the nonsafe term.

2. Adding all three crossings we have $s + t + u = 4{m_{\rm Z}}^2$ so the nonsafe terms cancel. Here we have assumed that all three of s,$|t|$ and $|u|$ are much larger than ${m_{\rm Z}}^2, {m_{\rm H}}^2$.

Solution to exercise 225
To have $m_{\rm W} = 0$ we need $c_{\rm W} = 0$ as well; and then $g_{\rm WWZ} = 0$ so the whole diagram disappears.

Solution to exercise 226

1. For the decay $H(p) \to t(q_1)\bar{t}(q_2)$ the amplitude is

$$\mathfrak{M} = i\hbar^{1/2}\, g_t\, \overline{u}(q_1)v(q_2) \;\;,\;\; g_t = \frac{e m_t}{2 s_{\rm W} m_{\rm W}}$$

   for the decay width we find

$$\begin{aligned} \left\langle |\mathfrak{M}|^2 \right\rangle &= \hbar g_t^2\, \mathrm{Tr}\left((\not{q}_1 + m_t)(\not{q}_2 - m_t)\right) = 2\hbar g_t^2 {m_{\rm H}}^2 \left(1 - \frac{4m_t^2}{{m_{\rm H}}^2}\right) \\ \Gamma(H \to t\bar{t}) &= \frac{\hbar g_t^2}{8\pi} m_{\rm H} \left(1 - \frac{4m_t^2}{{m_{\rm H}}^2}\right)^{3/2} \end{aligned}$$

   so this width is indeed linear in $m_{\rm H}$ when $m_{\rm H}$ becomes large.

2. For the decay $H(p) \to W^+(q_1, \epsilon_1)\, W^-(q_2, \epsilon_2)$ the amplitude is

$$\mathfrak{M} = i\hbar^{1/2} g_{\rm WWH}\left(\epsilon_1 \cdot \epsilon_2\right) \;\;,\;\; g_{\rm WWH} = \frac{2{m_{\rm W}}^2}{v}$$

   so this partial decay width is found by

$$\begin{aligned} \left\langle |\mathfrak{M}|^2 \right\rangle &= \hbar {g_{\rm WWH}}^2 \left(-g_{\alpha\beta} + \frac{q_{1\alpha} q_{1\beta}}{{m_{\rm W}}^2}\right)\left(-g^{\alpha\beta} + \frac{q_2^\alpha q_2^\beta}{{m_{\rm W}}^2}\right) \\ &= \frac{\hbar {g_{\rm WWH}}^2}{4} \frac{{m_{\rm H}}^4}{{m_{\rm W}}^4} \left(1 - 4\frac{{m_{\rm W}}^2}{{m_{\rm H}}^2} + 12\frac{{m_{\rm W}}^4}{{m_{\rm H}}^4}\right) \\ &= \frac{\hbar {m_{\rm H}}^4}{v^2} \left(1 - 4\frac{{m_{\rm W}}^2}{{m_{\rm H}}^2} + 12\frac{{m_{\rm W}}^4}{{m_{\rm H}}^4}\right) \end{aligned}$$

and so we find

$$\Gamma(H \to W^+W^-) = \frac{\hbar {m_{\rm H}}^3}{16\pi}\left(1 - 4\frac{{m_{\rm W}}^2}{{m_{\rm H}}^2} + 12\frac{{m_{\rm W}}^4}{{m_{\rm H}}^4}\right)\left(1 - \frac{4{m_{\rm W}}^2}{{m_{\rm H}}^2}\right)^{1/2}$$

$$\Gamma(H \to ZZ) = \frac{\hbar {m_{\rm H}}^3}{32\pi}\left(1 - 4\frac{{m_{\rm Z}}^2}{{m_{\rm H}}^2} + 12\frac{{m_{\rm Z}}^4}{{m_{\rm H}}^4}\right)\left(1 - \frac{4{m_{\rm Z}}^2}{{m_{\rm H}}^2}\right)^{1/2}$$

Note the symmetry factor in the ZZ decay! For $m_{\rm H} \gg m_{\rm Z}, m_{\rm W}$ these widths go as ${m_{\rm H}}^3$.

3. For large $m_{\rm H}$ the WW and ZZ decay modes are dominant, so

$$\Gamma_H \sim \frac{3}{32\pi}\frac{{m_{\rm H}}^3}{v^2} \quad , \quad m_{\rm H} \gg m_{\rm Z}, m_{\rm W}$$

Putting $\Gamma_H = m_{\rm H}$ gives $m_{\rm H} \sim\ 5.8\ v\ \sim\ 1.4\ \text{TeV}/c^2$.

