Solutions by

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to

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Solutions to Chapter 1

Exercise 1.1 Verify explicitly that $\|\cdot\|_1$ is a norm on $\mathbb{C}^n$

Solution: We need to verify those three conditions in Definition 1.13 for $\|\cdot\|_1$:

1. Since $\|x\|_1 = \sum_{j=1}^n |x_j|$, it is obvious that $\|x\|_1 \geq 0$. $\|x\|_1 = 0$ if and only if $x_j = 0$ for all $j = 1, \cdots, n$, i.e. $x = 0$;
2. $\|\alpha x\|_1 = \sum_{j=1}^n |\alpha x_j| = |\alpha| \sum_{j=1}^n |x_j| = |\alpha| \|x\|_1$ for all $\alpha \in \mathbb{F}$;
3. $\|x+y\|_1 = \sum_{j=1}^n |x_j + y_j|$. Since
$$|x_j + y_j| \leq |x_j| + |y_j|,$$
we have
$$\sum_{j=1}^n |x_j + y_j| \leq \sum_{j=1}^n \big(|x_j| + |y_j|\big) = \sum_{j=1}^n |x_j| + \sum_{j=1}^n |y_j| = \|x\|_1 + \|y\|_1,$$
i.e. $\|x+y\|_1 \leq \|x\|_1 + \|y\|_1$.

Exercise 1.2 Verify explicitly the steps in the proof of the Hölder inequality.

Solution: Starting from (1.3) that
$$|\sum_{j=1}^n x_j y_j| \leq \frac{\lambda^p}{p}\|x\|_p^p + \frac{\lambda^{-q}}{q}\|y\|_q^q,$$
where the minimum of the right hand side is achieved at $\lambda = \lambda_0 = \|y\|_q^{1/p}/\|x\|_p^{1/q}$. Since
$$\frac{1}{p} + \frac{1}{q} = 1,$$
we have
$$\frac{\lambda_0^p}{p}\|x\|_p^p = \frac{1}{p}\|x\|_p^{p(1-1/q)}\|y\|_q = \frac{1}{p}\|x\|_p\|y\|_q,$$
and
$$\frac{\lambda_0^{-q}}{q}\|y\|_q^q = \frac{1}{q}\|x\|_p\|y\|_q^{q(1-1/p)} = \frac{1}{q}\|x\|_p\|y\|_q.$$
Thus,
$$\frac{\lambda_0^p}{p}\|x\|_p^p + \frac{\lambda_0^{-q}}{q}\|y\|_q^q = \|x\|_p\|y\|_q.$$
Consequently, we have the Hölder inequality
$$|\sum_{j=1}^n x_j y_j| \leq \|x\|_p\|y\|_q.$$

Exercise 1.3 Prove Theorem 1.23.

Solution: Check those three conditions in Definition 1.13 for $\|\cdot\|_\infty$:

1. Since $\|x\|_\infty = \max_{j=1,\ldots,n}|x_j|$, it is obvious that $\|x\|_\infty \geq 0$. $\|x\|_\infty = 0$ if and only if $x = 0$;
2. $\|\alpha x\|_\infty = \max_{j=1,\ldots,n}|\alpha x_j| = |\alpha|\max_{j=1,\ldots,n}|x_j| = |\alpha|\|x\|_\infty$ for all $\alpha \in \mathbb{F}$;
3. $\|x+y\|_\infty = \max_{j=1,\ldots,n}|x_j+y_j|$. Since
$$|x_j + y_j| \leq |x_j| + |y_j|,$$
we have
$$\max_{j=1,\ldots,n}|x_j+y_j| \leq \max_{j=1,\ldots,n}\big(|x_j|+|y_j|\big) \leq \max_{i=1,\ldots,n}|x_i| + \max_{j=1,\ldots,n}|y_j| = \|x\|_\infty + \|y\|_\infty,$$
i.e. $\|x+y\|_\infty \leq \|x\|_\infty + \|y\|_\infty$.

Since
$$\lim_{p\to\infty}\left(\sum_{j=1}^n |x_j|^p\right)^{1/p} \leq \lim_{p\to\infty}\left(n \max_{j=1,\ldots,p}|x_j|^p\right)^{1/p} = \lim_{p\to\infty} n^{1/p}\max_{j=1,\ldots,p}|x_j| = \max_{j=1,\ldots,p}|x_j|,$$
and
$$\lim_{p\to\infty}\left(\sum_{j=1}^n |x_j|^p\right)^{1/p} \geq \lim_{p\to\infty}\left(\max_{j=1,\ldots,p}|x_j|^p\right)^{1/p} = \max_{j=1,\ldots,p}|x_j|,$$
we have $\lim_{p\to\infty}\|x\|_p = \|x\|_\infty$. For any $p > 1$,
$$\max_{j=1,\ldots,p}|x_j| = \left(\max_{j=1,\ldots,p}|x_j|^p\right)^{1/p} \leq \left(\sum_{j=1}^n |x_j|^p\right)^{1/p},$$
i.e. $\|x\|_\infty \leq \|x\|_p$.

Exercise 1.4 Prove Lemma 1.25.

Solution: Since $(|y_1|+|y_2|+\cdots+|y_n|)^p \geq |y_1|^p + |y_2|^p + \cdots + |y_n|^p$ is true for $n = 2$ by Lamma 1.24, we can prove it by induction. Suppose
$$(|y_1|+|y_2|+\cdots+|y_j|)^p \geq |y_1|^p + |y_2|^p + \cdots + |y_j|^p$$
is true for $n = 2, 3, \ldots, k$, where k is an upper bound integer. Thus, we have
$$(|y_1|+|y_2|+\cdots+|y_k|+|y_{k+1}|)^p \geq (|y_1|+|y_2|+\cdots+|y_k|)^p + |y_{k+1}|^p \quad \text{(case of } n=2\text{)},$$
and
$$(|y_1|+|y_2|+\cdots+|y_k|)^p \geq |y_1|^p + |y_2|^p + \cdots + |y_k|^p \quad \text{(case of } n=k\text{)}.$$
Consequently, we have
$$(|y_1|+|y_2|+\cdots+|y_k|+|y_{k+1}|)^p \geq |y_1|^p + |y_2|^p + \cdots + |y_k|^p + |y_{k+1}|^p,$$
i.e. the inequality is true for all $n \geq 2$.

Exercise 1.5 Show that there are "holes" in the rationals by demonstrating that $\sqrt{p}$ cannot be rational for a prime integer p. Hint: if p is rational then there are relatively prime integers m, n such that $\sqrt{p} = m/n$, so $m^2 = pn^2$.

Solution: Starting from the hint. If $m^2 = pn^2$, p divides m^2 which implies p divides m. Hence, there exists an integer k for which $m = pk$. Thus, we have $pk^2 = n^2$, which implies p also divides n. This is a contradiction to the assumption that m, n is relatively prime.

Exercise 1.6 Write the vector $w = (1, -2, 3)$ as a linear combination of each of the bases $\{e^{(i)}\}, \{u^{(i)}\}, \{v^{(i)}\}$ of Example 1.62.

Solution:

1. $w = e^{(1)} - 2e^{(2)} + 3e^{(3)}$;
2. $w = 3u^{(1)} - 5u^{(2)} + 3u^{(3)}$;
3. $w = v^{(1)} - v^{(2)} + v^{(3)}$;

Exercise 1.7 (a) Let V be a vector space. Convince yourself that V and $\{\Theta\}$ are subspaces of V and every subspace W of V is a vector space over the same field as V.

(b) Show that there are only two nontrivial subspaces of $\mathbb{R}^3$. (1) A plane passing through the origin and (2) A line passing through the origin.

Solution: (a) Since V is a vector space, it is obvious that $\alpha u + \beta v \in V$ for all $\alpha, \beta \in \mathbb{F}$ and $u, v \in V$. Hence, V is a subspace of V itself.
Since V is a vector space, $\{\Theta\} \subset V$. $\alpha\Theta + \beta\Theta = \Theta \in \{\Theta\}$ for all $\alpha, \beta \in \mathbb{F}$ and $\Theta \in \{\Theta\}$. Hence, $\{\Theta\}$ is a subspace of V.
If W is a subspace of V, using

$$\alpha u + \beta v \in W, \text{ for all } \alpha, \beta \in \mathbb{F} \text{ and } u, v \in W,$$

and the fact that V is a vector space, it is easy to verify that W satisfies the properties (Definition 1.1):

- For every pair $u, v \in W$, there is defined a unique vector $w = u + v \in W$;
- For every $\alpha \in \mathbb{F}, u \in W$, there is defined a unique vector $z = \alpha u \in W$;
- Commutative, Associative and Distributive laws
  1. $u + v = v + u$ (inherited from vector space V);
  2. $(u + v) + w = u + (v + w)$ (inherited from vector space V);
  3. $\Theta \in W$ (Let $\alpha = 0, \beta = 0$);
  4. For every $u \in W$ there is a $-u \in W$ (Let $\alpha = -1, \beta = 0$);
  5. $1u = u$ for all $u \in W$ (inherited from vector space V);
  6. $\alpha(\beta u) = (\alpha\beta)u$ for all $\alpha, \beta \in \mathbb{F}$ (inherited from vector space V);
  7. $(\alpha + \beta)u = \alpha u + \beta u$ (inherited from vector space V);

8. $\alpha(u+v) = \alpha u + \alpha v$ (inherited from vector space V).

Thus, W is a vector space.

(b) First, we show that (1) A plane passing through the origin and (2) A line passing through the origin are subspaces of $\mathbb{R}^3$.
(1) A plane passing through the origin can be represented as any vector $v = (x, y, z) \in \mathbb{R}^3$ satisfies

$$ax + by + cz = 0,$$

where $t = (a, b, c) \in \mathbb{R}^3$ is a non-zero vector. This is essentially saying that $v \in N(t)$. According to Lemma 1.81, $N(t)$ is a subspace of $\mathbb{R}^3$.
(2) A line passing through the origin can be represented as any vector $v = (x, y, z) \in \mathbb{R}^3$ satisfies

$$x = \alpha t \text{ for all } \alpha \in \mathbb{F},$$

in which $t = (a, b, c) \in \mathbb{R}^3$ is a non-zero vector. This is essentially saying that $v \in R(t)$. According to Lemma 1.81, $R(t)$ is a subspace of $\mathbb{R}^3$.
Next, we show that any nontrivial subspace of $\mathbb{R}^3$ is essentially (1) or (2). $\mathbb{R}^3$ can be represented by the span of any 3 of independent vector $u^{(1)}, u^{(2)}, u^{(3)}$ belong to $\mathbb{R}^3$. Therefore, any nontrivial subspace of $\mathbb{R}^3$ another than $\mathbb{R}^3$ and Θ can only be $R(u^{(i)})$ $(i = 1, 2, 3)$ or $R([u^{(i)}, u^{(j)}]) = N(u^{(6-i-j)})$ $(i, j = 1, 2, 3,\ i \neq j)$, of which the former is an instance of (1) and the latter is an instance of (2).

Exercise 1.8 (a) Prove or give a counterexample to the following statement: If $v^{(1)}, \ldots, v^{(k)}$ are elements of a vector space V and do not span V, then $v^{(1)}, \ldots, v^{(k)}$ are linearly independent.
(b) Prove that if $v^{(1)}, \ldots, v^{(m)}$ are linearly independent, then any subset $v^{(1)}, \ldots, v^{(k)}$ with $k < m$ is also linearly independent.
(c) Does the same hold true for linearly dependent vectors? If not, give a counterexample.

Solution: (a) False. For example, $v^{(1)} = (1, 0, 0), v^{(2)} = (0, 1, 0), v^{(3)} = (1, 1, 0)$ is linearly dependent and they do not span $\mathbb{R}^3$.

(b) Proof by contradiction. If $v^{(1)}, \ldots, v^{(k)}$ with $k < m$ is linearly dependent, i.e. there exists a non-zero solution $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ for

$$\alpha_1 v^{(1)} + \alpha_2 v^{(2)} + \cdots + \alpha_k v^{(k)} = 0,$$

then $(\alpha_1, \alpha_2, \ldots, \alpha_k, \alpha_{k+1}, \ldots, \alpha_m)$ with $\alpha_{k+1} = \cdots = \alpha_m = 0$ is a non-zero solution for

$$\alpha_1 v^{(1)} + \alpha_2 v^{(2)} + \cdots + \alpha_k v^{(m)} = 0,$$

which contradicts to the assumption that $v^{(1)}, \ldots, v^{(m)}$ is linearly independent.

(c) It is not true for linearly dependent vectors. In the example of (a), $v^{(1)}, v^{(2)}, v^{(3)}$ is linearly dependent, but $v^{(1)}, v^{(2)}$ is linearly independent.

Exercise 1.9 Show that the basic monomials $\{1, x, x^2, x^3, \dots, x^n\}$ are linearly independent in the space of polynomials in x. Hint: Use the Fundamental Theorem of Algebra which states that a non-zero polynomial of degree $n \geq 1$ has at most n distinct real roots (and exactly n complex roots, not necessarily distinct).

Solution: Starting from the hint. Since for non-zero vector $u = (\alpha_0, \alpha_1, \dots, \alpha_n)$, the solution of

$$\alpha_0 + \alpha_1 x + \cdots + \alpha_n x^n = 0$$

respected to x has at most n distinct values, which means that no non-zero $u \in \mathbb{C}^{n+1}$ can satisfy the above equation for arbitrary x. Thus, only $u = \Theta$ solve the above equation irrespectively of the value of x, i.e. $\{1, x, x^2, x^3, \dots, x^n\}$ are linearly independent.

Exercise 1.10 The Wronskian of a pair of differentiable, real-valued functions f and g is the scalar function

$$W[f(x), g(x)] = \det \begin{pmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{pmatrix} = f(x)g'(x) - f'(x)g(x).$$

(a) Prove that if f, g are linearly dependent, then $W[f(x), g(x)] \equiv 0$.

(b) Prove that if $W[f(x), g(x)] \not\equiv 0$, then f, g are linearly independent.

(c) Let $f(x) = x^3$ and $g(x) = |x|^3$. Prove that f and g are twice continuously differentiable on $\mathbb{R}$, are linearly independent but $W[f(x), g(x)] \equiv 0$. Thus (b) is sufficient but not necessary. Indeed, one can show the following: $W[f(x), g(x)] \equiv 0$ iff f, g satisfy a second-order linear ordinary differential equation.

Solution: (a) If f, g are linearly dependent, there exist $\alpha, \beta \in \mathbb{F}$ and α, β are not all zeros, satisfying

$$\alpha f(x) + \beta g(x) \equiv 0,$$

which also implies

$$\alpha f'(x) + \beta g'(x) \equiv 0.$$

Assume that $\alpha \neq 0$, then we have $f(x) = -\frac{\beta}{\alpha} g(x)$ and $f'(x) = -\frac{\beta}{\alpha} g'(x)$. Thus,

$$W[f(x), g(x)] = f(x)g'(x) - f'(x)g(x) = -\frac{\beta}{\alpha} g(x)g'(x) + \frac{\beta}{\alpha} g'(x)g(x) \equiv 0.$$

(b) This is the contrapositive of (a). Thus, it is true since (a) is true.

(c) $f(x)$ is obviously twice continuously differentiable on $\mathbb{R}$ since it is a polynomial function of order 3.

$$g(x) = \begin{cases} x^3, & x \geq 0 \\ -x^3, & x < 0 \end{cases}, \; g'(x) = \begin{cases} 3x^2, & x \geq 0 \\ -3x^2, & x < 0 \end{cases}, \; g''(x) = \begin{cases} 6x, & x \geq 0 \\ -6x, & x < 0 \end{cases}.$$

It is easy to check that

$$\lim_{x \to 0^+} \tilde{g}(x) = \lim_{x \to 0^-} \tilde{g}(x) = 0,$$

in which $\tilde{g}$ can be g, g' or g''. Hence, $g(x)$ is twice continuously differentiable on $\mathbb{R}$.

The equation for $\alpha f(x) + \beta g(x) \equiv 0$ can be considered separately for $x \geq 0$ and $x < 0$. For $x \geq 0$, the solution is $\alpha = -\beta$; For $x < 0$, the solution is $\alpha = \beta$. It implies that $-\beta = \beta$, thus $\alpha = \beta = 0$ is the only possible solution, i.e. f, g are linearly independent. We can directly check that

$$W[x^3, |x^3|] = \left\{ \begin{array}{ll} 3x^5 - 3x^5, & x \geq 0 \\ -3x^5 + 3x^5, & x < 0 \end{array} \right\} \equiv 0.$$

Exercise 1.11 Prove the following theorem. Suppose that V is an $n \geq 1$ dimensional vector space. Then the following hold.

(a) Every set of more than n elements of V is linearly dependent.

(b) No set of less than n elements spans V.

(c) A set of n elements forms a basis iff it spans V.

(d) A set of n elements forms a basis iff it is linearly independent.

Solution: (a) Suppose that one set is $T = [v^{(1)}, v^{(2)}, \ldots, v^{(m)}] \in \mathbb{C}^{n\times m}$, with $m > n$. According to Theorem 1.82 and Corollary 1.83, we have

$$\dim\bigl(N(T)\bigr) + \dim\bigl(R(T)\bigr) = m,$$

where $\dim(X)$ denotes the dimension of space X. Since the span of T is a subspace of V, we have $\dim\bigl(R(T)\bigr) = r \leq n$. Thus, $\dim\bigl(N(T)\bigr) = m - r > 0$, which means the null space of T is not $\{\Theta\}$, i.e. $\{v^{(1)}, v^{(2)}, \ldots, v^{(m)}\}$ is linearly dependent.

(b) In the example of (a), $\dim\bigl(N(T)\bigr)$ and $\dim\bigl(R(T)\bigr)$ are both non-negative integers. If $m < n$, we have $\dim\bigl(R(T)\bigr) \leq m < n$, which means the span of T can not be V itself.

(c) If a set of n elements $\{v^{(1)}, v^{(2)}, \ldots, v^{(n)}\}$ forms a basis, any arbitrary $u \in \mathbb{C}^n = V$ can be expressed by a linear combination of than set. Hence, $\{v^{(1)}, v^{(2)}, \ldots, v^{(n)}\}$ spans V.
Conversely, if $\{v^{(1)}, v^{(2)}, \ldots, v^{(n)}\}$ spans V, every elements in V can be expressed as a linear combination of $\{v^{(1)}, v^{(2)}, \ldots, v^{(n)}\}$, we need to prove that the expression is unique. Let $T = [v^{(1)}, v^{(2)}, \ldots, v^{(n)}]$. Since $\{v^{(1)}, v^{(2)}, \ldots, v^{(n)}\}$ spans V, we have $\dim\bigl(R(T)\bigr) = n$. Thus, $\dim\bigl(N(T)\bigr) = n - n = 0$, which indicates that the linear expression of any element in V by $\{v^{(1)}, v^{(2)}, \ldots, v^{(n)}\}$ is unique.

(d) Follow by the same notation in above solution, if $\{v^{(1)}, v^{(2)}, \ldots, v^{(n)}\}$ forms a basis, then $\dim\bigl(R(T)\bigr) = n$ and thereby $\dim\bigl(N(T)\bigr) = 0$. Thus, the null space of T is $\{\Theta\}$, i.e. $\{v^{(1)}, v^{(2)}, \ldots, v^{(n)}\}$ is linearly independent.
Conversely, if $\{v^{(1)}, v^{(2)}, \ldots, v^{(n)}\}$ is linearly independent, $\dim\bigl(N(T)\bigr) = 0$. Thus, $\dim\bigl(R(T)\bigr) = n$, which means the span of $\{v^{(1)}, v^{(2)}, \ldots, v^{(n)}\}$ is V. Therefore, every element in V can be expressed as a linear combination of $\{v^{(1)}, v^{(2)}, \ldots, v^{(n)}\}$, and the expression is unique due to $N(T) = \{\Theta\}$, i.e. $\{v^{(1)}, v^{(2)}, \ldots, v^{(n)}\}$ forms a basis.

Exercise 1.12 The Legendre polynomials $P_n(x), n = 0, 1, \ldots$ are the ON set of polynomials on the real space $L_2[-1, 1]$ with inner product

$$(f, g) = \int_{-1}^{1} f(x)g(x)dx,$$

obtained by applying the Gram-Schmidt process to the monomials $1, x, x^2, x^3, \ldots$ and defined uniquely by the requirement that the coefficient of t^n in $P_n(t)$ is positive. (In fact they form an ON *basis* for $L_2[-1, 1]$.) Compute the first four of these polynomial.

Solution: First, let $N_0(x) = 1$. $P_0(x)$ should have have the form $P_0(x) = c > 0$ and $(P_0, P_0) = 1$, where c is a constant. Hence, $P_0(x) = 1/\sqrt{2}$. By applying Gram-Schmidt process

$$N_1(x) = x - \frac{(x, N_0)}{(N_0, N_0)} N_0 = x.$$

Thus, $P_1(x) = N_1/\sqrt{(N_1, N_1)} = \sqrt{3/2}x$. Similarly,

$$N_2(x) = x^2 - \frac{(x^2, N_0)}{(N_0, N_0)} N_0 - \frac{(x^2, N_1)}{(N_1, N_1)} N_1 = x^2 - \frac{1}{3}.$$

Then,

$$P_2(x) = N_2/\sqrt{(N_2, N_2)} = \frac{3\sqrt{10}}{4}\left(x^2 - \frac{1}{3}\right).$$

$$N_3(x) = x^3 - \frac{(x^3, N_0)}{(N_0, N_0)} N_0 - \frac{(x^3, N_1)}{(N_1, N_1)} N_1 - \frac{(x^3, N_2)}{(N_2, N_2)} N_2 = x^3 - \frac{3}{5}x.$$

Thus,

$$P_3(x) = N_3/\sqrt{(N_3, N_3)} = \frac{5\sqrt{14}}{4}\left(x^3 - \frac{3}{5}x\right).$$

Exercise 1.13 Using the facts from the preceding exercise, show that the Legendre polynomials must satisfy a three-term recurrence relation

$$xP_n(x) = a_n P_{n+1}(x) + b_n P_n(x) + c_n P_{n-1}(x), \quad n = 0, 1, 2, \ldots$$

where we take $P_{-1}(x) \equiv 0$. (Note:If you had a general sequence of polynomials $\{p_n(x)\}$, where the highest-order term on $p_n(x)$ was a nonzero multiple of t^n, then the best you could say was that

$$xp_n(x) = \sum_{j=0}^{n+1} \alpha_j p_j(x).$$

What is special about orthogonal polynomials that leads to three-term relations?) What can you say about a_n, b_n, c_n without doing any detailed computations?

Solution: Suppose that

$$P_n(x) = k_n x^n + p_{n-1}(x),$$

in which k_n is the leading coefficient of the term x^n, and $p_{n-1}(x)$ is a polynomial of degree $\le n-1$. Then $P_{n+1}(x) - xk_{n+1}/k_n P_n(x)$ is a polynomial of degree $\le n$, and it can be uniquely expressed as the linearly combination of the ON basis $P_0(x), P_1(x), \ldots, P_n(x)$, i.e.

$$P_{n+1}(x) - \frac{k_{n+1}}{k_n} x P_n(x) = \sum_{j=0}^{n} \alpha_j P_j(x).$$

Due to the orthogonality property,

$$\Big(P_{n+1}(x) - \frac{k_{n+1}}{k_n} x P_n(x), P_j(x)\Big) = \alpha_j \Big(P_j(x), P_j(x)\Big) = \alpha_j$$

thus

$$\alpha_j = \Big(P_{n+1}(x) - \frac{k_{n+1}}{k_n} x P_n(x), P_j(x)\Big) = -\frac{k_{n+1}}{k_n}\Big(xP_n(x), P_j(x)\Big) = -\frac{k_{n+1}}{k_n}\Big(P_n(x), xP_j(x)\Big).$$

For $j < n-1$ we have the degree of $xP_j(x) < n$, which means $\big(P_n(x), xP_j(x)\big) = 0$. Hence, $\alpha_j = 0$ for $j < n-1$. Consequently, it results the three term recurrence relation

$$P_{n+1}(x) - \frac{k_{n+1}}{k_n} x P_n(x) = \alpha_n P_n(x) + \alpha_{n-1} P_{n-1}(x).$$

Furthermore, by applying inner product with $P_{n-1}(x)$ on both side of the above equation, we have

$$\alpha_{n-1} = -\frac{k_{n+1}}{k_n}\Big(xP_n(x), P_{n-1}(x)\Big) = -\frac{k_{n+1}}{k_n}\Big(P_n(x), xP_{n-1}(x)\Big),$$

in which

$$xP_{n-1}(x) = \frac{k_{n-1}}{k_n} P_n(x) + \sum_{j=0}^{n-1} \beta_j P_j(x).$$

Again, due to the orthogonality property,

$$\Big(P_n(x), xP_{n-1}(x)\Big) = \frac{k_{n-1}}{k_n}.$$

Thus,

$$\alpha_{n-1} = -\frac{k_{n-1}k_{n+1}}{k_n^2}.$$

Similarly,

$$\alpha_n = -\frac{k_{n+1}}{k_n}\Big(xP_n(x), P_n(x)\Big).$$

By writing in the form of

$$xP_n(x) = a_n P_{n+1}(x) + b_n P_n(x) + c_n P_{n-1}(x),$$

we have

$$a_n = \frac{k_n}{k_{n+1}}, \quad b_n = \Big(xP_n(x), P_n(x)\Big), \quad c_n = \frac{k_{n-1}}{k_n}.$$

Exercise 1.14 Let $L_2[\mathbb{R}, \omega(x)]$ be the space of square integrable functions on the real line, with respect to the weight function $\omega(x) = e^{-x^2}$. The inner product on this space is thus

$$(f, g) = \int_{-\infty}^{\infty} f(x)\overline{g(x)}\omega(x)dx$$

The Hermite polynomials $H_n(x), n = 0, 1, \dots$ are the ON set of polynomials on $L_2[R\mathbb{R}, \omega(x)]$, obtained by applying the Gram-Schmidt process to the monomials $1, x, x^2, x^3, \dots$ and defined uniquely by the requirement that the coefficient of x_n in $H_n(x)$ is positive. (In fact they form an ON basis for $L_2[\mathbb{R}, \omega(x)]$.) Compute the first four of these polynomials. NOTE: In the study of Fourier transforms we will show that $\int_{-\infty}^{\infty} e^{-isx}e^{-x^2}dt = \sqrt{\pi}e^{-s^2/4}$. You can use this result, if you wish, to simplify the calculations.

Solution: Let $h_n(x)$ be the unnormalized polynomial of $H_n(x)$, and $h_0(x) = 1$. Since $\big(H_0(x), H_0(x)\big) = 1$, we have $H_0(x) = 1/\sqrt[4]{\pi}$.

$$h_1(x) = x - \frac{(x, h_0)}{(h_0, h_0)}h_0 = x,$$

thus

$$H_1(x) = \frac{h_1}{\sqrt{(h_1, h_1)}} = \frac{\sqrt{2}}{\sqrt[4]{\pi}}x.$$

Then

$$h_2(x) = x^2 - \frac{(x^2, h_0)}{(h_0, h_0)}h_0 - \frac{(x^2, h_1)}{(h_1, h_1)}h_1 = x^2 - \frac{1}{2},$$

and

$$H_2(x) = \frac{h_2}{\sqrt{(h_2, h_2)}} = \frac{\sqrt{2}}{\sqrt[4]{\pi}}\big(x^2 - \frac{1}{2}\big).$$

Then

$$h_3(x) = x^3 - \frac{(x^3, h_0)}{(h_0, h_0)}h_0 - \frac{(x^3, h_1)}{(h_1, h_1)}h_1 - \frac{(x^3, h_2)}{(h_2, h_2)}h_2 = x^3 - \frac{3}{2}x,$$

and

$$H_3(x) = \frac{h_3}{\sqrt{(h_3, h_3)}} = \frac{2\sqrt{3}}{3\sqrt[4]{\pi}}\big(x^3 - \frac{3}{2}x\big).$$

Exercise 1.15 Note that in the last problem, $H_0(x), H_2(x)$ contained only even powers of x and $H_1(x), H_3(x)$ contained only odd powers. Can you find a simple proof, using only the uniqueness of the Gram-Schmidt process, of the fact that $H_n(x) = (-1)^n H_n(x)$ for all n?

Solution: We can prove it by induction. Since $H_0(x) = H_0(-x) = 1/\sqrt[4]{x}$ and $H_1(x) = -H_1(-x) = (\sqrt{2}/\sqrt[4]{\pi})x$, we assume that for $k = 0, 1, \dots, n$, $H_k(x) = (-1)^k H_k(x)$. By using the Gram-Schmidt process to get $h_{n+1}(-x)$, we have

$$h_{n+1}(-x) = (-x)^{n+1} - \sum_{k=0}^{n} \frac{\big((-x)^{n+1}, H_k(-x)\big)}{\big(H_k(-x), H_k(-x)\big)}H_k(-x),$$

in which

$$\big(H_k(-x), H_k(-x)\big) = (-1)^{2k}\big(H_k(x), H_k(x)\big) = \big(H_k(x), H_k(x)\big),$$

and

$$\begin{aligned}\big((-x)^{n+1}, H_k(-x)\big)H_k(-x) &= (-1)^{n+1}(-1)^{2k}\big(x^{n+1}, H_k(x)\big)H_k(x)\\ &= (-1)^{n+1}\big(x^{n+1}, H_k(x)\big)H_k(x).\end{aligned}$$

Therefore,

$$h_{n+1}(-x) = (-1)^{n+1}\Big(x^{n+1} - \sum_{k=0}^{n}\frac{\big(x^{n+1}, H_k(x)\big)}{\big(H_k(x), H_k(x)\big)}H_k(x)\Big) = (-1)^{n+1}h_{n+1}(x).$$

After the normalization we have

$$H_{n+1}(-x) = (-1)^{n+1}H_{n+1}(x).$$

Exercise 1.16 Use least squares to fit a straight line of the form $y = bx + c$ to the data

$$\begin{array}{rrrrr} x = & 0 & 1 & 3 & 4 \\ y = & 0 & 8 & 8 & 20 \end{array}$$

in order to estimate the value of y when $x = 2.0$. Hint: Write the problem in the form

$$\begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 3 & 1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} b \\ c \end{pmatrix}.$$

Solution: Let

$$v = \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 3 & 1 \\ 4 & 1 \end{pmatrix}, \quad w = \begin{pmatrix} b \\ c \end{pmatrix}.$$

The solution w to the least square problem is the solution w of the normal equation

$$A^T A w = A^T v.$$

Thus,

$$x = \begin{pmatrix} 4 \\ 1 \end{pmatrix}.$$

i.e. $b = 4.0, c = 1.0$. For $x = 2.0, y = bx + c = 9.0$.

Exercise 1.17 Repeat the previous problem to find the best least squares fit of the data to a parabola of the form $y = ax^2 + bx + c$. Again, estimate the value of y when $x = 2.0$

Solution: In this problem, each row of matrix A should be $[x^2 \;\; x \;\; 1]$ for each value of x. Thus

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 9 & 3 & 1 \\ 16 & 4 & 1 \end{pmatrix} \quad \text{and} \quad w = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

By solving the normal equation $A^T A w = A^T v$, we have $w = [2/3 \;\; 4/3 \;\; 2]^T$. For $x = 2.0,\; y = ax^2 + bx + c = 22/3$.

Exercise 1.18 Project the function $f(t) = t$ onto the subspace of $L_2[0, 1]$ spanned by the functions $\phi(t), \psi(t), \psi(2t), \psi(2t-1)$, where

$$\phi(t) = \begin{cases} 1, & \text{for } 0 \le t \le 1 \\ 0, & \text{otherwise} \end{cases} \qquad \psi(t) = \begin{cases} 1, & \text{for } 0 \le t \le 1/2 \\ -1, & \text{for } 1/2 \le t < 1 \\ 0, & \text{otherwise} \end{cases}$$

(This is related to the Haar wavelet expansion for f.)

Solution: If we can express the projection of $f(t)$ on the subspace as

$$\mathbf{proj}[f(t)] = a\phi(t) + b\psi(t) + c\psi(2t) + d\psi(2t-1),$$

then

$$a = \frac{\big(f(t), \phi(t)\big)}{\big(\phi(t), \phi(t)\big)} = \frac{1}{2}, \quad b = \frac{\big(f(t), \psi(t)\big)}{\big(\psi(t), \psi(t)\big)} = -\frac{1}{4}$$

$$c = \frac{\big(f(t), \psi(2t)\big)}{\big(\psi(2t), \psi(2t)\big)} = -\frac{1}{8}, \quad d = \frac{\big(f(t), \psi(2t-1)\big)}{\big(\psi(2t-1), \psi(2t-1)\big)} = -\frac{1}{8}$$

Exercise 1.19 A vector space pair $V := (V, \|\cdot\|)$ is called a quasi-normed space if for every $x, y \in V$ and $\alpha \in F$, there exists a map $\|\cdot\| : V \to [\,0, \infty)$ such that

(i) $\|x\| > 0$ if $x \neq 0$ and $\|0\| = 0$. (positivity)
(ii) $\|\alpha x\| = |\alpha|\|x\|$. (homogeneity)
(iii) $\|x + y\| \le C(\|x\| + \|y\|)$ for some $C > 0$ independent of x and y.

If $C = 1$ in (iii), then V is just a normed space since (iii) is just the triangle inequality. If $C = 1, \|x\| = 0$ but $x \not\equiv 0$, then V is called a semi-normed space.

(a) Let $n \ge 1$ and for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ let $(x, y) := \sum_{j=1}^n x_j y_j$. Recall and prove that $(\cdot, \cdot)$ defines the Euclidean inner product on the finite-dimensional vector space $\mathbb{R}^n$ with Euclidean norm $(x_1^2 + \cdots + x_n^2)^{1/2}$. Note that the Euclidean inner product is exactly the same as the dot product we know from the study of vectors. In this sense, inner product is a natural generalization of dot product.

(b) Use the triangle inequality to show that any metric is a continuous mapping. From this deduce that any norm and inner product are continuous mappings.

(c) Let $0 < p < \infty$, $[a, b] \subset \mathbb{R}$ and consider the infinite-dimensional vector space $L_p[a, b]$ of p integrable functions $f : [a, b] \to \mathbb{R}$, i.e., for which $\int_a^b |f(x)|^p dx < \infty$ where we identify functions equal if they are equal almost everywhere. Use the Cauchy-Schwarz inequality to show that $(f, g) = \int_a^b f(x)g(x)dx$ is finite and also that $(\cdot, \cdot)$, defines an inner product on $L_2[a, b]$. Hint: you need to show that

$$\int_a^b |f(x)g(x)|dx \leq \Big(\int_a^b |f(x)|^2 dx\Big)^{1/2} \Big(\int_a^b |g(x)|^2 dx\Big)^{1/2}$$

Let $1/p + 1/q = 1, p > 1$. The Hölder-Minkowski inequality

$$\int_a^b |f(x)g(x)|dx \leq \Big(\int_a^b |f(x)|^p dx\Big)^{1/p} \Big(\int_a^b |g(x)|^p dx\Big)^{1/q}$$

for $f \in L_p[a, b], g \in L_p[a, b]$, generalizes the former inequality. Prove this first for step functions using the Hölder inequality, Theorem 1.21, and then in general by approximating $L_p[a, b]$ and $L_q[a, b]$ functions by step functions. Note that without the almost everywhere identification above, $L_p[a, b]$ is a semi-normed space.

(d) Show that $L_p[a, b]$ is a complete metric space for all $0 < p < \infty$ by defining $d(f, g) := (\int_a^b |f(x) - g(x)|^p dx)^{1/p}$.

(e) Show that $L_p[a, b]$ is a complete normed space (Banach space) for all $1 \leq p < \infty$ and a quasi-normed space for $0 < p < 1$ by defining $\|f\| := (\int_a^b |f(x)|^p dx)^{1/p}$.

(f) Show that $L_p[a, b]$ is a complete inner product space (Hilbert space) iff $p = 2$.

(g) $L_\infty[a, b]$ is defined as the space of essentially bounded functions $f : [a, b] \to \mathbb{R}$ for which $\|f\|_\infty[a, b] := \sup_{x \in [a,b]} |f(x)| < \infty$. $C[a, b]$ is the space of continuous functions $f : [a, b] \to \mathbb{R}$ with the same norm. Show that both these spaces are Banach spaces.

Solution: (a) It is easy to check that $(\cdot, \cdot)$ satisfy

1. $(x + y, z) = (x, z) + (y, z)$
2. $(\alpha x, y) = \alpha(x, y)$ for $\alpha \in \mathbb{F}$
3. $(x, y) = (y, x)$
4. $(x, x) = \|x\|^2 \geq 0$ and equal iff $x = 0$

Thus, $(\cdot, \cdot)$ together with vector space $\mathbb{R}^n$ defines an inner product space.

(b) Suppose that $x, x' \in \mathbb{R}^n$ and x, x' are close to each other in the sense that

$$\max_{j=1,\ldots,n} (|x_j - x'_j|) < \varepsilon,$$

where ε is an arbitrary positive real number. Apparently, in $\mathbb{R}$ the metric of two scalar is $d(x_j, x'_j) = |x_j - x'_j|, j = 1, \ldots, n$. Let

$$z^{(j)} = (x'_1, x'_2, \ldots, x'_j, x_{j+1}, \ldots, x_n), j = 1, 2, \ldots, n - 1$$

and $d(x, x')$ defines a metric in $\mathbb{R}^n$. Thus, due to that a metric must satisfies the triangle inequality

$$\begin{aligned} d(x, x') &\le d(x, z^{(1)}) + d(z^{(1)}, y) \\ &\le d(x, z^{(1)}) + d(z^{(1)}, z^{(2)}) + d(z^{(2)}, y) \\ &\le d(x, z^{(1)}) + d(z^{(1)}, z^{(2)}) + \cdots + d(z^{(n-2)}, z^{(n-1)}) + d(z^{(n-1)}, x') \\ &\le \sum_{j=1}^{n} |z_j| \le n\varepsilon. \end{aligned}$$

Since n is a finite integer and ε can be arbitrarily small, we have $d(x, x') \to 0$ if $x' \to x$.
Suppose that we have $x, x', y, y' \in \mathbb{R}^n$,

$$\max_{j=1,\dots,n} (|x_j - x'_j|) < \varepsilon_1, \text{ and } \max_{j=1,\dots,n} (|y_j - y'_j|) < \varepsilon_2,$$

then

$$\begin{aligned} d(x, y) - d(x', y') &= d(x, y) - d(x', y) + d(x', y) - d(x', y') \\ &\le d(x, x') + d(y, y') \\ &\le n(\varepsilon_1 + \varepsilon_2), \end{aligned}$$

which means $d(x, y) - d(x', y') \to 0$ if $x' \to x$ and $y' \to y$. Thus, any metric is a continuous mapping.
It easy to check that the norm $\|x\| = \|x - 0\|$ define a metric $d(x, 0)$ since

1. $\|x - 0\| \ge 0$ and equal iff $x = 0$
2. $\|x - 0\| = \|0 - x\|$
3. $\|x + z\| \le \|x\| + \|z\|$

Thus, any norm is a continuous mapping.
(c) Let's define the step function $\hat{f}$ corresponding to $f \in L_p[a, b]$ as

$$\hat{f}(x) := f_j := f\left(\frac{(2j+1)(b-a)}{2n}\right) = f(x_j), \quad j = \left\lfloor \frac{nx}{b-a} \right\rfloor,$$

where n is the number of steps, $\lfloor * \rfloor$ is the rounding operator toward $-\infty$. Apparently, $|x - x_i| \le (b-a)/(2n)$ and $\lim_{n\to\infty} \hat{f}(x) = f(x)$. Thus, by using the Hölder's inequality, Theorem 1.21, we have

$$\int_a^b |\hat{f}(x)\hat{g}(x)| dx = \frac{b-a}{n} \sum_{j=1}^{n} |f_j g_j| \le \frac{b-a}{n} \left(\sum_{j=1}^{n} |f_j|^p\right)^{1/p} \left(\sum_{j=1}^{n} |g_j|^q\right)^{1/q},$$

where $p > 1,\ 1/p + 1/q = 1$. Let $p = q = 2$, then we have

$$\frac{b-a}{n} \sum_{j=1}^{n} |f_j g_j| \le \left(\frac{b-a}{n} \sum_{j=1}^{n} |f_j|^2\right)^{1/2} \left(\frac{b-a}{n} \sum_{j=1}^{n} |g_j|^2\right)^{1/2}.$$

since

$$\lim_{n\to\infty}\frac{b-a}{n}\sum_{j=1}^{n}|f_jg_j| = \int_a^b |f(x)g(x)|dx,$$

$$\lim_{n\to\infty}\frac{b-a}{n}\sum_{j=1}^{n}|f_j|^2 = \int_a^b |f(x)|^2dx,$$

$$\lim_{n\to\infty}\frac{b-a}{n}\sum_{j=1}^{n}|g_j|^2 = \int_a^b |g(x)|^2dx,$$

thus

$$\int_a^b |f(x)g(x)|dx \leq \Big(\int_a^b |f(x)|^2dx\Big)^{1/2}\Big(\int_a^b |g(x)|^2dx\Big)^{1/2} < \infty.$$

As $(f,g) = \int_a^b f(x)g(x)dx \leq \int_a^b |f(x)g(x)|dx$, therefore $\int_a^b f(x)g(x)dx < \infty$. We can easily check that

1. $(f,g) = (g,f)$
2. $(f+h,g) = (f,g) + (h,g)$
3. $(\alpha f, g) = \alpha(f,g)$, for all $\alpha \in \mathbb{R}$
4. $(f,f) \geq 0$, and $(f,f) = 0$ iff $f = 0$

Thus, $(\cdot,\cdot)$ defines an inner product on $L_2[a,b]$.

(d) If sequence $f_j(x) \in L_p[a,b], j = 1,2,\dots$ is a Cauchy sequence, namely for any $\varepsilon > 0$, there is an $n_\varepsilon \in \mathbb{N}$ such that for any $m \geq n_\varepsilon, n \geq n_\varepsilon$, we have

$$d(f_m,f_n) = \left(\int_a^b |f_m(x) - f_n(x)|^p dx\right)^{1/p} < \varepsilon, \quad 0 < p < \infty,$$

which implies that if n_ε is large enough, $f_m(x)$ is equal to $f_n(x)$ almost everywhere for $m,n \geq n_\varepsilon$. If $f_j(x),\ j = 1,2,\dots$ converges, let $f(x) = \lim_{j\to\infty} f_j(x)$. Thus,

$$d(f_m,0) - d(f,0) = \left(\int_a^b |f_m(x)|^p dx\right)^{1/p} - \left(\int_a^b |f(x)|^p dx\right)^{1/p} \leq d(f_m,f) < \varepsilon,$$

for $m \geq n_\varepsilon$. Since $f_m(x) \in L_p[a,b]$, thus $\int_a^b |f_m(x)|^p dx < \infty$. Consequently, $\int_a^b |f(x)|^p dx < \infty$, by definition, $f(x) \in L_p[a,b]$ and thereby $L_p[a,b]$ is a complete metric space.

(e) According to the definition of norm $\|f\| := (\int_a^b |f(x)|^p dx)^{1/p}$, we have $\|f-g\| = d(f,g)$, where $d(f,g)$ is defined in (d). Therefore, the conclusion of (d) tells us $L_p[a,b]$ is complete upon the equipped norm $\|\cdot\|$. For $1 \leq p < \infty$, the Minkowski inequality guarantees the triangle inequality

$$\|f+g\| \leq \|f\| + \|g\|,$$

thus, $\|f\|$ defines a norm and thereby $L_p[a,b]$ is a complete normed space. However, for $0<p<1$ we have

$$\|f+g\| \le 2^{(1-p)/p}(\|f\|+\|g\|) \le K(\|f\|+\|g\|),$$

for and $K \ge 2^{(1-p)/p}$. Thus $\|\cdot\|$ defines a quasi-norm and thereby $L_p[a,b]$ is a complete quasi-normed space.

(f) If $p=2$, the inner product is given by

$$(f,g) = \int_a^b f(x)g(x)dx,$$

thus

$$\|f\| = \sqrt{(f,f)} = \left(\int_a^b |f(x)|^2 dx\right)^{1/2},$$

is a norm and we can easily check that it satisfies the parallelogram law

$$\|f+g\|^2 + \|f-g\|^2 = 2\big(\|f\|^2+\|g\|^2\big).$$

Therefore, $L_2[a,b]$ is a Hilbert space.
If $L_p[a,b]$ is a Hilbert space, it must satisfies the parallelogram law that

$$\|f+g\|^2 + \|f-g\|^2 = 2\big(\|f\|^2+\|g\|^2\big),$$

for any $f,g \in L_p[a,b]$. Let

$$f(x) = \begin{cases} \left(\frac{b-a}{2}\right)^{-1/p}, & a \le x \le \frac{a+b}{2} \\ 0, & \text{otherwise} \end{cases}, \quad g(x) = \begin{cases} \left(\frac{b-a}{2}\right)^{-1/p}, & \frac{a+b}{2} \le x \le b \\ 0, & \text{otherwise} \end{cases},$$

then we have $\|f\|^2 = \|g\|^2 = 1$ and $\|f+g\|^2 = \|f-g\|^2 = 2^{2/p}$. Consequently, we must have

$$2^{2/p} + 2^{2/p} = 4,$$

for which only $p=2$ is valid.