<u>Solution to exercise</u> 227

We assume that the fermions have no colour, and that Higgs exchange dominates the scattering. Then for $F_1(p_1)\bar{F}_1(p_2) \to F_2(q_1)\bar{F}_2(q_2)$ the amplitude is, far above the Hiiggs mass, given by

$$\mathfrak{M} \sim -i\frac{\hbar g^2}{s}\ \overline{v}(p_1)u(p_2)\ \overline{u}(q_1)v(q_2)$$

$$\left\langle|\mathfrak{M}|^2\right\rangle = \hbar^2 g^4\,\beta^4 \quad , \quad \beta^2 = 1 - 4M^2/s$$

and the cross section is

$$\sigma = \frac{1}{2s\beta}\left\langle|\mathfrak{M}|^2\right\rangle\frac{4\pi\beta}{32\pi^2} = \frac{\hbar^2 g^4\,\beta^4}{16\pi s}$$

The unitarity bound in the $J = 0$ channel is, from Eq.(19.185)

$$\sigma = \frac{4\pi}{s\,\beta^2}$$

Therefore unitarity will be violated at high enough s if $g^4 > 64\pi^2$, or M larger than about $5v \approx 1.2$ TeV$/c^2$: a number remarkably similar to the 'unitarity bound' on the Higgs mass of exercise 226.

<u>Solution to exercise</u> 228

An elegant way to do it is to start with massless fermions, and to realize that a two-point vertex $-im/\hbar$ give the right propagator by Dyson summation. The combined two-point vertices are therefore

$$-i\frac{m}{\hbar} - i\frac{m}{\hbar}\frac{H}{v} = -i\frac{m}{\hbar}\frac{\Phi}{v}$$

The same combination $\Phi = H + v$ occurs.

Solution to exercise 229

Dropping terms linear in k and going over to $4-2\epsilon$ dimensions:

$$\frac{1}{(2\pi)^4}\int d^4k\frac{\gamma^\alpha(\not{k}+m)^2\gamma_\alpha}{(k^2-m^2+i\eta)^2(k^2+i\eta)}$$

$$\to \frac{(4-2\epsilon)\mu^{2\epsilon}}{(2\pi)^{4-2\epsilon}}\int_0^1 dx\;2x\int d^{4-2\epsilon}k\frac{k^2+m^2}{(k^2-xm^2+i\eta)^3}$$

$$= i\frac{(4-2\epsilon)(4\pi\mu^2)^\epsilon}{(4\pi)^2\Gamma(2-\epsilon)}\int_0^1 dx\;2x\int_0^\infty dt\,\frac{t^{2-\epsilon}-m^2\,t^{1-\epsilon}}{(t+xm^2)^3}$$

$$= i\frac{(4-2\epsilon)}{(4\pi)^2}\left(\frac{4\pi\mu^2}{m^2}\right)^\epsilon\Gamma(\epsilon)\int_0^1 dx\,x\left((2-\epsilon)x^{-\epsilon}-\epsilon\,x^{-1-\epsilon}\right)$$

$$= i\frac{(4-2\epsilon)}{(4\pi)^2}\left(\frac{4\pi\mu^2}{m^2}\right)^\epsilon\Gamma(\epsilon)\left(1-\frac{\epsilon}{1-\epsilon}\right)\approx\frac{4i}{(4\pi)^2}\left(R_\epsilon-\log m^2-\frac{3}{2}\right)$$

Solution to exercise 230

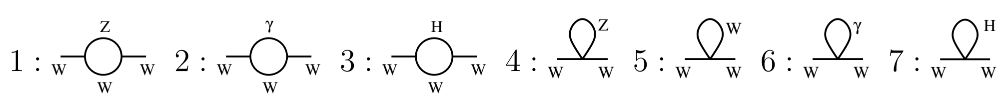
The diagram has 3 fermion propagators, leading to and integrand with up to 3 powers of momenta in the numerator and 6 in the denominator; the vertices must be of vector/axial-vector type so contain no momenta. Using the Feynman trick and shifting it is always possible to make the denominator look like $(k^2+A+i\eta))^3$, where k is the loop momentum and A a combination of the bosons' momenta and of the various masses in the problem. The leading power of loop momenta in the numerator of the integrand is therefore k^3, which vanishes by symmetry. The remaining terms go like $k^2/(k^2+A+i\eta)^3$ at most, leading to a logarithmic divergence $\Gamma(\epsilon)$ (see also exercise 229).

Solution to exercise 231

This exercise is actually rather a research project, so no further information is given.

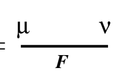
Solution to exercise 232

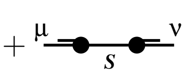
1. The 7 diagrams are:

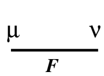


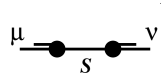
2. Naïve power counting tells us that for generic loop momentum t the propagators go as t^{-2} for γ, H, and as t^0 for W, Z; the vertices scale with t for the $WW\gamma, WWZ$ vertices, and as t^0 for the other one. This gives a sextic divergence for diagram 1; a quartic divergence for diagrams 2, 4, and 5; and a quadratic one for diagrams 3, 6, and 7.