(g) Suppose $f_j(x) \in L_\infty[a,b]$, $j=1,2,\ldots$ and $\lim_{j\to\infty} f_j(x) = f(x)$ is a Cauchy sequence, for any $\varepsilon>0$, there is an $n_\varepsilon \in \mathbb{N}$ such that for any $m \ge n_\varepsilon$, we have

$$\|f_m(x)-f(x)\|_\infty = \sup_{x\in[a,b]} |f_m(x)-f(x)| < \varepsilon.$$

Since $\sup_{x\in[a,b]}|f_m(x)| < \infty$, then we have $\sup_{x\in[a,b]}|f(x)|<\infty$, i.e. $f(x)\in L_\infty[a,b]$ and thereby $L_\infty[a,b]$ is a Banach space.
Similar proof can be made for that $C[a,b]$ is a Banach space.

Exercise 1.20 Prove that

$$(f,g) := \int_a^b [f(x)g(x) + f'(x)g'(x)]dx$$

defines an inner product on $C^1[a,b]$, the space of real–valued continuously differentiable function on the interval [a,b]. The induced norm is called the *Sobolev* H^1 norm.

Solution: We would need to verify (f,g) is well-defined on $C^1[a,b]$ and it satisfies the properties in definition 1.15 case two.
The operation is well defined because continuous function on a closed interval is integrable.

1. $(f,g) = (g,f)$ follows directly from the fact multiplication of real number is commutative.
2. $(f+h,g) = (f,g)+(h,g)$. Since both f and h $\in C^1[a,b]$, so is f+h, since differential operation is linear, and the sum of continuous function is continuous. Then the above result arrives easily by the fact integration is a linear operation.
3. $(\alpha f,g) = \alpha(f,g)$ follows by homogeneity of integration.
4. $(f,f) \geq 0$ and equal iff $f = 0$. As $(f,f) = \int_a^b [f(x)^2 + f'(x)^2]dx$, since $[f(x)^2 + f'(x)^2] \geq 0$, so $(f,f) \geq (b-a)0 = 0$, and the equality holds iff $f(x) = 0$, and $f'(x) = 0 \forall x \in [a,b]$, i.e $f = 0$.

Exercise 1.21 Let $1 \leq p \leq \infty$ and consider the infinite-dimensional space $l_p(\mathbb{R})$ of all real sequences $\{x_i\}_{i=1}^{\infty}$ so that $(\sum_{i=1}^{\infty} |x_i|^p)^{1/p} \leq \infty$.
Also define

$$l_\infty(\mathbb{R}) := \sup\{|x_1|, |x_2|, ...\}$$

. Then show the following.
(a) $(x_i, y_i) = \sum_{i=1}^{\infty} x_i y_i$ defines an inner product on $l_p(\mathbb{R})$ iff $p = 2$.

(b) $l_p(\mathbb{R})$ is a complete normed space for all $1 \leq p \leq \infty$.

(c) $l_p(\mathbb{R})$ is a quasi–normed, complete space for $0 < p < 1$.

(d) Recall the Hölder–Minkowski inequality for $l_p(\mathbb{R}), 1 < p < \infty$. See Theorem 1.21. In particular, prove that

$$(x_1 + x_2 + \cdots + x_n)^2 \leq n(x_1^2 + x_2^2 + \cdots + x_n^2)$$

for all real number $x_1, x_2, \ldots, x_n$. When does equality hold in the above?

Solution: (a) Let $\|x\|_p = (\sum_{i=1}^{\infty} |x_i|^p)^{1/p}$. If $(x_i, y_i) = \sum_{i=1}^{\infty} x_i y_i$ is the associated inner product of $l_p(\mathbb{R})$, namely $\|x\|_p = \sqrt{(x,x)}$ for $x \in l_p(\mathbb{R})$, then it must satisfy the parallelogram equality

$$\|x+y\|_p^2 + \|x-y\|_p^2 = 2(\|x\|_p^2 + \|y\|_p^2),$$

for $x, y \in l_p(\mathbb{R})$. Let $x = (1,0,0,0,\ldots)$ and $y = (0,1,0,0,\ldots)$, then

$$\|x+y\|_p^2 + \|x-y\|_p^2 = 2^{2/p} + 2^{2/p} = 2^{1+2/p},$$

while

$$2(\|x\|_p^2 + \|y\|_p^2) = 4.$$

Hence, we must have

$$2^{1+2/p} = 4,$$

where only $p = 2$ is valid.

(b) For $1 \le p \le \infty$, suppose $x_i \in l_p(\mathbb{R}), i = 1, 2, \ldots$ is a Cauchy sequence that $\lim_{i\to\infty} x_i = x$, namely

$$\lim_{i\to\infty} \|x_i - x\|_p = 0.$$

Since $\|x_i\|_p < \infty$ for all i, then we have $\|x\|_p < \infty$. Thus, by definition $x \in l_p(\mathbb{R})$ and thereby $l_p(\mathbb{R})$ is complete.

(c) Proof is similar to (b). However, for $0 < p < 1$, we have $\|x+y\|_p \le K(\|x\|_p + \|y\|_p)$ and $K > 1$, which makes $\|\cdot\|$ a quasi–norm. Thus, $l_p(\mathbb{R})$ is quasi–normed complete space.

(d) Let $x = [x_1, x_2, \ldots]$ and $y = [1, 1, \ldots] \in l_2(\mathbb{R})$. According to Hölder's inequality, we have

$$|(x, y)|^2 \le \|x\|_2^2 \cdot \|y\|_2^2,$$

where $|(x,y)|^2 = \left(\sum_{i=1}^n x_i\right)^2$, $\|y\|_2^2 = n$ and $\|x\|_2^2 = \sum_{i=1}^n x_i^2$. Thus, we have

$$(x_1 + x_2 + \cdots + x_n)^2 \le n(x_1^2 + x_2^2 + \cdots + x_n^2).$$

The equality holds if and only if $x_1 = x_2 = \cdots = x_n$.

Exercise 1.22 Let V be an inner product space with induced norm $\|\cdot\|$. If $x \perp y$ show that

$$\|x\| + \|y\| \le \sqrt{2}\|x + y\|, \quad x, y \in V.$$

Solution: Let $(\cdot, \cdot)$ denote the inner product in V, and thereby the induced norm for $x \in V$ is $\|x\| = \sqrt{(x,x)}$. Thus

$$\begin{aligned} 2\|x+y\|^2 - (\|x\| + \|y\|)^2 &= 2\big(\|x\|^2 + \|y\|^2 + 2(x,y)\big) - \big(\|x\|^2 + \|y\|^2 + 2\|x\|\|y\|\big) \\ &= \|x\|^2 + \|y\|^2 - 2\|x\|\|y\| + 4(x,y) \\ &= (\|x\| - \|y\|)^2 + 4(x,y) \end{aligned}$$

Since $x \perp y$, we have $(x,y) = 0$. Thus

$$2\|x+y\|^2 - (\|x\| + \|y\|)^2 = (\|x\| - \|y\|)^2 \ge 0.$$

Exercise 1.23 Let V be an inner product space with induced norm $\|\cdot\|$. Prove the parallelogram law

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2, \quad x, y \in V.$$

The parallelogram law does not hold for normed spaces in general. Indeed, inner product spaces are completely characterized as normed spaces satisfying the parallelogram law.

Solution: Let $(\cdot,\cdot)$ denote the inner product in V, and thereby the induced norm for $x \in V$ is $\|x\| = \sqrt{(x,x)}$. Thus

$$\begin{aligned}\|x+y\|^2 + \|x-y\|^2 &= (x+y, x+y) + (x-y, x-y)\\ &= (x,x) + (y,y) + 2(x,y) + (x,x) + (y,y) - 2(x,y)\\ &= 2\|x\|^2 + 2\|y\|^2.\end{aligned}$$

Exercise 1.24 Show that for $0 < p < \infty$, $f, g \in L_p[a,b]$ and $x_i, y_i \in l_p(\mathbb{F})$, the Minkowski and triangle inequalities yield the following:

- For all $f, g \in L_p[a,b]$, there exists $C > 0$ independent of f and g such that

$$\left(\int_a^b |f(x)+g(x)|^p dx\right)^{1/p} \le C\left[\left(\int_a^b |f(x)|^p dx\right)^{1/p} + \left(\int_a^b |g(x)|^p dx\right)^{1/p}\right].$$

- For all $x_i, y_i \in l_p(\mathbb{F})$, there exists $C' > 0$ independent of x_i and y_i such that

$$\left(\sum_{i=1}^{\infty} |x_i + y_i|^p\right)^{1/p} \le C'\left[\left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p} + \left(\sum_{i=1}^{\infty} |y_i|^p\right)^{1/p}\right]$$

Solution: For $f, g \in L_p[a,b]$, let $\|f\|_p$ denotes $\left(\int_a^b |f(x)|^p dx\right)^{1/p}$. If $1 \le p < \infty$, the Minkowski inequality holds, i.e.

$$\|f+g\|_p \le \|f\|_p + \|g\|_p,$$

thus, $C = 1$. If $0 < p < 1$, we have

$$\begin{aligned}\left(\int_a^b |f(x)+g(x)|^p dx\right)^{1/p} &\le \left(\int_a^b (|f(x)|^p + |g(x)|^p) dx\right)^{1/p} \quad (\text{due to } (a+b)^p \le a^p + b^p, \quad a, b \ge 0)\\ &\le 2^{\frac{1}{p}-1}\left(\left(\int_a^b |f(x)|^p dx\right)^{1/p} + \left(\int_a^b |g(x)|^p dx\right)^{1/p}\right) \quad (\text{convexity of } a^{1/p})\\ &\le 2^{\frac{1}{p}-1}(\|f\|_p + \|g\|_p).\end{aligned}$$

Hence, any constant $C \ge 2^{\frac{1}{p}-1}$ is valid.
Similar proof can be made for the case in $l_p(\mathbb{F})$ space.

Exercise 1.25 An $n \times n$ matrix K is called *positive definite* if it is symmetric, $K^T = K$ and K satisfies the positivity condition $x^T K x > 0$ for all $x \neq 0 \in \mathbb{R}^n$. K is called *positive semi definite* if $x^T K x \ge 0$ for all $x \in \mathbb{R}^n$. Show the following:

- Every inner product on $\mathbb{R}^n$ is given by

$$(x, y) = x^T K y, \quad x, y \in \mathbb{R}^n$$

  where K is symmetric and a positive definite matrix.

- If K is positive definite, then K is nonsingular.
- Every diagonal matrix is positive definite (semi positive definite) iff all its diagonal entries are positive (nonnegative). What is the associated inner product?

Solution: Suppose $x, y \in \mathbb{R}^n$ and let $\{e_1, e_2, \ldots, e_n\}$ denote the unit basis in $\mathbb{R}^n$, where

$$x = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n, \quad y = y_1 e_1 + y_2 e_2 + \cdots + y_n e_n,$$

then

$$(x, y) = \sum_{i,j=1}^{n} x_i y_j \ (e_i, e_j) = \sum_{i,j=1}^{n} x_i K_{ij} y_j,$$

where $K_{ij} = (e_i, e_j) = (e_j, e_i) = K_{ji}$. Let K to be the matrix whose (i, j)-th entries is K_{ij}, then we have

$$(x, y) = x^T K y,$$

and K is symmetric. The symmetric K must have n eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and their associated non-zeros eigenvectors $v_1, v_2, \ldots, v_n$. Consequently,

$$(v_i, v_i) = v_i^T K v_i = v_i^T \lambda_i v_i = \lambda_i (v_i^T v_i) > 0.$$

Since $(v_i^T v_i) > 0$, we must have $\lambda_i > 0, \ i = 1, 2, \ldots, n$. Thus, K is a positive definite matrix.
Conversely, suppose $(x, y) = x^T K y$, where K is a positive definite matrix. We can check that

1. $(y, x) = y^T K x = (y^T K x)^T = x^T K^T y = x^T K y = (x, y)$, since $K = K^T$.
2. $(\alpha x, y) = \alpha x^T K y = \alpha (x, y)$.
3. $(x + y, z) = (x + y)^T K z = x^T K z + y^T K z = (x, z) + (y, z)$.
4. $(x, x) = x^T K x > 0$ if $x \neq 0$, since by definition K is a positive definite matrix.

Thus, $(x, y) = x^T K y$ is a valid norm in $\mathbb{R}^n$.

If K is positive definite, it must have an eigen-decomposition

$$K = V \Lambda V^T,$$

in which $\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, with proper ordering so that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. $V = [v_1 \quad v_2 \quad \ldots \quad v_n]$ is a unitary matrix whose row vectors are the eigenvectors of K. Since $\lambda_i > 0, \ i = 1, 2, \ldots, n$. This eigen-decomposition is also a singular decomposition of K, whose singular values $\sigma_i, \ i = 1, 2, \ldots, n$ are also its eigenvalues. As $\sigma_i > 0, \ i = 1, 2, \ldots, n$, K is nonsingular.

For a diagonal matrix $D = \text{diag}(d_1, d_2, \ldots, d_n)$ and any $x \neq 0 \in \mathbb{R}^n$,

$$x^T D x = \sum_{i=1}^{n} d_i x_i^2.$$

If $d_i > 0$ or $d_i \geq 0,\ i = 1, 2, \ldots, n$, then we have $x^T D x > 0$ or $x^T D x \geq 0$. By definition, D is a positive or positive semi definite matrix.
Conversely, suppose a diagonal matrix $D = \text{diag}(d_1, d_2, \ldots, d_n)$ is positive or positive semi definite, we have

$$e_i^T D e_i = d_i > (\text{ or } \geq)\ 0, \quad i = 1, 2, \ldots, n,$$

in which $e_i = [0 \ \ 0 \ \ \ldots \ \ 0 \ \ 1 \ \ 0 \ \ \ldots \ \ \ldots \ \ 0]^T$ is the unit vector whose non-zero entry is i-th entry.
For $x, y \in \mathbb{R}^n$, the associated inner product respected to the diagonal matrix D is

$$(x, y) = x^T D y = \sum_{i=1}^{n} d_i x_i y_i,$$

where $d_i > 0,\ i = 1, 2, \ldots, n$.

Exercise 1.26 Let V be an inner product space and let $v_1, v_2, \ldots, v_n \in V$. The associated *Gram matrix*:

$$K := \begin{pmatrix} (v_1, v_1) & (v_1, v_2) & \ldots & (v_1, v_n) \\ (v_2, v_1) & (v_2, v_2) & \ldots & (v_2, v_n) \\ \vdots & \vdots & \vdots & \vdots \\ (v_n, v_1) & (v_n, v_2) & \ldots & (v_n, v_n) \end{pmatrix}$$

is the $n \times n$ matrix whose entries are the inner products between selective vector space elements. Show that K is positive semi definite and positive definite iff all $v_1, \ldots, v_n$ are linearly independent.

Given an $m \times n$ matrix $V, n \leq m$, show that the following are equivalent:
— The $m \times n$ Gram matrix $A^T A$ is positive definite.
— A has linearly independent columns.
— A has rank n.
— $N(A) = \{0\}$.

Solution: Let $\{e_1, e_2, \ldots, e_n\}$ denote the unit basis in $\mathbb{R}^n$, then $v_i = [v_i^{(1)} \ \ v_i^{(2)} \ \ \ldots \ \ v_i^{(n)}]^T$ can be expressed as

$$v_i = \sum_{p=1}^{n} v_i^{(p)} e_p.$$

Thus,

$$(v_i, v_j) = \sum_{p,q=1}^{n} v_i^{(p)} v_j^{(q)} (e_p, e_q) = v_i^T \underbrace{\begin{pmatrix} (e_1, e_1) & (e_1, e_2) & \ldots & (e_1, e_n) \\ (e_2, e_1) & (e_2, e_2) & \ldots & (e_2, e_n) \\ \vdots & \vdots & \vdots & \vdots \\ (e_n, e_1) & (e_n, e_2) & \ldots & (e_n, e_n) \end{pmatrix}}_{:=G} v_j.$$

Furthermore, let $V = [v_1 \ \ v_2 \ \ \ldots \ \ v_n] \in \mathbb{R}^{n \times n}$, K can be expressed as

$$K = V^T G V.$$

According to conclusion of Exercise 1.25, G is a positive definite matrix.

Suppose that K is a positive definite matrix, then $\text{rank}(K) = n$, which requires $\text{rank}(V) = n$, i.e. $v_1, v_2, \ldots, v_n$ must be linearly independent.
Conversely, suppose that $v_1, v_2, \ldots, v_n$ are linearly independent. For any $x \neq 0 \in \mathbb{R}^n$, $Vx = y \neq 0$ because V is nonsingular. Hence, $x^T K x = y^T G y > 0$, by definition K is a positive definite matrix.

For any $x \neq 0 \in \mathbb{R}^n$, $y = Ax \in \mathbb{R}^m$ is the non-zero linear combination of the columns of A. If A has linearly independent columns $\Leftrightarrow y \neq 0 \Leftrightarrow x^T A^T A x = y^T y > 0 \Leftrightarrow A^T A$ is positive definite.
A has linearly independent columns $\Leftrightarrow A$ has rank n.
$y = Ax \neq 0 \in \mathbb{R}^m$ for any $x \neq 0 \in \mathbb{R}^n \Leftrightarrow N(A) = \{0\}$.

Exercise 1.27 Given $n \geq 1$, a Hilbert matrix is of the form:

$$K(n) := \begin{pmatrix} 1 & 1/2 & \ldots & 1/n \\ 1/2 & 1/3 & \ldots & 1/(n+1) \\ \vdots & \vdots & \vdots & \vdots \\ 1/n & 1/(n+1) & \ldots & 1/(2n-1) \end{pmatrix}.$$

- Show that in $L_2[0,1]$, the Gram matrix from the monomials $1, x, x^2$ is the Hilbert matrix $K(3)$.
- More generally, show that the Gram matrix corresponding to the monomials $1, x, \ldots, x^n$ has entries $1/(i+j-1), \quad i, j = 1, \ldots, n+1$ and is $K(n+1)$.
- Deduce that $K(n)$ is positive definite and nonsingular.

Solution: Let $f_i(x) = x^{i-1} \in L_2[0,1], \ i = 1, 2, \ldots, n$, then we have

$$(f_i, f_j) = \int_0^1 x^{i+j-2} dx = \left. \frac{x^{i+j-1}}{i+j-1} \right|_0^1 = \frac{1}{i+j-1}.$$

thus, $(f_i, f_j) = [K(n+1)]_{ij}$, namely, $K(n+1)$ is the Gram matrix of $f_i(x), \ i = 1, 2, \ldots, n$.

Since the monomials $1, x, x^2, \ldots, x^{n-1}$ are linearly independent, according to the conclusion of Exercise 1.26, their Gram matrix $K(n)$ is positive definite and thereby nonsingular.

Exercise 1.28 Use Exercise 1.25 to prove: If K is a symmetric, positive definite $n \times n, n \geq 1$ matrix, $f \in \mathbb{R}^n$ and $c \in \mathbb{R}$, then the quadratic function:

$$x^T K x - 2x^T f + c$$

has a unique minimizer $c - f^T K^{-1} f = c - f^T x^* = c - (x^*)^T K x^*$ which is a solution to the linear system $Kx = f$.

Prove the following: Let $v_1, \ldots, v_n$ form a basis for a subspace $V \subset \mathbb{R}^m$, $n, m \geq 1$. Then given $b \in \mathbb{R}^m$, the closest point $v^* \in V$ is the solution $x^* = K^{-1}f$ of $Kx = f$ where K is the Gram matrix, whose (i,j)th entry is (v_i, v_j) and $f \in \mathbb{R}^n$ is the vector whose ith entry is the inner product (v_i, b). Show also that the distance

$$\|v^* - b\| = \sqrt{\|b\|^2 - f^T x^*}.$$

Solution: Since K is positive definite, K^{-1} exists and it is also positive definite. Hence,

$$\begin{aligned} x^T K x - 2x^T f + c &= x^T K x - 2x^T f + f^T K^{-1} f + c - f^T K^{-1} f \\ &= (Kx - f)^T (x - K^{-1} f) + c - f^T K^{-1} f \\ &= (Kx - f)^T K^{-1} (Kx - f) + c - f^T K^{-1} f. \end{aligned}$$

As K^{-1} is positive definite, $(Kx - f)^T K^{-1}(Kx - f)$ is an inner product of $(Kx - f)$ respected to K^{-1}, namely,

$$(Kx - f)^T K^{-1} (Kx - f) = (Kx - f, Kx - f)_{K^{-1}} \geq 0,$$

and the equality is reached only if $Kx - f = 0$. Thus, $x^* = K^{-1}f$ is the unique minimizer and the minimal is $c - f^T K^{-1} f$.

Let $b \in \mathbb{R}^m$ and $v \in V$. Since $v_1, v_2, \ldots, v_n$ form a basis for V, v can be expressed as a linear combination of $v_1, v_2, \ldots, v_n$ as

$$v = x_1 v_1 + x_2 v_2 + \cdots + x_n v_n = Ux,$$

in which $U = [v_1 \;\; v_2 \;\; \ldots \;\; v_n] \in \mathbb{R}^{m \times n}$ and $x = [x_1 \;\; x_2 \;\; \ldots \;\; x_n]^T \in \mathbb{R}^n$.

For $a, b \in \mathbb{R}^m$, suppose that the inner product in $\mathbb{R}^m$ is defined as

$$(a, b) = a^T G b,$$

where G is certain positive definite matrix, and the distance between a and b is defined as

$$\|a - b\| = \sqrt{(a - b, a - b)}.$$

Hence, the squared distance between v and b can be expressed as

$$\begin{aligned} \|Ux - b\|^2 &= (Ux - b)^T G (Ux - b) \\ &= x^T U^T G U x - 2b^T G U x + b^T G b. \end{aligned}$$

By definition, $U^T G U = K$, $b^T G U = f^T$ and $b^T G b = (b, b) = \|b\|^2$, because $v_i^T G v_j = (v_i, v_j)$ and $b^T G v_i = (b, v_i)$ for $i, j = 1, \ldots, n$. Hence,

$$\|Ux - b\|^2 = x^T K x - 2 f^T x + \|b\|^2,$$

and its minimizer is $x^* = K^{-1} f$ and the minimal is $\|b\|^2 - f^T K^{-1} f = \|b\|^2 - f^T x^*$. Thus, the minimal distance is the square root of the minimal, i.e.

$$\|v^* - b\| = \sqrt{\|b\|^2 - f^T x^*}.$$

Exercise 1.29 Suppose $w : \mathbb{R} \to (0, \infty)$ is continuous and for each fixed $j \geq 0$, the *moments*, $\mu_j := \int_{\mathbb{R}} x^j w(x)dx$ are finite. Use Gram–Schmidt to construct for $j \geq 0$, a sequence of orthogonal polynomials P_j of degree at most j such that $\int_{\mathbb{R}} P_i(x)P_j(x) = \delta_{ij}$ where δ_{ij} is 1 when $i = j$ and 0 otherwise. Recall the P_j are orthogonal if we only require $\int_{\mathbb{R}} P_i(x)P_j(x) = 0, i \neq j$. Now consider the *Hankel* matrix

$$K := \begin{pmatrix} \mu_0 & \mu_1 & \dots \\ \mu_1 & \mu_2 & \dots \\ \mu_2 & \mu_3 & \dots \\ \vdots & \vdots & \vdots \end{pmatrix}.$$

Prove that K is positive-semi definite and hence that P_j are unique. The sequence $\{P_j\}$ defines the unique orthonormal polynomials with respect to the weight w.

Solution: Suppose that $K \in \mathbb{R}^{n\times n}$ and any $z \in \mathbb{R}^n$, thus

$$\begin{aligned} z^T K z &= \sum_{i,j=1}^{n} z_i z_j K_{ij} \\ &= \sum_{k=0}^{2n-2} \mu_k \left(\sum_{\substack{i+j-2=k \\ 1\leq i,j\leq n}} z_i z_j \right) \\ &= \sum_{k=0}^{2n-2} \int_{\mathbb{R}} x^k w(x)dx \left(\sum_{\substack{i+j-2=k \\ 1\leq i,j\leq n}} z_i z_j \right) \\ &= \int_{\mathbb{R}} \left(\sum_{i,j=1}^{n} z_i x^{i-1} z_j x^{j-1} \right) w(x)dx \\ &= \int_{\mathbb{R}} \left(\sum_{i=1}^{n} z_i x^{i-1} \right)^2 w(x)dx \geq 0 \quad (\text{if } w(x) \geq 0). \end{aligned}$$

Thus, K is positive semi definite.

Exercise 1.30 For $n \geq 1$, let $t_1, t_2, \dots, t_{n+1}$ be $n+1$ distinct points in $\mathbb{R}$. Define the $1 \leq k \leq n+1$th *Lagrange interpolating polynomial* by

$$L_k(t) := \prod_{\substack{j=1 \\ j\neq k}}^{n+1} \frac{t - t_j}{t_k - t_j}, \quad t \in \mathbb{R}.$$

Verify that for each fixed k, L_k is of exact degree n and $L_k(t_i) = \delta_{ik}$.

Prove the following: Given a collection of points (say data points), $y_1, \dots, y_{n+1}$ in $\mathbb{R}$, the polynomial $p(t) := \sum_{j=1}^{n+1} y_j L_j(t), t \in \mathbb{R}$ is of degree at most n and is the unique polynomial of degree at most n interpolating the pairs (t_j, y_j), $j = 1, \dots, n+1$, i.e. $p(t_j) = y_j$ for every $j \geq 0$.

Now let y be an arbitrary polynomial of full degree n on $\mathbb{R}$ with coefficients $c_j, 0 \le j \le n$. Show that the total least squares error between $m \ge 1$ data points $t_1, \dots, t_m$ and the sample values $y(t_i), 1 \le i \le m$ is $\|y - Ax\|^2$ where $x = [c_1, \dots, c_m]^T, y = [y_1, \dots, y_m]^T$ and

$$A := \begin{pmatrix} 1 & t_1 & t_1^2 & \dots & t_1^n \\ 1 & t_2 & t_2^2 & \dots & t_2^n \\ \vdots & \vdots & \vdots & \vdots & \\ 1 & t_m & t_m^2 & \dots & t_m^n \end{pmatrix},$$

which is called the Vandermonde matrix.

Show that

$$det(A) = \prod_{1 \le i \le j \le n+1} (t_j - t_i)$$

Solution: Since there are n terms of product of degree–1 polynomials for L_k, L_k is of exact degree n. If $j \neq k$, $L_k(t_j)$ has the term $\frac{t_j - t_j}{t_k - t_j} = 0$, thus $L_k(t_j) = 0$. If $j = k$, $L_k(t_j)$ has the term $\frac{t_k - t_j}{t_k - t_j}$, which the numerator cancels the denominator and the rest parts are

$$L_k(t_k) = \prod_{\substack{j=1 \\ j \neq k}}^{n+1} \frac{t_k - t_j}{t_k - t_j} = 1.$$

Hence, $L_k(t_i) = \delta_{ik}$.

Since $L_j,\ j = 1, \dots, n+1$ is of exact degree n, $p(t) := \sum_{j=1}^{n+1} y_j L_j(t), t \in \mathbb{R}$ is the linear combination of $L_j,\ j = 1, \dots, n+1$ and thereby is at most of degree n. Moreover,

$$p(t_j) = \sum_{i=1}^{n+1} y_i \delta_{ij} = y_j.$$

Let $p(t) = \sum_{i=1}^{n+1} a_i t^{i-1}$, we have $p(t_j) = \sum_{i=0}^{n} a_i t_j^i = y_j$ for $j = 1, \dots, n+1$, which can be expressed as a linear system

$$\underbrace{\begin{pmatrix} 1 & t_1 & t_1^2 & \dots & t_1^n \\ 1 & t_2 & t_2^2 & \dots & t_2^n \\ \vdots & \vdots & \vdots & \vdots & \\ 1 & t_{n+1} & t_{n+1}^2 & \dots & t_{n+1}^n \end{pmatrix}}_{:=T \in \mathbb{R}^{(n+1)\times(n+1)}} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n+1} \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n+1} \end{pmatrix}.$$

Since $t_1, t_2, \dots, t_{n+1}$ are distinct, the row vectors in T are linearly independent. Thus, T is nonsingular, and for a given $[y_1 \ \ y_2 \ \ \dots \ \ y_{n+1}]^T$ there is a unique solution for the coefficients $[a_1 \ \ a_2 \ \ \dots \ \ a_{n+1}]^T$. Hence, $p(t)$ is unique.

Similarly, given polynomial $p(t) = \sum_{j=1}^{n+1} c_j t^{j-1}$, the least squares error between $p(t_i)$ and the sample value y_i for $i = 1, \dots, m$ is

$$\sum_{i=1}^{m} (p(t_i) - y_i)^2 = \sum_{i=1}^{m} \left(\sum_{j=1}^{n+1} c_j t_i^{j-1} - y_i \right)^2 = \|Ax - y\|^2,$$

where

$$A = \begin{pmatrix} 1 & t_1 & t_1^2 & \dots & t_1^n \\ 1 & t_2 & t_2^2 & \dots & t_2^n \\ \vdots & \vdots & \vdots & \vdots & \\ 1 & t_m & t_m^2 & \dots & t_m^n \end{pmatrix}, \quad x = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}.$$

To prove that $det(A) = \prod_{1\le i\le j\le n+1}(t_j - t_i)$ for square Vandermonde matrix A, we use the principle of mathematical induction.

For $n+1 = 2$, we can easily check that

$$det\left(\begin{bmatrix} 1 & t_1 \\ 1 & t_2 \end{bmatrix}\right) = t_2 - t_1 = \prod_{1\le i\le j\le 2}(t_j - t_i).$$

Suppose that $det(A) = \prod_{1\le i\le j\le n+1}(t_j - t_i)$ is true for some $n \ge 1$, then for the case of $n+1$,

$$A = \begin{pmatrix} 1 & t_1 & t_1^2 & \dots & t_1^n \\ 1 & t_2 & t_2^2 & \dots & t_2^n \\ \vdots & \vdots & \vdots & \vdots & \\ 1 & t_{n+1} & t_{n+1}^2 & \dots & t_{n+1}^n \end{pmatrix}.$$

By subtracting t_1 times the $(n-i)$th column of A to the $(n-i+1)$th row $(i = 0, \dots, n-1)$, we obtain the matrix

$$B = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & t_2 - t_1 & t_2(t_2 - t_1) & \dots & t_2^{n-1}(t_2 - t_1) \\ \vdots & \vdots & \vdots & \vdots & \\ 1 & t_{n+1} - t_1 & t_{n+1}(t_{n+1} - t_1) & \dots & t_{n+1}^{n-1}(t_{n+1} - t_1) \end{pmatrix}.$$

Considering the sub-matrix $C = B_{2:n+1,2:n+1}$, since each row has a linear factor $(t_i - t_1),\ i = 2, \dots, n+1$, we have

$$det(A) = det(B) = det(C) = det(D) \prod_{2\le i\le n+1}(t_i - t_1),$$

where

$$D = \begin{pmatrix} 1 & t_2 & t_2^2 & \dots & t_2^n \\ 1 & t_3 & t_3^2 & \dots & t_3^n \\ \vdots & \vdots & \vdots & \vdots & \\ 1 & t_{n+1} & t_{n+1}^2 & \dots & t_{n+1}^n \end{pmatrix}.$$

By the assumption, $det(D) = \prod_{2\le i\le j\le n+1}(t_j - t_i)$. Thus,

$$det(A) = \left(\prod_{2\le i\le j\le n+1}(t_j - t_i)\right)\left(\prod_{2\le i\le n+1}(t_i - t_1)\right) = \prod_{1\le i\le j\le n+1}(t_j - t_i).$$

Solutions to Chapter 2

Exercise 2.1 Use the integral test and Example (2.3) to show that $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^a}$ converges iff $a > 1$

Solution: Since

$$\int_2^s \frac{1}{x(\log x)^a} = \begin{cases} \log(\log s) - \log(\log 2), & a = 1; \\ \frac{(\log 2)^{1-a} - (\log s)^{1-a}}{a-1}, & a \neq 1. \end{cases}$$

Hence,

$$\lim_{s \to \infty} \int_2^s \frac{1}{x(\log x)^a} = \begin{cases} \infty, & a \leq 1; \\ \frac{(\log 2)^{1-a}}{a-1}, & a > 1. \end{cases}$$

and the integral converges iff $a > 1$. Therefore, $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^a}$ converges iff $a > 1$.

Exercise 2.2 Are $\int_{-\infty}^{\infty} e^{-|x|} dx$ and $\int_{\infty}^{\infty} x dx$ convergent? If so, evaluate them.

Solution: Since

$$\int_{s_1}^{s_2} e^{-|x|} dx = 2 - e^{s_1} - e^{-s_2},$$

thus,

$$\lim_{s_1 \to -\infty,\ s_2 \to \infty} \int_{s_1}^{s_2} e^{-|x|} dx = 2.$$

While

$$\int_{s_1}^{s_2} x dx = \frac{1}{2}(s_2^2 - s_1^2),$$

thus $\int_{\infty}^{\infty} x dx$ is not well defined, and thereby it is divergent.

Exercise 2.3 In Chapter 1, we defined completeness of a metric space. Formulate Lemma 2.7 into a statement about completeness of the real numbers.

Solution: For every $F : (a, x) \to \mathbb{R}$, if $\lim_{x \to \infty} F(x) = \widetilde{F}(x)$ exists and $\widetilde{F}(x) \in \mathbb{R}$, then F is a complete mapping to the real number $\mathbb{R}$.

Exercise 2.4 Prove Theorem 2.9 using the fact that a monotone (i.e. all terms have the same sign) bounded sequence converges. What is this limit?.

Solution: Since $g(x) \geq 0$, $\int_a^{\infty} g(x) dx$ is convergent, $\int_{C'}^{\infty} g(x) dx$ is convergent for any $C' \geq a$ and let $\int_{C'}^{\infty} g(x) dx = y$. If there exist nonnegative constants C, C' such that for $x \geq C'$, $|f(x)| \leq Cg(x)|$, then we have

$$\int_{C'}^{\infty} |f(x)| dx \leq \int_{C'}^{\infty} Cg(x) dx = Cy,$$

namely $\int_a^\infty |f(x)|dx$ is convergent. Since $0 \leq f(x) + |f(x)| \leq 2|f(x)|$, we have

$$0 \leq \int_a^s (f(x) + |f(x)|)dx \leq 2 \int_a^s |f(x)|dx \leq 2 \int_a^\infty |f(x)|dx,$$

for any $s \geq a$. Hence, $\int_a^s (f(x) + |f(x)|)dx$ is a monotone bounded sequence as $s \to \infty$ and thereby $\int_a^\infty (f(x) + |f(x)|)dx$ converges. Noting that $\int_a^\infty f(x)dx = \int_a^\infty (f(x) + |f(x)|)dx - \int_a^\infty |f(x)|dx$ is the difference of convergent integrals, thus $\int_a^\infty f(x)dx$ converges.

Exercise 2.5 Show, using Definition 2.13 that $\int_{-1}^1 \frac{dx}{\sqrt{1-x^2}} = \pi$.

Solution: Since

$$\lim_{s\to -1^+} \int_s^0 \frac{dx}{\sqrt{1-x^2}} = \lim_{s\to -1^+} -\arcsin(s) = \frac{\pi}{2},$$

and

$$\lim_{s\to 1^-} \int_s^0 \frac{dx}{\sqrt{1-x^2}} = \lim_{s\to 1^-} \arcsin(s) = \frac{\pi}{2},$$

we have

$$\int_{-1}^1 \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

Exercise 2.6 (a) Show that $\int_0^1 \frac{\sin(\ln x)}{\sqrt{x}} dx$ is absolutely convergent.

(b) Show that $\int_0^{\pi/2} 1/\big(\sqrt{t}\cos(t)\big)dt$ is divergent. (Hint: Split the integral into two integrals from $[0, \pi/4]$ and from $[\pi/4, \pi/2]$. For the first integral use that $\cos(t)$ is decreasing and for the second integral expand $cos(t)$ about $\pi/2$.)

(c) Show that $\int_{-1}^1 \frac{1}{x} dx$ is divergent.

Solution: (a) Since

$$\int_s^1 \left|\frac{\sin(\ln x)}{\sqrt{x}}\right| dx = \int_s^1 \frac{\sqrt{(\sin(\ln x))^2}}{\sqrt{x}} dx = \frac{4}{5} - \frac{2}{5}\sqrt{s}(1 - 2\cot(\ln s))\,|\sin(\ln s)|,$$

then we have

$$\frac{4}{5} - \frac{2}{5}\sqrt{s}(|\sin(\ln s)| + 2\cos(\ln s)) \leq \int_s^1 \left|\frac{\sin(\ln x)}{\sqrt{x}}\right| dx \leq \frac{4}{5} + \frac{2}{5}\sqrt{s}(|\sin(\ln s)| + 2\cos(\ln s)).$$

Hence

$$\lim_{s\to 0^+} \int_s^1 \left|\frac{\sin(\ln x)}{\sqrt{x}}\right| dx = \frac{4}{5}.$$

(b) Since

$$0 < \int_s^{\pi/4} \frac{1}{\sqrt{t}\cos(t)} dt < \frac{1}{\cos\left(\frac{\pi}{4}\right)} \int_s^{\pi/4} \frac{1}{\sqrt{t}} dt = \sqrt{2}(\sqrt{\pi} - 2\sqrt{s}),$$

$\lim_{s\to 0^+}\int_s^{\pi/4}\frac{1}{\sqrt{t}\cos(t)}dt$ converges. However, for $\pi/4 \le s \le \pi/2$, we have

$$\int_{\pi/4}^{s}\frac{1}{\sqrt{t}\cos(t)}dt > \int_{\pi/4}^{s}\frac{1}{\sqrt{t}(\pi/2-t)}dt = 2\sqrt{\frac{2}{\pi}}\left(-\operatorname{arcsinh}(1)+\operatorname{arctanh}\left(\sqrt{\frac{2s}{\pi}}\right)\right).$$

Hence, $\lim_{s\to(\pi/2)^-}\int_{\pi/4}^{s}\frac{1}{\sqrt{t}\cos(t)}dt$ diverges.

(c) Since

$$\int_{-1}^{s_1}\frac{1}{x}dx = \ln(-s_1), \quad \int_{s_2}^{1}\frac{1}{x}dx = -\ln(s_2),$$

$\int_{-1}^{1}\frac{1}{x}dx$ diverges.

Exercise 2.7 Test the following integrals for convergence/absolute convergence/conditional convergence:

(a) $\int_{10}^{\infty}\frac{1}{x(\ln x)(\ln\ln x)^a}dx,\ a>1.$

(b) $\int_1^{\infty}\frac{\cos x}{x}dx.$

(c) $\int_{-\infty}^{\infty}P(x)e^{-|x|}dx$, P is a polynomial.

(d) $\int_{-\infty}^{\infty}P(x)\ln(1+x^2)e^{-x^2}dx$, P is a polynomial.

(e) $\int_1^{\infty}\frac{\cos x\sin x}{x}dx.$

(f) $\int_{-\infty}^{\infty}\operatorname{sign}(x)dx.$

(g) $\int_1^{\infty}\frac{\cos 2x}{x}dx.$

(h) $\int_9^{\infty}\frac{\ln x}{x(\ln\ln x)^{100}}dx.$

Solution: (a) Since $\frac{1}{x(\ln x)(\ln\ln x)^a} > 0$ for $x\ge 10$,

$$\int_{10}^{s}\left|\frac{1}{x(\ln x)(\ln\ln x)^a}\right|dx = \int_{10}^{s}\frac{1}{x(\ln x)(\ln\ln x)^a}dx = \frac{(\ln\ln 10)^{1-a}-(\ln\ln s)^{1-a}}{a-1}.$$

Hence

$$\lim_{s\to\infty}\int_{10}^{s}\left|\frac{1}{x(\ln x)(\ln\ln x)^a}\right|dx = \frac{\ln\ln 10)^{1-a}}{a-1},$$

i.e. $\int_{10}^{\infty}\frac{1}{x(\ln x)(\ln\ln x)^a}dx,\ a>1$ is absolute convergence and thereby convergence.

(b) Since

$$\int_1^s\left|\frac{\cos x}{x}\right|dx > \int_1^s\frac{\cos^2 x}{x}dx = \frac{1}{2}(-\operatorname{Ci}(2)+\operatorname{Ci}(2s)+\ln s),$$

thus $\lim_{s\to\infty}\int_1^s\left|\frac{\cos x}{x}\right|dx$ diverges. However,

$$\int_1^{\infty}\frac{\cos x}{x}dx = -\operatorname{Ci}(1).$$

Hence, $\int_1^{\infty}\frac{\cos x}{x}dx$ is conditional convergence.

(c) Let $P(x) = \sum_{j=0}^{n} a_j x^j$, thus

$$\int_0^\infty \left|P(x)e^{-x}\right| dx \le \sum_{j=0}^{n} |a_j| \int_0^\infty x^j e^{-x} dx = \sum_{j=0}^{n} |a_j| \Gamma(j+1).$$

Similarly,

$$\int_{-\infty}^{0} |P(x)e^{x}| dx \le \sum_{j=1}^{n} |a_j| \Gamma(j+1).$$

Hence, $\int_{-\infty}^{\infty} P(x)e^{-|x|}dx$ is absolute convergence.

(d) Let $P(x) = \sum_{j=0}^{n} a_j x^j$. Since $|x| \ge \ln(1+x^2)$, we have

$$\int_{-\infty}^{\infty} |x|^j \ln\big(1+x^2\big)e^{-x^2} dx \le \int_{-\infty}^{\infty} |x|^{j+1} e^{-x^2} dx = \Gamma\left(1 + \frac{j}{2}\right).$$

Hence, $\int_{-\infty}^{\infty} P(x)\ln(1+x^2)e^{-x^2}dx$ is absolute convergence.

(e) Since $\cos x \sin x = \frac{1}{2}\sin 2x$, thus

$$\int_1^\infty \frac{\cos x \sin x}{x} dx = \int_1^\infty \frac{\sin 2x}{2x} dx = \frac{1}{2}\int_2^\infty \frac{\sin t}{t} dt.$$

Similar to that in (b), $\int_2^\infty \frac{\sin t}{t} dt$ is conditional convergence. Particularly,

$$\frac{1}{2}\int_2^\infty \frac{\sin t}{t} dt = \frac{1}{4}(\pi - 2\mathrm{Si}(2))$$

(f) Since

$$\int_{s_1}^{0} \mathrm{sign}(x) dx + \int_0^{s_2} \mathrm{sign}(x) dx = s_1 + s_2,$$

thus $\lim_{s_1\to-\infty, s_2\to\infty} \int_{s_1}^{s_2} \mathrm{sign}(x)dx$ is divergence.

(g) Since

$$\int_1^\infty \frac{\cos 2x}{x} dx = \int_2^\infty \frac{\cos t}{t} dt.$$

According to (b), $\int_2^\infty \frac{\cos t}{t} dt$ is conditional convergence. Particularly,

$$\int_2^\infty \frac{\cos t}{t} dt = -\mathrm{Ci}(2)$$

(h) Since

$$\int_9^s \left|\frac{\ln x}{x(\ln\ln x)^{100}}\right| dx = \int_9^s \frac{\ln x}{x(\ln\ln x)^{100}} dx = \frac{E_{100}(-2\ln\ln 9)}{(\ln\ln 9)^{99}} - \frac{E_{100}(-2\ln\ln s)}{(\ln\ln s)^{99}},$$

where

$$\frac{E_{100}(-2\ln\ln s)}{(\ln\ln s)^{99}} = \frac{-2^{99}(\ln\ln s)^{99}\Gamma(-99, -2\ln\ln s)}{(\ln\ln s)^{99}} = -2^{99}\Gamma(99, -2\ln\ln s).$$

thus, we have $\lim_{s\to\infty} \Gamma(99, -2\ln\ln s) = 0$, and thereby $\int_9^\infty \frac{\ln x}{x(\ln\ln x)^{100}} dx$ is absolute convergence.

Exercise 2.8 Let $f : [a, \infty) \to \mathbb{R}$ with $f \in R[a, s],\ \forall s > a$ and suppose that $\int_a^\infty f(x)dx$ converges. Show that $\forall \varepsilon > 0$, there exists $B_\varepsilon > 0$ such that

$$s \geq B_\varepsilon \ \text{ implies } \left| \int_s^\infty f(x)dx \right| < \varepsilon$$

Solution: According to Theorem 2.8, if $\int_a^\infty f(x)dx$ converges, there exists B_ε such that for $s, t \geq B_\varepsilon$, we have

$$\left| \int_a^s f(x)dx - \int_a^t f(x)dx \right| < \varepsilon.$$

Let $t \to \infty$, then we have

$$\left| \int_a^s f(x)dx - \int_a^\infty f(x)dx \right| = \left| \int_s^\infty f(x)dx \right| < \varepsilon.$$

Exercise 2.9 Show that if $f, g : [a, \infty) \to \mathbb{R}$ and $\int_a^\infty f(x)dx$ and $\int_a^\infty g(x)dx$ converges, then $\forall \alpha, \beta \in \mathbb{R}$, $\int_a^\infty [\alpha f(x) + \beta g(x)]dx$ converges to $\alpha \int_a^\infty f(x) + \beta \int_a^\infty g(x)dx$.