3. (a)

$$=\frac{i\hbar}{q^2-m^2}\left(-g^{\mu\nu}+\frac{q^\mu q^\mu}{m^2}\right)=$$


$+$


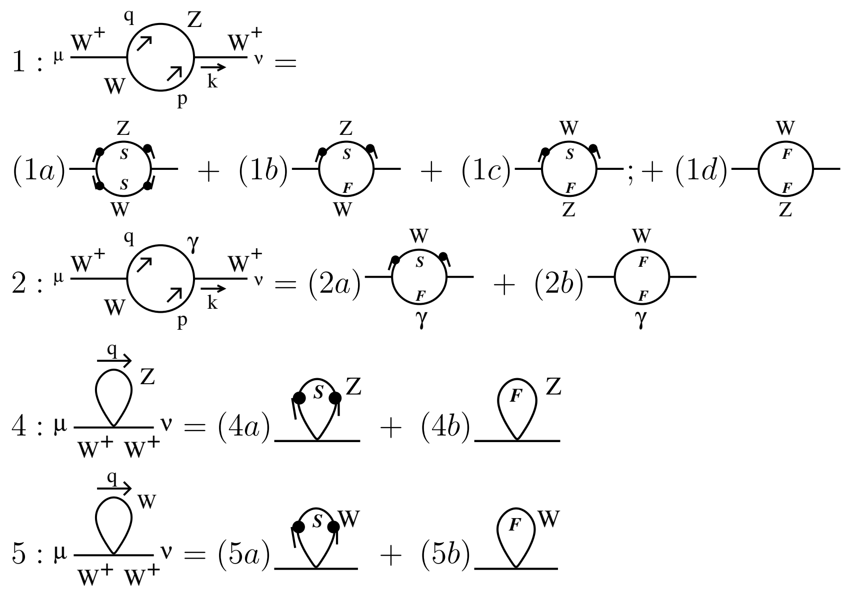


$$=\frac{-i\hbar g^{\mu\nu}}{q^2-m^2}\quad,$$


$$=\frac{-i\hbar}{q^2-m^2}\frac{1}{m^2}$$

The propagator $'S'$ has the wrong sign for a particle; this is necessary if we assume every handlebar to refer to an *outgoing* momentum.

(b) With the single loop momentum contained in both p and q, and k a fixed external momentum:



The integrand of diagram $(1a)$ has 6 powers of t in the numerator, and 4 in the denominator, and might appear, therefore, to contain a sextic divergence. Explicitly, however, it reads

$$\frac{{g_{\mathrm{WWZ}}}^2}{{m_{\mathrm{Z}}}^2 {m_{\mathrm{W}}}^2}\frac{A_1}{(q^2-{m_{\mathrm{Z}}}^2)(p^2-{m_{\mathrm{W}}}^2)}$$

$$\begin{aligned} A_1 &= Y(p,p;-k,\mu;q,q)Y(k,\nu;-p,-p;-q,-q) \\ &= (q_\alpha)(\Delta(k)^{\mu\alpha}-\Delta(q)^{\mu\alpha})(\Delta(q)^{\nu\beta}-\Delta(k)^{\nu\beta})(-q_\beta) \\ &= -k^2\, q^\mu \Delta(k)^{\nu\beta} q_\beta \end{aligned}$$

so that only two powers of the loop momentum survive in A_1. The diagram (1a) is thus quadratically divergent 'only'.

(c) We can combine diagrams $(1b)$ and $(4a)$, again at the integrand level:

$$\begin{aligned} (1b) &= \frac{{g_{\mathrm{WWZ}}}^2}{{m_{\mathrm{Z}}}^2}\frac{A_2}{(q^2-{m_{\mathrm{Z}}}^2)(p^2-{m_{\mathrm{W}}}^2)} \\ A_2 &= Y(p,\alpha;-k,\mu;q,q)Y(k,\nu;-p,\alpha;-q,-q) \\ &= (\Delta(p)^{\mu\alpha}-\Delta(k)^{\mu\alpha})(\Delta(k)^{\nu}_{\alpha}-\Delta(p)^{\nu}_{\alpha}) = p^2\Delta(p)^{\mu\nu}+(QD) \\ \Rightarrow\ (1b) &= \frac{{g_{\mathrm{WWZ}}}^2\,\Delta(p)^{\mu\alpha}}{{m_{\mathrm{Z}}}^2(q^2-{m_{\mathrm{Z}}}^2)}+(QD) \end{aligned}$$

where (QD) stands for terms that can only give a quadratic divergence.

$$\begin{aligned} (4a) &= -\frac{1}{2}\frac{{g_{\mathrm{WWZ}}}^2}{{m_{\mathrm{Z}}}^2(q^2-{m_{\mathrm{Z}}}^2)}X^{\mu\nu\alpha\beta}(q_\alpha)(-q_\beta) \\ &= -\frac{{g_{\mathrm{WWZ}}}^2\,\Delta(q)^{\mu\nu}}{{m_{\mathrm{Z}}}^2(q^2-{m_{\mathrm{Z}}}^2)} \end{aligned}$$

Note the symmetry factor 1/2 in diagram ($4a$)! Therefore

$$\begin{aligned}(1b)+(4a) &= \frac{g_{\mathrm{WWZ}}{}^2(\Delta(p)^{\mu\nu}-\Delta(q)^{\mu\nu})}{m_{\mathrm{Z}}{}^2(q^2-m_{\mathrm{Z}}{}^2)}+(QD)\\ &= \frac{g_{\mathrm{WWZ}}{}^2\ \Delta(k)^{\mu\nu}}{m_{\mathrm{Z}}{}^2(q^2-m_{\mathrm{Z}}{}^2)}+(QD)=(QD)\end{aligned}$$

where we note that here $\Delta(p)^{\mu\nu}-\Delta(q)^{\mu\nu}=\Delta(k)^{\mu\nu}$ since we can drop terms linear in q. The other quartic divergence threatens in diagrams ($1c$), ($2a$) and ($5a$). By the same method as above,

$$(1c)=\frac{g_{\mathrm{WWZ}}{}^2\ \Delta(q)^{\mu\nu}}{m_{\mathrm{Z}}{}^2(p^2-m_{\mathrm{W}}{}^2)}+(QD)\quad,\quad(2a)=\frac{Q_{\mathrm{W}}{}^2\ \Delta(q)^{\mu\nu}}{m_{\mathrm{Z}}{}^2(p^2-m_{\mathrm{W}}{}^2)}+(QD)$$

and, now without a symmetry factor since the W loop is oriented:

$$\begin{aligned}(5a) &= \frac{Q_{\mathrm{W}}{}^2}{s_{\mathrm{W}}{}^2m_{\mathrm{W}}{}^2(p^2-m_{\mathrm{W}}{}^2)}X^{\mu\alpha\nu\beta}(p_\alpha)(-p_\beta)\\ &= -\frac{Q_{\mathrm{W}}{}^2\ \Delta(p)^{\mu\nu}}{s_{\mathrm{W}}{}^2m_{\mathrm{W}}{}^2(p^2-m_{\mathrm{W}}{}^2)}\end{aligned}$$

Using

$$g_{\mathrm{WWZ}}{}^2+Q_{\mathrm{W}}{}^2=Q_{\mathrm{W}}{}^2\left(\frac{c_{\mathrm{W}}{}^2}{s_{\mathrm{W}}{}^2}+1\right)=\frac{Q_{\mathrm{W}}{}^2}{s_{\mathrm{W}}{}^2}$$

we find, as before,

$$(1c)+(2a)+(5a)=\frac{Q_{\mathrm{W}}{}^2(\Delta(q)^{\mu\nu}-\Delta(p)^{\mu\nu})}{s_{\mathrm{W}}{}^2m_{\mathrm{W}}{}^2(p^2-m_{\mathrm{W}}{}^2}+(QD)=(QD)$$

Diagrams ($1d$), ($2b$), ($4b$) and ($5b$) are themselves already (QD).

Note that the cencellations of the quartic divergences are, unsurprisingly, exactly those that ensure unitarity in the processes $WW\to ZZ$ and $WW\to WW$ for up to 2 longitudinal external polarizations.

<u>Solution to exercise</u> 233
The same remark applies as in exercise 231.

18 Exercises for chapter 18

<u>Solution to exercise</u> 234
The third term in brackets would appear with a minus sign instead of a plus sign. For the total cross section, this could mean an overestimate by as much as a factor of 7 around $s=5000$ GeV^2.

<u>Solution to exercise</u> 235

1. The process $\bar{D}(p_1)\,D(p_2)\ \rightarrow\ W^+(q_1,\epsilon_1)\,W^-(q_2,\epsilon_2)$

   The only difference with $e^+e^- \rightarrow W^+W^-$ is in the factor A_2:

$$A_2(\lambda) = -\frac{Q_{\rm W}Q_{\rm D}\,{m_{\rm Z}}^2}{s(s-{m_{\rm Z}}^2)} + \frac{4{g_{\rm W}}^2}{s-{m_{\rm Z}}^2}\delta_{\lambda,-}$$

   Since the first term goes as s^{-2}, at high energy it is only relevant for the double-longitudinal case. Writing $x = Q_{\rm D}/Q_{\rm W}$ we have

$$\begin{aligned}
\mathfrak{M}(+;0,0) &\approx \frac{\hbar e^2\,x}{2}\frac{{m_{\rm Z}}^2}{{m_{\rm W}}^2}(1-c^2)^{1/2} \approx 4\pi\alpha(1-c^2)^{1/2}\frac{x}{2{c_{\rm W}}^2}\\
\mathfrak{M}(-;0,0) &\approx \frac{\hbar e^2\,x}{2}(1-c^2)^{1/2}\left(\frac{1}{{s_{\rm W}}^2} - \frac{{m_{\rm Z}}^2}{{m_{\rm W}}^2}\left(\frac{x}{2{s_{\rm W}}^2}-1\right)\right)\\
&\approx 4\pi\alpha(1-c^2)^{1/2}\left(\frac{1}{4{s_{\rm W}}^2} + \frac{2x-1}{4{c_{\rm W}}^2}\right)
\end{aligned}$$

   For the actual d-type quarks, $x = 1/3$.

2. The process $\bar{U}(p_1)\,U(p_2)\ \rightarrow\ W^-(q_1,\epsilon^w)\,W^+(q_2,\epsilon^z)$

   The best strategy is simply to reinterpret, and assign q_1,ϵ_1 to the W^-, and q_2,ϵ_2 to the W^+, as indicated above. Then B_1 remains unchanged. On the other hand, A_2 changes sign (with now $x = Q_{\rm U}/Q_{\rm W}$. On the *other* other hand, also B_2 changes sign because of the antisymmetry of the Y function. Therefore, the same results as for the previous item are obtained, with $x = -2/3$ for the actual u-type quarks.