Solution: Apparently,

$$\int_a^s [\alpha f(x) + \beta g(x)]dx = \int_a^s \alpha f(x)dx + \int_a^s \beta g(x)dx = \alpha \int_a^s f(x)dx + \beta \int_a^s g(x)dx.$$

Hence,

$$\lim_{s\to\infty} \int_a^s [\alpha f(x) + \beta g(x)]dx = \alpha \lim_{s\to\infty} \int_a^s f(x)dx + \beta \lim_{s\to\infty} \int_a^s g(x)dx,$$

namely, $\int_a^\infty [\alpha f(x) + \beta g(x)]dx$ converges to $\alpha \int_a^\infty f(x) + \beta \int_a^\infty g(x)dx$.

Exercise 2.10 The substitution rule for improper integrals. Suppose that $g : [a, \infty) \to \mathbb{R}$ is monotone, increasing and continuously differentiable and $\lim_{t\to\infty} g(t) = \infty$. Suppose that $f : [g(a), \infty) \to \mathbb{R}$ and $f \in R[a, s], \forall s > g(a)$. Show that $\int_a^\infty f(g(t))g'(t)dt$ converges iff $\int_{g(a)}^\infty f(x)dx$ converges and if either converges, both are equal.

Solution: Suppose that F is an antiderivative of f, then we have

$$\begin{aligned}\int_a^s f(g(t))g'(t)dt &= \int_a^s F(g(t))'dt \\ &= F(g(s)) - F(g(a)) \\ &= \int_{g(a)}^{g(s)} f(x)dx.\end{aligned}$$

Since $s \to \infty$ gives $g(s) \to \infty$, if $\lim_{s\to\infty} \int_a^s f(g(t))g'(t)dt$ converges, we have the convergence of $\lim_{g(s)\to\infty} \int_{g(a)}^{g(s)} f(x)dx$ and

$$\int_a^\infty f(g(t))g'(t)dt = \int_{g(a)}^\infty f(x)dx.$$

Exercise 2.11 For what values of a do the series below converge?

$$\sum_{n=1}^{\infty}\frac{1}{n^a},\quad \sum_{n=2}^{\infty}\frac{1}{n(\ln n)^a},\quad \sum_{n=3}^{\infty}\frac{1}{n(\ln n)(\ln\ln n)^a}.$$

Solution: Since

$$\int_1^s \frac{1}{x^a} = \begin{cases} \ln s, & a=1; \\ \frac{1-s^{1-a}}{a-1}, & a\neq 1. \end{cases}$$

Hence, according to the integral test, $\sum_{n=1}^{\infty}\frac{1}{n^a}$ converges for $a>1$ and diverges for $a\leq 1$.

Since

$$\int_2^s \frac{1}{x(\ln x)^a} = \begin{cases} \ln\ln s - \ln\ln 2, & a=1; \\ \frac{(\ln 2)^{1-a}-(\ln 2)^{1-a}}{a-1}, & a\neq 1. \end{cases}$$

Hence, $\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^a}$ converges for $a>1$ and diverges for $a\leq 1$.

Since

$$\int_3^s \frac{1}{x(\ln x)(\ln\ln x)^a} = \begin{cases} \ln\ln\ln s - \ln\ln\ln 2, & a=1; \\ \frac{(\ln\ln 3)^{1-a}-(\ln\ln s)^{1-a}}{a-1}, & a\neq 1. \end{cases}$$

Hence, $\sum_{n=3}^{\infty}\frac{1}{n(\ln n)(\ln\ln n)^a}$ converges for $a>1$ and diverges for $a\leq 1$.

Exercise 2.12 Use integration by parts for the integral $\int_1^s \frac{\sin x}{x}dx$ to establish the convergence of the improper integral $\int_1^\infty \frac{\sin x}{x}dx$.

Solution: Since

$$\begin{aligned}\int_1^s \frac{\sin x}{x}dx &= -\int_1^s \frac{d(\cos x)}{x} \\ &= \frac{\cos x}{x}\Big|_s^1 - \int_1^s \frac{\cos x}{x^2}dx,\end{aligned}$$

thus,

$$\lim_{s\to\infty}\int_1^s \frac{\sin x}{x}dx = \cos(1) - \lim_{s\to\infty}\int_1^s \frac{\cos x}{x^2}dx.$$

Because $\int_1^\infty \left|\frac{\cos x}{x^2}\right|dx < \int_1^\infty \frac{1}{x^2}dx$ converges, $\int_1^\infty \frac{\sin x}{x}dx$ converges accordingly.

Exercise 2.13 If $f = f_1 + if_2$ is a complex-valued function on $[a,\infty)$, we define

$$\int_a^\infty f(x)dx = \int_a^\infty f_1(x)dx + i\int_a^\infty f_2(x)dx,$$

whenever both integrals on the right–hand side are defined and finite. Show that

$$\int_1^\infty \frac{e^{ix}}{x}$$

converges.

Solution: Since

$$\int_1^\infty \frac{e^{ix}}{x}dx = \int_1^\infty \frac{\cos x + i\sin x}{x}dx = \int_1^\infty \frac{\cos x}{x}dx + i\int_1^\infty \frac{\sin x}{x}dx,$$

in which $\int_1^\infty \frac{\cos x}{x}dx$ and $\int_1^\infty \frac{\sin x}{x}dx$ both converge, thus $\int_1^\infty \frac{e^{ix}}{x}$ converges.

Exercise 2.14 Suppose that $f : [-a, a] \to \mathbb{R}$ and $f \in R[-s, s]$, $\forall\ 0 < s < a$, but $f \notin R[-a, a]$. Suppose furthermore that f is even or odd. Show that $\int_{-a}^a f(x)dx$ converges iff $\int_{-a}^0 f(x)dx$ converges and

$$\int_{-a}^a f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & \text{if } f \text{ even;} \\ 0, & \text{if } f \text{ odd.} \end{cases}$$

Solution: If f is even or odd, for $0 < s < a$ we have

$$\int_{-s}^s f(x)dx = \int_{-s}^0 f(x)dx + \int_0^s f(x)dx = \begin{cases} 2\int_0^s f(x)dx, & \text{if } f(x) = f(-x) \text{ (even);} \\ 0, & \text{if } f(x) = -f(-x) \text{ (odd).} \end{cases}$$

Hence,

$$\lim_{s\to a^-} \int_{-s}^s f(x)dx = \begin{cases} 2\lim_{s\to a^-}\int_0^s f(x)dx, & \text{if } f \text{ even;} \\ \lim_{s\to a^-}\int_0^s f(x)dx - \lim_{s\to a^-}\int_0^s f(x)dx, & \text{if } f \text{ odd.} \end{cases}$$

Therefore, $\int_{-a}^a f(x)dx$ converges iff $\int_{-a}^0 f(x)dx$ converge and

$$\int_{-a}^a f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & \text{if } f \text{ even;} \\ 0, & \text{if } f \text{ odd.} \end{cases}$$

Exercise 2.15 Test the following integrals for convergence/absolute convergence/conditional convergence. Also say whether or not they are improper.

(a) $\int_{-1}^1 |x|^{-a}dx, a > 0$.

(b) $\int_0^1 \frac{\sin(1/x)}{x}dx$.

(c) $\int_0^1 (t(1-t))^{-a}dt,\ a > 0$.

(d) $\int_0^1 \frac{\sin x}{x}dx$.

(e) $\int_1^\infty e^{-x}(x-1)^{-2/3}dx$.

(f) $\int_{-\infty}^\infty e^{-x^2}|x|^{-1}dx$.

(g) The Beta function, Definition 33:

$$B(p, q) = \int_0^1 t^{p-1}(1-t)^{q-1}dt.$$

Solution: (a) Since

$$\int_{-1}^{1} ||x|^{-a}|dx = \int_{-1}^{1} |x|^{-a}dx = 2\lim_{s\to 0^+}\int_s^1 x^{-a}.$$

For $a > 0$ and $a \neq 1$,

$$2\lim_{s\to 0^+}\int_s^1 x^{-a} = 2\lim_{s\to 0^+}\frac{s^{1-a}-1}{a-1}.$$

Hence, $\int_{-1}^{1}|x|^{-a}dx$ is an improper integral. It converges if $0 < a < 1$ and diverges if $a > 1$.

For $a = 1$,

$$2\lim_{s\to 0^+}\int_s^1 x^{-a} = -2\lim_{s\to 0^+}\ln s.$$

Hence, $\int_{-1}^{1}|x|^{-1}dx$ diverges.

Therefore, $\int_{-1}^{1}|x|^{-a}dx$ is an improper integral. It is absolute convergence for $0 < a < 1$ and divergence for $a \geq 1$.

(b) For $0 < s < 1$,

$$\int_s^1 \left|\frac{\sin(1/x)}{x}\right|dx < \int_s^1 \frac{\sin^2(1/x)}{x}dx = \frac{1}{2}\left(\mathrm{Ci}(2) - \mathrm{Ci}\left(\frac{2}{s}\right) - \ln s\right).$$

Hence, $\int_0^1\left|\frac{\sin(1/x)}{x}\right|dx$ diverges. However,

$$\int_s^1 \frac{\sin(1/x)}{x}dx = \mathrm{Si}\left(\frac{1}{s}\right) - \mathrm{Si}(1),$$

thus $\lim_{s\to 0^+}\int_s^1\frac{\sin(1/x)}{x}dx = \pi/2 - \mathrm{Si}(1)$. $\int_0^1\frac{\sin(1/x)}{x}dx$ is an improper integral and conditional convergence.

(c) Since

$$\int_0^1 (t(1-t))^{-a}dt = \int_0^{1/2} (t(1-t))^{-a}dt + \int_{1/2}^1 (t(1-t))^{-a}dt,$$

and let $r = 1-t$,

$$\int_{1/2}^1 (t(1-t))^{-a}dt = \int_0^{1/2} (r(1-r))^{-a}dr,$$

we have

$$\int_0^1 \left|(t(1-t))^{-a}\right|dt = \int_0^1 (t(1-t))^{-a}dt = 2\int_0^{1/2} (t(1-t))^{-a}dt < 2^{a+1}\int_0^{1/2}\frac{1}{t^a}dt,$$

as well as

$$2\int_0^{1/2} (t(1-t))^{-a}dt > 2\int_0^{1/2}\frac{1}{t^a}dt.$$

According to (a), $\int_0^{1/2} \frac{1}{t^a} dt$ converges for $0 < a < 1$ and diverges for $a \geq 1$. Therefore, $\int_0^1 (t(1-t))^{-a} dt$ is an improper integral. It is absolute convergence for $0 < a < 1$ and divergence for $a \geq 1$.

(d) since

$$\int_0^1 \left| \frac{\sin x}{x} \right| dx = \int_0^1 \frac{\sin x}{x} dx = \mathrm{Si}(1),$$

and

$$\lim_{s \to 0^+} \frac{\sin x}{x} = 1,$$

$\int_0^1 \frac{\sin x}{x} dx$ is a regular integral and convergence.

(e) Since

$$\int_1^{\infty} \left| e^{-x}(x-1)^{-2/3} \right| dx = \int_1^{\infty} e^{-x}(x-1)^{-2/3} dx < e^{-1} \int_1^{\infty} x^{-1}(x-1)^{-2/3} dx,$$

$\int_1^{\infty} e^{-x}(x-1)^{-2/3} dx$ is an improper integral and absolute convergence. Actually,

$$\int_1^{\infty} e^{-x}(x-1)^{-2/3} dx = e^{-1} \Gamma\left(\frac{1}{3}\right).$$

(f) Since

$$\int_{-\infty}^{\infty} \left| e^{-x^2} |x|^{-1} \right| dx = \int_{-\infty}^{\infty} e^{-x^2} |x|^{-1} dx = 2 \lim_{s \to 0^+} \int_s^1 e^{-x^2} x^{-1} dx + 2 \int_1^{\infty} e^{-x^2} x^{-1} dx,$$

and

$$\lim_{s \to 0^+} \int_s^1 e^{-x^2} x^{-1} dx > e^{-1} \lim_{s \to 0^+} \int_s^1 x^{-1} dx,$$

where $\lim_{s \to 0^+} \int_s^1 x^{-1} dx$ diverges, thus $\int_{-\infty}^{\infty} e^{-x^2} |x|^{-1} dx$ is an improper integral and divergence.

(g) If $p, q \geq 1$, Beta function is a regular integral and thereby convergence. If either $0 < p < 1$ or $0 < q < 1$, very similar to (c), it is an improper integral and absolute convergence.

Exercise 2.16 Cauchy Principal Value. Sometimes, even when $\int_a^b f(x)dx$ diverges for a function f that is unbounded at some $c \in (a, b)$, we may still define what is called the *Cauchy Principal Value Integral*: Suppose that $f \in R[a, c-\varepsilon]$ and $f \in R[c+\varepsilon, b]$ for all small enough $\varepsilon > 0$ but f is unbounded at c. Define

$$\text{P.V.} \int_a^b f(x)dx := \lim_{\varepsilon \to)^+} \left[\int_a^{c-\varepsilon} f(x)dx + \int_{c+\varepsilon}^b f(x)dx \right]$$

if the limit exists. (P.V. stands for Principal Value).

(a) Show that $\int_{-1}^1 \frac{1}{x} dx$ diverges but P.V.$\int_{-1}^1 \frac{1}{x} dx = 0$.

(b) Show that P.V.$\int_a^b \frac{1}{x-c}dx = \ln\frac{b-c}{c-a}$ if $c \in (a,b)$.

(c) Show that if f is odd and $f \in R[\varepsilon, 1],\ \forall\ 0 < \varepsilon < 1$, then P.V.$\int_{-1}^1 f(x)dx = 0$.

Solution: (a) Since $\int_0^1 \frac{1}{x}dx$ and $\int_{-1}^0 \frac{1}{x}dx$ diverge, then $\int_{-1}^1 \frac{1}{x}dx$ diverges. However

$$\begin{aligned}\text{P.V.}\int_{-1}^1 \frac{1}{x}dx &= \lim_{\varepsilon\to 0^+}\left[\int_{-1}^{-\varepsilon}\frac{1}{x}dx + \int_{\varepsilon}^1 \frac{1}{x}dx\right]\\ &= \lim_{\varepsilon\to 0^+}(-\ln\varepsilon + \ln\varepsilon) = 0\end{aligned}$$

(b) For $c \in (a,b)$,

$$\begin{aligned}\text{P.V.}\int_a^b \frac{1}{x-c}dx &= \lim_{\varepsilon\to 0^+}\left[\int_a^{c-\varepsilon}\frac{1}{x-c}dx + \int_{c+\varepsilon}^b \frac{1}{x-c}dx\right]\\ &= \lim_{\varepsilon\to 0^+}(\ln\varepsilon - \ln(c-a) + \ln(b-c) - \ln\varepsilon) = \ln\left(\frac{b-c}{c-a}\right).\end{aligned}$$

(c) For $0 < \varepsilon < 1$ and f is odd,

$$\begin{aligned}\text{P.V.}\int_{-1}^1 f(x)dx &= \lim_{\varepsilon\to 0^+}\left[\int_{-1}^{-\varepsilon} f(x)dx + \int_{\varepsilon}^1 f(x)dx\right] \quad (\text{let } y = -x)\\ &= \lim_{\varepsilon\to 0^+}\left[-\int_{\varepsilon}^1 f(-y)d(-y) + \int_{\varepsilon}^1 f(x)dx\right]\\ &= \lim_{\varepsilon\to 0^+}\left[-\int_{\varepsilon}^1 f(x)dx + \int_{\varepsilon}^1 f(x)dx\right] = 0.\end{aligned}$$

Exercise 2.17 (a) Use integration by parts to show that

$$\Gamma(x+1) = x\Gamma(x),\ \ x > 0$$

This is called the **difference equation** for $\Gamma(x)$

(b) Show that $\Gamma(1) = 1$ and deduce that for $n \geq 0, \Gamma(n+1) = n!$. Thus Γ generalizes the factorial function.

(c) Show that for $n \geq 1$,

$$\Gamma(n+1/2) = (n-1/2)(n-3/2)\ldots(3/2)(1/2)\pi^{1/2}$$

Solution: (a)

$$\begin{aligned}\Gamma(x+1) &= \int_0^\infty t^x e^{-t}dt\\ &= -t^x e^{-t}\Big|_0^\infty + x\int_0^\infty t^x e^{-t}dt\\ &= 0 - 0 + x\Gamma(x-1) = x\Gamma(x-1).\end{aligned}$$

(b)

$$\Gamma(1) = \int_0^\infty e^{-t} dt = -e^{-t}\Big|_0^\infty = 1.$$

From (a) we have $\Gamma(n+1) = n\Gamma(n)$. By using the mathematical induction, we have $\Gamma(n+1) = n!$

(c) For $n \geq 1$,

$$\begin{aligned}
\Gamma(n+1/2) &= (n-1/2)\Gamma(n-1/2) \\
&= (n-1/2)(n-3/2)\Gamma(n-3/2) \\
&= \dots \\
&= (n-1/2)(n-3/2)\dots(3/2)(1/2)\Gamma(1/2),
\end{aligned}$$

where

$$\Gamma(1/2) = \int_0^\infty t^{-1/2} e^t dt = \sqrt{\pi}\,\mathrm{erf}(\sqrt{t})\Big|_0^\infty = \sqrt{\pi}.$$

Exercise 2.18 (a) Show that

$$B(p,q) = \int_0^\infty y^{p-1}(1+y)^{-p-q} dy$$

(b) Show that

$$B(p,q) = (1/2)^{p+q-1} \int_{-1}^1 (1+x)^{p-1}(1-x)^{q-1} dx.$$

Also $B(p,q) = B(q,p)$.

(c) Show that for

$$B(p,q) = B(p+1,q) + B(p,q+1).$$

(d) Write out $B(m,n)$ for m and n positive integers.

Solution: (a) Since

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1} dt,$$

let $t = y/(1+y)$, then

$$\begin{aligned}
B(p,q) &= \int_0^\infty \left(\frac{y}{1+y}\right)^{p-1} \left(1 - \frac{y}{1+y}\right)^{q-1} d\left(\frac{y}{1+y}\right) \\
&= \int_0^\infty y^{p-1}(1+y)^{-(p-1)-(q-1)-2} dy \\
&= \int_0^\infty y^{p-1}(1+y)^{-p-q} dy.
\end{aligned}$$

(b) Let $t = \frac{1}{2}(1+x)$, then

$$\begin{aligned}
B(p,q) &= \int_{-1}^{1} \left(\frac{1}{2}\right)^{p-1} (x+1)^{p-1} \left(1 - \frac{1}{2}(x+1)\right)^{q-1} d\left(\frac{1}{2}(x+1)\right) \\
&= \left(\frac{1}{2}\right)^{p-1+q-1+1} \int_{-1}^{1} (1+x)^{p-1}(1-x)^{q-1} dx \\
&= \left(\frac{1}{2}\right)^{p+q-1} \int_{-1}^{1} (1+x)^{p-1}(1-x)^{q-1} dx.
\end{aligned}$$

Similarly, let $t = \frac{1}{2}(x-1)$, then we have

$$B(q,p) = \left(\frac{1}{2}\right)^{p+q-1} \int_{-1}^{1} (1+x)^{p-1}(1-x)^{q-1} dx.$$

Hence, $B(q,p) = B(p,q)$.

(c)

$$\begin{aligned}
B(p+1,q) + B(p,q+1) &= \int_0^1 t^p (1-t)^{q-1} dt + \int_0^1 t^p (1-t)^q dt \\
&= \int_0^1 t^{p-1}(1-t)^{q-1}(t+1-t) dt \\
&= \int_0^1 t^{p-1}(1-t)^{q-1} dt = B(p,q).
\end{aligned}$$

(d) Since

$$\begin{aligned}
\Gamma(p)\Gamma(q) &= \int_0^\infty t^{p-1} e^{-t} dt \cdot \int_0^\infty r^{q-1} e^{-r} dr \\
&= \int_{t=0}^\infty \int_{r=0}^\infty t^{p-1} r^{q-1} e^{-t-r} dt dr
\end{aligned}$$

Let $t = yx$ and $r = y(1-x)$, we have

$$\begin{aligned}
\Gamma(p)\Gamma(q) &= \int_{y=0}^\infty \int_{x=0}^1 e^{-y} (yx)^{p-1} (y(1-x))^{q-1} y dx dy \\
&= \int_{y=0}^\infty e^{-y} y^{p+q-1} dy \cdot \int_{x=0}^1 x^{p-1}(1-x)^{q-1} dx \\
&= \Gamma(p+q) B(p,q).
\end{aligned}$$

Therefore,

$$\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} = B(p,q).$$

For positive integers m and n, according to Exercise 2.17(b), we have

$$B(m,n) = \frac{(m-1)!(n-1)!}{(m+n-1)!}.$$

Exercise 2.19 Large O and little o notation Let f and $h : [a, \infty) \to \mathbb{R}, g : [a, \infty) \to (0, \infty)$. We write $f(x) = O(g(x)), x \to \infty$ iff there exists $C > 0$ independent of x such that

$$|f(x)| < Cg(x), \ \forall \text{ large enough } x$$

We write $f(x) = o(g(x)), x \to \infty$ iff $\lim_{x\to\infty} f(x)/g(x) = 0$. Quite often, we take $g \equiv 1$. In this case, $f(x) = O(1), x \to \infty$ means $|f(x)| \le C$ for some C independent of x, i.e. f is uniformly bounded above in absolute value for large enough x and $f(x) = o(1), x \to \infty$ means $\lim_{x\to\infty} f(x) = 0$.

(a) Show that if $f(x) = O(1)$ and $h(x) = o(1)$ as $x \to \infty$, then $f(x)h(x) = o(1), x \to \infty$ and $f(x) + h(x) = o(1), x \to \infty$. Can we say anything about the ratio $h(x)/f(x)$ in general for large enough x?

(b) Suppose that $f, g \in R[a, s], \ \forall \ s > a$ and $\int_a^\infty g(x)dx$ converges. Show that if $f(x) = O(g(x)), x \to \infty$, then $\int_a^\infty f(x)dx$ converges absolutely.

(c) Show that if also h is nonnegative and

$$f(x) = O(g(x)), x \to \infty, g(x) = O(h(x)), x \to \infty$$

then this relation is transitive, that is

$$f(x) = O(h(x)), x \to \infty.$$

Solution: (a) If $f(x) = O(1)$ and $h(x) = o(1)$,

$$\lim_{x\to\infty} |f(x)g(x)| < C \lim_{x\to\infty} |g(x)| = C \cdot 0 = 0.$$

Thus, by definition $f(x)g(x) = o(1), x \to \infty$. Similarly,

$$\lim_{x\to\infty} |f(x) + g(x)| \le C + \lim_{x\to\infty} |g(x)| = C.$$

Thus, by definition $f(x) + h(x) = o(1), x \to \infty$. Since it is possible that $f(x) = o(1)$ if $f(x) = O(1)$ for $x \to \infty$, $h(x)/f(x), x \to \infty$ is undetermined.

(b) If $f(x) = O(g(x)), x \to \infty$, there exists a $x_0 > a$ and $C > 0$ that for $x > x_0$,

$$|f(x)| < Cg(x).$$

Therefore,

$$\begin{aligned}
\int_a^\infty |f(x)|dx &= \int_a^{x_0} |f(x)|dx + \int_{x_0}^\infty |f(x)|dx \\
&\le \int_a^{x_0} |f(x)|dx + C \int_{x_0}^\infty g(x)dx \\
&\le \int_a^{x_0} |f(x)|dx + C \int_a^\infty g(x)dx.
\end{aligned}$$

Since $\int_a^\infty g(x)dx$ converges, then $\int_a^\infty |f(x)|dx$ converges.

(c) If $f(x) = O(g(x)), x \to \infty, g(x) = O(h(x)), x \to \infty$, then there exists $B, C > 0$

$$|f(x)| \le Bg(x), \quad |g(x)| \le Ch(x), \quad h(x) \ge 0, \; x \to \infty.$$

Hence,

$$|f(x)| \le BCh(x), \; x \to \infty,$$

where $BC > 0$. By definition, $f(x) = O(h(x)), x \to \infty$.

Exercise 2.20 Show that

$$\prod_{n=2}^{\infty}\left(1 - \frac{2}{n(n+1)}\right) = \frac{1}{3}$$

Solution:

$$\begin{aligned}\prod_{n=2}^{\infty}\left(1 - \frac{2}{n(n+1)}\right) &= \prod_{n=2}^{\infty}\frac{(n+2)(n-1)}{n(n+1)} \\ &= \lim_{n\to\infty}\frac{(4\cdot 5\cdot \cdots \cdot (n+2))(1\cdot 2\cdot \cdots \cdot (n-1))}{(2\cdot 3\cdot \cdots \cdot n)(3\cdot 4\cdot \cdots \cdot (n+1))} \\ &= \lim_{n\to\infty}\frac{n+2}{n\cdot 3} = \frac{1}{3}.\end{aligned}$$

Exercise 2.21 Show that

$$\prod_{n=1}^{\infty}\left(1 + \frac{6}{(n+1)(2n+9)}\right) = \frac{21}{8}$$

Solution:

$$\begin{aligned}\prod_{n=1}^{\infty}\left(1 + \frac{6}{(n+1)(2n+9)}\right) &= \prod_{n=1}^{\infty}\frac{(2n+5)(n+3)}{(n+1)(2n+9)} \\ &= \lim_{n\to\infty}\frac{(7\cdot 9\cdot \cdots \cdot (2n+5))(4\cdot 5\cdot \cdots \cdot (n+3))}{(2\cdot 3\cdot \cdots \cdot (n+1))(11\cdot 13\cdot \cdots \cdot (2n+9))} \\ &= \lim_{n\to\infty}\frac{7\cdot 9\cdot (n+1)(n+2)}{2\cdot 3\cdot (2n+7)(2n+9)} = \frac{21}{8}.\end{aligned}$$

Exercise 2.22 Show that

$$\prod_{n=0}^{\infty}\left(1 + x^{2^n}\right) = \frac{1}{1-x}, \; |x| < 1.$$

Hint: Multiple the partial products by $1 - x$.

Solution:

$$\begin{aligned}(1-x)\prod_{n=0}^{\infty}\left(1+x^{2^n}\right) &= (1-x)(1+x)(1+x^2)(1+x^4)\dots\\ &= (1-x^2)(1+x^2)(1+x^4)\dots\\ &= (1-x^4)(1+x^4)\dots\\ &= \dots\\ &= \lim_{n\to\infty}(1-x^{2^{n+1}}) = 1, \quad |x|<1.\end{aligned}$$

Hence, $\prod_{n=0}^{\infty}\left(1+x^{2^n}\right) = \frac{1}{1-x},\ |x|<1$.

Exercise 2.23 Let $a, b \in \mathbb{C}$ and suppose that they are not negative integers. Show that

$$\prod_{n=1}^{\infty}\left(1+\frac{a-b-1}{(n+a)(n+b)}\right) = \frac{a}{b+1}.$$

Solution:

$$\begin{aligned}\prod_{n=1}^{\infty}\left(1+\frac{a-b-1}{(n+a)(n+b)}\right) &= \prod_{n=1}^{\infty}\frac{(n+a-1)(n+b+1)}{(n+a)(n+b)}\\ &= \lim_{n\to\infty}\frac{(1+a-1)(n+b+1)}{(n+a)(1+b)}\\ &= \frac{a}{b+1}\end{aligned}$$

Exercise 2.24 Is $\prod_{n=1}^{\infty}(1+n^{-1})$ convergent?

Solution: According the Theorem 2.27, since $\sum_{n=1}^{\infty} n^{-1}$ is not convergent, thus $\prod_{n=1}^{\infty}(1+n^{-1})$ is not convergent, too.

Exercise 2.25 Prove the inequality $0 \le e^u - 1 \le 3u$ for $u \in (0,1)$ and deduce that $\prod_{n=1}^{\infty}\left(1+\left(e^{1/n}-1\right)/n\right)$ converges.

Solution: Let $f(u) = (e^u-1)/(3u)$. Since $f'(u) > 0$ for $u \in (0,1)$, and

$$f(0) = \lim_{u\to 0^+} f(u) = \frac{e^u}{3} = \frac{1}{3}$$

$$f(1) = \frac{e-1}{3} \approx 0.572761.$$

Hence,

$$0 \le e^u - 1 \le 3u, \quad u \in (0,1)$$

Since

$$\frac{e^{1/n}-1}{n} = \frac{1}{n^2} + \frac{1}{2n^3} + \frac{1}{6n^4} + o\left(\frac{1}{n^5}\right),$$

$\sum_{n=1}^{\infty}(e^{1/n}-1)/n$ converges. Therefore, $\prod_{n=1}^{\infty}\left(1+\left(e^{1/n}-1\right)/n\right)$ converges.

Exercise 2.26 The **Euler–Mascheroni Constant**. Let

$$H_n = \sum_{k=1}^{n} \frac{1}{k}, \quad n \geq 1.$$

Show that $1/k \geq 1/x, x \in [k, k+1]$ and hence that

$$\frac{1}{k} \geq \int_k^{k+1} \frac{1}{x} dx.$$

Deduce that

$$H_n \geq \int_1^{n+1} \frac{1}{x} dx = \ln(n+1),$$

and that

$$\begin{aligned}
[H_{n+1} - \ln(n+1)] - [H_n - \ln n] &= \frac{1}{n+1} + \ln\left(1 - \frac{1}{n+1}\right) \\
&= -\sum_{k=2}^{\infty} \left(\frac{1}{n+1}\right)^k / k < 0.
\end{aligned}$$

Deduce that $H_n - \ln n$ decreases as n increases and thus has a non–negative limit γ, called the *Euler–Mascheroni constant* and with value approximately $0.5772\ldots$ not known to be either rational or irrational. Finally show that

$$\prod_{k=1}^{n} \left(1 + \frac{1}{k}\right) e^{-1/k} = \prod_{k=1}^{n} \left(1 + \frac{1}{k}\right) \prod_{k=1}^{n} e^{-1/k} = \frac{n+1}{n} \exp\left(-(H_n - \ln n)\right).$$

Deduce that

$$\prod_{k=1}^{\infty} \left(1 + \frac{1}{k}\right) e^{-1/k} = e^{-\gamma}.$$

Many functions have nice infinite product expansions which can be derived from first principles. The following examples, utilizing contour integration in the complex plane, illustrate some of these nice expansions.

Solution: Of course if $x \in [k, k+1] \geq k > 0$, then $1/k \geq 1/x$ and

$$\frac{1}{k} = \int_k^{k+1} \frac{1}{k} dx \geq \int_k^{k+1} \frac{1}{x} dx.$$

Hence,

$$H_n = \sum_{k=1}^{n} \int_k^{k+1} \frac{1}{k} dx \geq \sum_{k=1}^{n} \int_k^{k+1} \frac{1}{x} dx = \int_1^{n+1} \frac{1}{x} dx = \ln(n+1).$$

Thus,

$$\begin{aligned}
[H_{n+1} - \ln(n+1)] - [H_n - \ln n] &= \frac{1}{n+1} + \ln\left(1 - \frac{1}{n+1}\right) \\
&= -\sum_{k=2}^{\infty} \left(\frac{1}{n+1}\right)^k / k < 0.
\end{aligned}$$

Since the Taylor expansion of $\ln(1-1/(n+1))$ is

$$\ln\left(1-\frac{1}{n+1}\right) = -\sum_{k=1}^{\infty}\left(\frac{1}{n+1}\right)^k/k,$$

then

$$\frac{1}{n+1}+\ln\left(1-\frac{1}{n+1}\right) = -\sum_{k=2}^{\infty}\left(\frac{1}{n+1}\right)^k/k < 0.$$

Hence, $H_n - \ln n$ decreases as n increases.

$$\begin{aligned}\prod_{k=1}^{n}\left(1+\frac{1}{k}\right)e^{-1/k} &= \prod_{k=1}^{n}\left(1+\frac{1}{k}\right)\prod_{k=1}^{n}e^{-1/k}\\ &= \frac{2\cdot 3\cdot\cdots\cdot(n+1)}{1\cdot 2\cdot\cdots\cdot n}e^{-H_n}\\ &= (n+1)e^{-(H_n-\ln n)}e^{-\ln n}\\ &= \frac{n+1}{n}e^{-(h_n-\ln n)}.\end{aligned}$$

Hence,

$$\lim_{n\to\infty}\prod_{k=1}^{n}\left(1+\frac{1}{k}\right)e^{-1/k} = \lim_{n\to\infty}\frac{n+1}{n}e^{-(h_n-\ln n)} = e^{-\gamma}.$$

Exercise 2.28 Using the previous exercise, derive the following expansions for those arguments for which the RHS is meaningful.

(1) $\sin(x) = x\prod_{j=1}^{\infty}(1-(x/j\pi)^2)$.

(2) $\cos(x) = \prod_{j=1}^{\infty}\left(1-\left(\frac{x}{(j-1/2)\pi}\right)^2\right)$.

(3) Euler's product formula for the gamma function

$$x\Gamma(x) = \prod_{n=1}^{\infty}\left[(1+1/n)^x(1+x/n)^{-1}\right]$$

.

(4) Weierstrass's product formula for the gamma function

$$(\Gamma(x))^{-1} = xe^{x\gamma}\prod_{k=1}^{\infty}\left[(1+x/k)e^{-x/k}\right].$$

Here, γ is the *Euler–Mascheroni constant.*

Solution: (1) Since

$$\frac{\sin(\pi x)}{\pi x} = \frac{1}{x\Gamma(x)\Gamma(1-x)} = \frac{1}{\Gamma(1+x)\Gamma(1-x)},$$

and

$$\Gamma(x) = \frac{1}{x}\prod_{j=1}^{\infty} \frac{(1+1/j)^x}{1+x/j},$$

then we have

$$\begin{aligned}
\frac{1}{\Gamma(1+x)\Gamma(1-x)} &= (1-x^2)\prod_{j=1}^{\infty} \frac{(1+1/j+x/j)(1+1/j-x/j)}{(1+1/j)^{1+x+1-x}} \\
&= (1-x^2)\prod_{j=1}^{\infty} \frac{(1+1/j)^2-(x/j)^2}{(1+1/j)^2} \\
&= (1-x^2)\prod_{j=1}^{\infty} \left(1-\left(\frac{x}{j+1}\right)^2\right) \\
&= \prod_{j=0}^{\infty} \left(1-\left(\frac{x}{j+1}\right)^2\right) = \prod_{j=1}^{\infty} \left(1-\left(\frac{x}{j}\right)^2\right).
\end{aligned}$$

Hence,

$$\frac{\sin(x)}{x} = \prod_{j=1}^{\infty} \left(1-\left(\frac{x}{j\pi}\right)^2\right),$$

and thereby $\sin(x) = x\prod_{j=1}^{\infty}\left(1-(x/j\pi)^2\right)$.

(2)

$$\begin{aligned}
\cos(x) &= \frac{\sin(2x)}{2\sin(x)} \\
&= \frac{2x\prod_{j=1}^{\infty}\left(1-(2x/j\pi)^2\right)}{2x\prod_{j=1}^{\infty}\left(1-(x/j\pi)^2\right)} \\
&= \frac{2x\prod_{j=1}^{\infty}\left(1-(2x/j\pi)^2\right)}{2x\prod_{j=1}^{\infty}\left(1-(2x/2j\pi)^2\right)} \\
&= \prod_{j=1}^{\infty} \left(1-\left(\frac{2x}{2(j-1)\pi}\right)^2\right) = \prod_{j=1}^{\infty} \left(1-\left(\frac{x}{(j-1/2)\pi}\right)^2\right).
\end{aligned}$$

(3) Let $g(x) = \ln\Gamma(x)$. By using $\Gamma(x+1) = x\Gamma(x)$ repeatedly, we have

$$g(x+n) = \sum_{j=0}^{n-1} \ln(x+j) + g(x),$$

and

$$L_n(x) = \sum_{j=1}^{n-1} \ln(j) + x\ln(x+n-1) \le g(x+n) \le \sum_{j=1}^{n-1} \ln(j) + x\ln n = U_n(x).$$

Since

$$\begin{aligned}
\lim_{n\to\infty} U_n(x) - L_n(x) &= \lim_{n\to\infty} x(\ln n - \ln(x+n-1)) \\
&= \lim_{n\to\infty} x \ln\left(\frac{1}{1+(x-1)/n}\right) = 0,
\end{aligned}$$

then

$$g(x) = \lim_{n\to\infty}\left(U_n(x) - \sum_{j=0}^{n-1}\ln(x+j)\right) = \lim_{n\to\infty}\left(\sum_{j=1}^{n-1}(\ln j - \ln(x+j)) - \ln x + x\ln n\right),$$

and

$$\begin{aligned}
\Gamma(x) &= \lim_{n\to\infty} \exp\left[\left(\sum_{j=1}^{n-1}(\ln j - \ln(x+j)) - \ln x + x\ln n\right)\right] \\
&= \lim_{n\to\infty} \frac{n^x (n-1)!}{x(x+1)\dots(x+n-1)} \\
&= \lim_{n\to\infty} \frac{(n+1)^x n!}{x(x+1)\dots(x+n)} \\
&= \frac{1}{x}\lim_{n\to\infty}\prod_{j=1}^{n}\frac{(1+1/j)^x}{1+x/j}
\end{aligned}$$

(4) Following (3), we have

$$\begin{aligned}
\frac{1}{\Gamma(x)} &= x\prod_{j=1}^{\infty}\frac{1+x/j}{(1+1/j)^x} \\
&= xe^{x\gamma}e^{-x\gamma}\prod_{j=1}^{\infty}\frac{1+x/j}{(1+1/j)^x} \\
&= xe^{x\gamma}\prod_{j=1}^{\infty}(1+1/j)^x e^{-x/j}\prod_{j=1}^{\infty}\frac{1+x/j}{(1+1/j)^x} \\
&= xe^{x\gamma}\prod_{j=1}^{\infty}(1+x/j)e^{-x/j}.
\end{aligned}$$

Solutions to Chapter 3

Exercise 3.1 Let $g(a) = \int_a^{2\pi+a} f(t)dt$ and give a new proof of lemma 3.1 based on a computation of the derivative g'(a).

Solution: By fundamental theorem of calculus, for all a:

$$g(a) = \int_a^{2\pi+a} f(t)dt = F(2\pi + a) - F(a)$$

Thus

$$g'(a) = F'(2\pi + a) - F'(a) = f(2\pi + a) - f(a) = 0$$

which means g(a) is a constant function

Exercise 3.2 Verify lemma 3.3

Solution: Euler's formula $e^{iy} = \cos(y) + i\sin(y)$

1. $e^{z_1}e^{z_2} = e^{x_1+iy_1}e^{x_2+iy_2} = e^{x_1+iy_1+x_2+iy_2} = e^{z_1+z_2}$
2. $|e^z| = |e^x||\cos(y) + i\sin(y)| = e^x(\cos^2(y) + \sin^2(y)) = e^x$
3. $\overline{e^z} = e^x(\overline{\cos(y) + i\sin(y)}) = e^x(\cos(y) - i\sin(y)) = e^x e^{-iy} = e^{\overline{z}}$

Exercise 3.4 Suppose f is piecewise continuous and 2π-periodic. For any point t define the right-hand derivative $f'_R(t)$ and the left-hand derivative $f'_L(t)$ of f by

$$f'_R(t) = \lim_{u \to t+} \frac{f(u) - f(t+0)}{u - t}$$

$$f'_L(t) = \lim_{u \to t-} \frac{f(u) - f(t-0)}{u - t}$$

respectively. Show that in the proof of Theorem 3.12 we can drop the requirement for f' to be piecewise continuous and the conclusion of the theorem will still hold at any point t such that both $f'_R(t)$ and $f'_L(t)$ exist.
Solution: All we need to show under the new condition

$$\lim_{x \to 0+} \frac{f(t+x) + f(t-x) - 2f(t)}{2\sin\left(\frac{x}{2}\right)}$$

still exists
We can split the expression into

$$\lim_{x \to 0+} [\frac{f(t+x) - f(t)}{2\sin\left(\frac{x}{2}\right)} - \frac{f(t) - f(t-x)}{2\sin\left(\frac{x}{2}\right)}]$$

$$\lim_{x\to 0+}[\frac{f(t+x)-f(t)}{2\sin\left(\frac{x}{2}\right)}] - \lim_{x\to 0+}[\frac{f(t)-f(t-x)}{2\sin\left(\frac{x}{2}\right)}]$$

For each t, Define $h_t^+(x) = f(t+x) - f(t)$,and $h_t^-(x) = f(t) - f(t-x)$
By assumption f is piecewise continuous, we have

$$\lim_{x\to 0+} h_t^+(x) = \lim_{x\to 0+} h_t^-(x) = 0$$

Also $f'_R(t)$ and $f'_L(t)$ exist, which can be rewritten (change of variable) as

$$f'_R(t) = \lim_{x\to 0+} \frac{f(x+t) - f(t+0)}{x}$$

$$f'_L(t) = \lim_{x\to 0+} \frac{f(t) - f(t-x)}{x}$$

Then by L'Hopital's rule the limit we are looking at exists

Exercise 3.5 Show that if f and f' are piecewise continuous then for any point t we have $f'(t+0) = f'_R(t)$ and $f'(t-0) = f'_L(t)$

Solution: Starting from the hint. for $u > t$ and u sufficient close to t there is a point c such that $t < c < u$ and

$$\frac{f(u) - f(t+0)}{u-t} = f'(c)$$

Then take the limit $u \to t+$ and limit $u \to t-$ we obtained the result desired

Exercise 3.6 Let

$$f(t) = \{ \begin{array}{ll} t^2\sin\left(\frac{1}{t}\right), & for\, t \neq 0 \\ 0, & for\, t = 0 \end{array}$$

Show that f is continuous for all t and that $f'_R(t) = f'_L(t) = 0$. Show that $f'(t+0)$ and $f'(t-0)$ do not exist for $t = 0$. Hence argue that f' is not a piecewise continuous function.

Solution: By definition

$$f'_R(0) = \lim_{u\to 0+} \frac{f(u) - f(0+0)}{u-0} = \lim_{u\to 0+} \frac{u^2(\sin(1/u)) - 0}{u} = \lim_{u\to 0+} u(\sin(1/u))$$

Since sin function is bounded by -1 and 1, by comparison test, we have $f'_R(0) = 0$ and by similar argument we have $f'_L(0) = 0$

$$f'(0+0) = \lim_{t\to 0+} f'(t) = \lim_{t\to 0+} [t\sin\left(\frac{1}{t}\right) - \cos\left(\frac{1}{t}\right)]$$

which does not exist, identical argument can be made regard $f'(0-0)$

Exercise 3.7 let

$$f(t) = \{ \begin{array}{ll} \frac{2\sin\left(\frac{t}{2}\right)}{t}, & for\, 0 < |t| \le \pi \\ 1, & for\, t = 0 \end{array}$$

Extend f to be a 2π-periodic function on the entire real line. Verify that f satisfies the hypotheses of Theorem 3.12 and is continuous at t=0. Apply the localization Theorem(3.12) to f at $t = 0$ to give a new evaluation of the improper integral $\int_0^\infty \frac{\sin(x)}{x} dx = \frac{\pi}{2}$.