3. The process $\bar{D}(p_1)\,U(p_2)\ \rightarrow\ W^+(q_1,\epsilon_1)\,Z^0(q_0,\epsilon_0)$

   (a) The fermions now always have negative helicity. The amplitude consists of 3 diagrams at tree level:

$$\begin{aligned}
\mathfrak{M}(\lambda_1,\lambda_0) &= \sum_{j=1}^{3} A_j\,B_j(\lambda_1,\lambda_0)\\
A_1 &= -2i\hbar g_{\rm W}(v_u+a_u)\frac{1}{t}\ \ ,\ \ t=(q_1-p_1)^2\\
A_2 &= -2i\hbar g_{\rm W}(v_d+a_d)\frac{1}{u}\ \ ,\ \ u=(q_0-p_1)^2\\
A_3 &= +2i\hbar g_{\rm W}g_{\rm WWZ}\frac{1}{s-{m_{\rm W}}^2}\ \ ,\ \ s=P^2=4E^2\ \ ,\ \ P^\mu = p_1^\mu+p_2^\mu\\
B_1(\lambda_1,\lambda_0) &= \overline{u}_-(p_1)\not\epsilon^w_{\lambda_1}(\not q_1-\not p_1)\not\epsilon^z_{\lambda_0}u_-(p_2)\\
B_2(\lambda_1,\lambda_0) &= \overline{u}_-(p_1)\not\epsilon^z_{\lambda_0}(\not q_0-\not p_1)\not\epsilon^w_{\lambda_1}u_-(p_2)\\
B_3(\lambda_1,\lambda_0) &= \overline{u}_-(p_1)\gamma_\alpha u_-(p_2)\,Y(q_1,\epsilon^w_{\lambda_1};-P,\alpha;q_0,\epsilon^z_{\lambda_0})
\end{aligned}$$

   (b) Kinematics and polarization:

$$p_1^\mu = (E,0,0,E)\ \ ,\ \ p_2^\mu = (E,0,0,-E)$$

$$q_1^\mu = (q_1^0, q\vec{e}) \ \ , \ \ q_0^\mu = (q_0^0, -q\vec{e}) \ \ , \ \ a^\mu = (1, \vec{e}) \ \ , \ \ b^\mu = (1, -\vec{e})$$
$$q_1^0 = E - \frac{\delta_m}{4E} \ \ , \ \ q_0^0 = E + \frac{\delta_m}{4E} \ \ , \ \ q = \beta E$$
$$\beta^2 = 1 - 2\frac{\sigma_m}{s} + \frac{\delta_m^2}{s^2} \ \ \ , \ \ \delta_m = {m_\mathrm{Z}}^2 - {m_\mathrm{W}}^2 \ \ , \ \ \sigma_m = {m_\mathrm{Z}}^2 + {m_\mathrm{W}}^2$$
$$\epsilon_\pm^\mu = \frac{1}{\sqrt{8}} u_\pm(a)\gamma^\mu u_\pm(b)$$
$$(\epsilon_0^w)^\mu = \frac{1}{\beta m_\mathrm{W}}\left((1 - \delta_m/s)q_1^\mu - \frac{2{m_\mathrm{W}}^2}{s}P^\mu\right)$$
$$(\epsilon_0^z)^\mu = \frac{1}{\beta m_\mathrm{Z}}\left((1 + \delta_m/s)q_0^\mu - \frac{2{m_\mathrm{Z}}^2}{s}P^\mu\right)$$

(c) Amplitudes: define

$$K_\pm = \overline{u}_-(p_1)\not{\epsilon}_- u_-(p_2) \quad , \quad K_0 = \overline{u}_-(p_1)\not{\epsilon}_0 u_-(p_2)$$

Then

$$\begin{aligned}
B_{1,2}(+,+) &= -\tfrac{1}{2}\, s_-(p_1,b)^2 s_+(a,p_1)s_+(a,p_2) \\
B_{1,2}(-,-) &= -\tfrac{1}{2}\, s_-(p_1,a)^2 s_+(b,p_1)s_+(b,p_2) \\
B_1(+,-) &= +\frac{1}{\beta^2 s}(2t + \sigma_m - \beta\delta_m - \delta_m^2/s)K_0 \\
B_1(-,+) &= +\frac{1}{\beta^2 s}(2t + \sigma_m + \beta\delta_m - \delta_m^2/s)K_0 \\
B_1(0,+) &= +\frac{1}{\beta m_\mathrm{W}}(t(1 - \delta_m/s) + {m_\mathrm{W}}^2(1 - \beta + \delta_m/s))K_+ \\
B_1(0,-) &= +\frac{1}{\beta m_\mathrm{W}}(t(1 - \delta_m/s) + {m_\mathrm{W}}^2(1 + \beta + \delta_m/s))K_- \\
B_1(+,0) &= -\frac{1}{\beta m_\mathrm{Z}}(t(1 + \delta_m/s) + {m_\mathrm{Z}}^2(1 - \beta - \delta_m/s))K_+ \\
B_1(-,0) &= -\frac{1}{\beta m_\mathrm{Z}}(t(1 + \delta_m/s) + {m_\mathrm{Z}}^2(1 + \beta - \delta_m/s)K_- \\
B_1(0,0) &= +\frac{1}{\beta^2 m_\mathrm{W} m_\mathrm{Z}}(t(1 - \delta_m^2/s^2) + 4{m_\mathrm{Z}}^2{m_\mathrm{W}}^2/s)K_0
\end{aligned}$$