Solution: First verify the properties

1. f is piecewise continuous on $[-\pi, \pi]$
2. f' is piecewise continuous on $[-\pi, \pi]$

For (1) since f is differentiable everywhere except at $t = 0$, so all we need to verify is f is continuous at $t = 0$. By L'Hopital's Rule

$$\lim_{t\to 0} f(t) = \lim_{t\to 0} \frac{2\sin\left(\frac{t}{2}\right)}{t} = \lim_{t\to 0} \frac{\cos\left(\frac{t}{2}\right)}{1} = 1 = f(0)$$

For (2) for $t \neq 0, f'(t) = \frac{t\cos\left(\frac{t}{2}\right) - 2\sin\left(\frac{t}{2}\right)}{t^2}$ is differentiable, so all we need to verify is f' is continuous at $t = 0$. Apply L'Hopital's Rule

$$\lim_{t\to 0} f'(t) = \lim_{t\to 0} \frac{t\cos\left(\frac{t}{2}\right) - 2\sin\left(\frac{t}{2}\right)}{t^2} = \lim_{t\to 0} \frac{\cos\left(\frac{t}{2}\right) - \frac{\sin\left(\frac{t}{2}\right)}{2} - \cos\left(\frac{t}{2}\right)}{2t} = 0$$

Exists Follow Theorem 3.11

$$\lim_{k\to\infty} S_k(0) = \lim_{k\to\infty} \frac{1}{\pi} \int_{-\delta}^{\delta} D_k(-x) f(x) dx = \lim_{k\to\infty} \frac{1}{\pi} \int_{-\delta}^{\delta} \frac{\sin[(k+\frac{1}{2})x]}{x} dx$$

Since function integrating is even function, and we can do a change of integrating variable with $x' = (k + \frac{1}{2})x$, we have:

$$\lim_{k\to\infty} S_k(0) = \frac{2}{\pi} \lim_{k\to\infty} \int_0^{k\delta+\frac{1}{2}} \frac{\sin(x')}{x'} dx' = \int_0^\infty \frac{\sin(x)}{x} dx$$

By Theorem 3.12

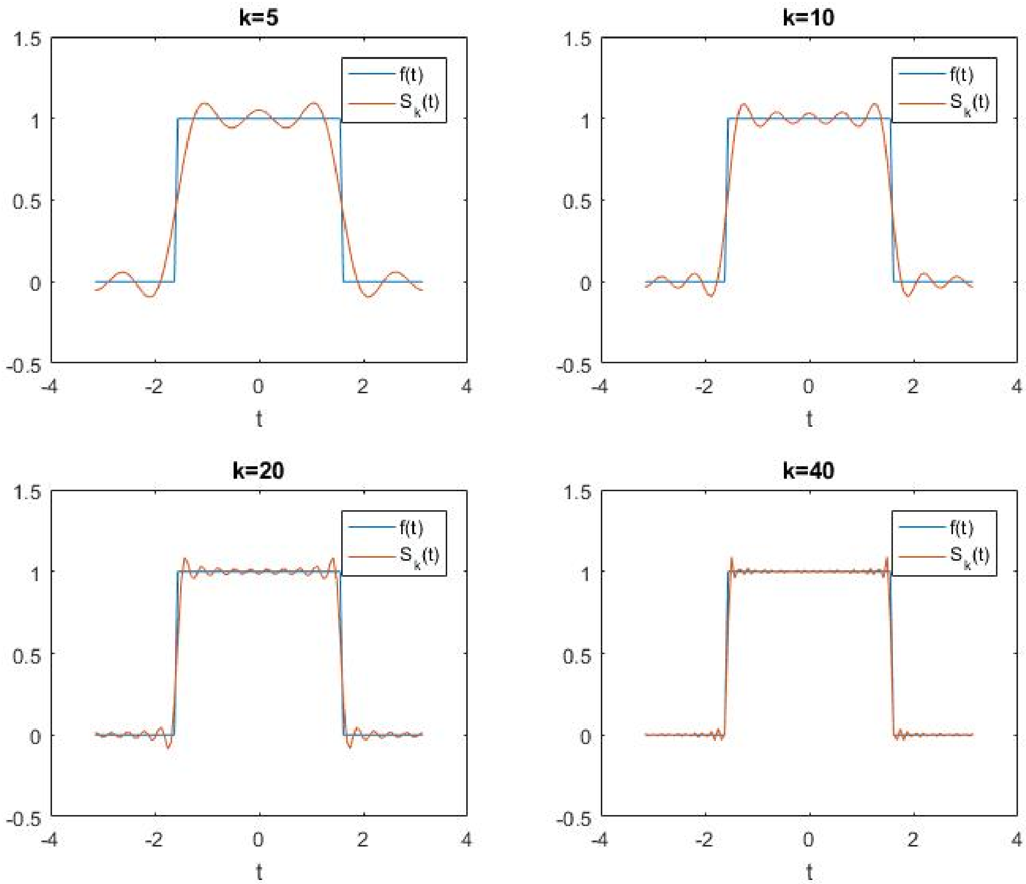
$$1 = f(0) = \lim_{k\to\infty} S_k(0)$$

After simple substitution, we obtained result we desire.

Exercise 3.8 Let

$$f(x) = \{ \begin{array}{ll} 0 & for\, -\pi < x \le -\frac{\pi}{2} \\ 1 & for\, -\frac{\pi}{2} < x \le \frac{\pi}{2} \\ 0 & for\, \frac{\pi}{2} < x \le \pi \end{array}$$

in a Fourier series on the interval $-pi \le x \le \pi$. Plot both f and the partial sums S_k for $k = 5, 10, 20, 40$.Observe how the partial sums approximate f. What accounts for the slow rate of convergence?



Solution: Above are the plots generated by matlab By definition

$$a_n = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(nx)dx = \frac{1}{n\pi}\sin(nx)|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{2\sin\left(\frac{n\pi}{2}\right)}{n\pi}$$

$$b_n = 0 \quad (Odd\,Function)$$

$$a_0 = \pi$$

$$S_k(t) = \frac{\pi}{2} + \sum_{n=1}^{k} a_n \cos(nt)$$

Note: a_n is essentially an alternating decreasing sequence.
The major reason for the slow convergence is the discontinuity at $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$, so Gibbs Phenomena applies

Below is the matlab code used to generate the plots

```
clear;
k=10;
x =linspace(-pi,pi);
```

```
f =ones(100,1);
f(1:25)=0;
f(76:end)=0;
idx=1:k;
A=zeros(k,100);
a=2*sin(pi*idx/2)./(pi*idx);
for i=1:k
    for j=1:100
        A(i,j)=cos(x(j)*i);
    end
end
S_k=1/2+a*A;
plot(x,f,x,S_k);
```

Exercise 3.9 Consider Example 3.9: $f(t) = t, 0 \leq \ t \leq \pi$. Is it permissible to differentiate the Fourier sine series for this function term-by-term? If so, say what the differentiated series converges to at each point. Is it permissible to differentiate the Fourier cosine series for the function $f(t)$ term-by-term? If so, say what the differentiated series converges to at each point.

Solution:

Fourier sin:

$$t \approx \sum_{n=1}^{\infty} \frac{2(-1)^{(n+1)} \sin(nt)}{n}$$

Fourier cos:

$$t \approx \frac{\pi}{2} - \frac{\pi}{4} \sum_{j=1}^{\infty} \frac{\cos[(2j-1)t]}{(2j-1)^2}$$

Differentiate term-by-term we have

Fourier sin:

$$1 \approx \sum_{n=1}^{\infty} 2(-1)^{(n+1)} \cos(nt)$$

Which does not converge, since $2(-1)^{(n+1)} \cos(nt)$ does not go to 0

Fourier cos:

$$1 \approx \frac{\pi}{4} \sum_{j=1}^{\infty} \frac{\sin[(2j-1)t]}{(2j-1)} = \frac{\pi}{4} \sum_{n=1}^{\infty} \frac{\sin(nt)}{n} \approx 2 - \frac{t}{\pi}$$

Exercise 3.10 Derive Lemma 3.24 from expression (3.13) and Lemma 3.23.

Solution:

$$
\begin{aligned}
\theta_k(t) &= \frac{1}{\pi}\int_0^{2\pi} F_k(t-x)f(x)dx \quad (\mathit{Expression}\, 3.13)\\
&= \frac{1}{\pi}\int_0^{2\pi} \frac{F_k(t-x)+F_k(t+x)}{2} f(x)dx\\
&= \frac{2}{\pi}\int_0^{\pi} \frac{F_k(t-x)+F_k(t+x)}{2} f(x)dx\\
&= \frac{2}{k\pi}\int_0^{\pi} \frac{F_k(t-x)+F_k(t+x)}{2} [\frac{\sin(\frac{kx}{2})}{\sin\frac{x}{2}}]^2 dx \quad (\mathit{Lemma}\, 3.23)\\
&= \frac{2}{k\pi}\int_0^{\frac{\pi}{2}} (F_k(t-2x)+F_k(t+2x))[\frac{\sin(kx)}{\sin(x)}]^2 dx \quad (\mathit{Change\ of\ variable})
\end{aligned}
$$

Exercise 3.11 Expand

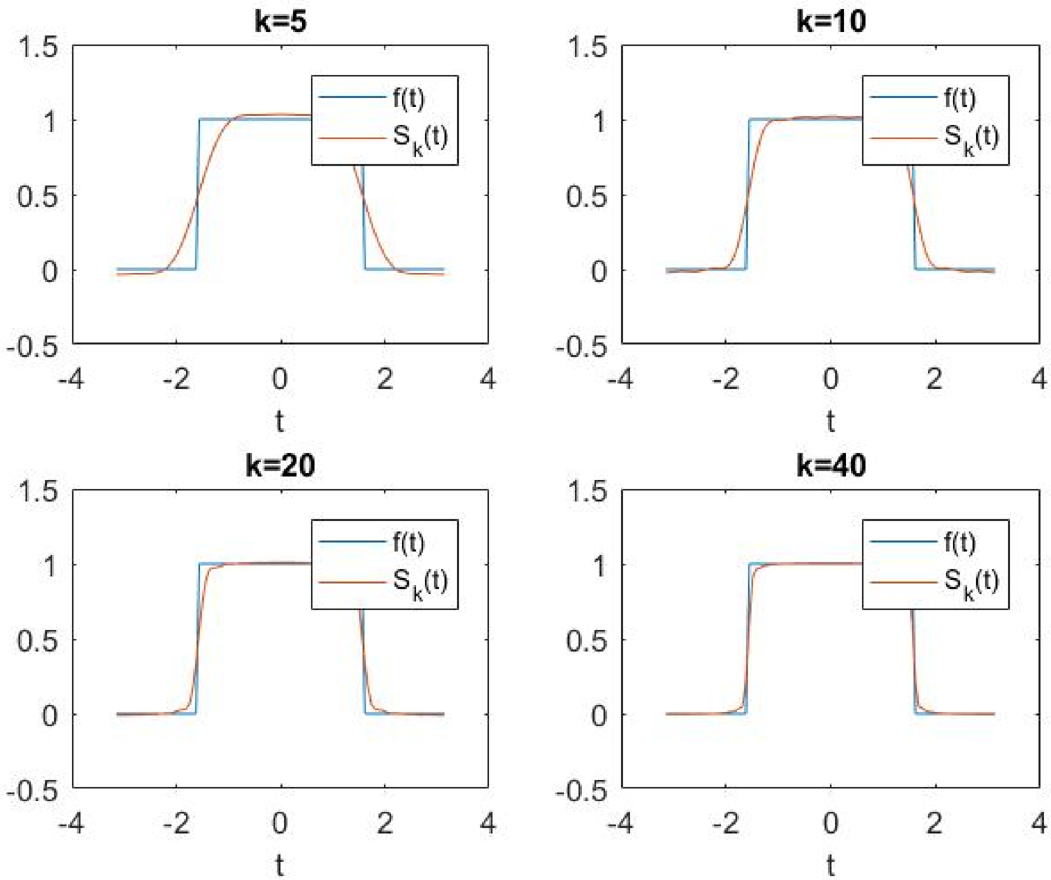
$$
f(x) = \{ \begin{array}{ll} 0 & for\ -\pi < x \le -\frac{\pi}{2} \\ 1 & for\ -\frac{\pi}{2} < x \le \frac{\pi}{2} \\ 0 & for\ \frac{\pi}{2} < x \le \pi \end{array}
$$

in a Fourier series on the interval $-\pi \le x \le \pi$. Plot both f and the arithmetic sum θ_k for $k = 5, 10, 20, 40$. Observe how the arithmetic sums approximate f. Compare with Exercise 3.8.

Solution:

As you can see in the figure below, the new approximate has a slower convergence rate with respect to mean-square norm (due to the fact we are taking mean values), but better point wise convergence result(no more Gibbs Phenomenon). Below is the matlab code used to generate the plots

```
k=20;
theta_k=0;
x =linspace(-pi,pi);
f =ones(100,1);
f(1:25)=0;
f(76:end)=0;
for n=1:k
idx=1:n;
A=zeros(n,100);
a=2*sin(pi*idx/2)./(pi*idx);
for i=1:n
    for j=1:100
        A(i,j)=cos(x(j)*i);
    end
end
theta_k=theta_k+1/2+a*A;
end
theta_k=theta_k/k;
subplot(2,2,3);
plot(x,f,x,theta_k);
title('k=20');
```


k=5
k=10
k=20
k=40
f(t)
S_k(t)
1.5
1
0.5
0
-0.5
-4
-2
0
2
4
t

```
xlabel('t');
legend('f(t)','S_k(t)')
```

Exercise 3.12 let $f(x) = x, x \in [-\pi, \pi]$.

(a) Show that $f(x)$ has the Fourier series

$$\sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n-1} \sin(nx).$$

(b) Let $\alpha > 0$ Show that $f(x) = exp(\alpha x), x \in [-\pi, \pi]$ has the Fourier series

$$(\frac{e^{\alpha\pi} - e^{-\alpha\pi}}{\pi})(\frac{1}{2\alpha} + \sum_{k=1}^{\infty}(-1)^k \frac{\alpha\cos(kx) - k\sin(kx)}{\alpha^2 + k^2}$$

(c) Let α be any real number other than an integer. Let $f(x) = cos(\alpha x), x \in [-\pi, \pi]$. Show that f has a Fourier series

$$\frac{\sin(\alpha\pi)}{\alpha\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty}[\frac{\sin(\alpha+n)\pi}{\alpha+n} + \frac{\sin(\alpha-n)\pi}{\alpha-n}]\cos(nx).$$

(d) Find the Fourier series of $f(x) = -\alpha\sin(\alpha x), x \in [-\pi, \pi]$. Do you notice any relationship to that in (c)?

(e) Find the Fourier series of $f(x) = |x|, x \in [-\pi, \pi]$.

(f) Find the Fourier series of

$$f(x) = sign(x) = \{ \begin{array}{ll} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{array}$$

Do you notice any relationship to that in (e)?

Solution:

(a) $f(x)$ is odd function, so $a_n = 0$

$$b_n = \frac{2}{\pi}\int_0^{\pi} x\sin(nx)dx$$

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi} x\sin(nx)dx \\ &= -\frac{2}{\pi n}x\cos(nx)\Big|_0^{\pi} + \frac{2}{\pi n}\int_0^{\pi}\cos(nx)dx \quad (Integration\ by\ Part) \\ &= -\frac{2}{n}\cos(n\pi) \\ &= \frac{2}{n}(-1)^{n-1} \end{aligned}$$

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos(nx) + b_n \sin(nx))$
(b)By Euler's formula we have $\sin(nx) = \frac{e^{inx}-e^{-inx}}{2i}$ and $\cos(nx) = \frac{e^{inx}+e^{-inx}}{2}$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{\alpha x}dx = \frac{e^{\alpha\pi} - e^{-\alpha\pi}}{\alpha\pi}$$

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{\alpha x}\cos(nx)dx \\ &= \frac{1}{2\pi}[\int_{-\pi}^{\pi} e^{(\alpha+in)x}dx + \int_{-\pi}^{\pi} e^{(\alpha-in)x}dx] \\ &= (-1)^n\frac{\alpha}{\alpha^2+n^2} \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{\alpha x}\sin(nx)dx \\ &= \frac{1}{2\pi i}[\int_{-\pi}^{\pi} e^{(\alpha+in)x}dx - \int_{-\pi}^{\pi} e^{(\alpha-in)x}dx] \\ &= (-1)^n\frac{-n}{\alpha^2+n^2} \end{aligned}$$

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos(nx) + b_n \sin(nx))$
(c) we can take part (b)'s result, and plug in αi instead of α, and take the real part. The remaining is purely algebraic manipulation.
(d) Notice that $f(x) = -\alpha\sin(\alpha x) = \frac{d}{dx}(\cos(\alpha x))$, and it's piecewise smooth so the Fourier series is:

$$-\frac{1}{\pi}\sum_{n=1}^{\infty}[\frac{n\sin(\alpha+n)\pi}{\alpha+n} + \frac{n\sin(\alpha-n)\pi}{\alpha-n}]\sin(nx)$$

(e) Even function, so $b_n = 0$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} xdx = \pi$$

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi} x\cos(nx)dx \\ &= \frac{2}{\pi n}x\sin(nx)\Big|_0^{\pi} - \frac{2}{\pi n}\int_0^{\pi}\sin(nx)dx \quad (Integration\ by\ Part) \\ &= \frac{2}{\pi n^2}[(-1)^n - 1] \end{aligned}$$

$$f(x) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n\cos(nx) - \cos(nx)}{n^2}$$

(f) Notice that $f(x)$ is the derivative of the function in (e), and it's piecewise smooth, so the Fourier series is:

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\sin(nx) + \sin(nx)}{n}$$

Exercise 3.13 Sum the series $\sum_{n=0}^{\infty} 2^{-n}\cos(nt)$, and $\sum_{n=0}^{\infty} 2^{-n}\sin(nt)$.
Hint: Take the real and imaginary parts of the $\sum_{n=0}^{\infty} 2^{-n}e^{int}$ You should be able to sum this last series directly. **Solution**:

$$\begin{aligned}\sum_{n=0}^{\infty} 2^{-n}e^{int} &= \frac{1}{1-\frac{e^{it}}{2}} \quad (Sum\,of\,GeometricSeries)\\ &= \frac{2}{2-\cos(t)-i\sin(t)}\\ &= \frac{4-2\cos(t)+2i\sin(t)}{5-4\cos(t)}\end{aligned}$$

So the cos sum is:

$$\frac{4-2\cos(t)}{5-4\cos(t)}$$

The sin sum is:

$$\frac{2\sin(t)}{5-4\cos(t)}$$

Exercise 3.14 Find the $L_2[-\pi,\pi]$ projection of the function $f_1(x) = x^2$ onto the $(2n+1)$-dimensional subspace spanned by the ON set

$$\frac{1}{\sqrt{2\pi}}, \frac{\cos(kx)}{\sqrt{\pi}}, \frac{\sin(kx)}{\sqrt{\pi}} : k = 1, ..., n$$

for n=1. Repeat for n=2,3. Plot these projections along with f_1. Repeat the whole exercise for $f_2(x) = x^3$. Do you see any marked differences between the graphs in the two cases?
Solution:

For $f_1(x)$, it is an even function, so $b_n = 0$,

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2 dx = \frac{2\pi^2}{3}$$

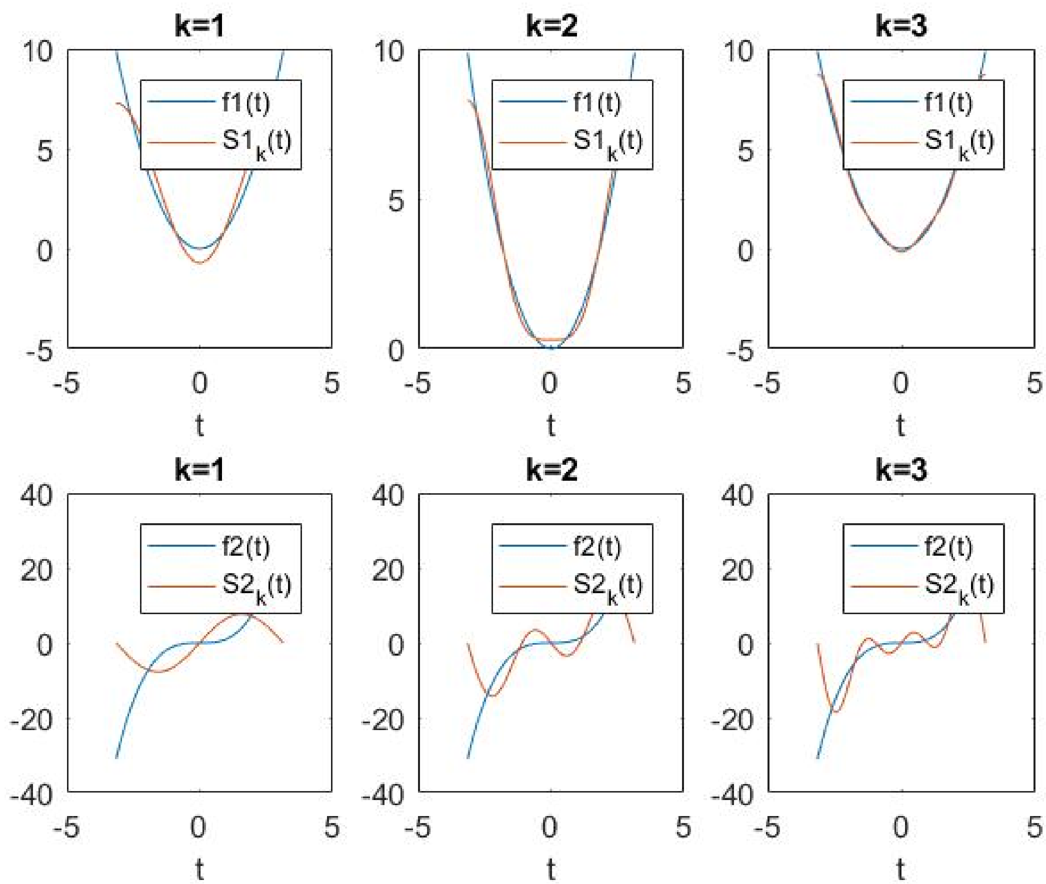
Similar to calculation with Excercise 3.12(a), but this time with two-steps of integration by part, we have:

$$\begin{aligned}a_n &= \frac{2}{\pi}\int_0^{\pi} x^2\cos(nx)dx\\ &= \frac{4}{n^2}(-1)^n\end{aligned}$$

For $f_2(x)$, it is an odd function, so $a_n = 0$, Similar to calculation above, but this time with three-steps of integration by part, we have:

$$\begin{aligned}b_n &= \frac{2}{\pi}\int_0^{\pi} x^3\sin(nx)dx\\ &= \frac{12}{n^3}(-1)^n - \frac{2\pi^2(-1)^n}{n}\end{aligned}$$

Below is the matlab code used to generate the plots



k=1
k=2
k=3
f1(t)
$S1_k(t)$
f2(t)
$S2_k(t)$
t
10
5
0
-5
40
20
-20
-40

```
x=linspace(-pi,pi);
f1=x.^2;
f2=x.^3;
k=3;
idx=1:k;
A1=zeros(k,100);
A2=zeros(k,100);
a1=4*(-1).^idx./(idx.^2);
a2=12*((-1).^idx)./(idx.^3)-2*pi^2*((-1).^idx)./idx;
for i=1:k
    for j=1:100
        A1(i,j)=cos(x(j)*i);
        A2(i,j)=sin(x(j)*i);
    end
end
S1_k=pi^2/3+a1*A1;
S2_k=a2*A2;

subplot(2,3,3);
plot(x,f1,x,S1_k);
title('k=3');
xlabel('t');
legend('f1(t)','S1_k(t)')
subplot(2,3,6);
plot(x,f2,x,S2_k);
title('k=3');
xlabel('t');
legend('f2(t)','S2_k(t)')
```

Notice that we can interpret $f_1(x) = \frac{1}{3}(\frac{d}{dx}(f_2(x)))$ Because $f_1(x)$ is continuous, however $f_x(x)$ has a point of discontinuity at the edge, we observe Gibbs Phenomenon in $f_2(x)$ but not in $f_1(x)$

Exercise 3.15 By substituting special values of x in convergent Fourier series, we can often deduce interesting series expansions for various numbers, or just find the sum of important series.

(a) Use the Fourier series for $f(x) = x^2$ to show that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}$$

(b) Prove that Fourier series for $f(x) = x$ converges in $(-\pi, \pi)$ and hence that

$$\sum_{l=0}^{\infty} \frac{(-1)^l}{2l+1} = \frac{\pi}{4}$$

(c) Show that the Fourier series of $f(x) = exp(\alpha x), x \in [-\pi, \pi]$ converges in $[-\pi, pi)$ to $exp(\alpha x)$ and at $x = \pi$ to $\frac{exp(\alpha pi)+exp(-\alpha pi)}{2}$. Hence show that

$$\frac{\alpha\pi}{\tanh(\alpha\pi) = 1 + \sum_{k=1}^{\infty} \frac{2\alpha^2}{k^2+\alpha^2}}$$

(d) Show that

$$\frac{\pi}{4} = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{4n^2 - 1}.$$

(e) Plot both $f(x) = x^2$ and the partial sums

$$S_k(x) = \frac{a_0}{2} + \sum_{n=0}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

for k=1,2,5,7. Observe how the partial sums approximate f.
Solution:

(a) The Fourier series is already derived in Exercise 3.14, since $f(x)$ has smooth derivative, it's Fourier Series converges point-wise in $(-\pi, \pi)$, take $x = 0$, we obtained the desired equality.
(b) The Fourier series is already derived in Exercise 3.12(a), since $f(x)$ has smooth derivative in $(-\pi, \pi)$, the series converge within the same range. Then take $x = \frac{\pi}{2}$ we obtained the desired equality.
(c) The Fourier series is already derived in Exercise 3.12(b), since $f(x)$ has smooth derivative in $(-\pi, \pi)$, the series converge within the same range. And since its derivative is right-continuous at $-\pi$ and left-continuous at π, then at point π it converges to the average of the two limit, which is $\frac{exp(\alpha pi)+exp(-\alpha pi)}{2}$.
Then take $x = \pi$ we obtained the desired equality.
(d) Follow the Fourier series obtained in Exercise 3.12(c), take $\alpha = \frac{1}{2}$ and $x = 0$, we obtained the desired equality (e) Identical Result to Exercise 3.14

Exercise 3.16 Let $f_1(x) = x$ and $f_2(x) = \pi^2 - 3x^2, -\pi \leq t \leq \pi$. Find the Fourier series of f_1 and f_2 and use them alone to sum the following series:

1. $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1}$
2. $\sum_{n=1}^{\infty} \frac{1}{n^2}$
3. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$
4. $\sum_{n=1}^{\infty} \frac{1}{n^4}$

Solution: Fourier series $f_1(x)$ is already obtained as in Exercise 3.12(a) For $f_2(x)$ it's a modified version from Exercise 3.14, we can obtained it's Fourier series easily:

$$f_2(x) = \sum_{n=1}^{\infty} \frac{12(-1)^{n-1}}{n^2}$$

1. Directly obtained from Excercise 3.15(b), it's equal to $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1} = \frac{pi}{4} - 1$
2. Take $x = \pi$ for $f_2(x)$, we have $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi}{6}$
3. Multiply by -1 to Exercise 3.15(a), $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}$
4. Apply Parseval's identity to $f_2(x)$, we have $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$

Exercise 3.17 By Applying Parseval's identity to suitable Fourier Series:

(a) show that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

(b) Evaluate

$$\sum_{n=1}^{\infty} \frac{1}{\alpha^2 + n^2}.$$

(c) Show that

$$\sum_{l=0}^{\infty} \frac{1}{(2l+1)^4} = \frac{\pi^4}{96}.$$

Solution:

(a) Same as Exercise 3.16.4
(b) From Exercise 3.15 (c):

$$\sum_{n=1}^{\infty} \frac{1}{\alpha^2 + n^2} = \frac{\pi}{2\alpha \tanh(\alpha\pi)} - \frac{1}{2\alpha^2}$$

(c) Apply Parseval's identity to $f(x) = |x|$, note $(-1)^n - 1 = 0 for n even$ and $(-1)^n - 1 = -2 for n odd$

$$\frac{\pi^2}{2} + \sum_{l=0}^{\infty} \frac{16}{(2l+1)^4} = \frac{2\pi^2}{3}$$

Thus we obtained the quality desired

Exercise 3.18 Is

$$\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^{1/2}}$$

The Fourier series of some square integrable function f on $[-\pi, \pi]$?
The same question for the series

$$\sum_{n=1}^{\infty} \frac{\cos(nx)}{\log(n+2)}$$

Solution:

Neither of them are.
Prove By contradition, assume they are, then they will satisfy Parseval's identity
However, neither of the series sum converge (in fact they diverge to infinity): $\sum_{n=1}^{\infty} \frac{1}{n}$ note: Famous divergent series, and can apply integral test to prove it's a divergent series)
$\sum_{n=1}^{\infty} \frac{1}{[\log(n+2)]^2}$, apply comparison test with the series above so neither of them can be the Fourier series of some square integrable function

Solutions to Chapter 4

Exercise 4.1 Verify the rules for Fourier transforms listed above.

Solution:

$$\begin{aligned}
\mathcal{F}[af+bg] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(af(t)+bg(t))e^{-i\lambda t}dt \\
&= a\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(t)e^{-i\lambda t}dt + b\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}g(t)e^{-i\lambda t}dt \\
&= a\mathcal{F}[f]+b\mathcal{F}[g].
\end{aligned}$$

$$\begin{aligned}
\mathcal{F}^*[af+bg] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(af(\lambda)+bg(\lambda))e^{i\lambda t}d\lambda \\
&= a\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(\lambda)e^{i\lambda t}d\lambda + b\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}g(\lambda)e^{i\lambda t}d\lambda \\
&= a\mathcal{F}^*[f]+b\mathcal{F}^*[g].
\end{aligned}$$

$$\begin{aligned}
\mathcal{F}[t^n f(t)](\lambda) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}t^n f(t)e^{-i\lambda t}dt \\
&= \frac{1}{\sqrt{2\pi}}\frac{1}{(-i)^n}\frac{d^n}{d\lambda^n}\int_{-\infty}^{\infty}f(t)e^{-i\lambda t}dt \\
&= i^n\frac{d^n}{d\lambda^n}\mathcal{F}[f](\lambda).
\end{aligned}$$

$$\begin{aligned}
\mathcal{F}^*[\lambda^n f(\lambda)](t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\lambda^n f(\lambda)e^{i\lambda t}d\lambda \\
&= \frac{1}{\sqrt{2\pi}}\frac{1}{i^n}\frac{d^n}{dt^n}\int_{-\infty}^{\infty}f(\lambda)e^{i\lambda t}d\lambda \\
&= (-i)^n\frac{d^n}{d\lambda^n}\mathcal{F}[f](\lambda).
\end{aligned}$$

Since

$$\begin{aligned}
\mathcal{F}[f'](\lambda) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f'(t)e^{-i\lambda t}dt \\
&= \frac{1}{\sqrt{2\pi}}f(t)e^{-i\lambda t}\Big|_{t=-\infty}^{\infty} - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(-i\lambda)f(t)e^{-i\lambda t}dt \\
&= i\lambda\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(t)e^{-i\lambda t}dt = i\lambda\mathcal{F}[f](\lambda),
\end{aligned}$$

thus

$$\mathcal{F}[f^{(n)}](\lambda) = (i\lambda)^n\mathcal{F}[f](\lambda).$$

Since

$$\begin{aligned}\mathcal{F}^*[f'](t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f'(\lambda)e^{i\lambda t}d\lambda \\ &= \frac{1}{\sqrt{2\pi}}f(\lambda)e^{i\lambda t}\Big|_{\lambda=-\infty}^{\infty} - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(it)f(\lambda)e^{i\lambda t}d\lambda \\ &= -it\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\lambda)e^{i\lambda t}d\lambda = -it\mathcal{F}[f](\lambda),\end{aligned}$$

thus

$$\mathcal{F}^*[f^{(n)}](t) = (-it)^n\mathcal{F}^*[f](t).$$

Let $bt-a=r$, then $t=r/b+a/b, dt=dr/b$ and

$$\begin{aligned}\mathcal{F}[f(bt-a)](\lambda) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(bt-a)e^{-i\lambda t}dt \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{b}f(r)e^{-i(r/b+a/b)\lambda}dr \\ &= \frac{1}{b}e^{-i\lambda a/b}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(r)e^{-i\lambda r/b}dr = \frac{1}{b}e^{-i\lambda a/b}\mathcal{F}[f](\frac{\lambda}{b}).\end{aligned}$$

Exercise 4.2 Let $f,g,h:\mathbb{R}\to\mathbb{R}$. Let $a,b\in\mathbb{R}$, show that:

(i) Convolution is linear:

$$(af+bg)*h = a(f*h)+b(g*h).$$

(ii) Convolution is commutative:

$$f*g=g*f$$

(iii) Convolution is associative:

$$(f*g)*h = f*(g*h)$$

Solution: (i)

$$\begin{aligned}(af+bg)*h &= \int_{-\infty}^{\infty}(af(x)+bg(x))h(t-x)dx \\ &= a\int_{-\infty}^{\infty} f(x)h(t-x)dx + b\int_{-\infty}^{\infty} g(x)h(t-x)dx \\ &= a(f*h)+b(g*h).\end{aligned}$$

(ii) Let $y=t-x$, then $x=t-y, dx=-dy$ and

$$\begin{aligned}f*g &= \int_{-\infty}^{\infty} f(x)g(t-x)dx \\ &= -\int_{\infty}^{-\infty} f(t-y)g(y)dy \\ &= \int_{-\infty}^{\infty} f(t-y)g(y)dy = g*f.\end{aligned}$$

(iii)

$$(f*g)*h = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(x)g(y-x)dx\right)h(t-y)dy$$

Let $y = x + z$, then

$$\begin{aligned}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(x)g(y-x)dx\right)h(t-y)dy &= \int_{-\infty}^{\infty} f(x)\left(\int_{-\infty}^{\infty} g(z)h(t-x-z)dz\right)dx \\ &= \int_{-\infty}^{\infty} f(x)(g*h)(t-x)dx \\ &= f*(g*h).\end{aligned}$$

Exercise 4.3 Let

$$\Pi(t) = \begin{cases} 1 & \text{for } -\frac{1}{2} < t < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases},$$

the box function on the real line. We will often have the occasion to express the Fourier transform of $f(at+b)$ in terms of the Fourier transform $\hat{f}(\lambda)$ of $f(t)$, where a, b are real parameters. This exercise will give you practice in correct application of the transform.

1. Sketch the graphs of $\Pi(t)$, $\Pi(t-3)$ and $\pi(2t-3) = \Pi(2(t-3/2))$.

2. Sketch the graphs of $\Pi(t)$, $\Pi(2t)$ and $\Pi(2(t-3))$. *Note*: In the first part a right 3-translate is followed by a 2-dilate; but in the second part a 2-dilate is followed by a right 3-translate. The results are not the same.

3. Find the Fourier transforms of $g_1(t) = \Pi(2t-3)$ and $g_2(t) = \Pi(2(t-3))$ from parts 1 and 2.

4. Set $g(t) = \Pi(2t)$ and check your answers to part 3 by applying the translation rule to

$$g_1(t) = g(t-\frac{3}{2}),\ g_2(t) = g(t-3),\ \text{noting } g_2(t) = g_1(t-\frac{3}{2}).$$

Solution: 1. See Figure 1.

2. See Figure 2.

3.

$$\begin{aligned}\mathcal{F}[g_1](\lambda) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g_1(t)e^{-i\lambda t}dt \\ &= \frac{1}{\sqrt{2\pi}}\int_{5/4}^{7/4} e^{-i\lambda t}dt \\ &= \frac{2e^{-3i\lambda/2}\sin(\lambda/4)}{\sqrt{2\pi}\lambda}\end{aligned}$$

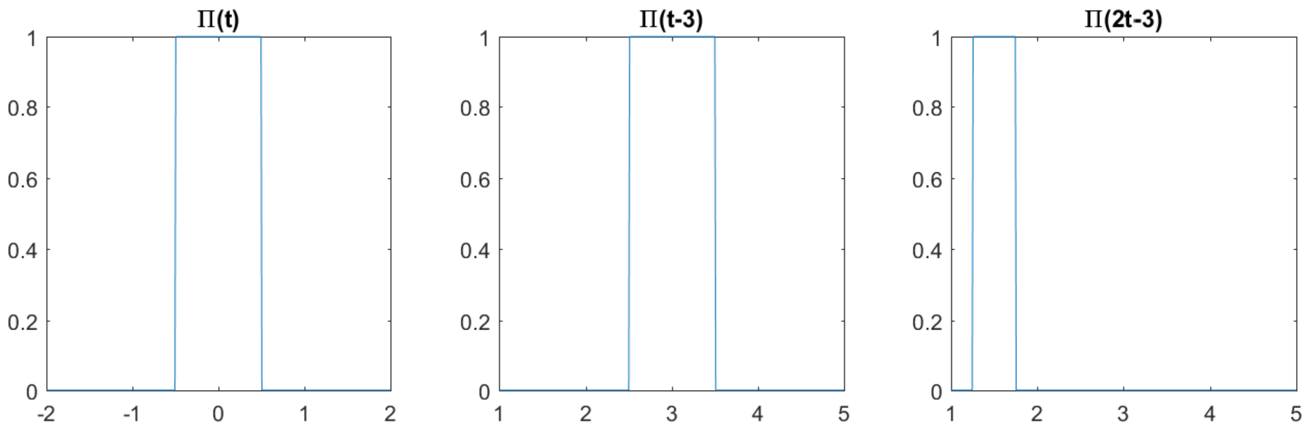


Figure 1:

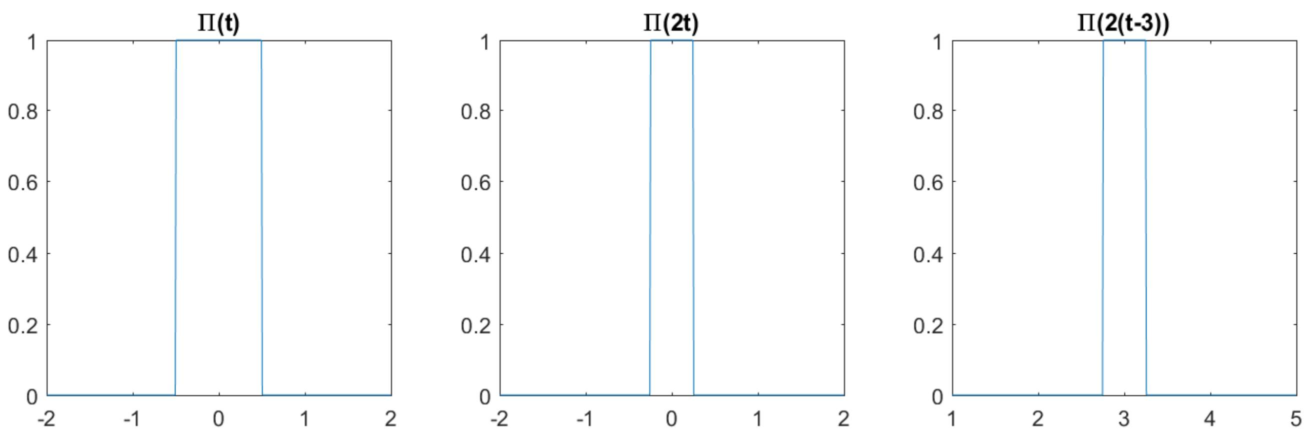


Figure 2:

$$
\begin{aligned}
\mathcal{F}[g_2](\lambda) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g_1(t) e^{-i\lambda t} dt \\
&= \frac{1}{\sqrt{2\pi}} \int_{11/4}^{13/4} e^{-i\lambda t} dt \\
&= \frac{2e^{-3i\lambda} \sin(\lambda/4)}{\sqrt{2\pi}\lambda}
\end{aligned}
$$

4. Since

$$\mathcal{F}[g](\lambda) = \frac{2\sin(\lambda/4)}{\sqrt{2\pi}\lambda},$$

then

$$\mathcal{F}[g_1](\lambda) = e^{-3i\lambda/2}\mathcal{F}[g](\lambda), \quad \mathcal{F}[g_2](\lambda) = e^{-3i\lambda}\mathcal{F}[g](\lambda).$$

Exercise 4.4 Assuming that the improper integral $\int_0^\infty (\sin x/x)dx = I$ exists, establish its value (2.1) by first using the Riemann-Lebesgue lemma for Fourier series to show that

$$I = \lim_{k\to\infty}\int_0^{(k+1/2)\pi}\frac{\sin x}{x}dx = \lim_{k\to\infty}\int_0^{\pi} D_k(u)du$$

where $D_k(u)$ is the Dirichlet kernel function. Then use Lemma 3.11.

Solution: Let $x=(k+1/2)u$, then $dx=(k+1/2)du$ and

$$\begin{aligned}\lim_{k\to\infty}\int_0^{(k+1/2)\pi}\frac{\sin x}{x}dx &= \lim_{k\to\infty}\int_0^{\pi}\frac{\sin(k+1/2)u}{u}du\\ &= \lim_{k\to\infty}\int_0^{\delta}\frac{\sin(k+1/2)u}{u}du + \lim_{k\to\infty}\int_{\delta}^{\pi}\frac{\sin(k+1/2)u}{u}du.\end{aligned}$$

The second part of the integral is zero and it can be replaced as

$$\lim_{k\to\infty}\int_{\delta}^{\pi}\frac{\sin(k+1/2)u}{u}du = 0 = \lim_{k\to\infty}\int_{\delta}^{\pi}\frac{\sin(k+1/2)u}{2\sin(u/2)}du.$$

If δ is sufficient small and $u\in[0,\delta]$, $u\leftrightarrow 2\sin(u/2)$, thus the first part of the integral can be replaced as

$$\lim_{k\to\infty}\int_0^{\delta}\frac{\sin(k+1/2)u}{u}du = \lim_{k\to\infty}\int_0^{\delta}\frac{\sin(k+1/2)u}{2\sin(u/2)}du.$$

Consequently, we have

$$\lim_{k\to\infty}\int_0^{\pi}\frac{\sin(k+1/2)u}{u}du = \lim_{k\to\infty}\int_0^{\pi}\frac{\sin(k+1/2)u}{2\sin(u/2)}du,$$

where

$$\int_0^{\pi}\frac{\sin(k+1/2)u}{2\sin(u/2)}du = \int_0^{\pi}\left(\frac{1}{2}+\sum_{n=1}^{k}\cos nu\right)du = \frac{\pi}{2}.$$

Exercise 4.5 Define the *right-hand derivative* $f'_R(t)$ and *the left-hand derivative* $f'_L(t)$ of f by

$$f'_R(t) = \lim_{u\to t+}\frac{f(u)-f(t+0)}{u-t},\quad f'_L(t) = \lim_{u\to t-}\frac{f(u)-f(t-0)}{u-t},$$

respectively, as in Exercise 3.4. Show that in the proof of Theorem 4.17 we can drop the requirements 3 and 4, and the right-hand side of (4.12) will converge to $\frac{(t+0)+f(t-0)}{2}$ at any point t such that both $f'_R(t)$ and $f'_L(t)$ exist.

Solution: In the proof of Theorem 4.17

$$\int_0^c (f(t+x)+f(t-x))\frac{\sin Lx}{\pi x}dx = \int_0^c f(t+x)\frac{\sin Lx}{\pi x}dx + \int_0^c f(t-x)\frac{\sin Lx}{\pi x}dx,$$

when $L\to\infty$, in the right hand side, the first part will converge to $f(t+0)/2$ if $f'_R(t)$ exist and the second part will converge to $f(t-0)/2$ if $f'_L(t)$ exist. Hence, the total result converge to $\frac{(t+0)+f(t-0)}{2}$.

Exercise 4.6 Let $a > 0$. Use the Fourier transforms of sinc (x) and sinc $^2(x)$, together with the basic tools of Fourier transform theory, such as Parseval's equation, substitution, ... to show the following. (Use only rules from Fourier transform theory. You shouldn't do any detailed computation such as integration by parts.)