$$\begin{aligned}
B_2(+,-) &= -\frac{1}{\beta^2 s}(2u + \sigma_m + \beta\delta_m - \delta_m^2/s)K_0 \\
B_2(-,+) &= -\frac{1}{\beta^2 s}(2u + \sigma_m - \beta\delta_m - \delta_m^2/s)K_0 \\
B_2(0,+) &= -\frac{1}{\beta m_\mathrm{W}}(u(1 - \delta_m/s) + {m_\mathrm{W}}^2(1 + \beta + \delta_m/s))K_+ \\
B_2(0,-) &= -\frac{1}{\beta m_\mathrm{W}}(u(1 - \delta_m/s) + {m_\mathrm{W}}^2(1 - \beta + \delta_m/s))K_- \\
B_2(+,0) &= +\frac{1}{\beta m_\mathrm{Z}}(u(1 + \delta_m/s) + {m_\mathrm{Z}}^2(1 + \beta - \delta_m/s))K_+
\end{aligned}$$

$$
\begin{aligned}
B_2(-,0) &= +\frac{1}{\beta m_{\rm Z}}(u(1+\delta_m/s)+{m_{\rm Z}}^2(1-\beta-\delta_m/s))K_- \\
B_2(0,0) &= -\frac{1}{\beta^2 m_{\rm W} m_{\rm Z}}(u(1-\delta_m^2/s^2)+4{m_{\rm Z}}^2{m_{\rm W}}^2/s)K_0
\end{aligned}
$$

$$
\begin{aligned}
B_3(+,+) &= B_3(-,-) \;=\; 0 \\
B_3(+,-) &= B_3(-,+) \;=\; -2K_0 \\
B_3(0,\pm) &= -\frac{1}{\beta m_{\rm W}}(s-2\sigma_m+\delta_m^2/s)K_\pm \;=\; -B_3(\pm,0) \\
B_3(0,0) &= -\frac{1}{m_{\rm W} m_{\rm Z}}(s+\sigma_m)K_0
\end{aligned}
$$

(d) High-energy limit: $q_1^\mu \sim q\,a^\mu$, $q_0^\mu \sim q\,b^\mu$

$$
\begin{aligned}
& s_-(b,p_1)^2 s_+(a,p_1)s_+(a,p_2) \doteq 4u\sqrt{tu}/s \\
& s_-(a,p_1)^2 s_+(b,p_1)s_+(b,p_2) \doteq 4t\sqrt{tu}/s \\
& K_0 \doteq \sqrt{tu-{m_{\rm W}}^2{m_{\rm Z}}^2} \;\;,\;\; K_+ \doteq u\sqrt{2/s} \;\;,\;\; K_- \doteq t\sqrt{2/s}
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{M}(+,+) &\doteq \frac{4\hbar g_{\rm W}}{s}\big((v_u+a_u)u+(v_d+a_d)t\big)\sqrt{u/t} \\
\mathfrak{M}(-,-) &\doteq \frac{4\hbar g_{\rm W}}{s}\big((v_u+a_u)u+(v_d+a_d)t\big)\sqrt{t/u} \\
\mathfrak{M}(+,-) &\sim \mathfrak{M}(-,+) \sim \mathfrak{M}(0,\pm) \sim \mathfrak{M}(\pm,0) \sim 0 \\
\mathfrak{M}(0,0) &\sim 2\hbar g_{\rm W} g_{\rm WWZ} c_{\rm W}\frac{\sqrt{tu}}{s}
\end{aligned}
$$

4. The process $\bar{D}(p_1)\,U(p_2) \;\to\; W^+(q_1,\epsilon^w)\,\gamma(q_2,\epsilon^0)$

   This is obtained from the previous case by

$$
m_{\rm Z}\to 0 \;\;,\;\; (v_u+a_u)\to Q_{\rm U} \;\;,\;\; (v_d+a_d)\to Q_{\rm D} \;\;,\;\; g_{\rm WWZ}\to Q_{\rm W}
$$

   and disregarding the longitudinal Z polarizations. Only $\mathfrak{M}(+,+)$ and $\mathfrak{M}(-,-)$ survive in the high-energy limit.