- $\int_{-\infty}^{\infty} \left(\frac{\sin ax}{x}\right)^3 dx = \frac{3a^2\pi}{4}$
- $\int_{-\infty}^{\infty} \left(\frac{\sin ax}{x}\right)^4 dx = \frac{2a^3\pi}{3}$

Solution:

$$\begin{aligned}
\int_{-\infty}^{\infty} \left(\frac{\sin ax}{x}\right)^3 dx &= a^3 \int_{-\infty}^{\infty} \left(\frac{\sin ax}{ax}\right)^2 \overline{\left(\frac{\sin ax}{ax}\right)} dx \\
&= a^3 \int_{-\infty}^{\infty} \mathcal{F}\left[\left(\frac{\sin ax}{ax}\right)^2\right](\lambda) \overline{\mathcal{F}\left[\frac{\sin ax}{ax}\right]}(\lambda) d\lambda \\
&= a^3 \int_{-\infty}^{\infty} \frac{\sqrt{\pi}}{\sqrt{2a^2}} \mathrm{tri}\left(\frac{\lambda}{2a}\right) \frac{\sqrt{\pi}}{\sqrt{2a^2}} \mathrm{rect}\left(\frac{\lambda}{2a}\right) d\lambda \\
&= \frac{\pi a}{2} \int_{-a}^{a} \mathrm{tri}\left(\frac{\lambda}{2a}\right) d\lambda \\
&= \pi a \int_{0}^{a} \left(1 - \frac{\lambda}{2a}\right) d\lambda = \frac{3a^2\pi}{4}.
\end{aligned}$$

$$\begin{aligned}
\int_{-\infty}^{\infty} \left(\frac{\sin ax}{x}\right)^4 dx &= a^4 \int_{-\infty}^{\infty} \left(\frac{\sin ax}{ax}\right)^2 \overline{\left(\frac{\sin ax}{ax}\right)^2} dx \\
&= a^4 \int_{-\infty}^{\infty} \mathcal{F}\left[\left(\frac{\sin ax}{ax}\right)^2\right](\lambda) \overline{\mathcal{F}\left[\left(\frac{\sin ax}{ax}\right)^2\right]}(\lambda) d\lambda \\
&= a^4 \int_{-\infty}^{\infty} \frac{\sqrt{\pi}}{\sqrt{2a^2}} \mathrm{tri}\left(\frac{\lambda}{2a}\right) \frac{\sqrt{\pi}}{\sqrt{2a^2}} \mathrm{tri}\left(\frac{\lambda}{2a}\right) d\lambda \\
&= \frac{\pi a^2}{2} \int_{-2a}^{2a} \mathrm{tri}^2\left(\frac{\lambda}{2a}\right) d\lambda \\
&= \pi a^2 \int_{0}^{2a} \left(1 - \frac{\lambda}{2a}\right)^2 d\lambda = \frac{2a^3\pi}{3}.
\end{aligned}$$

Exercise 4.7 Show that the n-translates of sinc are orthonormal:

$$\int_{-\infty}^{\infty} \mathrm{sinc}\ (x-n) \cdot \mathrm{sinc}\ (x-m) dx = \begin{cases} 1 & \text{for } n = m \\ 0 & \text{otherwise,} \end{cases} \quad n, m = 0, \pm 1, \ldots$$

Solution:

$$
\begin{aligned}
\int_{-\infty}^{\infty} \text{sinc }(x-n)\cdot \text{sinc }(x-m)dx &= \int_{-\infty}^{\infty} \mathcal{F}[\text{sinc }(x-n)](\lambda)\cdot \overline{\mathcal{F}[\text{sinc }(x-m)]}(\lambda)d\lambda \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \text{rect}^2\left(\frac{\lambda}{2\pi}\right)e^{-i(n-m)\lambda}d\lambda \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i(n-m)\lambda}d\lambda.
\end{aligned}
$$

Hence,

$$
\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i(n-m)\lambda}d\lambda = \begin{cases} 1 & \text{for } n=m \\ 0 & \text{otherwise,} \end{cases} \quad n,m = 0,\pm 1,\dots
$$

Exercise 4.8 Let

$$
f(t) = \begin{cases} 1 & -2 \le t \le -1 \\ 1 & 1 \le t \le 2 \\ 0 & \text{otherwise.} \end{cases}
$$

- Compute the Fourier transform $\hat{f}(\lambda)$ and sketch the graphs of f and $\hat{f}$.
- Compute and sketch the graph of the function with Fourier transform $\hat{f}(-\lambda)$.
- Compute and sketch the graph of the function with Fourier transform $\hat{f}'(\lambda)$.
- Compute and sketch the graph of the function with Fourier transform $\hat{f} * \hat{f}(\lambda)$.
- Compute and sketch the graph of the function with Fourier transform $\hat{f}(\lambda/2)$.

Solution:

- $f(t) = \text{rect}(t+3/2) + \text{rect}(t-3/2)$, thus

  $$
  \hat{f}(\lambda) = (e^{-3i\lambda/2} + e^{3i\lambda/2})\text{sinc }(\frac{\lambda}{2\pi}) = 2\cos(3\lambda/2)\text{sinc }(\frac{\lambda}{2\pi}).
  $$

  See Figure 3.
- $\hat{f}(-\lambda) = \overline{\hat{f}(\lambda)} = \hat{f}(\lambda)$.
- $$
  \hat{f}'(\lambda) = \sqrt{2\pi}\mathcal{F}[-itf(t)](\lambda) = \frac{2(-\lambda\cos(\lambda) + 2\lambda\cos(2\lambda) + \sin(\lambda) - \sin(2\lambda))}{\lambda^2}
  $$

  See Figure 4.
- $\hat{f} * \hat{f}(\lambda) = (2\pi)^{3/2}\mathcal{F}[f^2(t)](\lambda) = (2\pi)^{3/2}\mathcal{F}[f(t)](\lambda) = 2\pi\hat{f}(\lambda)$. See Figure 5.
- $\hat{f}(\lambda/2) = 2\sqrt{2\pi}\mathcal{F}[f(2t)](\lambda)$. See Figure 6.

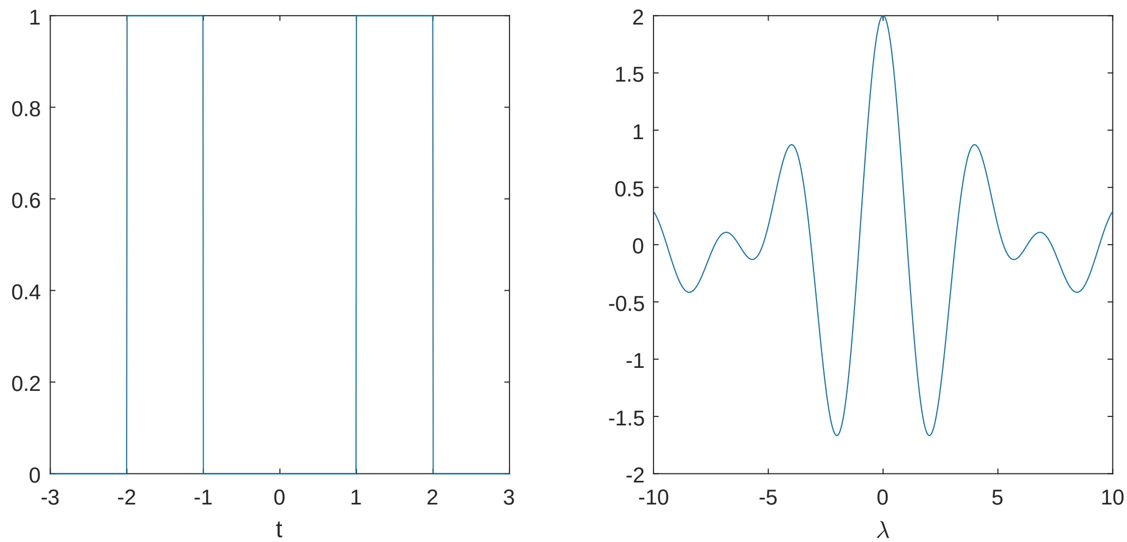


Figure 3: $f(t)$ and $\hat{f}(\lambda)$

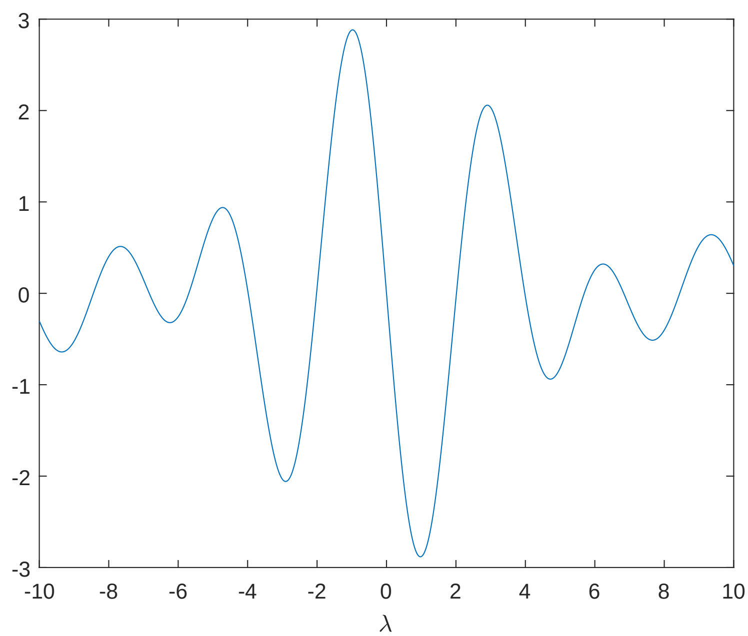


Figure 4: $\hat{f}'(\lambda)$

Exercise 4.9 Deduce what you can about the Fourier transform $\hat{f}(\lambda)$ if you know that $f(t)$ satisfies the dilation equation

$$f(t) = f(2t) + f(2t - 1)$$

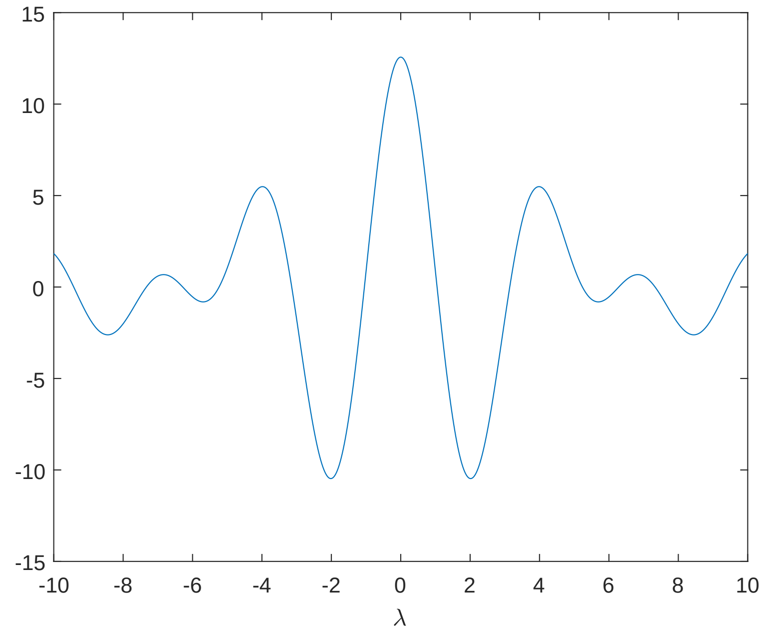


Figure 5: $\hat{f} * \hat{f}(\lambda)$

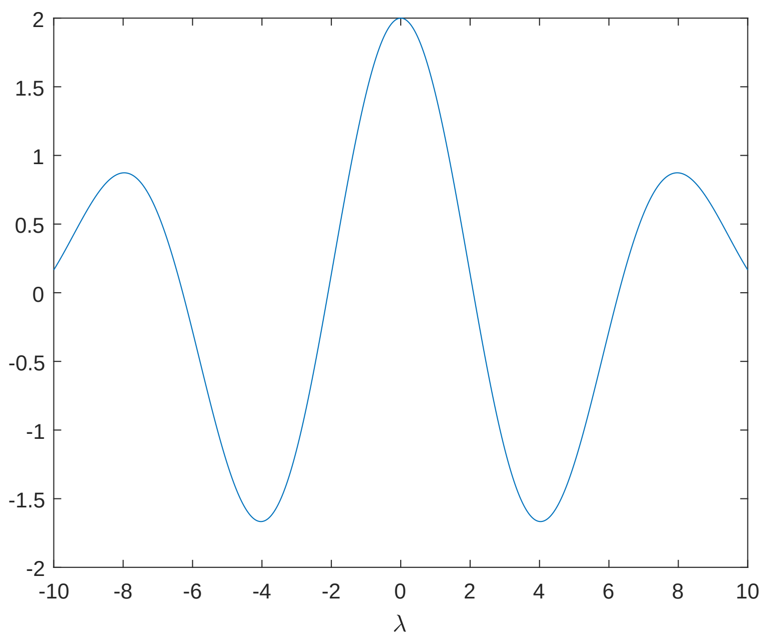


Figure 6: $\hat{f}(\lambda/2)$

Solution:

$$\hat{f}(\lambda) = \frac{1}{2}\hat{f}\left(\frac{\lambda}{2}\right) + \frac{e^{-i\lambda/2}}{2}\hat{f}\left(\frac{\lambda}{2}\right) = \frac{1+e^{-i\lambda/2}}{2}\hat{f}\left(\frac{\lambda}{2}\right)$$

Exercise 4.10 Just as Fourier series have a complex version and a real version, so does the Fourier transform. Under the same assumptions as Theorem 4.17 set

$$\hat{C}(\alpha) = \frac{1}{2}\left[\hat{f}(\alpha) + \hat{f}(-\alpha)\right], \quad \hat{S}(\alpha) = \frac{1}{2i}\left[-\hat{f}(\alpha) + \hat{f}(-\alpha)\right], \ \alpha \geq 0$$

and derive the expansion

$$f(t) = \frac{1}{\pi}\int_0^\infty \left(\hat{C}(\alpha)\cos\alpha t + \hat{S}(\alpha)\sin\alpha t\right)d\alpha,$$

$$\hat{C}(\alpha) = \int_{-\infty}^\infty f(s)\cos(\alpha s)ds, \quad \hat{S}(\alpha) = \int_{-\infty}^\infty f(s)\sin(\alpha s)ds.$$

Show that the transform can be written in a more compact form as

$$f(t) = \frac{1}{\pi}\int_0^\infty d\alpha \int_{-\infty}^\infty f(s)\cos\alpha(s-t)ds$$

Solution:

$$\begin{aligned}
&\frac{1}{\pi}\int_0^\infty \left(\hat{C}(\alpha)\cos\alpha t + \hat{S}(\alpha)\sin\alpha t\right)d\alpha \\
=&\frac{1}{2\pi}\int_0^\infty \left(\hat{f}(\alpha)(\cos\alpha t + i\sin\alpha t) + \hat{f}(-\alpha)(\cos(-\alpha t) + i\sin(-\alpha t))\right)d\alpha \\
=&\frac{1}{2\pi}\int_{-\infty}^\infty \hat{f}(\alpha)(\cos\alpha t + i\sin\alpha t)d\alpha \\
=&\frac{1}{2\pi}\int_{-\infty}^\infty \hat{f}(\alpha)e^{i\alpha t}d\alpha = f(t).
\end{aligned}$$

$$\begin{aligned}
\hat{C}(\alpha) &= \frac{1}{2}\left[\int_{-\infty}^\infty f(s)e^{-i\alpha s}ds + \left(\int_{-\infty}^\infty f^*(s)e^{-i\alpha s}ds\right)^*\right] \\
&= \frac{1}{2}\left[\int_{-\infty}^\infty f(s)e^{-i\alpha s}ds + \int_{-\infty}^\infty f(s)e^{i\alpha s}\right] \\
&= \frac{1}{2}\int_{-\infty}^\infty f(s)\left(e^{-i\alpha s} + e^{i\alpha s}\right)ds \\
&= \int_{-\infty}^\infty f(s)\cos(\alpha s)ds
\end{aligned}$$

$$\begin{aligned}
\hat{S}(\alpha) &= \frac{1}{2i}\left[-\int_{-\infty}^\infty f(s)e^{-i\alpha s}ds + \left(\int_{-\infty}^\infty f^*(s)e^{-i\alpha s}ds\right)^*\right] \\
&= \frac{1}{2i}\left[-\int_{-\infty}^\infty f(s)e^{-i\alpha s}ds + \int_{-\infty}^\infty f(s)e^{i\alpha s}\right] \\
&= \frac{1}{2i}\int_{-\infty}^\infty f(s)\left(-e^{-i\alpha s} + e^{i\alpha s}\right)ds \\
&= \int_{-\infty}^\infty f(s)\sin(\alpha s)ds
\end{aligned}$$

Substitute the expansion of $\hat{C}(\alpha)$ and $\hat{S}(\alpha)$ into the expansion of $f(t)$, then we have

$$\begin{aligned}
f(t) &= \frac{1}{\pi}\int_0^\infty \left(\int_{-\infty}^\infty f(s)\cos(\alpha s)\cos(\alpha t)ds + \int_{-\infty}^\infty f(s)\sin(\alpha s)\sin(\alpha t)ds\right)d\alpha \\
&= \frac{1}{\pi}\int_0^\infty \left(\int_{-\infty}^\infty f(s)(\cos(\alpha s)\cos(\alpha t) + \sin(\alpha s)\sin(\alpha t))ds\right)d\alpha \\
&= \frac{1}{\pi}\int_0^\infty d\alpha \int_{-\infty}^\infty f(s)\cos\alpha(s-t)ds
\end{aligned}$$

Exercise 4.11 There are also Fourier integral analogs of the Fourier cosine series and the Fourier sine series. Let $f(t)$ be defined for all $t \geq 0$ and extend it to an even function on the real line, defined by

$$F(t) = \begin{cases} f(t), & \text{if } t \geq 0 \\ f(-t), & \text{if } t < 0 \end{cases}$$

By applying the results of Exercise 4.10 show that, formally,

$$f(t) = \frac{2}{\pi}\int_0^\infty \cos\alpha t\ d\alpha \int_0^\infty f(s)\cos\alpha s\ ds, \quad t \geq 0$$

Find conditions on $f(t)$ such that this pointwise expansion is rigorously correct.

Solution: According to the result of Exercise 4.10,

$$\begin{aligned}
F(t) &= \frac{1}{\pi}\int_0^\infty d\alpha \int_{-\infty}^\infty F(s)\cos\alpha(s-t)ds \\
&= \frac{1}{\pi}\int_0^\infty \cos\alpha t\ d\alpha \int_{-\infty}^\infty F(s)\cos\alpha s\ ds + \frac{1}{\pi}\int_0^\infty \sin\alpha t\ d\alpha \int_{-\infty}^\infty F(s)\sin\alpha s\ ds.
\end{aligned}$$

Since $F(s), \cos\alpha s$ are even functions and $\sin\alpha s$ is odd function, we have

$$\int_{-\infty}^\infty F(s)\cos\alpha s\ ds = 2\int_0^\infty F(s)\cos\alpha s\ ds = 2\int_0^\infty f(s)\cos\alpha s\ ds,$$

$$\int_{-\infty}^\infty F(s)\sin\alpha s\ ds = 0.$$

Thus, for $t \geq 0$

$$f(t) = F(t) = \frac{2}{\pi}\int_0^\infty \cos\alpha t\ d\alpha \int_0^\infty f(s)\cos\alpha s\ ds.$$

If $f(t)$ satisfies the assumptions in Theorem 4.17, this pointwise expansion is rigorously correct.

Exercise 4.12 Let $f(t)$ be defined for all $t > 0$ and extend it to an odd function on the real line, defined by

$$G(t) = \begin{cases} f(t), & \text{if } t > 0 \\ -f(-t), & \text{if } t < 0 \end{cases}$$

By applying the results of Exercise 4.10 show that, formally

$$f(t) = \frac{2}{\pi}\int_0^\infty \sin\alpha t \, d\alpha \int_0^\infty f(s)\sin\alpha s \, ds, \quad t > 0$$

Find conditions on $f(t)$ such that this pointwise expansion is rigorously correct.

Solution: According to the result of Exercise 4.10,

$$\begin{aligned} G(t) &= \frac{1}{\pi}\int_0^\infty d\alpha \int_{-\infty}^\infty F(s)\cos\alpha(s-t)ds \\ &= \frac{1}{\pi}\int_0^\infty \cos\alpha t \, d\alpha \int_{-\infty}^\infty G(s)\cos\alpha s \, ds + \frac{1}{\pi}\int_0^\infty \sin\alpha t \, d\alpha \int_{-\infty}^\infty G(s)\sin\alpha s \, ds. \end{aligned}$$

Since $\cos\alpha s$ is even function and $G(s), \sin\alpha s$ are odd functions, we have

$$\int_{-\infty}^\infty G(s)\cos\alpha s \, ds = 0,$$

$$\int_{-\infty}^\infty G(s)\sin\alpha s \, ds = 2\int_0^\infty G(s)\sin\alpha s \, ds = 2\int_0^\infty f(s)\sin\alpha s \, ds.$$

Thus, for $t > 0$

$$f(t) = G(t) = \frac{2}{\pi}\int_0^\infty \sin\alpha t \, d\alpha \int_0^\infty f(s)\sin\alpha s \, ds.$$

If $f(t)$ satisfies the assumptions in Theorem 4.17, this pointwise expansion is rigorously correct.

Exercise 4.13 Find the Fourier Cosine and Sine transforms of the following functions:

$$f(t) := \begin{cases} 1, & t \in [0, a] \\ 0, & t > a \end{cases}$$

$$f(t) := \begin{cases} \cos(at), & t \in [0, a] \\ 0, & t > a \end{cases}$$

Solution: For the first function,

$$\begin{aligned} \hat{f}^c(\alpha) &= \int_{-\infty}^\infty f(t)\cos(\alpha t)dt \\ &= \int_0^a \cos(\alpha t)dt = \frac{1}{\alpha}\sin(a\alpha) \end{aligned}$$

$$\begin{aligned}\hat{f}^s(\alpha) &= \int_{-\infty}^{\infty} f(t)\sin(\alpha t)dt \\ &= \int_0^a \sin(\alpha t)dt = \frac{1}{\alpha}(1-\cos(a\alpha))\end{aligned}$$

For the second function,

$$\begin{aligned}\hat{f}^c(\alpha) &= \int_{-\infty}^{\infty} f(t)\cos(\alpha t)dt \\ &= \int_0^a \cos(at)\cos(\alpha t)dt \\ &= \frac{a\cos(a\alpha)\sin(a^2) - \alpha\cos(a^2)\sin(a\alpha)}{a^2-\alpha^2}\end{aligned}$$

$$\begin{aligned}\hat{f}^s(\alpha) &= \int_{-\infty}^{\infty} f(t)\sin(\alpha t)dt \\ &= \int_0^a \cos(at)\sin(\alpha t)dt \\ &= \frac{\alpha\cos(a^2)\cos(a\alpha) + a\sin(a^2)\sin(a\alpha) - \alpha}{a^2-\alpha^2}\end{aligned}$$

Exercise 4.14 Use the fact that $f(t) = \text{sinc }(t+a)$ is frequency bandlimited with $B = \pi$ to show that

$$\text{sinc }(t+a) = \sum_{j=-\infty}^{\infty} \text{sinc }(j+a)\text{sinc }(t-j)$$

for all a, t. Derive the identity

$$1 = \sum_{j=-\infty}^{\infty} \text{sinc }^2(t-j).$$

Solution: Since we have the Fourier transform

$$f(t+a) \longleftrightarrow e^{ia\lambda}\hat{f}(\lambda),$$

Substitute $f(t)$, $\hat{f}(\lambda)$ and B in the proof of Theorem 4.18 with $f(t+a)$, $e^{ia\lambda}\hat{f}(\lambda)$ and π, respectively. Then, we have

$$e^{ia\lambda}\hat{f}(\lambda) = \sum_{k=-\infty}^{\infty} c_k e^{ik\lambda}, \quad c_k = \frac{1}{2\pi}\int_{\pi}^{\pi} \hat{f}(\lambda)e^{i(a-k)\lambda},$$

thus,

$$f(t+a) = \text{sinc }(t+a) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ia\lambda}\hat{f}(\lambda)e^{i\lambda t}d\lambda = \frac{1}{2\pi}\sum_{-\infty}^{\infty} c_k \int_{-\pi}^{\pi} e^{i(k+t)\lambda}d\lambda = \sum_{k=-\infty}^{\infty} c_k\text{sinc }(t+k),$$

where

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{pi} \hat{f}(\lambda)e^{i(a-k)\lambda}d\lambda = f(-k+a) = \text{sinc }(-k+a).$$

Hence, setting $k = -j$,

$$\text{sinc }(t+a) = \sum_{j=-\infty}^{\infty} \text{sinc }(j+a)\text{sinc }(t-j).$$

Let $a = -t$, then sinc $(t+a) =$ sinc $(0) = 1$, and sinc $(j-t) =$ sinc $(t-j)$ since sinc (x) is an even function. Thus, we have

$$1 = \sum_{j=-\infty}^{\infty} \text{sinc }^2(t-j).$$

Exercise 4.15 Suppose $f(t)$ satisfies the conditions of Theorem 4.18. Derive the Parseval formula

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(\lambda)|^2 d\lambda = \frac{\pi}{B}\sum_{k=-\infty}^{\infty} |f(\frac{\pi k}{B})|^2$$

Solution:

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(\lambda)|^2 d\lambda$$

is valid from the property of Fourier transform. We only prove that

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{\pi}{B}\sum_{-\infty}^{\infty} |f(\frac{\pi k}{B})|^2.$$

Since

$$f(t) = \sum_{j=-\infty}^{\infty} f(\frac{j\pi}{B})\text{sinc }(\frac{Bt}{\pi} - j),$$

then

$$\begin{aligned}\int_{-\infty}^{\infty} |f(t)|^2 dt &= \int_{-\infty}^{\infty} \left(\sum_{j=-\infty}^{\infty} f(\frac{j\pi}{B})\text{sinc }(\frac{Bt}{\pi} - j)\right)\left(\sum_{i=-\infty}^{\infty} f(\frac{i\pi}{B})\text{sinc }(\frac{Bt}{\pi} - i)\right)^* dt \\ &= \sum_{i,j=-\infty}^{\infty} f(\frac{j\pi}{B})f^*(\frac{i\pi}{B})\int_{-\infty}^{\infty} \text{sinc }(\frac{Bt}{\pi} - j)\text{sinc }(\frac{Bt}{\pi} - i)dt.\end{aligned}$$

Due to the orthogonal property that

$$\int_{-\infty}^{\infty} \text{sinc }(\frac{Bt}{\pi} - j)\text{sinc }(\frac{Bt}{\pi} - i)dt = \begin{cases} \frac{B}{\pi}, & i = j \\ 0, & i \neq j \end{cases},$$

then we have

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{\pi}{B}\sum_{k=-\infty}^{\infty} |f(\frac{\pi k}{B})|^2.$$

Exercise 4.16 Let

$$f(t) = \begin{cases} \left(\frac{\sin t}{t}\right)^2, & \text{for } t \neq 0 \\ 1, & \text{for } t = 0. \end{cases}$$

Then from Equation (2.5) we have

$$\hat{f}(\lambda) = \begin{cases} \pi\left(1 - \frac{|\lambda|}{2}\right), & \text{for } |\lambda| \leq 2 \\ 0, & \text{for } |\lambda| \geq 2. \end{cases}$$

1. Choose $B = 1$ and use the sampling theorem to write $f(t)$ as a series.
2. Graph the sum of the first 20 terms in the series and compare with the graph of $f(t)$.
3. Repeat the last two items for $B = 2$ and $B = 3$.

Solution: 1.

$$f(t) = \sum_{j=-\infty}^{\infty} \left(\frac{\sin(j\pi)}{j\pi}\right)^2 \text{sinc}\,\left(\frac{t}{\pi} - j\right)$$

. 2. See Figure 7

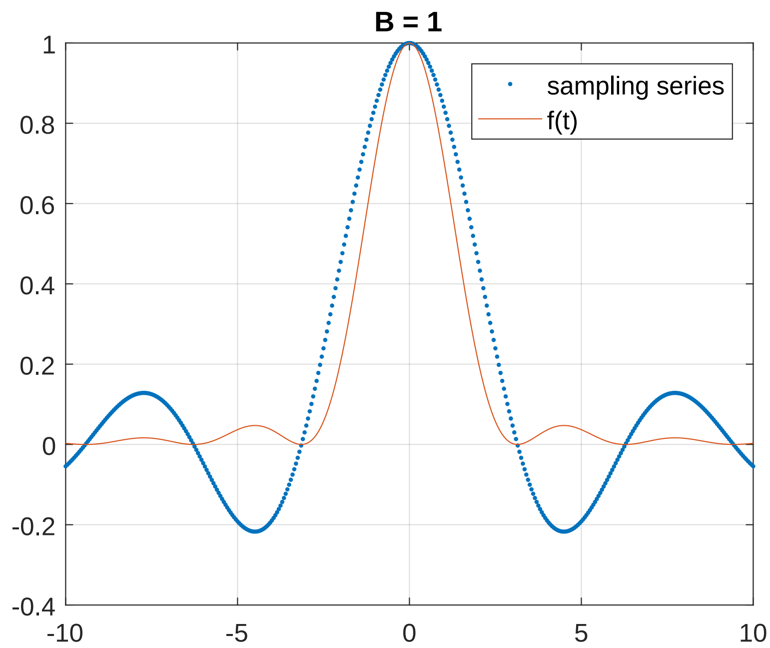


Figure 7: $B = 1$

3. See Figure 8 and Figure 9.

Exercise 4.17 If the continuous-time band limited signal is $x(t) = \cos t$, what is the period T that gives sampling exactly at the Nyquist (minimal B) rate? What samples $x(nT)$ do you get at this rate? What samples do you get from $x(t) = \sin t$?

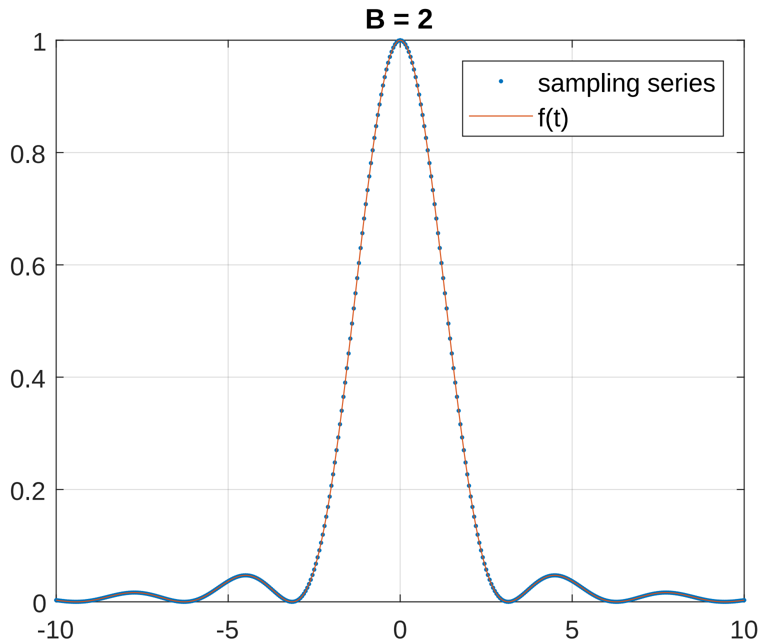


Figure 8: $B = 2$

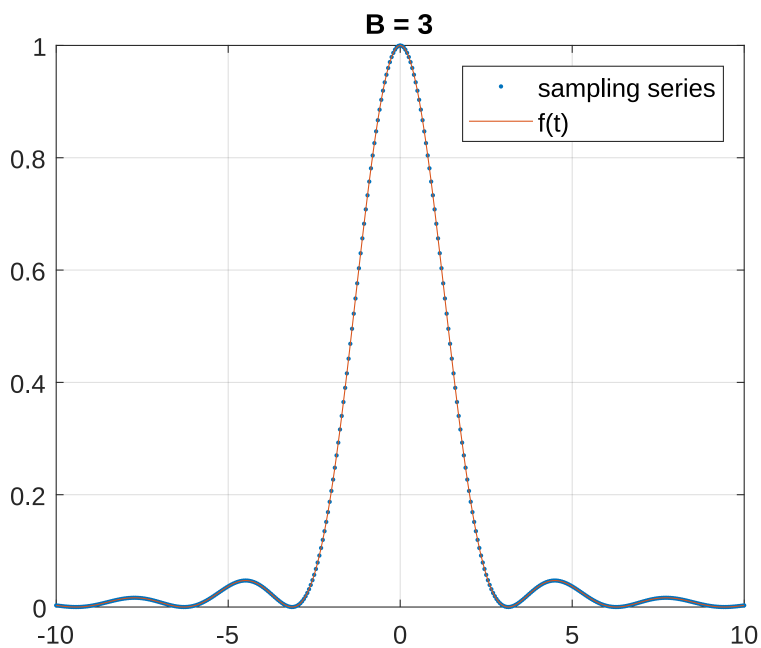


Figure 9: $B = 3$

Solution: For $x(t) = \cos t$, $\hat{x}(\lambda) = \pi(\delta(\lambda - 1) + \delta(\lambda + 1))$. Thus, $B_{\min} = 1$ and $T = \pi/B_{\min} = \pi$. Consequently, $x(nT) = x(n\pi) = \cos(n\pi)$.

Similarly, for $x(t) = \sin t$, $\hat{x}(\lambda) = -i\pi(\delta(\lambda - 1) - \delta(\lambda + 1))$. Thus, $B_{\min} = 1$ and $T = \pi/B_{\min} = \pi$. Consequently, $x(nT) = x(n\pi) = \sin(n\pi)$.

Exercise 4.18 Show that the function

$$h(\lambda) = \begin{cases} \exp\left(\frac{1}{1-\lambda^2}\right), & \text{if} -1 < \lambda < 1 \\ 0, & \text{if} |\lambda| \geq 1, \end{cases}$$

is infinitely differentiable with compact support. In particular compute the derivatives $\frac{d^n}{d\lambda^n} h(\pm 1)$ for all n.

Solution: $\lim_{\lambda \to \pm 1} \frac{d^n}{d\lambda^n} h(\lambda) \Rightarrow \pm\infty$

Exercise 4.19 Construct a function $\hat{g}(\lambda)$ which (1) is arbitrarily differentiable, (2) has support contained in the interval $[-4, 4]$ and (3) $\hat{g}(\lambda) = 1$ for $\lambda \in [1, 1]$. Hint: Consider the convolution $\frac{1}{2c} R_{[-2,2]} * h_2(\lambda)$ where $R_{[-2,2]}$ is the rectangular box function on the interval $[-2, 2]$, $h_2(\lambda) = h(\lambda/2)$, h is defined in Exercise 4.18 and $c = \int_{-\infty}^{\infty} h(\lambda) d\lambda$.

Solution: Problematic because of Exercise 4.18, and $c \to \infty$ is indefinite.

Exercise 4.20 Let $f(t) = \frac{a}{t^2+a^2}$ for $a > 0$.

- Show that $\hat{f}(t) = \pi e^{-a|\lambda|}$. Hint: It is easier to work backwards.
- Use the Poisson summation formula to derive the identity

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + a^2} = \frac{\pi}{a} \frac{1 + e^{-2\pi a}}{1 - e^{-2\pi a}}$$

  What happens as $a \to 0+$? Can you obtain the value of $\sum_{n=1}^{\infty} \frac{1}{n^2}$ from this?

Solution:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\lambda) e^{i\lambda t} d\lambda = \int_{0}^{\infty} e^{-a\lambda + i\lambda t} d\lambda = \left. \frac{e^{-a\lambda+i\lambda t}}{-a + it} \right|_{\lambda=0}^{\infty} = \frac{a + it}{t^2 + a^2},$$

thus,

$$\mathrm{Re}\left(\frac{a + it}{t^2 + a^2}\right) = \frac{a}{t^2 + a^2} = f(t).$$

By using the Poisson summation,

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(2\pi n),$$

we have

$$\sum_{n=-\infty}^{\infty}\frac{1}{n^2+a^2}=\sum_{n=-\infty}^{\infty}\frac{\pi}{a}e^{-a|2\pi n|}$$
$$=\frac{\pi}{a}+\frac{2\pi}{a}\sum_{n=1}^{\infty}e^{-2na\pi}$$
$$=\frac{\pi}{a}+\frac{2\pi}{a}\frac{e^{-2\pi a}}{1-e^{-2\pi a}}=\frac{\pi}{a}\frac{1+e^{-2\pi a}}{1-e^{-2\pi a}}.$$

Since

$$\sum_{n=-\infty}^{\infty}\frac{1}{n^2+a^2}=\frac{1}{a^2}+2\sum_{n=1}^{\infty}\frac{1}{n^2+a^2},$$

then

$$\sum_{n=1}^{\infty}\frac{1}{n^2+a^2}=\frac{1}{2}\left(\frac{\pi}{a}\frac{1+e^{-2\pi a}}{1-e^{-2\pi a}}-\frac{1}{a^2}\right)$$
$$=\frac{1+\pi a+e^{2\pi a}(-1+\pi a)}{2a^2(-1+e^{2\pi a})}.$$

Hence,

$$\lim_{a\to 0+}\frac{1+\pi a+e^{2\pi a}(-1+\pi a)}{2a^2(-1+e^{2\pi a})}=\lim_{a\to 0+}\frac{\frac{2}{3}\pi^3a^3+O(a^4)}{4\pi a^3+O(a^4)}=\frac{\pi^2}{6}=\sum_{n=1}^{\infty}\frac{1}{n^2}$$

Exercise 4.21 Verify Equations (4.22).

Solution: As

$$\|f\|^2=\int_{-\infty}^{\infty}|f(t)|^2dt=\int_{-\infty}^{\infty}f(t)\cdot\overline{f(t)}dt,$$

then

$$\|g\|^2=\int_{-\infty}^{\infty}g(t)\cdot\overline{g(t)}dt=\int_{-\infty}^{\infty}f(t+t_0)\cdot\overline{f(t+t_0)}e^{-i\lambda_0 t+i\lambda_0 t}d(t+t_0)$$
$$=\int_{-\infty}^{\infty}f(s)\overline{f(s)}ds=\|f\|^2.$$

According to Plancherel identity

$$2\pi\|f\|^2=\|\hat{f}\|^2,\quad 2\pi\|g\|^2=\|\hat{g}\|^2,$$

then we have

$$\|\hat{f}\|^2=\|\hat{g}\|^2.$$

Consequently,

$$D_0g=\int_{-\infty}^{\infty}t^2\frac{|g(t)|^2}{\|g\|^2}dt=\int_{-\infty}^{\infty}t^2\frac{|f(t+t_0)|^2}{\|f\|^2}dt\overset{s:=t+t_0}{=}\int_{-\infty}^{\infty}(s-t_0)^2\frac{|f(s)|^2}{\|f\|^2}ds=D_{t_0}f.$$

Since

$$\hat{g}(t) = \hat{f}(\lambda + \lambda_0)e^{it_0\lambda},$$

then

$$D_0\hat{g} = \int_{-\infty}^{\infty} \lambda^2 \frac{|\hat{g}(\lambda)|^2}{\|\hat{g}\|^2} d\lambda = \int_{-\infty}^{\infty} \lambda^2 \frac{|\hat{f}(\lambda + \lambda_0)|^2}{\|\hat{f}\|^2} d\lambda \overset{s:=\lambda+\lambda_0}{=} \int_{-\infty}^{\infty} (s-\lambda_0)^2 \frac{|\hat{f}(s)|^2}{\|\hat{f}\|^2} ds = D_{\lambda_0}\hat{f}.$$

Exercise 4.22 Let

$$f(t) = \begin{cases} 0, & \text{if } t < 0 \\ \sqrt{2}e^{-t}, & \text{if } t \geq 0 \end{cases}$$

Compute $(D_{t_0}f)(D_{\lambda_0}\hat{f})$ for any $t_)$, $\lambda_0 \in \mathbf{R}$ and compare with Theorem 4.21.

Solution: Since $\|f\|^2 = \int_0^\infty 2e^{-2t}dt = 1 = \frac{1}{2\pi}\|\hat{f}\|^2$, and $\hat{f}(\lambda) = \sqrt{2}\frac{1-i\lambda}{\lambda^2+1}$ then

$$D_{t_0}f = \int_0^\infty (t - t_0)^2 2e^{-2t}dt = t_0^2 - t_0 + \frac{1}{2},$$

$$D_{\lambda_0}\hat{f} = \int_{-\infty}^{\infty} \frac{(\lambda - \lambda_0)^2}{\pi(\lambda^2 + 1)} d\lambda = \text{undefined}(\to \infty).$$

Hence, $(D_{t_0}f)(D_{\lambda_0}\hat{f}) \to \infty \geq 1/4$.

Exercise 4.23 Show that the eigenvalues of a covariance matrix are nonnegative.

Solution: Since $C = E\big((\mathbf{t} - \mathbf{u})(\mathbf{t} - \mathbf{u})^T\big)$, for arbitrary vector $v \neq 0$,

$$v^T C v = E\big(v^T(\mathbf{t} - \mathbf{u})(\mathbf{t} - \mathbf{u})^T v\big) = E(s^2) \geq 0,$$

where $s = v^T(\mathbf{t} - \mathbf{u}) = (\mathbf{t} - \mathbf{u})^T v$ is a scalar random viable. Therefore, C is semidefinite positive and its eigenvalues are nonnegative.

Exercise 4.24 Show that C is in fact the covariance matrix of the distribution (4.24). This takes some work and involves making an orthogonal change of coordinates where the new coordinate vectors are the orthonormal eigenvectors of C.

Solution: Since C is semidefinite positive, we have the eigen-decomposition $C = V\Lambda V^T$, where V is an orthonormal matrix consisted the eigenvectors of C, and Λ is a diagonal matrix that $\Lambda = \text{diag}(\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2)$ with $\sigma_i^2 \geq 0$.

Let s be the new coordinates of random variables t under the coordinate transform V, then we have $t = Vs$ and $\mu = E(t) = E(Vs) = V \cdot E(s) = V\nu$, or alternatively $s = V^T t$ and $\nu = V^T\mu$. Due to the eigen-decomposition, under this new coordinate system we have

$$E((s_i - \nu_i)(s_j - \nu_j)) = \begin{cases} \sigma_i^2, & i = j \\ 0, & i \neq j \end{cases},$$

which means that $\{s_1, s_2, \ldots, s_n\}$ are independent. Thus, the multivariate normal distribution under the new coordinate system is

$$\begin{aligned}
\rho(s) &= \frac{1}{\sqrt{(2\pi)^n \prod_{i=1}^{n} \sigma_i^2}} \exp\left[\sum_{i=1}^{n}\left(-\frac{(s_i-\nu_i)^2}{2\sigma_i^2}\right)\right] \\
&= \frac{1}{\sqrt{(2\pi)^n \det(C)}} \exp\left[\frac{1}{2}(s-\nu)^T \operatorname{diag}(\sigma_1^{-2}, \sigma_2^{-2}, \ldots, \sigma_n^{-2})(s-\nu)\right] \\
&= \frac{1}{\sqrt{(2\pi)^n \det(C)}} \exp\left[\frac{1}{2}(t-\mu)^T V \Lambda^{-1} V^T (t-\mu)\right] \\
&= \frac{1}{\sqrt{(2\pi)^n \det(C)}} \exp\left[\frac{1}{2}(t-\mu)^T C^{-1} (t-\mu)\right] = \rho(t).
\end{aligned}$$

Exercise 4.25 If the random variables are independently distributed, show that

$$C(t_i, t_j) = \sigma_i^2 \delta_{ij}$$

Solution: If the random variables are independently distributed, the covariance of any pair (t_i, t_j) is zero if $i \neq j$. This is because that

$$E((t_i - \mu_i)(t_j - \mu_j)) = E(t_i t_j) - E(t_i)E(t_j).$$

If t_i, t_j are independent, $E(t_i t_j) = E(t_i)E(t_j)$ and thus $E((t_i - \mu_i)(t_j - \mu_j)) = 0$. If $i = j$, $E((t_i - \mu_i)(t_j - \mu_j)) = \sigma_i^2$

Exercise 4.26 Show that $E(T_i T_j) = \mu^2 + \sigma^2 \delta_{ij}$

Solution: If $i \neq j$, since T_i and T_j are independent, $E(T_i T_j) = E(T_i)E(T_j) = \mu^2$.
If $i = j$, $\sigma^2 = E(T_i^2) - E^2(T_i) = E(T_i^2) - \mu^2$, thus $E(T_i^2) = \sigma^2 + \mu^2$.

Exercise 4.27 Let $f(x) = \exp(-sx^2)$ for fixed $s > 0$ and all x. Verify that $\hat{f}(\lambda) = \sqrt{\pi/s}\exp(-\lambda^2/4s)$. Hint: By differentiating under the integral sign and integration by parts, establish that

$$\frac{d\hat{f}(\lambda)}{d\lambda} = -\frac{\lambda}{2s}\hat{f}(\lambda), \quad \text{so } \hat{f}(\lambda) = C\exp\bigl(-\lambda^2/4s\bigr)$$

for some constant C. To compute C, note that

$$\left[\int_{-\infty}^{\infty} e^{-sx^2} dx\right]^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-s(x^2+y^2)} dx dy = \int_0^{2\pi} d\theta \int_0^{\infty} e^{-sr^2} r dr$$

Solution: Since

$$\hat{f}(\lambda) = \int_{-\infty}^{\infty} e^{-(sx^2 + i\lambda x)} dx,$$

then we have

$$\frac{d\hat{f}(\lambda)}{d\lambda} = \int_{-\infty}^{\infty} -xe^{-sx^2} \cdot ie^{-i\lambda x}dx$$
$$= \frac{i}{2s}e^{-sx^2-i\lambda x}\Big|_{x=-\infty}^{\infty} - \frac{\lambda}{2s}\int_{-\infty}^{\infty} e^{-(sx^2+i\lambda x)}dx = -\frac{\lambda}{2s}\hat{f}(\lambda).$$

Hence,

$$\hat{f}(\lambda) = Ce^{-\lambda^2/4s}.$$

As

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} e^{-2sx^2}dx = \sqrt{\frac{\pi}{2s}},$$

and

$$\int_{-\infty}^{\infty} |\hat{f}(\lambda)|^2 d\lambda = C^2\int_{-\infty}^{\infty} e^{-\lambda^2/2s}d\lambda = C^2\sqrt{2\pi s},$$

according to the Plancherel identity

$$2\pi\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |\hat{f}(\lambda)|^2 d\lambda,$$

we have $C = \sqrt{\frac{\pi}{s}}$. Hence

$$\hat{f}(\lambda) = \sqrt{\frac{\pi}{s}}e^{-\lambda^2/4s}.$$

Exercise 4.28 Let $g : \mathbb{R} \to \mathbb{R}$. Find a function H such that for all x,

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} g(t)dt = (H * g)(x).$$

(H is called the Heaviside function)

Solution:

$$(H * g)(x) = \int_{-\infty}^{\infty} H(x-t)g(t)dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} g(t)dt.$$

so H could be

$$H(t) = \begin{cases} \frac{1}{\sqrt{2\pi}}, & t \geq 0 \\ 0, & t < 0 \end{cases}.$$

Exercise 4.29 Let $f, g : \mathbb{R} \to \mathbb{R}$. Let f' exist. Assuming the convergence of the relevant integrals below, show that

$$(f * g)'(x) = f'(x) * g(x).$$

Solution:

$$(f * g)'(x) = \int_{-\infty}^{\infty} [f(x-t)g(t)]'_x dt = \int_{-\infty}^{\infty} f'(x-t)g(t)dt = f'(x) * g(x).$$

Exercise 4.30 For $a \in \mathbb{R}$, let

$$f_a(t) = \begin{cases} 0, & t < a \\ 1, & t \geq a. \end{cases}$$

Compute $f_a * f_b$ for $a, b \in \mathbb{R}$. Deduce that

$$(f_a * f_{-a})(x) = \frac{x f_0(x)}{\sqrt{2\pi}}.$$

Does $f_a * (1 - f_b)$ exist? For $a \in \mathbb{R}$, let

$$g_a(t) := \begin{cases} 0, & t < 0 \\ \exp(-at), & t \geq 0. \end{cases}$$

Compute $g_a * g_b$.

Solution:

$$(f_a * f_b)(x) = \int_{-\infty}^{\infty} f_a(x-t) f_b(t) dt = \begin{cases} x - (a+b), & x \geq a+b \\ 0, & x < a+b. \end{cases}$$

Thus

$$(f_a * f_{-a})(x) = \left\{ \begin{array}{ll} x, & x \geq 0 \\ 0, & x < 0. \end{array} \right\} = x f_0(x)$$

. Since $1 - f_b(t) = f_{-b}(-t)$, then

$$(f_a * (1 - f_b))(x) = \int_{-\infty}^{\infty} f_a(x-t) f_{-b}(-t) dt = \int_{-\infty}^{\min(b, x-a)} dt.$$

Hence, $f_a * (1 - f_b)$ does not exist.

$$(g_a * g_b)(x) = \int_{-\infty}^{\infty} g_a(x-t) g_b(t) dt = \begin{cases} \int_0^x e^{-(a+b)t} dt = \frac{1 - \exp(-(a+b)x)}{a+b}, & x \geq 0 \\ 0, & x < 0. \end{cases}$$

Exercise 4.31 Fourier transforms are useful in "deconvolution" or solving "convolution integral equations." Suppose that we are given functions g, h and are given that

$$f * g = h.$$

Over task is to find f in terms of g, h.

(i) Show that

$$\mathcal{F}[f] = \mathcal{F}[h]/\mathcal{F}[g]$$

and hence, if we can find a function k such that

$$\mathcal{F}[h]/\mathcal{F}[g] = \mathcal{F}[k]$$

then $f = k$.

(ii) As an example, suppose that

$$f * \exp\bigl(-t^2/2\bigr) = (1/2)t\exp\bigl(-t^2/4\bigr).$$

Solution: (i) If $f * g = h$, then we have $\mathcal{F}[f]\cdot\mathcal{F}[g] = \mathcal{F}[h]$. Thus, $\mathcal{F}[f] = \mathcal{F}[h]/\mathcal{F}[g]$.
(ii)

$$\mathcal{F}[f * \exp\bigl(-t^2/2\bigr)] = \mathcal{F}[f]\cdot\mathcal{F}[\exp\bigl(-t^2/2\bigr)] = \mathcal{F}[(1/2)t\exp\bigl(-t^2/4\bigr)],$$

thus

$$\begin{aligned} &\mathcal{F}[f]\cdot e^{-\lambda^2/2} = i\sqrt{2}\lambda e^{-\lambda^2} \\ \Rightarrow &\mathcal{F}[f] = i\sqrt{2}\lambda e^{-\lambda^2/2} \\ \Rightarrow &f = \sqrt{2}te^{-t^2/2}. \end{aligned}$$

Exercise 4.32 (i) The *Laplace* transform of a function $f : [0;\infty) \to \mathbf{R}$ is defined as

$$\mathcal{L}[f](p) = \int_0^\infty f(t)\exp(-pt)dt$$

whenever the right-hand side makes sense. Show formally, that if we set

$$g(x) := \begin{cases} f(x), & x \geq 0 \\ 0, & x < 0 \end{cases}$$

then

$$\mathcal{L}[f](p) := \sqrt{2\pi}\mathcal{F}[g](-ip).$$

(ii) Let $h : \mathbf{R} \to \mathbf{R}$ and define:

$$h_+(x) := \begin{cases} h(x), & x \geq 0 \\ 0, & x < 0 \end{cases}$$

and

$$h_-(x) := \begin{cases} h(-x), & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Show that $h(x) = h_+(x) + h_x(x)$ and express $\mathcal{F}[h]$ in terms of $\mathcal{L}[h_+]$ and $\mathcal{L}[h_-]$.

Solution: (i)

$$\begin{aligned} \mathcal{L}[f](p) &= \int_0^\infty f(t)e^{-pt}dt \\ &= \int_{-\infty}^\infty g(t)e^{-i(-ip)t}dt \\ &= \hat{g}(-ip) = \sqrt{2\pi}\mathcal{F}[g](-ip). \end{aligned}$$

(ii)

$$
\begin{aligned}
\mathcal{F}[h](\lambda) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(x) e^{-ix\lambda} dx \\
&= \frac{1}{\sqrt{2\pi}} \left(\int_0^{\infty} h(-x) e^{-x(-i\lambda)} dx + \int_0^{\infty} h(x) e^{-x(i\lambda)} dx \right) \\
&= \frac{1}{\sqrt{2\pi}} (\mathcal{L}[h_-](-i\lambda) + \mathcal{L}[h_+](i\lambda)).
\end{aligned}
$$

Solutions to Chapter 5

Exercise 5.1 Why is the row of 1's needed in the sample matrix for example (5.2)?

Solution: The row of 1's is particularly helpful for the 0th sample whose 2nd $\sim m$th rows are $[0\ \ 0\ \ \ldots\ \ 0]^T$, to distinguish it from the sample of no input signal, as well as keep all n samples the same length m.

Exercise 5.2 Using the method of expression (5.2) design a sample matrix to detect a 1-sparse signal of length $n = 16$. Apply your matrix to a signal with a spike $x_9 = -2$.

Solution: The Φ matrix is

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

The sample for $x_9 = -2$ is $[-2\quad -2\ \ 0\ \ 0\quad -2]^T$.

Exercise 5.3 Show that even though Σ_k is not a vector space, it can be expressed as the set–theoretical union of subspaces $X_{T_k} = \{x : \text{supp}(x) \subset T_k\}$ where T_k runs over all k-element subsets of the integers $\{1, 2, \ldots, n\}$, i.e.

$$\Sigma_k = \cup_{T_k} X_{T_k}$$

Solution: In the view of set, which considers the n elements of a vector outside the vector context, Σ_k defined in (5.3) is essentially interpreted by $\Sigma_k = \cup_{T_k} X_{T_k}$.

Exercise 5.4 Verify that the matrix $\Phi_T^*\Phi_T$ is self-adjoint

Solution:

$$(\Phi_T^*\Phi_T)^* = \Phi_T^*(\Phi_T^*)^* = \Phi_T^*\Phi_T$$

Exercise 5.5 In the proof of Theorem 5.4 show in detail why $\Phi(-h_{S^c}) = \Phi(h_S) = \Theta$.

Solution: If $h_S \neq \Theta$, since $\Sigma_{2k} \cap N(\Phi) = \{\Theta\}$ and apparently $h_S \in \Sigma_{2k}$, then $h_S \notin N(\Phi)$ and thereby $\Phi(-h_{S^c}) = \Phi(h_S) \neq \Theta$.

Exercise 5.6 In the proof of Theorem 5.4 show why the ℓ_1 norm rather than some other ℓ_p norm is essential.

Solution: Because in the proof of Theorem 5.4 we used the result of the Definition 5.3, which is based on ℓ_1 norm.

Exercise 5.7 Show that if Φ satisfies the null space property of order k then it also satisfies the null space property for orders $1, 2, \ldots, k-1$.

Solution: Since $\Sigma_2 p \subset \Sigma_{2(p+1)}$ for any integer $1 \leq p < \lfloor n/2 \rfloor$, then $\Sigma_{2p} \cap N(\Phi) = \{\Theta\}$ is valid for any $1 \leq p \leq k-1$ if $\Sigma_{2p} \cap N(\Phi) = \{\Theta\}$ is valid, i.e. Φ satisfies the null space property of order k.

Exercise 5.8 We order the components of $x \in \mathbb{C}^n$ in terms of the magnitude of the absolute value, so that

$$|x_{i_1}| \geq |x_{i_2}| \geq \cdots \geq |x_{i_n}|.$$

Show that $\sigma_k(x) = \sum_{j=k+1}^{n} |x_{i_j}|$, i.e., $\sigma_k(x)$ is the sum of the absolute values of the $n-k$ smallest components of x. Show that

$$\text{Argmin} \inf_{z \in \Sigma_k} \|x - z\|_1 = x^k$$

where x^k has index set $T = \{i_1, i_2, \ldots, i_k\}$. Thus, x^k is the closest k-sparse approximation of x with respect to the ℓ_1 norm.

Solution: If $z \in \Sigma_k$, we denote the k indices with nonzero values in z by S, and the $n-k$ indices with zero values in z by S^c, thus

$$\begin{aligned}
\sigma_k(x) = \inf_{z \in \Sigma_k} \|x - z\|_1 &= \inf_{z \in \Sigma_k} \sum_{i=1}^{n} |x_i - z_i| \\
&= \inf_{z \in \Sigma_k} \left(\sum_{i \in S} |x_i - z_i| + \sum_{i \in S^c} |x_i - 0| \right) \qquad (\text{let } z_i = x_i \text{ for } i \in S) \\
&= \inf_{z \in \Sigma_k} \sum_{i \in S^c} |x_i|.
\end{aligned}$$

Hence, choosing S^c to be the indices where the elements have the smallest absolute value can result in the infimum of $\sigma_k(x)$. Meanwhile, the z results in the infimum of $\sigma_k(x)$ have the form of

$$z_{i_j} = x_{i_j} \text{ for } j = 1, \ldots, k \text{ and } z_{i_j} = 0 \text{ for } j = k+1, \ldots, n.$$

Exercise 5.9 If

$$x = (3, -2, 6, 0, 3, -1, 2, -5) \in \mathbb{C}^8,$$

find x^k and $\sigma_k(x)$ for $k = 1, 2, \ldots, 7$.

Solution: $x^1 = (0, 0, 6, 0, 0, 0, 0, 0)$, $\sigma_1(x) = 16$;

$x^2 = (0, 0, 6, 0, 0, 0, 0, -5)$, $\sigma_2(x) = 11$;

$x^3 = (3, 0, 6, 0, 0, 0, 0, -5)$ or $(0, 0, 6, 0, 3, 0, 0, -5)$ $\sigma_3(x) = 8$;

$x^4 = (3, 0, 6, 0, 3, 0, 0, -5)$, $\sigma_4(x) = 5$;

$x^5 = (3, -2, 6, 0, 3, 0, 0, -5)$ or $(3, 0, 6, 0, 3, 0, 2, -5)$ $\sigma_5(x) = 3$;

$x^6 = (3, -2, 6, 0, 3, 0, 2, -5)$, $\sigma_6(x) = 1$;

$x^7 = (3, -2, 6, 0, 3, -1, 2, -5)$, $\sigma_7(x) = 0$;

Exercise 5.10 Show that if Φ satisfies the null space property of order k, Definition 5.3, then there exists a $\rho \in (0,1)$ such that Φ satisfies the null space property of order k with constant ρ.

Solution: If Φ satisfies the null space property of order k, Definition 5.3, then for every index set S with $\#(S) = k$ we have

$$\|h_S\|_1 < \|h_{S^c}\|_1, \quad \forall\, h \in N(\Phi), h \neq \Theta,$$

then we have

$$\|h_S\|_1 \leq \rho\|h_{S^c}\|_1, \quad \forall\, h \in N(\Phi), h \neq \Theta,$$

where $\rho \in (0,1)$. For any index set $T \subseteq S$, let $R = T - S$. then

$$\|h_S\|_1 = \|h_T\|_1 + \|h_R\|_1 \leq \rho\|h_{S^c}\|_1 = \rho(\|h_{T^c}\|_1 - \|h_R\|_1).$$

Hence

$$\|h_T\|_1 \leq \rho\|h_{T^c}\|_1 - (1+\rho)\|h_R\|_1 \leq \rho\|h_{T^c}\|_1.$$

Exercise 5.11 Give the details of the derivation of inequalities (5.18).

Solution: Any k-sparse x can be decomposed as the linear combination of the eigenvectors $v_1, v_2, \ldots, v_r$ of $\Phi_T^*\Phi_T$, as

$$x = a_1v_1 + a_2v_2 + \cdots + v_ra_r,$$

where $r \leq k$ and $v_1, v_2, \ldots, v_r$ are orthonormal,corresponding to the eigenvalues $\lambda_{\max}(T) = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r = \lambda_{\min}(T) > 0$ of $\Phi_T^*\Phi_T$. Thus,

$$\|x\|_2^2 = |a_1|^2 + |a_2|^2 + \cdots + |a_r|^2,$$

and we have

$$\|\Phi_Tx\|_2^2 = x^*\Phi_T^*\Phi_Tx = \lambda_1|a_1|^2 + \lambda_2|a_2|^2 + \cdots + \lambda_r|a_r|^2.$$

Apparently,

$$\lambda_{\min}(T)\|x\|_2^2 \leq \|\Phi_Tx\|_2^2 \leq \lambda_{\max}(T)\|x\|_2^2.$$

Since $\lambda_{\min} \leq \lambda_{\min}(T)$ and $\lambda_{\max} \geq \lambda_{\max}(T)$ for over all index sets T with $\leq k$ indices, then

$$\lambda_{\min}\|x\|_2^2 \leq \|\Phi x\|_2^2 \leq \lambda_{\max}\|x\|_2^2.$$

Exercise 5.12 Show that (1) $h_{T_\ell} \in \Sigma_k$, (2) $h_{T_\ell} \cdot h_{T_{\ell'}} = 0$ for $\ell \neq \ell'$ and (3)

$$h = \sum_{\ell=0}^{a} h_{T_\ell}.$$

Solution: (1) By definition, $T_\ell, \ell = 0, 1, \ldots, a-1$ are k-index sets and T_a is r-index set where $r < k$. Therefore, $h_{T_\ell} \in \Sigma_k$.

(2) Since T_ℓ and $T_{\ell'}$ are disjoint sets as $\ell \neq \ell'$, then $T_\ell \cdot T_{\ell'} = 0$.

(3) By definition, $h = \sum_{\ell=0}^{a} h_{T_\ell}$

Exercise 5.13 Verify the details of the proof of Lemma 5.10.

Solution: Since

$$2(1-\delta_{2k}) \le \|\Phi(z+z')\|_2^2 \le 2(1+\delta_{2k}),$$
$$2(1-\delta_{2k}) \le \|\Phi(z-z')\|_2^2 \le 2(1+\delta_{2k}),$$

then

$$\frac{2(1-\delta_{2k})-2(1+\delta_{2k})}{4} \le \frac{1}{4}\big(\|\Phi(z+z')\|_2^2 - \|\Phi(z-z')\|_2^2\big) \le \frac{2(1+\delta_{2k})-2(1-\delta_{2k})}{4},$$

namely

$$-\delta_{2k} \le \frac{1}{4}\big(\|\Phi(z+z')\|_2^2 - \|\Phi(z-z')\|_2^2\big) \le \delta_{2k}.$$

Therefore,

$$|\langle \Phi z, \Phi z'\rangle| \le \delta_{2k},$$

for $\|z\|_2 = \|z'\|_2 = 1$. By renormalizing, we have

$$|\langle \Phi x, \Phi x'\rangle| \le \delta_{2k}\|x\|_2\|x'\|_2$$

Exercise 5.14 If $\delta_{2k} = 1/4$ for some k and the sample matrix Φ, verify from Theorem 5.13 that the estimate $\|\hat{x}-x\|_1 \le 5:5673\|x-x^k\|_1$ holds for the approximation of a signal x by a k-sparse signal $\hat{x}$.

Solution: Can not find Theorem 5.13.

Exercise 5.15 Show that even though RIP implies the null space property, the converse is false. Hint: Given an $m \times n$ sample matrix Φ let $\Xi = A\Phi$ be a new sample matrix, where A is an invertible $m \times m$ matrix. If Φ satisfies the null space property, then so does Ξ. However, if Φ satisfies RIP we can choose A so that Ξ violates RIP.

Solution: Suppose that a sample matrix Φ satisfies RIP, and thereby it is implied to satisfy the null space property. A is an invertible matrix and $\Xi = A\Phi$. It is clear that $N(\Phi) = N(A\Phi)$, thus Ξ also satisfies the null space property. Φ satisfying RIP indicates that

$$\|\Phi(x)\|_2^2 \le (1+\delta_k)\|x\|_2^2,$$

for $x \in \Sigma_k$ with $\delta_k < 1$, which implies that

$$\|\Phi(x)\|_2^2 < 2\|x\|_2^2,$$

if Φ satisfies RIP. For arbitrary $x_0 \in \Sigma_k$ and $\|x_0\|_2 = 1$, let $\alpha = \|\Phi(x_0)\|_2^2$. Then, we can form A as a diagonal matrix with all diagonal entries to be $\sqrt{\beta}$ where $\beta > 2/\alpha$. Hence

$$\|\Xi x_0\|_2^2 = \|A\Phi x_0\|_2^2 = \beta\alpha > 2,$$

which means that Ξ violates RIP. Therefore, we can have a sample matrix Ξ which satisfies the null space property but violates RIP.

Exercise 5.16 Verify the inequality (5.39) via the following elementary argument.

1. Using upper and lower Darboux sums for $\int_1^n \ln x dx$, establish the inequality

$$\sum_{j=1}^{n-1} \ln j \leq \int_1^n \ln x dx \leq \sum_{j=2}^{n} \ln j.$$

2. Show that

$$(n-1)! \leq e\Big(\frac{n}{e}\Big)^n \leq n!.$$

3. Verify that

$$\binom{n}{k} \equiv \frac{n!}{k!(n-k)!} \leq \frac{n^k}{k!} \leq \Big(\frac{ne}{k}\Big)^k$$

Solution: 1. Since

$$\ln j \leq \int_j^{j+1} \ln x dx \leq \ln(j+1),$$

for $j \geq 1$. Hence

$$\sum_{j=1}^{n-1} \ln j \leq \sum_{j=1}^{n-1} \int_j^{j+1} \ln x dx = \int_1^n \ln x dx \leq \sum_{j=2}^{n} \ln j.$$

2. Since

$$\int_1^n \ln x dx = n \ln\left(\frac{n}{e}\right) + 1,$$

and from 1. we have

$$\exp\left(\sum_{j=1}^{n-1} \ln j\right) \leq \exp\left(\int_1^n \ln x dx\right) \leq \exp\left(\sum_{j=2}^{n} \ln j\right),$$

thus

$$(n-1)! \leq e\Big(\frac{n}{e}\Big)^n \leq n!.$$

3. Because

$$\frac{n!}{(n-k)!} = n \cdot (n-1) \cdot \cdots \cdot (n-k+1) \leq n^k,$$

and from 2. we have

$$\frac{1}{k!} \leq \frac{1}{e}\Big(\frac{e}{k}\Big)^k,$$

thus

$$\frac{n!}{k!(n-k)!} \leq \frac{n^k}{k!} \leq \frac{1}{e}\Big(\frac{ne}{k}\Big)^k \leq \Big(\frac{ne}{k}\Big)^k.$$

Exercise 5.17 Verify the identity (5.43).

Solution:

$$\begin{aligned}\sum_{i=1}^{m}\sum_{j=1}^{n}T_{ij}^2 &= \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{n}t_{ik}^2 O_{kj}^2\\ &= \sum_{i=1}^{m}\sum_{k=1}^{n}t_{ik}^2\left(\sum_{j=1}^{n}O_{kj}^2\right)\\ &= \sum_{i=1}^{m}\sum_{k=1}^{n}t_{ik}^2\end{aligned}$$

Exercise 5.18 Show that $E\big(\|\Phi x\|_{\ell_2^n}\big) = \int_0^\infty \tau p(\tau)d\tau = 1$ by evaluating the integral explicitly.

Solution: Let $x = m\tau/2$, then

$$\begin{aligned}\int_0^\infty \tau^{m/2}e^{-m\tau/2}d\tau &= \left(\frac{2}{m}\right)^{m/2+1}\int_0^\infty x^{m/2}e^{-x}dx\\ &= \left(\frac{2}{m}\right)^{m/2+1}\Gamma(m/2+1)\\ &= \left(\frac{2}{m}\right)^{m/2+1}\left(\frac{m}{2}\right)\Gamma(m/2) = \left(\frac{2}{m}\right)^{m/2}\Gamma(m/2).\end{aligned}$$

Hence,

$$\int_0^\infty \tau p(\tau)d\tau = 1.$$

Exercise 5.19 Verify the right-hand inequality in (5.46) by showing that the maximum value of the function $f(\epsilon) = (1+\epsilon)\exp(-\epsilon+\epsilon^2/2-\epsilon^3/3)$ on the interval $0 < \epsilon$ is 1.

Solution: Since

$$f'(\epsilon) = -x^3\exp\big(-\epsilon+\epsilon^2/2-\epsilon^3/3\big),$$

$f'(\epsilon) \le 0$ for $\epsilon \ge 0$. Hence, $f_{\max}$ appears at $f(\epsilon = 0) = 1$ for $\epsilon \ge 0$. Therefore,

$$f^{m/2}(\epsilon) = \frac{[e^{-\epsilon}(1+\epsilon)]^{m/2}}{e^{-m(\epsilon^2/4-\epsilon^3/6)}} \le 1,$$

and thereby

$$\big[e^{-\epsilon}(1+\epsilon)\big]^{m/2} \le e^{-m(\epsilon^2/4-\epsilon^3/6)}.$$

Exercise 5.20 Verify the right-hand inequality in (5.47) by showing that $g(\epsilon) = (1-\epsilon)e^{\epsilon+\epsilon^2/2-\epsilon^3/3} \le 1$ for $0 < \epsilon < 1$.

Solution: since

$$g'(\epsilon) = (x-2)x^2\exp\big(\epsilon+\epsilon^2/2-\epsilon^3/3\big),$$

$g'(\epsilon) \leq 0$ for $0 \leq \epsilon \leq 1$. Hence, $g_{\max}$ appears at $g(\epsilon = 0) = 1$ for $0 \leq \epsilon \leq 1$. Therefore,

$$g^{m/2}(\epsilon) = \frac{[e^{\epsilon}(1-\epsilon)]^{m/2}}{e^{-m(\epsilon^2/4-\epsilon^3/6)}} \leq 1,$$

and thereby

$$[e^{\epsilon}(1-\epsilon)]^{m/2} \leq e^{-m(\epsilon^2/4-\epsilon^3/6)}.$$

Exercise 5.21 Using Lemmas 5.19, 5.20 and the mean value theorem, show that for the uniform probability distribution ρ_1 the "worst case" is $w = (1, \dots, 1)/\sqrt{n}$.

Solution: Lemma 5.19 shows that the "worst case" must occur for all $x_j \geq 0$. Lemma 5.20 shows that when all $x_j \geq 0$, the "worst case" must occur for $\theta = \pi/4$, viz., any two entries of x must equal. Therefore, the "worst case" is $x = w = (1, \dots, 1)/\sqrt{n}$.

Exercise 5.22 Demonstrate the roughly logarithmic dependence of required sample size on n for fixed k by verifying that if $n = 10^{10}, k = 6, m = 22000$ then the estimate (5.64) guarantees that the method will work with a probability of failure less than 6×10^{-22}. Thus squaring n requires only that m be doubled for success.

Solution: According to Theorem 5.17, the method will work with a probability of failure less than e^{-mc_2}, where

$$c_2 = c_0\left(\frac{\delta}{2}\right) - \frac{k}{m}\left[\ln\left(\frac{n}{k}\right) + 1 + \ln\left(\frac{12}{\delta}\right)\right] - \frac{\ln 2}{m},$$

in which $\delta \approx 0.4142$ and $c_0(\epsilon) = \epsilon^2/4 - \epsilon^3/6$. Thus $c_2 \approx 0.0022293$ and then $e^{-mc_2} \approx 5.0143 \times 10^{-22} < 6 \times 10^{-22}$.

Exercise 5.23 Work out explicitly the action of the Vandermonde coder/decoder Example 5.2 for (a) $k = 1$, and (b) $k = 2$.

Solution: (a) for $k = 1$,

$$\Phi = \begin{pmatrix} 1 & 1 & \dots & 1 \\ a_1 & a_2 & \dots & a_n \end{pmatrix}.$$

Thus, any $2k \times 2k$ submatrix is

$$\Phi_{\text{sub}} = \begin{pmatrix} 1 & 1 \\ a_i & a_j \end{pmatrix},$$

where $1 \leq i \neq j \leq n$, and $\text{Det}(\Phi_{\text{sub}}) = a_j - a_i \neq 0$.

(b) for $k = 2$,

$$\Phi = \begin{pmatrix} 1 & 1 & \dots & 1 \\ a_1 & a_2 & \dots & a_n \\ a_1^2 & a_2^2 & \dots & a_n^2 \\ a_1^3 & a_2^3 & \dots & a_n^3 \end{pmatrix}.$$

Thus, any $2k \times 2k$ submatrix is

$$\Phi = \begin{pmatrix} 1 & 1 & 1 & 1 \\ a_{k_1} & a_{k_2} & a_{k_3} & a_{k_4} \\ a_{k_1}^2 & a_{k_2}^2 & a_{k_3}^2 & a_{k_4}^2 \\ a_{k_1}^3 & a_{k_2}^3 & a_{k_3}^3 & a_{k_4}^3 \end{pmatrix},$$

where $1 \le k_i \ne k_j \le n$ for $i, j = 1, 2, 3, 4$. Thus, $\mathrm{Det}(\Phi_{\mathrm{sub}}) = \pm \prod_{i>j}(a_{k_i} - a_{k_j}) \ne 0$.

Exercise 5.24 (a) Show that the minimization problems (5.7) and (5.8) are equivalent. (b) Show that the minimization problems (5.9) and (5.10) are equivalent.

Solution: (a) $x \in F(y) \Leftrightarrow \Phi x = y \Leftrightarrow y_k = r^{(k)} \cdot x$ for $k = 1, \ldots, m$. And

$$\mathrm{Argmin}_x \|x\|_1 \Leftrightarrow \mathrm{Argmin}_x \sum_{j=1}^{n} |x_j| \Leftrightarrow \mathrm{Argmin}_u \sum_{j=1}^{n} u_j \quad \text{for} \quad -u_j \le x_j \le u_j.$$

(b)

$$\mathrm{Argmin}_x \sum_{i=1}^{m} |y_i - r^{(i)} \cdot x| \Leftrightarrow \mathrm{Argmin}_u \sum_{i=1}^{m} u_i \quad \text{for} \quad -u_i \le y_i - r^{(i)} \cdot x \le u_i.$$

Exercise 5.25 Show that the ℓ_∞ minimization problems analogous to (5.7) and (5.9) can be expressed as problems in linear programming.

Solution: The linear programming problem for the ℓ_∞ analogous problem of (5.7) is

$$\min t, \quad \text{such that} \quad y_k = r^{(k)} \cdot x, \quad -t \le x_i \le t.$$

The linear programming problem for the ℓ_∞ analogous problem of (5.9) is

$$\min t, \quad \text{such that} \quad -t \le y_k - r^{(k)} \cdot x \le t.$$

Exercise 5.26 Let Φ be a sample matrix with identical columns $\Phi = (c, c, \ldots, c)$ where $\|c\|_2 = 1$. Show that $\delta_1 = 0$, so this matrix does not satisfy RIP.

Solution: If x is 1-sparse, then $\|\Phi x\|_2^2 = \|x\|_2^2 \|c\|_2^2 = \|x\|_2^2$. Thus, the satisfied δ_1 for

$$(1 - \delta_1)\|x\|_2^2 \le \|\Phi x\|_2^2 \le (1 + \delta_1)\|x\|_2^2,$$

is $\delta_1 = 0$. Therefore, this matrix does not satisfy RIP.

Exercise 5.27 Use the eigenvalue property (5.18) to show that for any sample matrix Φ it is not possible for $\delta_k = 0$ if $k \ge 2$.

Solution: If $\delta_k = 0$ for $k \ge 2$, then we have $\lambda_{\min} = \lambda_{\max} = 1$, which means any $\Phi_T^* \Phi_T$ is an identity matrix, namely, the columns of Φ_T are orthonormal, over all index set T with $\le k$ indices where $k \ge 2$.

However, for any $\Phi \in \mathbb{C}^{m\times n}$ with $m < n$, we can always find two columns in Φ which are not orthogonal, then we can form a submatrix Φ_T that containing such non-orthogonal columns, and obtain an invertible $\Phi_T^*\Phi_T \neq I$, for which it has $\lambda_{\min}(T) < \lambda_{\max}(T)$ strictly. Hence, we always have $\lambda_{\min} < \lambda_{\max}$ over all index set T with $\leq k$ indices where $k \geq 2$, and thereby $\delta_k > 0$ for $k \geq 2$.

Exercise 5.28 Use the $m \times n$ sample matrix with constant elements $\Phi_{ij} = 1/\sqrt{m}$ and k-sparse signals x such that $x_j = 1$ for $x \in T$ to show that this sample matrix is not RIP for $k \geq 2$.

Solution: If x is k-sparse and has exact k non-zero entries, then $\|x\|_2^2 = k$ and $\|\Phi x\|_2^2 = k^2$. Apparently, $\|\Phi x\|_2^2 \geq 2\|x\|_2^2$ if $k \geq 2$, which violates the RIP condition $\|\Phi x\|_2^2 \leq (1+\delta_k)\|x\|_2^2$ for $\delta_k < 1$.

Exercise 5.29 Derive (5.53).

Solution: Since

$$E(t^{2k}) = \int_{-\infty}^{\infty} t^{2k}\sqrt{\frac{m}{2\pi}}e^{-mt^2/2}dt = 2\sqrt{\frac{m}{2\pi}}\int_0^{\infty} t^{2k}e^{-mt^2/2}dt.$$

Let $x = mt^2/2$, then $t = \sqrt{2x/m}, dt = dx\sqrt{1/2mx}$. Hence,

$$\begin{aligned}
E(t^{2k}) &= 2\sqrt{\frac{m}{2\pi}}\sqrt{\frac{1}{2m}}\int_0^{\infty}\frac{2^k}{m^k}x^{k-\frac{1}{2}}e^{-x}dx \\
&= \frac{2^k}{m^k\sqrt{\pi}}\Gamma\left(k+\frac{1}{2}\right) \\
&= \frac{2^k}{m^k\sqrt{\pi}}\cdot\frac{(2k)!}{4^k k!}\sqrt{\pi} \\
&= \frac{(2k)!}{k!(2m)^k}.
\end{aligned}$$

Because

$$\frac{(2k)!}{k!2^k} = (2k-1)!! \geq 1,$$

thus

$$\frac{(2k)!}{k!(2m)^k} \geq \frac{1}{m^k}.$$

Exercise 5.30 Verify the right-hand inequality in (5.58).

Solution: Since $u = m\epsilon(1+\epsilon)/2$,

$$e^{u(1-\epsilon)}\left[1-\frac{u}{m}+\frac{3u^2}{2m^2}\right]^m = e^{m\epsilon(1-\epsilon^2)/2}\left[1-\epsilon/2-\epsilon^2/8+3\epsilon^3/4+3\epsilon^4/8\right]^m.$$

Thus,

$$\begin{aligned}f(\epsilon) &= \frac{e^{\epsilon(1-\epsilon^2)/2}[1-\epsilon/2-\epsilon^2/8+3\epsilon^3/4+3\epsilon^4/8]}{e^{-(\epsilon^2/4-\epsilon^3/6)}} \\ &= e^{\epsilon/2+\epsilon^2/4-2\epsilon^3/3}\big[1-\epsilon/2-\epsilon^2/8+3\epsilon^3/4+3\epsilon^4/8\big].\end{aligned}$$

Actually, we have $-5.66922\times 10^{-8} \le f(\epsilon)-1 \le 0$ for $0 \le \epsilon \le 0.0294155$, which let the inequality

$$e^{u(1-\epsilon)}\left[1-\frac{u}{m}+\frac{3u^2}{2m^2}\right]^m \le e^{-m(\epsilon^2/4-\epsilon^3/6)},$$

to be satisfied in a very weak sense.

Exercise 5.31 Verify the inequality (5.62).

Solution: For $\ell = 0, 1, \ldots, j-1, \quad j = 1, 2, \ldots,$ we have

$$\begin{aligned}4j\binom{2j-2}{2\ell}\Big/2\binom{2j}{2\ell+1} &= \frac{4j(2j-2-2\ell+1)(2\ell+1)}{2\cdot 2j(2j-1)} \\ &= \frac{(2j-2\ell-1)(2\ell+1)}{2j-1} \\ &= \frac{mn}{m+n-1}\end{aligned}$$

where $m = 2j-2\ell-1, n = 2\ell+1$. Apparently, if $m+n-1 \equiv k$, then $(mn)_{\min} = k$ when $m = k, n = 1$ or $m = 1, n = k$. Hence,

$$4j\binom{2j-2}{2\ell} \ge 2\binom{2j}{2\ell+1}.$$

Exercise 5.32 Verify the inequality (5.63).

Solution: Let

$$f(k) = \frac{(2k)!}{k!(2m)^k}\Big/\frac{3^k}{(2k+1)m^k} = \frac{(2k)!}{k!(2k+1)6^k}.$$

$f(x) > 1$ when $k \ge 6$. Hence,

$$E_{N(0,\frac{1}{\sqrt{m}})}(t^{2k}) > E_{\rho_1}(t^{2k}),$$

if $k \ge 6$.

Solutions to Chapter 6

Exercise 6.1 Use the theorems above to show the following:

(i) If $x = \{ka^k\}$ then $X[z] = \frac{az}{(z-a)^2}$.

(ii) If $x = \{ka^{k-4}\}$ then $X[z] = \frac{a^{-3}z}{(z-a)^2}$.

Solution:

(i) $X[z] = \sum_{k=0}^{\infty} ka^k z^{-k} = -z\partial\big(\sum_{k=0}^{\infty} a^k z^{-k}\big)/\partial z = -z\partial\big(\frac{z}{z-a}\big)/\partial z = \frac{az}{(z-a)^2}$.

(ii) $X[z] = a^{-4}\sum_{k=0}^{\infty} ka^k z^{-k} = a^{-4}[\frac{az}{(z-a)^2}] = \frac{a^{-3}z}{(z-a)^2}$

Exercise 6.2 Verify Theorem 6.4.

Solution: (i)

$$X^{(-l)}[z] = \sum_{k=0}^{\infty} x_{k-l}z^{-k} = \sum_{k=l}^{\infty} x_{k-l}z^{-k} = z^{-l}\sum_{k=l}^{\infty} x_{k-l}z^{-(k-l)} = \frac{1}{z^l}X[z].$$

(ii)

$$X^{(l)}[z] = \sum_{k=0}^{\infty} x^{k+l}z^{-k} = z^l\left(\sum_{k=0}^{\infty} x^{k+l}z^{-(k+l)}\right) = z^l\left(\sum_{k=0}^{\infty} x^k z^{-k} - \sum_{j=0}^{l-1} x_j z^{-j}\right) = z^l\left(X[z] - \sum_{j=0}^{l-1} x_j z^{-j}\right)$$

Exercise 6.3 Let $x_k = a^k, k \geq 0, a \in \mathbb{R}$. Recalling that

$$Z[\{a^k\}] = \frac{z}{z-a}$$

show that

$$X^{(-2)}[z] = \frac{1}{z(z-a)}$$

and that

$$X^{(3)}[z] = \frac{z^3(a/z)^3}{1-a/z}$$

.

Solution:

$$X^{(-2)} = z^{-2}\frac{z}{z-a} = \frac{1}{z(z-a)}.$$

$$X^{(3)} = z^3\left(\sum_{k=3}^{\infty}\frac{a^k}{z^k}\right) = \frac{z^3(a/z)^3}{1-a/z}.$$

Exercise 6.4 Consider the space of $2N$-tuples $f' = (f'[0], \ldots, f'[2N-1])$. Let $\mu = e^{-2\pi i/2N}$, define the appropriate inner product $(f', g')_{2N}$ for this space and verify that the vectors $e^{(\ell)'} = (e^{(\ell)'}[0], \ldots, (e^{(\ell)'}[2N-1], \ell = 0, \ldots, 2N-1$, form an ON basis for the space, where $e^{(\ell)'}[n] = \mu^{-\ell n}$.

Solution: The inner product $(f', g')_{2N}$ is defined as

$$(f', g')_{2N} = \frac{1}{2N} \sum_{n=0}^{2N-1} f'[n]\bar{g}'[n].$$

Since $e^{(\ell)'}[n] = \mu^{-\ell n} = e^{2\ell n\pi i/2N}$, let $2N = M$ and according to Lemma 6.14 and Lemma 6.15, we conclude that $e^{(\ell)'} = (e^{(\ell)'}[0], \ldots, (e^{(\ell)'}[2N-1], \ell = 0, \ldots, 2N-1$, form an ON basis for the space.

Exercise 6.5 1. Show that a $2N$-tuple f' satisfies the relation $f'[n] = f'[2N-n-1]$ for all integers n if and only if $f' = Rf$, (6.10), for some N-tuple f.

2. Show that $f' = Rf$ for f an N-tuple if and only if f' is a linear combination of $2N$-tuples $E^{(k)}, k = 0, 1, \ldots, N-1$ where

$$E^{(k)}[n] = \frac{1}{2}\left(\mu^{-k/2}e^{(k)'}[n] + \mu^{k/2}e^{(-k)'}[n]\right) = \cos\left[\frac{\pi}{N}(n+1/2)k\right],$$

and $n = 0, 1, \ldots, 2N-1$. Recall that $e^{(-k)'}[n] = e^{(2N-k)'}[n]$.

Solution: 1. If a $2N$-tuple f' satisfies the relation $f'[n] = f'[2N-n-1]$ for all integers n, let $f, f[n] = f'[N+n], n = 0, \ldots, N-1$ to be an N-tuple, then we have $f' = Rf$. Conversely, if f is an N-tuple and $f' = Rf$, then we have $f'[n] = f'[2N-n-1]$ for $n = 0, \ldots, 2N-1$.

2. Since

$$\begin{aligned}
E^{(k)}[2N-n-1] &= \cos[\frac{\pi}{N}(2N-n-1+1/2)k] \\
&= \cos[2\pi - \frac{\pi}{N}(n+1/2)k] \\
&= \cos[\frac{\pi}{N}(n+1/2)k] \\
&= E^{(k)}[n],
\end{aligned}$$

then if f' is a linear combination of $E^{(k)}$, we have $f'[n] = f'[2N-n-1]$. Conversely, let $E^{(k)}_{1/2} = (E^{(k)}[0], \ldots, E^{(k)}[N-1])$ be the N-tuple consisted of the first N entries of the $2N$-tuple of $E^{(k)}$. Since $e^{(k)'}, k = 1, \ldots, 2N-1$ form an ON basis, then

$$
\begin{aligned}
< E^{(k)}_{1/2}, E^{(h)}_{1/2} >_N &=< E^{(k)}, E^{(h)} >_{2N} \\
&= \frac{1}{2N}\sum_{n=0}^{2N-1}\frac{1}{4}\Big(\mu^{-k/2}e^{(k)'}[n] + \mu^{k/2}e^{(2N-k)'}[n]\Big)\Big(\mu^{h/2}e^{(2N-h)'}[n] + \mu^{-h/2}e^{(h)'}[n]\Big) \\
&= \frac{1}{8N}\sum_{k'=\pm k, h'=\pm h}\sum_{n=0}^{2N-1}\mu^{-(k'+h')/2}e^{(k')'}[n]e^{(h')'}[n] \\
&= \frac{1}{4}\sum_{k'=\pm k, h'=\pm h}\mu^{-(k'+h')/2} < e^{(k')'}, e^{(-h')'} >_{2N}.
\end{aligned}
$$

For $k, h = 0, \ldots, N-1$, if $k \neq h$, $< E^{(k)}_{1/2}, E^{(h)}_{1/2} >_N = 0$ and if $k = h$, $< E^{(k)}_{1/2}, E^{(h)}_{1/2} >_N \neq 0$. Thus, $E^{(k)}_{1/2}, k = 0, \ldots, N-1$ form an orthogonal basis, and any f is a linear combination of $E^{(k)}_{1/2}$ and therefore any $f' = Rf$ is a linear combination of $E^{(k)}$.

Exercise 6.6 Verify the orthogonality relations

$$
< E^{(k)}, E^{(h)} >_{2N} = \begin{cases} \frac{1}{2} & \text{if } h = k \neq 0 \\ 1 & \text{if } h = k = 0 \\ 0 & \text{if } h \neq k \end{cases}
$$

Solution: According to the result of Exercise 6.5, if $k \neq h$, $< e^{(k')'}, e^{(h')'} >_{2N} = 0 \Rightarrow < E^{(k)}, E^{(h)} >_{2N} = 0$.If $k = h$, $< E^{(k)}, E^{(h)} >_{2N} = \frac{1}{4}\sum_{k'=\pm k, h'=\pm h}\mu^{-(k'+h')/2}$, so when $k = h = 0$, $< E^{(k)}, E^{(h)} >_{2N} = 1$; when $k = h \neq 0$, $\mu^{\pm(k-k)/2} < e^{(k)'}, e^{(k)'} >_{2N} = 1$ and $\mu^{\pm(k+k)/2} < e^{(k)'}, e^{(-k)'} >_{2N} = 0$, thus $< E^{(k)}, E^{(h)} >_{2N} = 1/2$.

Exercise 6.7 Using the results of the preceding exercises and the fact that $f[n] = f[N+n]$ for $0 \leq n \leq N-1$, establish the discrete cosine transform and its inverse:

$$
F[k] = \omega[k]\sum_{n=0}^{N-1} f[n]\cos\left[\frac{\pi}{N}(n+1/2)k\right], \quad k = 0, \ldots, N-1,
$$

$$
f[n] = \sum_{k=0}^{N-1}\omega[k]F[k]\cos\left[\frac{\pi}{N}(n+1/2)k\right], \quad n = 0, \ldots, N-1,
$$

where $\omega[0] = 1/\sqrt{N}$ and $\omega[k] = \sqrt{2/N}$ for $1 \leq k \leq N-1$.

Solution: We assume that the N-tuple $\{f[n]\}$ is given and the N-tuple $\{F[k]\}$ is defined by the discrete cosine transform. Since

$$
\cos\left[\frac{\pi}{N}(n+\frac{1}{2})k\right] = -\cos\left[\frac{\pi}{N}(n+\frac{1}{2})(2N-k)\right], \quad k = 0, \ldots N-1,
$$

and $\cos\left[\frac{\pi}{N}(n+\frac{1}{2})N\right]=0$, we can extend F to a $2N$-tuple by defining $F[2N-k]=-F[k]$, $\omega[2N-k]=\omega[k]$, for $k=0,\dots N$, (so $F[N]=0$ and we assume $\omega[N]\neq 0$). In analogy with Exercise 6.6, it is straightforward to derive the orthogonality relation

$$\sum_{k=1}^{2N-1}\cos\left[\frac{\pi}{N}(n+\frac{1}{2})k\right]\cos\left[\frac{\pi}{N}(m+\frac{1}{2})k\right]=\delta_{n,m}N.$$

Thus

$$\sum_{k=0}^{2N-1}\frac{F[k]}{\omega[k]}\cos\left[\frac{\pi}{N}(m+\frac{1}{2})k\right]=$$

$$\sum_{n=0}^{N-1}f[n]\sum_{k=0}^{2N-1}\cos\left[\frac{\pi}{N}(m+\frac{1}{2})k\right]\cos\left[\frac{\pi}{N}(n+\frac{1}{2})k\right]=N\,f[m],$$

so

$$f(m)=\sum_{k=0}^{2N-1}\frac{F[k]}{N\omega[k]}\cos\left[\frac{\pi}{N}(m+\frac{1}{2})k\right]=\frac{F(0)}{N\omega[0]}+2\sum_{k=1}^{N-1}\frac{F[k]}{N\omega[k]}$$

$$=\sum_{k=0}^{N-1}\omega[k]F[k]\cos\left[\frac{\pi}{N}(m+\frac{1}{2})k\right],\quad m=0,\dots,N-1,$$

the desired inverse discrete cosine transform.

Exercise 6.8 Find the Z transforms of the following sequences:

(i) $\{(1/4)^k\}_0^\infty$.

(ii) $\{(-7i)^k\}_0^\infty$.

(iii) $\{8(-7i)^k+4(1/4)^k\}_0^\infty$.