<u>Solution to exercise</u> 236

Since we are aiming for a zero result we can drop all overall factors. The following FORM program does the hard work:

```
nwrite statistics;
V k,q1,q2,e1,e2,h1,h2,r,v1,v2; S m,x,y,z,r2,fi,mh;
L N = (g_(1,k)+m)*g_(1,e2)*(g_(1,k)-g_(1,q2)+m)*
      (g_(1,k)+g_(1,q1)+m)*g_(1,e1)
    - (g_(1,k)+m)*g_(1,h2)*(g_(1,k)-g_(1,q2)+m)*
      (g_(1,k)+g_(1,q1)+m)*g_(1,h1);  trace4,1;
id k = z*r - x*q1 + y*q2; id z^2 = 1; .sort
id z = 0; id r.v1?*r.v2? = r2*v1.v2/fi;
```

```
id h1.q1 = 0; id h1.q2 = 0; id h2.q1 = 0; id h2.q2 = 0;
id e1.q1 = 0; id e2.q2 = 0; .sort
id h1 = e1 - q1*e1.q2*2/mh^2; id h2 = e2 - q2*e2.q1*2/mh^2; .sort
id q1.q1 = 0; id q2.q2 = 0; id q1.q2 = mh^2/2;
id r.r = r2; id q1.e2*q2.e1 = 1; b r2;
print;
.end
```

The loop integrand is now seen to be

$$\begin{aligned} N &\propto \hat{k}^2\left(\frac{32}{4-2\epsilon}-8\right)\frac{M}{{m_{\rm H}}^2}-8Mxy+8\frac{M^3}{{m_{\rm H}}^2} \\ &\propto \frac{\epsilon}{2-\epsilon}\hat{k}^2+(M^2-{m_{\rm H}}^2xy) \end{aligned}$$

After using Wick rotation and the t-shell formula, and with $a = M^2 - {m_{\rm H}}^2xy$, we have the integral

$$\int\limits_0^\infty dt\,\left(\frac{\epsilon}{2-\epsilon}t^{2-\epsilon}-a\right)(t+a)^{-3} =$$
$$\frac{\epsilon}{2-\epsilon}\left(a^{-\epsilon}\frac{\Gamma(\epsilon)\Gamma(3-\epsilon)}{2}\right)-a\left(a^{-1-\epsilon}\frac{\Gamma(1+\epsilon)\Gamma(2-\epsilon)}{2}\right)=0$$

<u>Solution to exercise</u> 237

1. This follows directly from the cutting rule if we take the state i to be the Higgs, state f to be the two gluons, and restrict the intermediate state k to the fermion-antifermion pair.

2. The expression is obtained by simply app;lying the Feynman rules. Note that the cut propagators of p_1 and p_2 are on-shell, leading to relatively simple denominators. Also recall that to the right of the cutting line we have to also complex conjugate the coupling constant i's. The factor 1/2 is, surprisingly, the symmetry factor for indistinguishable gluons![6] Using $q_{1,2}\cdot\eta_{1,2}=0$, $(p_1+p_2\cdot\eta_{1,2})=0$ and$(q_1+q_2\cdot p_1-p_2)=0$ we have

$$\mathcal{T}(q_1,q_2)=2M(\eta_1\cdot\eta_2)(t\cdot r)+4M(r\cdot\eta_1)(r\cdot\eta_2)$$

   with $r^\mu = p_1^\mu - p_2^\mu$ and $t^\mu = q_1^\mu - q_2^\mu$.

3. The phase space can be written as

$$dV=\frac{\beta}{16\pi}dc\ \ ,\ \ \beta^2=1-\frac{4M^2}{{m_{\rm H}}^2}<1$$

   where c is the cosine of the angle between $\vec{p}_1$ and $\vec{q}_1$ in the rest frame:

$$(r\cdot t)=2(p_1\cdot q_1)-2(p_1\cdot q_2)=-{m_{\rm H}}^2\beta c\ \ ,\ \ (p_1\cdot q_1)={m_{\rm H}}^2(1-\beta c)$$

[6]To understand this, consider computing the cross section for $e^+e^-\to\gamma\gamma$ where we keep the *photon* directions fixed and integrate over the *fermion* solid angle.

Therefore

$$Y = \int dV \, \frac{r \cdot t}{p_1 \cdot q_1} = \int dV \, \frac{-4\beta c}{1 - \beta c}$$
$$= \frac{\beta}{4\pi} \int_{-1}^{1} dc \left(1 - \frac{1}{1 - \beta c} \right) = \frac{1}{4\pi} \left(2\beta - \log \left(\frac{1+\beta}{1-\beta} \right) \right)$$

The second integral is a bit more involved: we can write, with $P^\mu = q_1^\mu + q_2^\mu$,

$$Y^{\mu\nu} = \int dV \, \frac{r^\mu r^\nu}{p_1 \cdot q_1} = A \left(g^{\mu\nu} - \frac{P^\mu P^\nu}{{m_{\rm H}}^2} \right) + B \, t^\mu t^\nu$$
$$A = \frac{1}{2} \left(g_{\mu\nu} + \frac{t_\mu t_\nu}{{m_{\rm H}}^2} \right) Y^{\mu\nu} = \frac{\beta}{32\pi} \int_{-1}^{1} dc \, \frac{(r \cdot r) + (t \cdot r)^2 / {m_{\rm H}}^2}{(p_1 q_1)}$$
$$= \frac{\beta}{8\pi} \int_{-1}^{1} dc \left(-1 - \beta c + \frac{1 - \beta^2}{1 - \beta c} \right) = \frac{1}{4\pi} \left(-\beta + \frac{1 - \beta^2}{2} \log \left(\frac{1+\beta}{1-\beta} \right) \right)$$