(iv) $\{5k\}_0^\infty$.

(v) $\{k^2 3^k\}_0^\infty$.

(vi) $\{2^k/k!\}_0^\infty$.

(vii) $\{x_j\}_0^\infty$ where

$$x_j=\begin{cases}(-1)^{(j-1)/2}, & j \text{ odd}\\ 0, & j \text{ even}\end{cases}$$

(viii) $\{\cos(k\theta)\}_0^\infty$

Solution: For $x=\{a^k\}_0^\infty$, $X[z]=z/(z-a)$, thus

(i) $X[z]=z/(z-1/4)$;

(ii) $X[z]=z/(z+7i)$;

(iii) $X[z]=8z/(z+7i)+4z/(z-1/4)$;

For $x=\{ka^k\}_0^\infty$, $X[z]=az/(z-a)^2$, thus

(iv) $X[z]=5z/(z-1)^2$;

For $x = \{k^2 a^k\}_0^\infty$, $X[z] = -z\partial[az/(z-a)^2]/\partial z = az(z+a)/(z-a)^3$, thus

(v) $X[z] = 3z(z+3)/(z-3)^3$;

(vi) $X[z] = e^{2/z}$;

(vii) $X[z] = z/(1+z^2)$;

(viii) $X[z] = z(z-\cos\theta)/(1+z^2-2z\cos\theta)$.

Exercise 6.9 Let $x = \{x_k\}_0^\infty, y, z$ be sequences. Show that

(i) $x * y = y * x$.

(ii) $(ax+by) * z = a(x*z) + b(y*z), a, b \in \mathbb{R}$.

Solution: (i) Let $w_1 = x * y$ and $w_2 = y * x$, then $W_1[z] = X[z]Y[z] = Y[z]X[z] = W_2[z]$, by the uniqueness of Z transform we have $x * y = y * x$.

(ii) Similar to (i), proved by using the linearity and uniqueness of Z transform.

Exercise 6.10 Let $x(t)$ be the signal defined on the integers t and such that

$$x(t) = \begin{cases} 1 & \text{if } t = 0, 1 \\ 0 & \text{otherwise.} \end{cases}$$

1. Compute the convolution $x_2(t) = x * x(t)$.

2. Compute the convolution $x_3(t) = x * (x * x)(t) = x * x_2(t)$.

3. Evaluate the convolution $x_n(t) = x * x_{n-1}(t)$ for $n = 2, 3, \ldots$. Hint: Use Pascal's triangle relation for binomial coefficients.

Solution: 1. $x_2(0) = 1, x_2(1) = 2, x_2(2) = 1$ and $x_2(t) = 0$ for $t \geq 3$.

2. $x_3(0) = 1, x_3(1) = 3, x_3(2) = 3, x_3(3) = 1$ and $x_2(t) = 0$ for $t \geq 4$.

3. $x_n(t) = C_t^n = \frac{n!}{t!(n-t)!}$ for $0 \leq t \leq n$, and $x_n(t) = 0$ for $t \geq n+1$.

Exercise 6.11 Find the inverse Z transforms of the the following:

(i) $\frac{z}{z-i}$.

(ii) $\frac{z}{3z+1}$.

(iii) $\frac{4z}{3z+1}$.

(iv) $\frac{1}{3z+1}$.

(v) $\frac{4z+7}{3z+1}$.

(vi) $\frac{1}{z(z-i)}$.

(vii) $\frac{3z^2+4z-5/z-z^{-2}}{z^3}$.

Solution:

(i) $Z^{-1}\left(\frac{z}{z-i}\right) = \{i^k\}_0^\infty$.

(ii) $Z^{-1}\left(\frac{z}{3z+1}\right) = \{(1/3)(-1/3)^k\}_0^\infty$.

(iii) $Z^{-1}\left(\frac{4z}{3z+1}\right) = \{(4/3)(-1/3)^k\}_0^\infty$.

(iv) $Z^{-1}\left(\frac{1}{3z+1}\right) = \{(-1)^{k+1}(1/3)^k u[k-1]\}0^\infty$.

(v) $Z^{-1}\left(\frac{4z+7}{3z+1}\right) = \{7(-1)^{k+1}(1/3)^k u[k-1]\}_0^\infty + \{(4/3)(-1/3)^k\}_0^\infty$.

(vi) $Z^{-1}\left(\frac{1}{z(z-i)}\right) = i\delta(k-1) - \{i^k\}_1^\infty = \{-i^k u[k-2]\}_0^\infty$.

(vii) $Z^{-1}\left(\frac{3z^2+4z-5/z-z^{-2}}{z^3}\right) = \{0, 3, 4, -5, -1, 0, 0, \dots\}$.

Exercise 6.12 Prove that if

$$X[z] = \sum_{k=0}^{\infty} x_k z^{-k}$$

is a power series in $1/z$ and ℓ is a positive integer, then the inverse Z transform of $z^{-\ell}X[z]$ is the sequence $\{x_{k-\ell}\}_0^\infty$.

Solution: Since

$$z^{-\ell}X[z] = \sum_{k=0}^{\infty} x_k z^{-k-\ell} = \sum_{j=0}^{\infty} x_{j-\ell} z^{-j},$$

then

$$Z^{-1}\big(z^{-\ell}X[z]\big) = \{x_{k-\ell}\, u[k-\ell]\}_0^\infty.$$

Exercise 6.13 Using partial fractions, find the inverse z transforms of the following functions:

(i) $\frac{z}{(z-1)(z-2)}$.

(ii) $\frac{z}{z^2-z+1}$.

(iii) $\frac{2z+1}{(z+1)(z-3)}$.

(iv) $\frac{z^2}{(2z+1)(z-1)}$.

(v) $\frac{z^2}{(z-1)^2(z^2-z+1)}$.

(vi) $\frac{2z^2-7z}{(z-1)^2(z-3)}$.

Solution:

(i) $\frac{z}{(z-1)(z-2)} = \frac{2}{z-2} - \frac{1}{z-1}$, thus $Z^{-1}\left(\frac{2}{z-2} - \frac{1}{z-1}\right) = \{(2^k-1)u[k-1]\}_0^\infty = \{2^k-1\}_0^\infty$.

(ii) $\frac{z}{z^2-z+1} = \frac{1}{\sin(\pi/3)}\frac{\sin(\pi/3)z}{z^2-2\cos(\pi/3)z+1}$, thus $Z^{-1}\left(\frac{1}{\sin(\pi/3)}\frac{\sin(\pi/3)z}{z^2-2\cos(\pi/3)z+1}\right) = \{\frac{2\sqrt{3}}{3}\sin(k\pi/3)\}_0^\infty$.

(iii) $\frac{2z+1}{(z+1)(z-3)} = \frac{7}{4(z-3)} + \frac{1}{4(z+1)}$, thus $Z^{-1}\left(\frac{7}{4(z-3)} + \frac{1}{4(z+1)}\right) = \{(\frac{7}{4}3^{k-1} + \frac{1}{4}(-1)^{k-1})u[k-1]\}_0^\infty$.

(iv) $\frac{z^2}{(2z+1)(z-1)} = \frac{z}{3(2z+1)} + \frac{z}{3(z-1)}$, thus $Z^{-1}\left(\frac{z}{3(2z+1)} + \frac{z}{3(z-1)}\right) = \{\frac{1}{6}(-2)^{-k} + \frac{1}{3}\}_0^\infty$.

(v) $\frac{z^2}{(z-1)^2(z^2-z+1)} = \frac{1}{(z-1)^2} + \frac{1}{z-1} - \frac{z}{z^2-z+1}$, thus $Z^{-1}\left(\frac{1}{(z-1)^2} + \frac{1}{z-1} - \frac{z}{z^2-z+1}\right) = \{(k-1)u[k-1] + u[k-1] - \frac{2\sqrt{3}}{3}\sin(k\pi/3)\}_0^\infty = \{k - \frac{2\sqrt{3}}{3}\sin(k\pi/3)\}_0^\infty$.

(vi) $\frac{2z^2-7z}{(z-1)^2(z-3)} = -\frac{3}{4(z-3)} + \frac{5}{2(z-1)^2} + \frac{11}{4(z-1)}$, thus $Z^{-1}\left(-\frac{3}{4(z-3)} + \frac{5}{2(z-1)^2} + \frac{11}{4(z-1)}\right) = \{-\frac{3}{4}3^{k-1}u[k-1] + \frac{5}{2}(k-1)u[k-1] + \frac{11}{4}u[k-1]\}_0^\infty = \{-3^k/4 + \frac{5}{2}k + \frac{1}{4}\}_0^\infty$.

Exercise 6.14 Solve the following difference equations, using Z transform methods:

(i) $8y_{k+2} - 6y_{k+1} + y_k = 9, y_0 = 1, y_1 = 3/2$.

(ii) $y_{k+2} - 2y_{k+1} + y_k = 0, y_0, y_1 = 1$.

(iii) $y_{k+2} - 5y_{k+1} - 6y_k = (1/2)^k, y_0 = 1, y_1 = 2$.

(iv) $2y_{k+2} - 3y_{k+1} - 2y_k = 6k + 1, y_0 = 1, y_1 = 2$.

(v) $y_{k+2} - 4y_k = 3k - 5, y_0 = 1, y_1 = 0$.

Solution:

(i) Since

$$8z^2\left(Y[z] - 1 - \frac{3}{2}z^{-1}\right) - 6z(Y[z]-1) + Y[z] = \frac{9z}{(z-1)},$$

we have

$$Y[z] = \frac{3z}{z-1} - \frac{8z}{2z-1} + \frac{8z}{4z-1},$$

and thus

$$y_k = 3 + 2^{1-2k} - 2^{2-k}, k \geq 0.$$

(ii) Since

$$z^2\left(Y[z] - 1 - z^{-1}\right) - 2z(Y[z]-1) + Y[z] = 0,$$

we have

$$Y[z] = \frac{z}{z-1},$$

and thus

$$y_k = 1, k \geq 0.$$

(iii) Since

$$z^2Y[z] - 5zY[z] - 6Y[z] = \frac{2z}{(2z-1)},$$

we have

$$Y[z] = \frac{12}{77(z-6)} - \frac{2}{21(z+1)} + \frac{4}{33(2z-1)},$$

and thus

$$y_k = 2\left(\frac{(-1)^k}{21} - \frac{2^{1-k}}{33} + \frac{6^k}{77}\right), k \geq 0.$$

(iv) Since

$$2z^2\left(Y[z] - 1 - 2z^{-1}\right) - 3z(Y[z]-1) - 2Y[z] = \frac{6z}{(z-1)^2} + \frac{z}{z-1},$$

we have

$$Y[z] = \frac{12z}{5(z-2)} - \frac{2z}{(z-1)^2} - \frac{z}{z-1} - \frac{4z}{5(2z+1)},$$

and thus

$$y_k = -1 - 2k - \frac{2(-2)^{-k}}{5} + \frac{3(2)^{2+k}}{5}, k \geq 0.$$

(v) Since

$$z^2Y[z] - 4Y[z] = \frac{3z}{(z-1)^2} - \frac{5z}{z-1},$$

we have

$$Y[z] = -\frac{z}{2(z-2)} - \frac{z}{(z-1)^2} + \frac{z}{z-1} - \frac{z}{2(z+2)},$$

and thus

$$y_k = 1 - k + (-2)^{k-1} - 2^{k-1}, k \geq 0.$$

Exercise 6.15 (i) Let a, b, c, d, e be real numbers. Consider the difference equation:

$$ay_{k+3} + by_{k+2} + cy_{k+1} + dy_k = e.$$

(ii) Let $Y[z]$ be as usual. Show that

$$Y[z](az^3 + bz^2 + cz + d) = z^3(ay_0) + z^2(ay_1 + by_0) + z(ay_2 + by_1 + cy_0) + \frac{ez}{z-1}.$$

Deduce that when $y_0 = y_1 = 0$,

$$Y[z] = \frac{z(ay_2) + \frac{ez}{z-1}}{az^3 + bz^2 + cz + d}.$$

(iii) Use (i) - (ii) to solve the difference equation

$$y_{k+3} + 9y_{k+2} + 26y_{k+1} + 24y_k = 60$$

subject to $y_0 = y_1 = 0, y_2 = 60$.

Solution: Take Z transform to the difference equation of (i),

$$az^3(Y[z] - y_0 - z^{-1}y_1 - z^{-2}y_2) + bz^2(Y[z] - y_0 - z^{-1}y_1) + cz(Y[z] - y_0) + dY[z] = \frac{ez}{z-1}.$$

After some rearrangement, we have

$$Y[z](az^3 + bz^2 + cz + d) = z^3(ay_0) + z^2(ay_1 + by_0) + z(ay_2 + by_1 + cy_0) + \frac{ez}{z-1}.$$

By letting $y_0 = y_1 = 0$, we get

$$Y[z] = \frac{z(ay_2) + \frac{ez}{z-1}}{az^3 + bz^2 + cz + d}.$$

For $a = 1, b = 9, c = 26, d = 24, e = 60, y_2 = 60$, we have

$$Y[z] = \frac{60z + \frac{60z}{z-1}}{z^3 + 9z^2 + 26z + 24} = \frac{1}{z-1} - \frac{40}{z+2} + \frac{135}{z+3} - \frac{96}{z+4}.$$

Therefore, by taking the inverse Z transform, we get

$$y_k = 1 + 5(-2)^{2+k} + 3(-1)^k 2^{3+2k} - 5(-3)^{2+k}, k \geq 0.$$

Exercise 6.16 Let $N = 6$ and $f[n] = n, n = 0, \ldots, 5$. Compute the DFT $F[n]$ explicitly and verify the recovery of f from its transform

Solution:

$$F[k] = \sum_{n=0}^{N} f[n] e^{-i2\pi kn/N},$$

thus, $F[0] = 15, F[1] = -3 + 5.19615i, F[2] = -3 + 1.73205i, F[3] = -3, F[4] = -3 - 1.73205i, F[5] = -3 - 5.19615i$. By using

$$f[n] = \frac{1}{N} \sum_{k=0}^{N} F[k] e^{i2\pi kn/N},$$

we can recover $f[n] = n$.

Exercise 6.17 (i) Let $N = 4$ and $f[n] = a_n, n = 0, \ldots, 3$. Compute the DFT $F[n]$ explicitly and verify the recovery of f from its transform.

(ii) Let $N = 4$ and $f[n] = a_n, n = 0, \ldots, 3$. Compute the DCT $F[n]$ explicitly and verify the recovery of f from its transform.

Solution: (i) $F[0] = a_0 + a_1 + a_2 + a_3, F[1] = a_0 - ia_1 - a_2 + ia_3, F[2] = a_0 - a_1 + a_2 - a_3, F[3] = a_0 + ia_1 - a_2 - ia_3$. It is easy to check the recovery process.

(ii) the DCT and inverse DCT are

$$F[k] = \omega[k] \sum_{n=0}^{N-1} f[n] \cos\left[\frac{\pi}{N}(n + 1/2)k\right], \quad k = 0, \ldots, N-1,$$

$$f[n] = \sum_{k=0}^{N-1} \omega[k] F[k] \cos\left[\frac{\pi}{N}(n + 1/2)k\right], \quad n = 0, \ldots, N-1,$$

where $\omega[0] = 1/\sqrt{N}$ and $\omega[k] = \sqrt{2/N}$ for $1 \leq k \leq N-1$.

Thus, $F[0] = \frac{1}{2}(a_0 + a_1 + a_2 + a_3), F[1] = \frac{1}{\sqrt{2}}(\cos(\pi/8)(a_0 - a_3) + \sin(\pi/8)(a_1 - a_2)), F[2] = \frac{1}{2}(a_0 - a_1 - a_2 + a_3), F[3] = \frac{1}{\sqrt{2}}(\cos(\pi/8)(a_2 - a_1) + \sin(\pi/8)(a_0 - a_3))$.

We can check the inverse process directly.

Exercise 6.18 This problem investigates the use of the FFT in denoising. Let

$$f = \exp\bigl(-t^2/10\bigr)(\sin(t) + 2\cos(5t) + 5t^2) \quad 0 < t \le 2\pi.$$

Discretize f by setting $f_k = f(2k\pi/256), k = 1, \dots, 256$. Suppose the signal has been corrupted by noise, modeled by the addition of the vector

```
x=2*rand(1,2^8)-0.5
```

in MATLAB, i.e., by random addition of a number between -2 and 2 at each value of k, so that the corrupted signal is $g = f + x$. Use MATLAB's fft command to compute $\hat{g}_k$ for $0 \le k \le 255$. (Note that $g_{n-k} = \bar{g}_k$. Thus the low-frequency coeœcients are $\hat{g}_0, \dots, \hat{g}_m$ and $\hat{g}_{256-m}, \dots, \hat{g}_{256}$ for some small integer m). Filter out the high-frequency terms by setting $\hat{g}_k = 0$ for $m \le k \le 255-m$ with $m = 10$. Then apply the inverse FFT to these filtered $\hat{g}_k$ to compute the filtered g_k. Plot the results and compare with the original unfiltered signal. Experiment with several different values of m.

The following sample MATLAB program is an implementation of the denoising procedure in Exercise 6.18.

```
t=linspace(0,2*pi,2^8);
x=2*(rand(1,2^8)-0.5);
f=exp(-t.^2/10).*(sin(t)+2*cos(5*t)+5*t.^2);
g=f+x;
m=10; % Filter parameter.
hatg=fft(g);
%If $hatg$ is FFT of $g$, can filter out high frequency
%components from $hatg$ with command such as

denoisehatg=[hatg(1:m) zeros(1,2^8-2*m) hatg(2^8-m+1:2^8)];

figure
subplot(2,2,1)
plot(t,f) % The signal.
title ('The signal')
axis([0 2*pi 0 20])
xlabel('t')
ylabel('f')

subplot(2,2,2)
plot(t,x) % The noise.
title ('The noise')
axis([0 2*pi -2 2])
xlabel('t')
ylabel('x')

subplot(2,2,3)
```

```
plot(t,g) % Signal plus noise.
title ('Signal plus noise')
axis([0 2*pi 0 20])
xlabel('t')
ylabel('g')

subplot(2,2,4)
plot(t,ifft(denoisehatg)) % Denoised signal.
title ('Denoised signal')
axis([0 2*pi 0 20])
xlabel('t')
ylabel('g')
```

Solution:

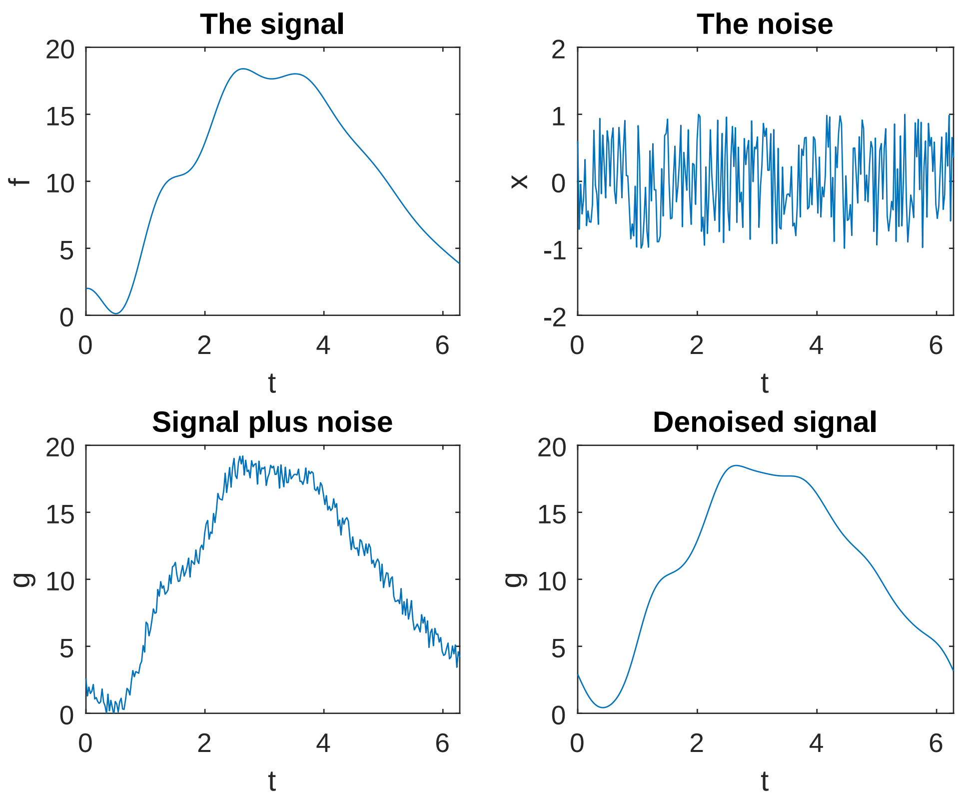


Figure 1: Exercise 6.18

Exercise 6.19 Consider the signal

$$f(t) = \exp\bigl(-t^2/10\bigr)\bigl(\sin(t) + 2\cos(2t) + \sin(5t) + \cos(8t) + 2t^2\bigr) \quad 0 < t \leq 2\pi.$$

Discretize f by setting $f_k = f(2k\pi/256), k = 1, \ldots, 256$. Suppose the signal has been corrupted by noise, modeled by the addition of the vector

```
x=2*rand(1,2^8)-0.5
```

Discretize f by setting $f_k = f(2k\pi/256), k = 1, \ldots, 256$, as in Exercise 6.18. Thus the information in f is given by prescribing 256 ordered numbers, one for each value of k. The problem here is to compress the information in the signal f by expressing it in terms of only $256c$ ordered numbers, where $0 < c < 1$ is the compression ratio. This will involve throwing away some information about the original signal, but we want to use the compressed data to reconstitute a facsimile of f that retains as much information about f as possible. The strategy is to compute the FFT $\hat{f}$ of f and compress the transformed signal by zeroing out the fraction $1 - c$ of the terms $\hat{f}_k$ with smallest absolute value. Apply the inverse FFT to the compressed transformed signal to get a compressed signal f_{c_k}. Plot the results and compare with the original uncompressed signal. Experiment with several different values of c. A measure of the information lost from the original signal is L2error= $\|f - f_c\|_2 = \|f\|_2$, so one would like to have c as small as possible while at the same time keeping the $L2$ error as small as possible. Experiment with c and with other signals, and also use your inspection of the graphs of the original and compressed signals to judge the effectiveness of the compression.

The following sample MATLAB program is an implementation of the compression procedure in Exercise 6.19.

```
% Input: time vector t, signal vector f, compression rate c,
%(between 0 and 1)
t=linspace(0,2*pi,2^8);
f=exp(-t.^2/10).*(sin(t)+2*cos(2*t)+sin(5*t)+cos(8*t)+2*t.^2);
c=.5;
hatf=fft(f);
% Input vector f and ratio c: 0<= c <=1.
% Output is vector fc in which smallest
% 100c% of terms f_k, in absolute value, are set
% equal to zero.
N=length(f); Nc=floor(N*c);
ff=sort(abs(hatf));
cutoff=abs(ff(Nc+1));

hatfc=(abs(hatf)>=cutoff).*hatf;
fc=ifft(hatfc);
L2error=norm(f-fc,2)/norm(f); %Relative L2 information loss

subplot(1,2,1)
plot(t,f) %Graph of f
title('Original')
axis([0 2*pi 0 9])
```

```
xlabel('t')
ylabel('f')


subplot(1,2,2)
plot(t, fc,'r') % Graph of compression of f
title(['Compressed, c=0.5, err=',num2str(L2error)])
axis([0 2*pi 0 9])
xlabel('t')
ylabel('f_c')
```

Solution:

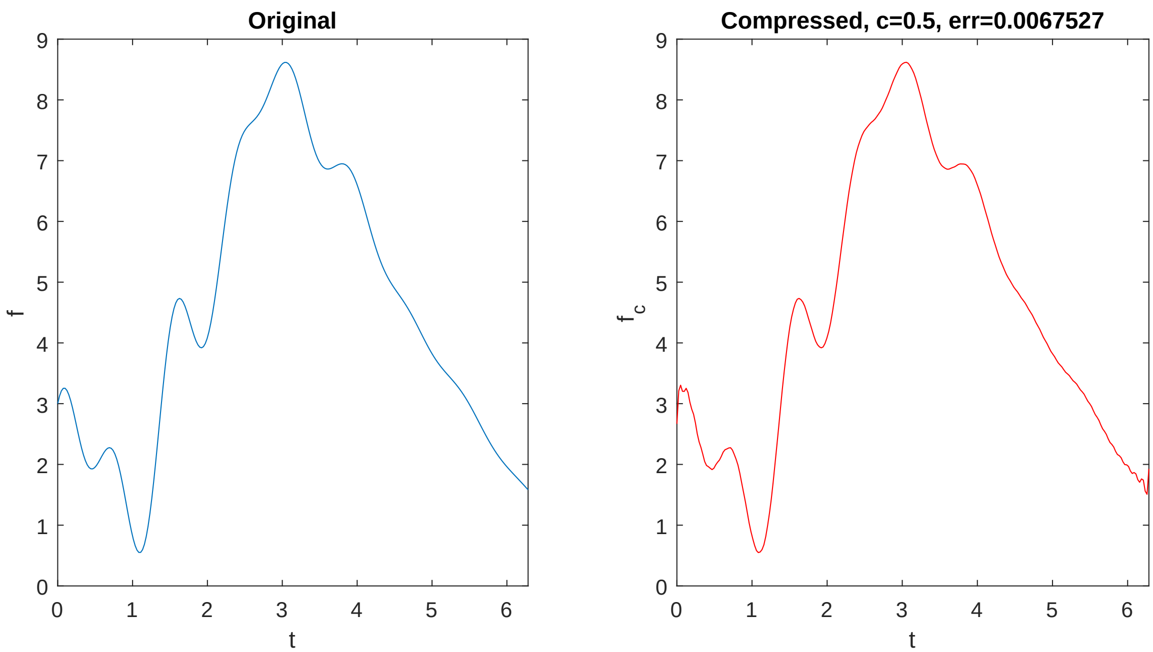


Figure 2: Exercise 6.19

Solutions to Chapter 7

Exercise 7.1 Verify that each of these four examples $y^{(j)}(n)$ defines a time-independent, causal, FIR filter, compute the associated impulse response vector $h^{(j)}(k)$ and determine the number of taps.

Solution: $y^{(1)}(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-i)$, where $h(k) = 0$ except for $h(1) = h(2) = 1/2$. $h(k)$ is independent of n, thus it is time-independent. $h(k) = 0$ for $k < 0$, thus it is causal. $h(k) \neq 0$ for only finite $k = 0, 1$, thus it is FIR. The number of taps is 2.

$y^{(2)}(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-i)$, where $h(k) = 0$ except for $h(0) = 1/2, h(1) = -1/2$. $h(k)$ is independent of n, thus it is time-independent. $h(k) = 0$ for $k < 0$, thus it is causal. $h(k) \neq 0$ for only finite $k = 0, 1$, thus it is FIR. The number of taps is 2.

$y^{(3)}(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-i)$, where $h(k) = 0$ except for $h(i) = 1/150, i = 0, \ldots, 149$. $h(k)$ is independent of n, thus it is time-independent. $h(k) = 0$ for $k < 0$, thus it is causal. $h(k) \neq 0$ for only finite $k = 0, \ldots, 149$, thus it is FIR. The number of taps is 150.

$y^{(4)}(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-i)$, where $h(k) = 0$ except for $h(0) = 149/150, h(i) = -1/150, i = 1, \ldots, 149$. $h(k)$ is independent of n, thus it is time-independent. $h(k) = 0$ for $k < 0$, thus it is causal. $h(k) \neq 0$ for only finite $k = 0, \ldots, 149$, thus it is FIR. The number of taps is 150.

Exercise 7.2 Suppose the only nonzero components of the input vector x and the impulse response vector h are $x_0 = 1, x_1 = 2$ and $h_0 = 1/4, h_1 = 1/2, h_2 = 1/4$. Compute the outputs $y_n = (x * h)_n$. Verify in the frequency domain that $Y(\omega) = H(\omega)X(\omega)$.

Solution: $y_0 = h_0x_0 = 1/4, y_1 = x_0h_1 + x_1h_0 = 1, y_2 = x_0h_2 + x_1h_1 = 5/4, y_3 = x_1h_2 = 1/2$.

$Y(\omega) = 1/4 + e^{-i\omega} + 5e^{-2i\omega}/4 + e^{-3i\omega}/2$.

$H(\omega) = 1/4 + e^{-i\omega}/2 + e^{-2i\omega}/4$.

$X(\omega) = 1 + 2e^{-i\omega}$

It is apparent to see that $Y(\omega) = H(\omega)X(\omega)$.

Exercise 7.3 Iterate the averaging filter $\mathbf{H}$ of Exercise 7.2 four times to get $\mathbf{K} = \mathbf{H}^4$. What is $K(\omega)$ and what is the impulse response k_n?

Solution: $K(\omega) = H^4(\omega) = 1/256 + e^{-i\omega}/32 + 7e^{-2i\omega}/64 + 7e^{-3i\omega}/32 + 35e^{-4i\omega}/128 + 7e^{-5i\omega}/32 + 7e^{-6i\omega}/64 + e^{-7i\omega}/32 + e^{-8i\omega}/256$.

Thus, $k_0 = 1/256, k_1 = 1/32, k_2 = 7/64, k_3 = 7/32, k_4 = 35/128, k_5 = 7/32, k_6 = 7/64, k_7 = 1/32, k_8 = 1/256$.

Exercise 7.4 Consider a filter with finite impulse response vector h. The problem is to verify the convolution rule $Y = HX$ in the special case that $h_1 = 1$ and all other $h_n = 0$. Thus

$$h = (\dots, 0, 0, 0, 1, 0, \dots).$$

It will be accomplished in the following steps:

1. What is $y = h * x = h * (\dots, x_{-1}, x_0, x_1, \dots)$?
2. What is $H(\omega)$?
3. Verify that $\sum y_n e^{-in\omega}$ agrees with $H(\omega)X(\omega)$.

Solution: 1. $y_k = x_{k-1}$.

2. $H(\omega) = e^{-i\omega}$.

3. $Y(\omega) = \sum_k x_k e^{-i(k+1)\omega}$, and $X(\omega) = \sum_k x_k e^{-ik\omega}$. Thus, it is easy to see that $Y(\omega) = H(\omega)X(\omega)$.

Exercise 7.5
- Write down the infinite matrix $(\downarrow 3)$ that executes the downsampling: $(\downarrow 3)x_n = x_{3n}$.
- Write down the infinite matrix $(\uparrow 3)$ that executes the upsampling:

$$(\uparrow 3)y_n = \begin{cases} y_{\frac{n}{3}}, & \text{if 3 divides } n \\ 0, & \text{otherwise} \end{cases}.$$

- Multiply the matrices $(\uparrow 3)(\downarrow 3)$ and $(\downarrow 3)(\uparrow 3)$. Describe the output for each of these product operations.

Solution:

$$(\downarrow 3) = \begin{bmatrix} \cdots & & & & & & & & \cdots \\ \cdots & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\ \cdots & & & & & & & & \cdots \end{bmatrix}.$$

$$(\uparrow 3) = \begin{bmatrix} \cdots & & & & \cdots \\ \cdots & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 1 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 1 & \cdots \\ \cdots & & & & \cdots \end{bmatrix}.$$

$$
(\uparrow 3)(\downarrow 3) = \begin{bmatrix}
\cdots & & & & & & & & \cdots \\
\cdots & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\
\cdots & & & & & & & & \cdots
\end{bmatrix}.
$$

$$
(\downarrow 3)(\uparrow 3) = I.
$$

Exercise 7.6 Show that if $x \in \ell_1$ then $x \in \ell_2$.

Solution: If $x \in \ell_1$, then

$$
\sum_{k=-\infty}^{\infty} |x_k| < \infty.
$$

Thus, we have

$$
\sum_{k=-\infty}^{\infty} |x_k|^2 \le \left(\sum_{k=-\infty}^{\infty} |x_k| \right)^2 < \infty,
$$

which indicates $x \in \ell_2$.

Exercise 7.7 Verify the property (7.4) for the Z transform of the composition of two FIR filters.

Solution: Since the Z transform of $y = \mathbf{\Phi} x = h * y$ is $Y[z] = H[z]X[z]$, and the Z transform of $w = \mathbf{\Psi} y = k * y$ is $W[z] = K[z]Y[z]$, thus $W[z] = K[z]H[z]X[z]$, which is the Z transform of $w = \mathbf{\Psi}(\mathbf{\Phi} x) = k * (h * x)$.

Exercise 7.8 For each of the operators $\mathbf{R}, \mathbf{L}, (\downarrow 2), (\uparrow 2), \mathbf{A}, \mathbf{F}, \mathbf{AIF}$ determine which of the properties of time invariance, causality and finite impulse response are satisfied.

Solution: **R**: time invariance, causality, finite impulse response.

L: time invariance, non-causality, finite impulse response.

$(\downarrow 2)$: time invariance, non-causality, infinite impulse response.

$(\uparrow 2)$: time invariance, non-causality, infinite impulse response.

AIF: time-dependent, non-causality, infinite impulse response.

Exercise 7.9 A direct approach to the convolution rule $Y = HX$. What is the coefficient of z^{-n} in $\left(\sum h_k z^{-k}\right)\left(\sum x_\ell z^{-\ell}\right)$? Show that your answer agree with $\sum h_k x_{n-k}$.

Solution: The coefficient of z^{-n} in $\left(\sum h_k z^{-k}\right)\left(\sum x_\ell z^{-\ell}\right)$ is

$$
a = \sum_{k+\ell=n} h_k x_\ell = \sum_k h_k x_{n-k}.
$$

Exercise 7.10 Let **H** be a causal filter with six taps, i.e., only coefficients $h_0, h_1, \ldots, h_5$ can be nonzero. We say that such a filter is **anti-symmetric** if the reflection of h about its midpoint (5/2 in this case) takes h to its negative: $h_k = -h_{5-k}$. Can such an antisymmetric filter ever be low pass? Either give an example or show that no example exists.

Solution: No, it cannot be a low pass filter. Since $h_k = -h_{5-k}$, we have $H(0) = \sum_{k=0}^{5} h_k = 0$.

Solutions to Chapter 8

Exercise 8.1 Show directly that $\mathbf{PP}* = \mathbf{I}$ on V'.

Solution: Let $\phi \in V'$. Using (8.6), (8.7), (8.8) we have

$$\begin{aligned}\mathbf{PP}^*\phi = \mathbf{P}\Big(\int_0^1 \phi(x_1,y)dy\Big) &= \sum_{n=-\infty}^{e^{2\pi i x_2}} \int_0^1 \phi(n+x_1,y)dy \\ &= \sum_{n=-\infty}^{\infty} e^{2\pi i n x_2} \int_0^1 e^{-2\pi i n y}\phi(x_1,y)dy = \phi(x_1,x_2),\end{aligned}$$

so $\mathbf{PP}^* = \mathbf{I}$ on V'.

Exercise 8.2 Using the results of Exercise 4.15, show that the Shannon-Whittaker sampling procedure can be interpreted to define a tight frame. What is the frame bound?

Solution: Let the inner product be $(f,g) = \int_{-\infty}^{\infty} f(t)\bar{g(t)}dt$ and $f^{(j)}(t) = \delta(t - \frac{\pi j}{B})$, a Dirac delta function. From Exercise 4.15 we have

$$\|f\|^2 = \frac{\pi}{B}\sum_j |(f, f^{(j)})|^2,$$

so $\{f^{(j)}\}$ is a tight frame with frame bound B/π.

Exercise 8.3 Using the definition (8.22) of $\mathbf{T}^*$ show that if $\|\mathbf{T}\| \le K$ for some constant K then $\|\mathbf{T}^*\| \le K$. Hint: Choose $f = \mathbf{T}^*\xi$.

Solution: Let $f = \mathbf{T}^*\xi$, then

$$\begin{aligned}\|\mathbf{T}^*\xi\|^2 &= (\mathbf{T}^*\xi, \mathbf{T}^*\xi) \\ &=< \xi, \mathbf{TT}^*\xi > \\ &\le \|\xi\| \cdot \|\mathbf{TT}^*\xi\| \\ &\le \|\xi\|^2 \cdot \|\mathbf{TT}^*\|,\end{aligned}$$

thus

$$\frac{\|\mathbf{T}^*\xi\|^2}{\|\xi\|^2} \le \|\mathbf{TT}^*\| \le \|\mathbf{T}\| \cdot \|\mathbf{T}^*\|,$$

which means

$$\|\mathbf{T}^*\|^2 \le \|\mathbf{T}\| \cdot \|\mathbf{T}^*\|.$$

Therefore, we have

$$\|\mathbf{T}^*\| \le \|\mathbf{T}\| \le K.$$

Exercise 8.4 Verify that the requirement $\int_{\infty}^{\infty} w(t)dt = 0$ ensure that C is finite, where $C = \int |\hat{w}(\lambda)|^2 \frac{d\lambda}{|\lambda|}$ in the Plancherel formula for continuous wavelets.

Solution: If C is finite, we must have

$$\int_{-\epsilon}^{0} \frac{|\hat{w}(\lambda)|^2}{|\lambda|} d\lambda < \infty \text{ and } \int_{0}^{\epsilon} \frac{|\hat{w}(\lambda)|^2}{|\lambda|} d\lambda < \infty,$$

where ϵ is an arbitrary positive real number and $\epsilon \to 0^+$. If $\hat{w}(0) \neq 0$, we can always find positive real number a and b such that

$$\int_{-\epsilon}^{0} \frac{|\hat{w}(\lambda)|^2}{|\lambda|} d\lambda > a \int_{-\epsilon}^{0} \frac{1}{|\lambda|} d\lambda \to \infty \text{ and } \int_{0}^{\epsilon} \frac{|\hat{w}(\lambda)|^2}{|\lambda|} d\lambda > b \int_{0}^{\epsilon} \frac{1}{|\lambda|} d\lambda \to \infty,$$

Thus, $\hat{w}(0) = \int_{\infty}^{\infty} w(t) e^{-0t} dt = \int_{\infty}^{\infty} w(t)dt$ must be zero to ensure C to be finite. Here, we only prove that $\int_{\infty}^{\infty} w(t)dt = 0$ is the necessary condition for C being finite.

Exercise 8.5 Consider the Haar mother wavelet

$$w(t) = \begin{cases} -1, & 0 \le t < \frac{1}{2} \\ 1, & \frac{1}{2} \le t < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Compute the explicit expression for the wavelet transform. Find the explicit integral expression for C. Can you determine its exact value?

Solution: The continuous wavelet transform is

$$\begin{aligned} F_w(a,b) &= |a|^{-1/2} \left(\int_{b}^{b+|a|/2} -f(t)dt + \int_{b+|a|/2}^{b+|a|} f(t)dt \right) \\ &= |a|^{1/2} \left(\int_{0}^{1/2} -f(au+b)du + \int_{1/2}^{1} f(au+b)du \right). \end{aligned}$$

Since

$$\begin{aligned} \hat{w}(\lambda) &= \int_{-\infty}^{\infty} w(t) e^{-i\lambda t} dt = \int_{0}^{1/2} -e^{-i\lambda t} dt + \int_{1/2}^{1} e^{-i\lambda t} dt \\ &= \frac{i - 2ie^{-i\lambda/2} + ie^{-i\lambda}}{\lambda} \\ &= -4 \frac{\sin^2(\lambda/4)(\sin(\lambda/2) + i\cos(\lambda/2))}{\lambda}, \end{aligned}$$

we have

$$|\hat{w}(\lambda)|^2 = \hat{w}(\lambda) * \overline{\hat{w}(\lambda)} = \frac{16\sin^4(\lambda/4)}{\lambda^2}.$$

Therefore

$$C = \int_{\infty}^{\infty} \frac{|\hat{w}(\lambda)|^2}{|\lambda|} d\lambda = 2\int_{0}^{\infty} \frac{16\sin^4(\lambda/4)}{\lambda^3} d\lambda = 2\ln 2$$

Exercise 8.6 Show that the derivatives of the Gaussian distribution

$$w^{(n)}(t) = K_n \frac{d^n}{dt^n} G(t), \quad G(t) = \frac{1}{\sqrt{2\pi}\sigma} \exp\bigl(-t^2/2\sigma^2\bigr), \quad n = 1, 2, \ldots$$

each satisfy the conditions for reconstruction with a single wavelet in the case $a > 0$, provided the constant K_n is chosen so that $\|w^{(n)}\| = 1$. In common use is the case $n = 2$, the Mexican hat wavelet. Graph this wavelet to explain the name.

Solution: Since

$$\hat{w}^{(n)}(\lambda) = K_n (i\lambda)^n \exp\bigl(-\sigma^2 \lambda^2\bigr),$$

we have $\hat{w}^{(n)}(0) \equiv 0$ to satisfy the conditions for reconstruction with a single wavelet. For $n = 2$,

$$w^{(2)}(t) = \frac{K_2}{\sqrt{2\pi\sigma^5}} (t^2 - \sigma^2) \exp\bigl(-t^2/2\sigma^2\bigr).$$

If $\|w^{(2)}\| = 1$, we should have $K_2 = -(2/3)\sqrt{6}\pi^{1/4}\sigma^{5/2}$. Let $\sigma = 1$, then we have

$$w^{(2)}(t) = \frac{2\sqrt{3}}{3} \pi^{-1/4} (1 - t^2) \exp\bigl(-t^2/2\bigr).$$

The plot of $w^{(2)}(t)$ is shown in Figure 1, whose shape is similar to a Mexican hat.

Exercise 8.7 Show that the continuous wavelets $w^{(n)}$ are insensitive to signals t^k of order $k < n$. Of course, polynomials are not square integrable on the real line. However, it follows from this that $w^{(n)}$ will filter out most of the polynomial portion (of order $< n$) of a signal restricted to a finite interval.

Solution: Let's prove it by induction. Obviously, $\int_{-\infty}^{\infty} \bar{w}^{(n)}(t) dt = \int_{-\infty}^{\infty} w^{(n)}(t) dt = 0$. Assume that for certain $n \geq 1$, we have

$$F_{n,k}(a, b) = |a|^{1/2} \int_{-\infty}^{\infty} (at + b)^k \bar{w}^{(n)}(t) dt = 0, \quad \text{for } 0 \leq k < n.$$

Thus,

$$\begin{aligned}
F_{n+1,k}(a, b) &= |a|^{1/2} \int_{-\infty}^{\infty} (at + b)^k \bar{w}^{(n+1)}(t) dt \qquad (1 \leq k < n + 1) \\
&= |a|^{1/2} \int_{-\infty}^{\infty} (at + b)^k d\bigl(\bar{w}^{(n)}(t)\bigr) \\
&= |a|^{1/2} \left((at + b)^k \bar{w}^{(n)}(t) \Big|_{-\infty}^{\infty} - ak \int_{-\infty}^{\infty} (at + b)^{k-1} \bar{w}^{(n)}(t) dt \right).
\end{aligned}$$

Since $\bar{w}^{(n)}(t)$ has the form of $P_n(t)e^{-t^2/2\sigma^2}$, where $P(t)$ is a n-th polynomial respected to t, we always have $(at + b)^k \bar{w}^{(n)}(t)\Big|_{-\infty}^{\infty} = 0$ for the first part. Furthermore, $\int_{-\infty}^{\infty} (at + b)^{k-1} \bar{w}^{(n)}(t) dt = 0$ is obtained by the assumption. Hence, we prove that $F_{n+1,k}(a, b) = 0$ for $0 \leq k < n + 1$.