Therefore

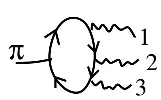
$$\int dV \, \frac{\mathcal{T}(q_1, q_2)}{p_1 \cdot q_1} = 2M(Y + 2A)(\eta_1 \cdot \eta_2) = -\frac{\beta^2 M}{2\pi {m_{\rm H}}^2} \log \left(\frac{1+\beta}{1-\beta} \right) {F_1}^{\mu\nu} F_{2\mu\nu}$$

Combining all factors we find precisely the right result, once we realize that, in Eq.(18.58),

$$\Im \Big(\mathrm{Li}_2(1/x_+) + \mathrm{Li}_2(1/x_-) \Big) = \pi \log \left(\frac{1+\beta}{1-\beta} \right) \quad , \quad 0 < \beta < 1$$

<u>Solution to exercise</u> 238

1. One diagram is



   To otain the other ones we can either consider all permutations of $(1, 2, 3)$, or only the *cyclic* permutations, and take both directions for the fermion line. This last choice is simplest. If the pion were a vector particle, Furry's theorem would assert that both fermion flows give the same result; but since the pion is scalar, Furry's theorem tells us that the fermion flows give an *opposite* result. The three cyclic permutations of the photons therefore vanish separately.

2. Assuming conservation of charge conjugation, and with $C = (-1)$ for each photon, the pion must have $C = (-1)^2 = (+1)$. A final state of three photons has charge conjugation $C = (-1)^3 = (-1)$, and the decay is forbidden. Note that, from the simple fact that $\pi^0 \to \gamma\gamma$ occurs but $\pi^0 \to \gamma\gamma\gamma$ does not, we might in this way *conclude* that photons have $C = (-1)$ and pions $C = (+1)$.

3. For any odd number of photons the same charge conjugation argument applies, as does the Furry argument.

Solution to exercise 239

Using ${F_j}^{\mu\nu} = q_j^\mu \epsilon^\nu - q_j^\nu \epsilon^\mu$ we have

$$\begin{aligned}
\mathfrak{M}_\pi &= \frac{\alpha}{4\pi f_\pi}\epsilon_{\mu\nu\rho\sigma}{F_1}^{\mu\nu}{F_2}^{\rho\sigma} \doteq \frac{\alpha}{\pi f_\pi}\epsilon_{\mu\nu\rho\sigma}\epsilon_1^\mu\epsilon_2^\nu q_1^\rho q_2^\sigma \\
\langle|\mathfrak{M}_\pi|^2\rangle &= \left(\frac{\alpha}{\pi f_\pi}\right)^2\left(-2\left\{\begin{array}{cc} q_1 & q_2 \\ q_1 & q_2 \end{array}\right\}\right) = \frac{\alpha^2 m_\pi^4}{2\pi^2 f_\pi^2} \\
\Gamma(\pi^0 \to \gamma\gamma) &= \frac{1}{2m_\pi}\frac{\alpha^2 m_\pi^4}{2\pi^2 f_\pi^2}\frac{4\pi}{32\pi^2}\frac{1}{2} \quad = \quad \frac{\alpha^2 m_\pi^3}{64\pi^3 f_\pi^2}
\end{aligned}$$

Solution to exercise 240

If $s = 0$ the loop integrals in A_1 and A_3 become simple. Using the definitions for $J_{2\omega,n}$ of Eq.(19.193) we have in PV regularization, with $L(\mu) = \mu^2 \log(\Lambda_0^2/\mu^2)$ and dropping finite terms:

$$\begin{aligned}
A_1 &= \frac{i}{(4\pi)^2}\,4g_f^2\left(\Lambda_0^2 - 3L(m)\right) \\
A_2 &= -\frac{i}{(4\pi)^2}\,\lambda\left(2\Lambda_0^2 - L(\tilde{m}_1) - L(\tilde{m}_2)\right) \\
A_3 &= \frac{i}{(4\pi)^2}\,g_s^2\left(\frac{1}{\tilde{m}_1^2}L(\tilde{m}_1) + \frac{1}{\tilde{m}_2^2}L(\tilde{m}_2)\right)
\end{aligned}$$

The quadratic divergence will disappear if we take $\lambda = 2g_f^2$, with no constraint on the masses, as in Eq.(18.84) (in dimensional regularization, we actually have the *weaker* constraint

$$2g_f^2(m^2)^{1-\epsilon} = \lambda((\tilde{m}_1^2)^{1-\epsilon} + (\tilde{m}_2^2)^{1-\epsilon})$$

if this is to hold for 'general' ϵ then we need $m = \tilde{m}_1 = \tilde{m}_2$ so again $\lambda = 2g_f^2$). Let us now take $\tilde{m}_{1,2} = \tilde{m}$ for simplicity, and write $g_s = 2k\tilde{m}$. Then the logarithmic divergence will also drop out provided

$$\frac{m^2}{\tilde{m}^2} = \frac{1}{3} + \frac{2}{3}\left(\frac{k}{g_f}\right)^2$$