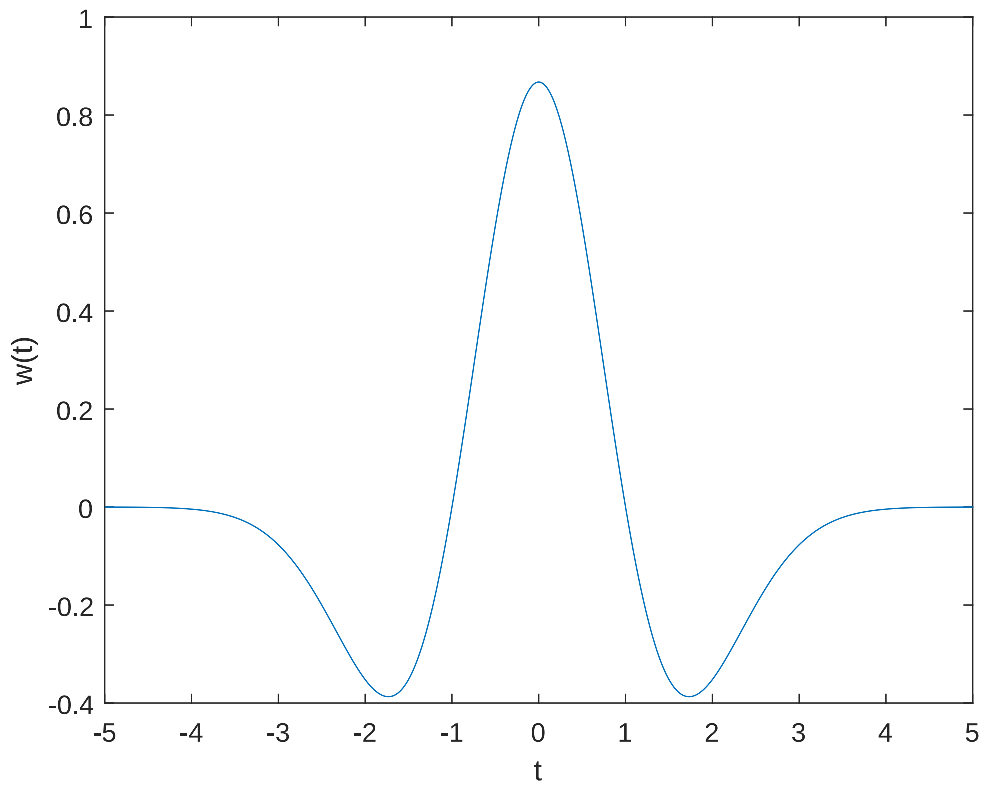


Figure 1: Exercise 8.6

Exercise 8.8 For students with access to the MATLAB wavelet toolbox. The MATLAB function cwt computes $F(a,b)$, (called $C_{a,b}$ by MATLAB) for a specified discrete set of values of a and for b ranging the length of the signal vector. Consider the signal $f(t) = e^{-t^2/10}(\sin t + 2\cos 2t + \sin(5t + \cos 8t + 2t^2 2), 0 \le t < 2\pi$. Discretize the signal into a 512 component vector and use the Mexican hat wavelet $gau2$ to compute the continuous wavelet transform $C_{a,b}$ for a taking integer values from 1 to 32 A simple code that works is

```
t=linspace(0,2*pi,2^8);
f=exp(-t.^2/10).*(sin(t)+2*cos(2*t)+sin(5*t)+cos(8*t)+2*t.^2);
c=cwt(f,1:32,'gau2')
```

For color graphical output an appropriate command is

```
cc=cwt(f,1:32,'gau2','plot')
```

This produces four colored graphs whose colors represent the magnitudes of the real and the imaginary parts of $C_{a,b}$, the absolute value of $C_{a,b}$ and the phase angle of $C_{a,b}$, respectively, for the matrix of values (a,b). Repeat the analysis for the Haar wavelet $'db1'$ or $'haar'$ and for $'cgau3'$. Taking Exercise 8.7 into account, what is your interpretation of the information about the signal obtained from these graphs?

Solution: The above Matlab command can only work with the wavelet $'cgau3'$ in Matlab of newer versions. The output is shown in Figure 2. Those graphs indicate that the signal has weak oscillation in high frequency regime ($a \to 1.0$), but noticeable oscillation in intermediate and low frequency regimes ($a \to 32.0$).

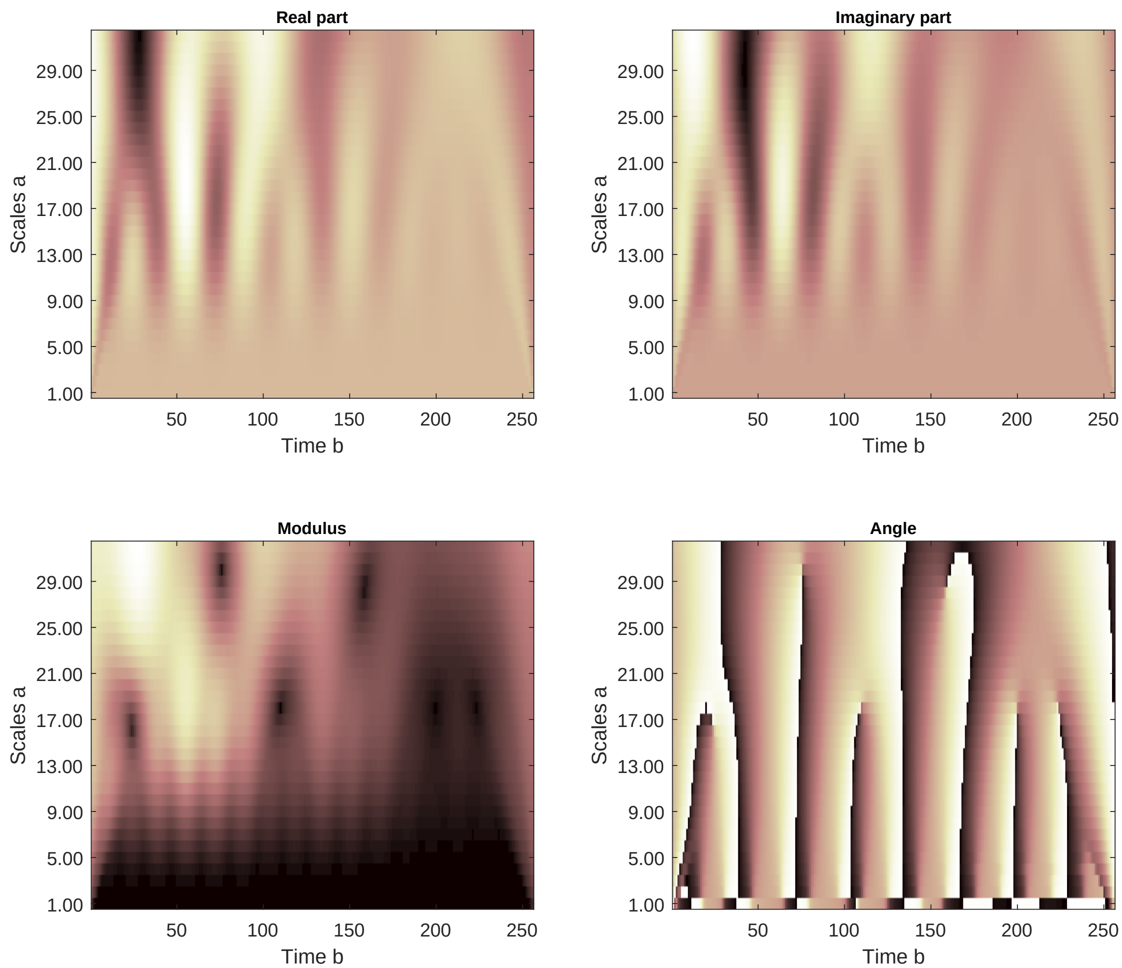


Figure 2: The output with '$cgau3'$ wavelet of Exercise 8.8.

Exercise 8.9 Show that the affine translations $w^{(a,b)}(t)$ span $L_2(\mathbb{R})$ where $w(t) = G(t)$, the Gaussian distribution.

Solution: If $f(t) \in L_2(\mathbb{R})$ and $\langle f, w^{(a,b)} \rangle = 0$ for all $a > 0$ and $b \in \mathbb{R}$. According to

Plancherel theorem, we have

$$\int_{-\infty}^{\infty} f(t)\overline{w^{(a,b)}(t)}dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\lambda)\overline{\hat{w}^{(a,b)}(\lambda)}d\lambda$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} a\hat{f}(\lambda)e^{-(a\sigma\lambda)^2/2}e^{ib\lambda}d\lambda \equiv 0,$$

where the RHS is the inverse Fourier transform of $a\hat{f}(\lambda)e^{-(a\sigma\lambda)^2/2}$ respected to b. Due to the uniqueness of Fourier/inverse Fourier transform, we can conclude that $a\hat{f}(\lambda)e^{-(a\sigma\lambda)^2/2} \equiv 0$ and therefore $\hat{f}(\lambda) \equiv 0$. Consequently, $f(t) \equiv 0$, which means $w^{(a,b)}(t)$ span $L_2(\mathbb{R})$.

Exercise 8.10 Show that the set of affine translations of a complex Gaussian $w_\omega^{(n)}(t)$ span $L_2(\mathbb{R})$. Hint: We already know that the affine translations of the real Gaussian (8.45) span $L_2(\mathbb{R})$.

Solution: Similar to Exercise 8.9, if $f(t) \in L_2(\mathbb{R})$ and $\langle f, \left(w_\omega^{(n)}\right)^{(a,b)}\rangle = 0$ for all $a > 0$ and $b \in \mathbb{R}$. According to Plancherel theorem, we have

$$\int_{-\infty}^{\infty} f(t)\overline{\left(w_\omega^{(n)}\right)^{(a,b)}(t)}dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\lambda)\overline{\left(\hat{w}_\omega^{(n)}\right)^{(a,b)}(\lambda)}d\lambda$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} a\hat{f}(\lambda)e^{-\sigma^2(\omega-a\lambda)^2/2}\overline{(i\lambda)^n}e^{ib\lambda}d\lambda \equiv 0,$$

where the RHS is the inverse Fourier transform of $a\hat{f}(\lambda)e^{-\sigma^2(\omega-a\lambda)^2/2}\overline{(i\lambda)^n}$ respected to b. Due to the uniqueness of Fourier/inverse Fourier transform, we can conclude that $a\hat{f}(\lambda)e^{-\sigma^2(\omega-a\lambda)^2/2}\overline{(i\lambda)^n} \equiv 0$ and therefore $\hat{f}(\lambda) \equiv 0$. Consequently, $f(t) \equiv 0$, which means $\left(w_\omega^{(n)}\right)^{(a,b)}(t)$ span $L_2(\mathbb{R})$.

Exercise 8.11 This is a continuation of Exercise 8.5 on the Haar mother wavelet. Choosing the lattice $a_0 = 2^j, b_0 = 1$, for fixed integer $j \neq 0$, show that the discrete wavelets w^{mn} for $m, n = 0, \pm 1, \pm 2, \ldots$ form an ON set in $L_2(\mathbb{R})$. (In fact it is an ON basis.)

Solution: If the discrete wavelets w^{mn} for $m, n = 0, \pm 1, \pm 2, \ldots$ based on Haar mother wavelet form an ON set in $L_2(\mathbb{R})$, the Haar mother wavelet should be defined in the internal $(0, 1]$ as

$$w(t) = \begin{cases} 1, & \frac{1}{2} \le t < 1 \\ -1, & 0 \le t < \frac{1}{2} \\ 0, & \text{otherwise.} \end{cases}$$

Thus,

$$(w^{mn}, w^{mn}) = (2^{-mj/2})^2 \int_{n2^{mj}}^{(n+1/2)2^{mj}} dt + (2^{-mj/2})^2 \int_{(n+1/2)2^{mj}}^{(n+1)2^{mj}} dt = \frac{1}{2} + \frac{1}{2} = 1.$$

For $m \neq m'$ or $n \neq n'$, w^{mn} and $w^{m'n'}$ are either have non-overlapping domains of definition, or the domain of definition of w^{mn} ($w^{m'n'}$) is completely encompassed in either arm of the domain of definition of $w^{m'n'}$ (w^{mn}). Therefore,

$$\left(w^{mn}, w^{m'n'}\right) \equiv 0, \quad \text{for } m \neq m' \text{ or } n \neq n'.$$

Exercise 8.12 For lattice parameters $a_0 = 2, b_0 = 1$, choose $\hat{w}_+ = \chi_{[1,2)}$ and $\hat{w}_- = \chi_{(-2,-1]}$. Show that w_+ generates a tight frame for $\mathcal{H}^+$ with $A = B = 1$ and w_- generates a tight frame for $\mathcal{H}^-$ with $A = B = 1$, so $\{w_+^{mn}, w_-^{mn}\}$ is a tight frame for $L_2(\mathbb{R})$. (Indeed, one can verify directly that $w_\pm^{mn}$ is an ON basis for $L_2(\mathbb{R})$.)

Solution: For any integer m' and any $1 \leq \nu < 2$, we can have a $\omega = \nu/(2^{m'}2\pi) > 0$ which is ranged in $(0, \infty]$, such that $1 \leq a^{m'}2\pi\omega = 2^{m'}2\pi\omega < 2$, and it is obvious that $a^{m'-1}2\pi\omega < 1$ and $a^{m'+1}2\pi\omega \geq 2$. Thus

$$\begin{aligned}\sum_m |\mathcal{F}w_+(a_0^m\omega)|^2 &= \sum_m |\hat{w}_+(a_0^m 2\pi\omega)|^2 \\ &= \ldots 0 + 0 + 1 + 0 + 0 \ldots \\ &= 1\end{aligned}$$

According to Lemma 8.21, $\{w_+^{mn}\}$ is a tight frame for $\mathcal{H}^+$ with $A = B = 1/b_0 = 1$. Similarly, we can show that $\{w_-^{mn}\}$ is a tight frame for $\mathcal{H}^-$ with $A = B = 1/b_0 = 1$. Hence, $\{w_+^{mn}, w_-^{mn}\}$ is a tight frame for $L_2(\mathbb{R})$.

Exercise 8.13 Let w be the function such that

$$\mathcal{F}w(\omega) = \frac{1}{\sqrt{\ln a}} \begin{cases} 0, & \text{if } \omega \leq \ell \\ \sin \frac{\pi}{2} v\left(\frac{\omega - \ell}{\ell(a-1)}\right), & \text{if } \ell < \omega \leq a\ell \\ \cos \frac{\pi}{2} v\left(\frac{\omega - a\ell}{a\ell(a-1)}\right), & \text{if } a\ell < \omega \leq a^2\ell \\ 0, & \text{if } \omega > a^2\ell \end{cases}$$

where $v(x)$ is defined as in (8.27). Show that $\mathcal{F}w(\omega)$ is a tight frame for $\mathcal{H}^+$ with $A = B = 1/b \ln a$. Furthermore, for $w^+ = w$ and $w_- = \bar{w}$ show that $\{w_\pm^{mn}\}$ is a tight frame for $L_2(\mathbb{R})$.

Solution: For any $\omega > 0$, there must be certain m' such that $\ell < a^{m'}\omega \leq a\ell$. Let $\omega' = a^{m'}\ell$, then

$$\begin{aligned}\sum_m |\mathcal{F}w(a^m\omega)|^2 &= \ldots 0 + 0 + \frac{1}{\ln a}\sin^2 \frac{\pi}{2} v\left(\frac{\omega' - \ell}{\ell(a-1)}\right) + \frac{1}{\ln a}\cos^2 \frac{\pi}{2} v\left(\frac{a\omega' - a\ell}{a\ell(a-1)}\right) + 0 + 0 \ldots \\ &= \frac{1}{\ln a}\left(\sin^2 \frac{\pi}{2} v\left(\frac{\omega' - \ell}{\ell(a-1)}\right) + \cos^2 \frac{\pi}{2} v\left(\frac{\omega' - \ell}{\ell(a-1)}\right)\right) \\ &= \frac{1}{\ln a}.\end{aligned}$$

According to Lemma 8.21, $\{w^{mn}\}$ is a tight frame for $\mathcal{H}^+$ with $A = B = 1/b \ln a$. For $w_- = \bar{w}$, we have $\mathcal{F}w_-(\omega) = \overline{\mathcal{F}w(-\omega)} = \mathcal{F}w(-\omega)$, which has the support in the interval $[-a^2\ell, -\ell]$. Thus, $\{w_-^{mn}\}$ is a tight frame for $\mathcal{H}^-$ with $A = B = 1/b \ln a$. Consequently, $\{w_\pm^{mn}\}$ is a tight frame for $L_2(\mathbb{R})$.

Exercise 8.14 Verify formulas (8.2), using the definitions (8.3).

Solution:

$$\begin{aligned}
\int_{-\infty}^{\infty} t|g^{[x_1,x_2]}(t)|^2 dt &= \int_{-\infty}^{\infty} t|e^{2\pi i t x_2} g(t+x_1)|^2 dt \\
&= \int_{-\infty}^{\infty} t|g(t+x_1)|^2 dt \\
&= \int_{-\infty}^{\infty} (r-x_1)|g(r)|^2 dr \\
&= \int_{-\infty}^{\infty} r|g(r)|^2 dr - x_1 \int_{-\infty}^{\infty} |g(r)|^2 dr \\
&= t_0 - x_1
\end{aligned}$$

$$\begin{aligned}
\int_{-\infty}^{\infty} \omega|\tilde{g}^{[x_1,x_2]}(\omega)|^2 d\omega &= \int_{-\infty}^{\infty} \omega|\mathcal{F}\big[e^{2\pi i t x_2} g(t+x_1)\big]|^2 d\omega \\
&= \int_{-\infty}^{\infty} \omega|e^{2\pi i \omega x_1} \tilde{g}(t-x_2)|^2 dt \\
&= \int_{-\infty}^{\infty} (\nu+x_2)|\tilde{g}(\nu)|^2 d\nu \\
&= \int_{-\infty}^{\infty} \nu|\tilde{g}(\nu)|^2 d\nu + x_2 \int_{-\infty}^{\infty} |\tilde{g}(\nu)|^2 d\nu \\
&= w_0 + x_2
\end{aligned}$$

Exercise 8.15 Derive the inequality (8.5).

Solution: Since $\tilde{g}(\omega) = \hat{g}(2\pi\omega)$ and $\|g\|^2 = \|\tilde{g}\|^2 = \frac{1}{2\pi}\|\hat{g}\|^2 = 1$, we have

$$\begin{aligned}
\int_{-\infty}^{\infty} (t-t_0)^2|g(t)|^2 dt \int_{-\infty}^{\infty} (\omega-\omega_0)^2|\tilde{g}(\omega)|^2 d\omega &= \int_{-\infty}^{\infty} (t-t_0)^2 \frac{|g(t)|^2}{\|g\|^2} dt \int_{-\infty}^{\infty} (\omega-\omega_0)^2 \frac{2\pi|\hat{g}(2\pi\omega)|^2}{\|\hat{g}\|^2} d\omega \\
&= (D_{t_0} g) \int_{-\infty}^{\infty} (\frac{\lambda}{2\pi} - \omega_0)^2 \frac{2\pi|\hat{g}(\lambda)|^2}{\|\hat{g}\|^2} \frac{d\lambda}{2\pi} \\
&= \frac{1}{4\pi^2} (D_{t_0} g) \int_{-\infty}^{\infty} (\lambda - 2\pi\omega_0)^2 \frac{\hat{g}(\lambda)|^2}{\|\hat{g}\|^2} d\lambda \\
&= \frac{1}{4\pi^2} (D_{t_0} g)(D_{2\pi\omega_0} \hat{g}).
\end{aligned}$$

The Heisenberg's inequality ensures that

$$(D_{t_0} g)(D_{2\pi\omega_0} \hat{g}) \geq \frac{1}{4},$$

thus

$$\int_{-\infty}^{\infty}(t-t_0)^2|g(t)|^2dt\int_{-\infty}^{\infty}(\omega-\omega_0)^2|\tilde{g}(\omega)|^2d\omega \geq \frac{1}{16\pi^2}.$$

Exercise 8.16 Given the function

$$g(t)=\phi(t)=\begin{cases} 1, & 0\leq t<1 \\ 0, & \text{otherwise,} \end{cases}$$

i.e., the Haar scaling function, show that the set $\{g^{[m,n]}\}$ is an ON basis for $L_2(\mathbb{R})$. Here, m,n run over the integers. Thus $g^{[x_1,x_2]}$ is overcomplete.

Solution: Since $g^{[m,n]}(t)=e^{2\pi itn}g(t+m)$, we have

$$\left(g^{[m,n]},g^{[m',n']}\right)=\int_{[-m,-m+1]\cup[-m',-m'+1]}e^{2\pi it(n-n')}g(t+m)g(t+m')dt.$$

If $m=m'$ and $n=n'$, $\left(g^{[m,n]},g^{[m',n']}\right)=1$;

If $m\neq m'$, $g(t+m)g(t+m')\equiv 0$ and therefore $\left(g^{[m,n]},g^{[m',n']}\right)=0$;

If $m=m'$ and $n\neq n'$, $\left(g^{[m,n]},g^{[m',n']}\right)=\int_0^1 e^{2\pi it(n-n')}dt=0$.

Hence, the set $\{g^{[m,n]}\}$ is an ON basis for $L_2(\mathbb{R})$.

Exercise 8.17 Verify expansion (8.19) for a frame.

Solution: According to the definition of $\mathbf{T}^*$, we have

$$(\mathbf{T}^*\xi,f)=<\xi,\mathbf{T}f>=\sum_j\xi_j\overline{(f,f^{(j)})}=\sum_j\xi_j(f^{(j)},f)=\left(\sum_j\xi_jf^{(j)},f\right).$$

Thus, $\mathbf{T}^*\xi=\sum_j\xi_jf^{(j)}$.

Exercise 8.18 Show that the necessary and sufficient condition for a frame $f^{(1)},\ldots,f^{(M)}$ in the finite-dimensional inner product space $\mathcal{V}_n$ is that the $N\times M$ matrix Ω, (8.21), be of rank N. In particular we must have $M\geq N$.

Solution: The condition that $\{f^{(j)}\}$ be a frame is that there exist real numbers $0<A\leq B$ such that

$$A\bar{\phi}^{\rm tr}\phi\leq\bar{\phi}^{\rm tr}\Omega\Omega^*\phi\leq B\bar{\phi}^{\rm tr}\phi,$$

for all N-tuples ϕ.

Let $\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_N\geq 0$ be the N eigenvalues of $\Omega\Omega^*$. Apparently, $A\leq\lambda_N$ and $B\geq\lambda_1$. If such $0<A\leq B$ exists, we have $\lambda_N\geq A>0$, which means $\text{rank}(\Omega\Omega^*)=N$ and therefore $\text{rank}(\Omega)=N$.

Conversely, if $\text{rank}(\Omega)=N$, then we have $\text{rank}(\Omega\Omega^*)=N$ and consequently $\lambda_N>0$. Therefore, there exist real numbers A,B with $0<A\leq\lambda_N\leq\lambda_1\leq B$ such that

$$A\bar{\phi}^{\rm tr}\phi\leq\bar{\phi}^{\rm tr}\Omega\Omega^*\phi\leq B\bar{\phi}^{\rm tr}\phi,$$

for all N-tuples ϕ, which means that $\{f^{(j)}\}$ is a frame.

Exercise 8.19 Show that the necessary and sufficient condition for a frame $f^{(1)}, \ldots, f^{(M)}$ in the finite-dimensional inner product space $\mathcal{V}_n$ to be a Riesz basis is that $M = N$.

Solution: According to Exercise 8.18, the necessary and sufficient condition for a frame $f^{(1)}, \ldots, f^{(M)}$ in the finite-dimensional inner product space $\mathcal{V}_n$ is that the $N \times M$ matrix Ω be of rank N. In addition, to be a Riesz basis, $f^{(1)}, \ldots, f^{(M)}$ must be linearly independent. Therefore, the necessary and sufficient condition for a frame $f^{(1)}, \ldots, f^{(M)}$ to be a Riesz basis is $\text{rank}(\Omega) = N$ and $M = N$.

Exercise 8.20 Show that the $N \times M$ matrix defining the dual frame is $(\Omega\Omega^*)^{-1}\Omega$.

Solution: According to the definition of dual frame $\{\mathbf{S}^{-1} f^{(j)}\}$, where $\mathbf{S} = \mathbf{T}^*\mathbf{T} = \Omega\Omega^*$, it is obviously that the dual frame is $(\Omega\Omega^*)^{-1}\Omega$.

Exercise 8.21 Verify that $\lambda_j = \sigma_j^2$, for $j = 1, \ldots, N$ where the σ_j are the singular values of the matrix Ω, (8.21). Thus optimal values for A and B can be computed directly from the singular value decomposition of Ω.

Solution: Let $U\Sigma V^* = \Omega$ be the singular value decomposition of Ω, where $\Sigma = \text{diag}[\sigma_1, \sigma_2, \ldots, \sigma_N]$. Therefore, $U\Sigma^2 U^* = \Omega\Omega^*$ is the eigenvalue decomposition of $\Omega\Omega^*$, where $\Sigma^2 = \text{diag}[\sigma_1^2, \sigma_2^2, \ldots, \sigma_N^2] = \text{diag}[\lambda_1, \lambda_2, \ldots, \lambda_N]$. Hence, $\lambda_j = \sigma_j^2$, for $j = 1, \ldots, N$.

Exercise 8.22 Show that expansion (8.21) for a frame in the finite-dimensional real inner product space $\mathcal{V}_N$ corresponds to the matrix identity $\phi = \Omega\Omega^*(\Omega\Omega^*)^{-1}\phi$

Solution: don't know what exactly does the question mean.

Exercise 8.23 Find the matrix identity corresponding to the expansion (8.21) for a frame in the finite-dimensional real inner product space $\mathcal{V}_N$.

Solution: don't know what exactly does the question mean.

Exercise 8.24 By translation in t if necessary, assume that $\int_{-\infty}^{\infty} t|w_\pm(t)|^2 dt = 0$. Let

$$k_+ = \int_0^\infty y|\mathcal{F}w_+(y)|^2 dy, \ k_- = \int_{-\infty}^0 y|\mathcal{F}w_-(y)|^2 dy.$$

Then $w_\pm$ are centered about the origin in position space and about $k_\pm$ in momentum space. Show that

$$\int_{-\infty}^\infty t|w_\pm^{(a,b)}(t)|^2 dt = -b, \quad \pm\int_0^\infty y|\mathcal{F}w_\pm^{(a,b)}(\pm y)|^2 dy = a^{-1}k_\pm,$$

so that $w_\pm^{(a,b)}$ are centered about $-b$ in time and $a^{-1}k_\pm$ in frequency.

Solution: Since $w^{(a,b)}(t) = |a|^{-1/2} w\big(\frac{t-b}{a}\big)$, thus

$$\begin{aligned}
\int_{-\infty}^{\infty} t|w_{\pm}^{(a,b)}(t)|^2 dt &= \int_{-\infty}^{\infty} t\big||a|^{-1/2} w_{\pm}(\frac{t-b}{a})\big|^2 dt \\
&= \int_{-\infty}^{\infty} \frac{(au+b)}{a} \big|w_{\pm}(u)\big|^2 a du \\
&= a \int_{-\infty}^{\infty} u\big|w_{\pm}(u)\big|^2 du + b \int_{-\infty}^{\infty} \big|w_{\pm}(u)\big|^2 du \\
&= b.
\end{aligned}$$

Moreover, $\mathcal{F}w^{(a,b)}(y) = e^{-2\pi iby}|a|^{1/2}\mathcal{F}w(ay)$, thus

$$\begin{aligned}
\pm \int_{0}^{\infty} y|\mathcal{F}w_{\pm}^{(a,b)}(\pm y)|^2 dy &= \pm \int_{0}^{\infty} ay|\mathcal{F}w_{\pm}(\pm ay)|^2 dy \\
&= \pm \int_{0}^{\infty} z|\mathcal{F}w_{\pm}(\pm z)|^2 \frac{dz}{a} \\
&= a^{-1}k_{\pm}.
\end{aligned}$$

Exercise 8.25 Show that if the support of w_+ is contained in an interval of length ℓ in the time domain then the support of $w_+^{(a,b)}$ is contained in an interval of length $|a|l$. Similarly, show that if the support of $\mathcal{F}w_+$ is contained in an interval of length L in the frequency domain then the support of $\mathcal{F}w_+^{(a,b)}$ is contained in an interval of length $|a|^{-1}L$. This implies that the length and width of the "window" in time-frequency space will be rescaled by a and a^{-1} but the area of the window will remain unchanged.

Solution: Since $w^{(a,b)}(t) = |a|^{-1/2} w\big(\frac{t-b}{a}\big)$ and $\mathcal{F}w^{(a,b)}(y) = e^{-2\pi iby}|a|^{1/2}\mathcal{F}w(ay)$, it is obvious that there intervals are rescaled by $|a|$ and $|a|^{-1}$, respectively.

Solutions to Chapter 10

Exercise 10.1 For each of the Figures 10.1, 10.2, 10.3, determine the significance of the horizontal and vertical scales

Solution: For Figure 10.1, 10.2 and 10.3, the horizontal scales can be mapped to the support internal $[0,3], [0,5]$ and $[0,7]$ for D_4, D_6 and D_8. The vertical scales are adjusted according to the number of discretized points of the Daubechies wavelets in the support internals, so that the norm of the vectors for the wavelets' values is unit.

Exercise 10.2 Apply the cascade algorithm to the inner product vector of the ith iterate of the cascade algorithm with the scaling function itself

$$b_k^{(i)} = \int_{-\infty}^{\infty} \phi(t)\bar{\phi}^{(i)}(t-k)dt,$$

and derive, by the method leading to (10.1), the result

$$b^{(i+1)} = Tb^{(i)}.$$

Solution:

$$\begin{aligned} b_s^{(i+1)} &= \int_{-\infty}^{\infty} \phi(t)\bar{\phi}^{(i+1)}(t-s)dt = \int_{-\infty}^{\infty} \phi(t+s)\bar{\phi}^{(i+1)}(t)dt \\ &= 2\sum_{k,\ell} c_\ell \bar{c}_k \int_{-\infty}^{\infty} \phi(2t+2s-\ell)\bar{\phi}^{(i)}(2t-k)dt \\ &= \sum_{k,\ell} c_\ell \bar{c}_k \int_{-\infty}^{\infty} \phi(t+2s-\ell)\bar{\phi}^{(i)}(t-k)dt \\ &= \sum_{k,\ell} c_\ell \bar{c}_k b^{(i)}_{2s-\ell+k} = \sum_{m,j} c_{2s+m}\bar{c}_{m+j} b_j^{(i)}, \end{aligned}$$

thus

$$b^{(i+1)} = Tb^{(i)} = (\downarrow 2)C\bar{C}^{\mathrm{tr}} b^{(i)}.$$

Exercise 10.3 The application of Theorem 10.1 to the Haar scaling function leads to an identity for the sinc function. Derive it.

Solution: If $\phi(t)$ is the Haar scaling function, then $\hat{\phi}(t) = e^{-i\omega/2}\mathrm{sinc}\ (\omega/(2\pi))$ and we have the identity

$$A(\omega) = \sum_{n=-\infty}^{\infty} \left|\mathrm{sinc}\left(\frac{\omega}{2\pi}+n\right)\right|^2 = \sum_{n=-\infty}^{\infty} \left|\frac{\sin(\omega/2+n\pi)}{\omega/2+n\pi}\right|^2 \equiv 1.$$

Exercise 10.4 The finite T matrix for the dilation equation associated with a prospective family of wavelets has the form

$$T = \begin{pmatrix} 0 & -\frac{1}{16} & 0 & 0 & 0 \\ 1 & \frac{9}{16} & 0 & -\frac{1}{16} & 0 \\ 0 & \frac{9}{16} & 1 & \frac{9}{16} & 0 \\ 0 & -\frac{1}{16} & 0 & \frac{9}{16} & 1 \\ 0 & 0 & 0 & -\frac{1}{16} & 0 \end{pmatrix}$$

Designate the components of the low pass filter defining this matrix by $c_0, \ldots, c_N$.

1. What is N in this case? Give a reason for your answer.
2. The vector $a_k = \int \phi(t)\overline{\phi(t-k)}dt$, for $k = -2, -1, 0, 1, 2$ is an eigenvector of the T matrix with eigenvalue 1. By looking at T can you tell if these wavelets are ON?
3. The Jordan canonical form for T is

$$J = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{8} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & 1 \\ 0 & 0 & 0 & 0 & \frac{1}{4} \end{pmatrix}$$

   What are the eigenvalues of T?
4. Does the cascade algorithm converge in this case to give an L_2 scaling function $\phi(t)$? Give a reason for your answer.
5. What polynomials $P(t)$ can be expanded by the $\phi(t-k)$ basis, with no error. (This is equivalent to determining the value of p.) Give a reason for your answer.

Solution: $[-1/16, 0, 9/16, 1, 9/16, 0, -1/16]$ are the coefficients of the halfband filter $P[z]$ for the Daubechies wavelet D_4. Thus, $c_0 = \frac{1+\sqrt{3}}{4\sqrt{2}}, c_1 = \frac{3+\sqrt{3}}{4\sqrt{2}}, c_2 = \frac{3-\sqrt{3}}{4\sqrt{2}}, c_3 = \frac{1-\sqrt{3}}{4\sqrt{2}}$.

1. The dimension of T is $2N - 1 = 5$, so $N = 3$.
2. These wavelets are ON because for $P[z] = -\frac{1}{16}z^3 + \frac{9}{16}z + 1 + \frac{9}{16}z^{-1} - \frac{1}{16}z^{-3}$, $P[z] + P[-z] = 2$.
3. The eigenvalues of T are $1, \frac{1}{2}, \frac{1}{4}$ and $\frac{1}{8}$.
4. Yes, because T has all its eigenvalues $|\lambda| < 1$ except for the simple eigenvalue $\lambda = 1$.
5. Any polynomial with order $p \leq 2$ can be expanded by the $\phi(t-k)$ basis with no error, since $N = 2p - 1 = 5$ and therefore $p = 2$.

Solutions to Chapter 11

Exercise 11.1 Verify that the 2-spline scaling function $\phi_2(t)$ can be embedded in a 5/3 filter bank defined by

$$S^{(0)}[z] = \left(\frac{1+z^{-1}}{2}\right)^2, \quad H^{(0)}[z] = \left(\frac{1+z^{-1}}{2}\right)^2 [-1+4z^{-1}-z^{-2}],$$

i.e. $p = 2$ This corresponds to Daubechies D_4 (which would be 4/4). Work out the filter coefficients for the analysis and synthesis filters. Show that the convergence criteria are satisfied, so that this defines a family of biorthogonal wavelets.

Solution: The synthesis filter coefficient is $s_0 = \frac{1}{4}, s_1 = \frac{1}{2}, s_2 = \frac{1}{4}$.

The analysis filter coefficient is $h_0 = -\frac{1}{4}, h_1 = \frac{1}{2}, h_2 = \frac{3}{2}, h_3 = \frac{1}{2}, h_4 = -\frac{1}{4}$.

Its halfband filter is $P^{(0)}[z] = S^{(0)}[z]H^{(0)}[z] = -\frac{1}{16}z^3 + \frac{9}{16}z + 1 + \frac{9}{16}z^{-1} - \frac{1}{16}z^{-3}$. Thus its finite T matrix is

$$T = \begin{pmatrix} 0 & -\frac{1}{16} & 0 & 0 & 0 \\ 1 & \frac{9}{16} & 0 & -\frac{1}{16} & 0 \\ 0 & \frac{9}{16} & 1 & \frac{9}{16} & 0 \\ 0 & -\frac{1}{16} & 0 & \frac{9}{16} & 1 \\ 0 & 0 & 0 & -\frac{1}{16} & 0 \end{pmatrix}$$

and its eigenvalues are $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$, which satisfies the convergence criteria of Theorem 10.2.

Exercise 11.2 Verify Lemma 11.14.

Solution: Since $(\uparrow M)$ means up sampling by M times, by definition we have

$$x_k = \begin{cases} y_{\frac{k}{M}}, & \text{if } M \text{ divides } k \\ 0, & \text{otherwise} \end{cases}$$

In frequency domain, as $Y(\omega) = \sum_k y_k e^{-jk\omega}$, we have $X(\omega) \sum_k x_k e^{-jk\omega} = \sum_k y_k e^{-jkM\omega} = Y(M\omega)$.

For the Z transform, as $Y[z] = \sum_k y_k z^{-k}$, we have $X[z] = \sum_k x_k z^{-k} = \sum_k y_k z^{-Mk} = Y[z^M]$.

For $u = (\downarrow M)(\uparrow M)x$, as

$$\frac{1}{M}\sum_{k=0}^{M-1} X[ze^{2\pi ik/M}] = \sum_n x_n \frac{\left(\sum_{k=0}^{M-1} e^{-2\pi ikn/M}\right)}{M} z^{-n},$$

where

$$\sum_{k=0}^{M-1} e^{-2\pi ikn/M} = \begin{cases} M, & \text{if } M \text{ divides } n \\ 0, & \text{otherwise} \end{cases}$$

Therefore,

$$\frac{1}{M}\sum_{k=0}^{M-1} X[ze^{2\pi ik/M}] = \sum_k x_{kM}z^{-kM},$$

which is a subset of $X[z]$ only taking the n-th term when M divides n.

Exercise 11.3 Let $x = (x_0, x_1, x_2, x_3)$ be a finite signal. Write down the infinite signals obtained from x by (a) zero padding, (b) constant padding with $c = 1$, (c) wraparound, (d) whole-point refection, (e) halfpoint refection.

Solution: (a)

$$x_k = \begin{cases} x_k, & 0 \le k \le 3 \\ 0, & \text{otherwise} \end{cases}.$$

(b)

$$x_k = \begin{cases} x_k, & 0 \le k \le 3 \\ 1, & \text{otherwise} \end{cases}.$$

(c)

$$x_n = x_{n \bmod 3}.$$

(d)

$$x_n = x_{n \bmod 6}, \text{ where } x_4 = x_2, x_5 = x_1.$$

(e)

$$x_n = x_{n \bmod 8}, \text{ where } x_4 = x_3, x_5 = x_2, x_6 = x_1, x_7 = x_0.$$

Exercise 11.4 Verify explicitly that the circulant matrix Φ of Example 11.18 is diagonal in the DFT frequency space and compute its diagonal elements.

Solution:

$$D^{\Phi} = F^{-1}\Phi F = \frac{1}{M}\overline{F}\Phi F = \begin{pmatrix} 11 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & -2-3i & 0 \\ 0 & 0 & 0 & -2+3i \end{pmatrix}.$$

Solutions to Chapter 12

Exercise 12.1 Consider the image immediately above. Use a contrast stretch to construct this image from the image

$$\begin{bmatrix} 5 & 8 & 8 & 6 \\ 0 & 0 & 0 & 0 \\ 7 & 9 & 9 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution: By using

$$b[m,n] = \frac{234(a[m,n] - \min[a(m,n)])}{\max[a(m,n)] - \min[a(m,n)]},$$

we have

$$b = \begin{bmatrix} 130 & 208 & 208 & 156 \\ 0 & 0 & 0 & 0 \\ 182 & 234 & 234 & 156 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Exercise 11.2 Assume we have a system producing a dynamic range of 256, i.e., 8 bits and the system adds two percent (of the dynamic range) random noise to the image. Suppose we measure the dark image first and then add two percent random noise to it as shown. Show that the contrast stretching has no effect on the dynamic range/noise ratio.

Solution: Since $I_{\text{noise}} = I_{\text{signal+noise}} - I_{\text{signal}}$ and the stretching operator is

$$I'[m,n] = \frac{(2^N - 1)(I[m,n] - \min[I(m,n)])}{\max[I(m,n)] - \min[I(m,n)]},$$

we have

$$I'_{\text{singal+noise}}[m,n] = \frac{(2^N - 1)(I_{\text{singal}}[m,n]}{\max(I_{\text{signal+nosie}}) - \min(I_{\text{signal+nosie}})}\,\delta(I_{\text{noise}})$$

Exercise 12.3 Consider the examples already studied. For a suitable structuring element K of your choice, calculate (1) The boundary of the element below using the structuring element shown. (2) The boundary of the nuts and bolts example using a dilation of 2. (3) Study the thickened boundary $(AdK) \cap (AeK)$. What do you find?

Solution: (1) Let

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \ K = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

then the boundary of A is

$$\partial A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The thickened boundary $(AdK) \cap (A^c dK)$ is

$$(AdK) \cap (A^c dK) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Exercise 12.4 Apply the skeletal algorithm above to the nuts and bolts example above.

Solution: For another picture with circles, the comparison of the original and skeletonized pictures is shown in Figure 1.

Exercise 12.5 Take FTs of two images. Add them using a suitable morphology operator for example a blend, and take the inverse Fourier Transform of the sum. Explain the result.

Solution: Two pictures for blending is shown in Figure 2. The blended picture is shown in Figure 3.

Exercise 12.6 Add different sorts of noise to two images and compare the FT's. Investigate if the Fourier Transform is distributive over multiplication.

Solution:

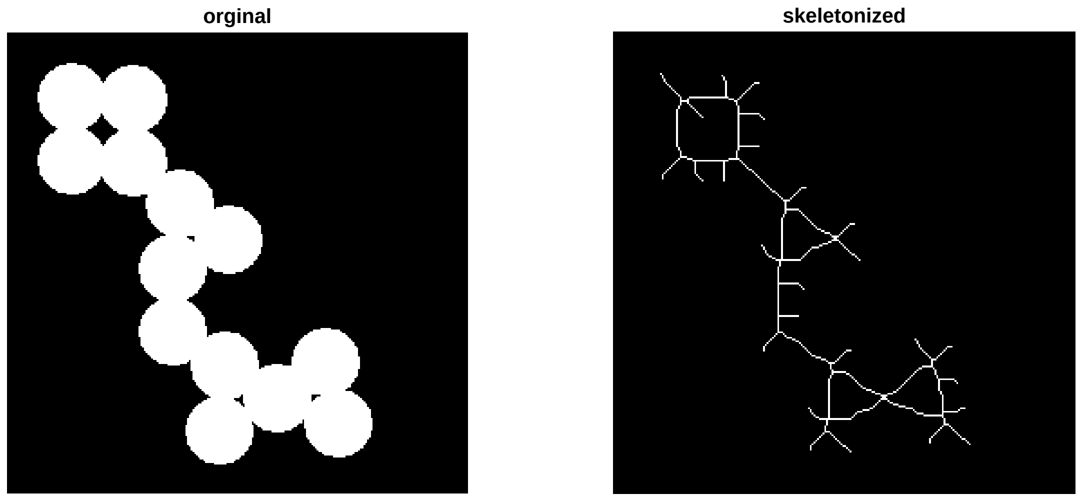


Figure 1: Exercise 12.4.

Figure 2: Two pictures used in Exercise 12.5.

Exercise 12.7 Show that $P(B|C)$ is a probability set function on the subset of C.

Solution: Since $B \cap C$ is a subset of C, and $P(B|C)$ is the probability of $B \cap C$ in the space C. Hence, $P(B|C)$ is a probability set function on the subset of C.

Exercise 12.8 Prove (12.1) and give its interpretation for the handwriting problem.

Solution: Since $A_j, j = 0, \ldots, 9$ are mutually disjoint, thus

$$P(B) = \sum_{j=0}^{9} P(B \cap A_j) = \sum_{j=0}^{9} P(B)P(B|A_j).$$

It means that the occurrence of a subset B of the handwriting characters is the sum of the occurrences of handwriting characters in each class A_j from B.

Figure 3: Blended picture for Exercise 12.5.

Exercise 12.9 Suppose the set of training vectors is such that $a = 1, c = 5$ in the proof of Theorem 12.6 and we start the Perceptron algorithm with $w^{(1)}$ as the zero $n+1$-tuple. Find the smallest k such that the algorithm is guaranteed to terminate in at most k steps.

Solution: If $a = 1, c = 5$ and $w^{(1)} = 0$, $\|z'\|' = 1$, we have

$$k^2 \le \left\| \sum_{j=1}^{k} x'^{(j)} \right\|^2 \le \sum_{j=1}^{k} \|x'^{(j)}\|^2 \le 5k.$$

Thus

$$k^2 - 5k \le 0 \;\Rightarrow\; k \le 5.$$

There,the algorithm is guaranteed to terminate in approximately 5 steps.

Exercise 12.10 Find an example in two dimensions and an example in three dimensions of Class 1 and Class 2 training vectors that are not linearly separable.

Solution: In two dimension

$$\text{Class } 1 = \big\{ v = (x, y) \mid x^2 + y^2 < r^2 \big\}, \;\; \text{Class } 2 = \big\{ v = (x, y) \mid x^2 + y^2 > r^2 \big\},$$

is not linearly separable.

In three dimension

$$\text{Class } 1 = \big\{ v = (x, y, z) \mid x^2 + y^2 + z^2 < r^2 \big\}, \;\; \text{Class } 2 = \big\{ v = (x, y, z) \mid x^2 + y^2 + z^2 > r^2 \big\},$$

is not linearly separable.

Exercise 12.11 Show that $E(C^S) = C$, i.e., the expectation of C^S is C.

Solution:

$$\begin{aligned}
E\left(C^S_{i,j}\right) &= E\left(\frac{1}{n-1}\sum_{k=1}^{m}\left(T_i^{(k)} - \bar{T}_i\right)\left(T_j^{(k)} - \bar{T}_j\right)\right) \\
&= \frac{1}{n-1}\sum_{k=1}^{m}\left(T_i^{(k)} - E(T_i)\right)\left(T_j^{(k)} - E(T_j)\right) \\
&= E[(T_i - E(T_i))(T_j - E(T_j))] = C_{i,j}
\end{aligned}$$

Exercise 12.12 Verify the following properties of the $n \times n$ sample covariance matrix C^S and the $n \times m$ matrix (12.6). (1) $C^S = X^{\mathrm{tr}}X$, (2) $(C^S)^{\mathrm{tr}} = C^S$. (3) The eigenvalues λ_j of C^S are nonnegative.

Solution: (1) Since

$$\begin{aligned}
C^S_{i,j} &= \frac{1}{n-1}\sum_{k=1}^{m}\left(T_i^{(k)} - \bar{T}_i\right)\left(T_j^{(k)} - \bar{T}_j\right) \\
&= \sum_{k=1}^{m}\frac{1}{\sqrt{n-1}}\left(T_i^{(k)} - \bar{T}_i\right)\frac{1}{\sqrt{n-1}}\left(T_j^{(k)} - \bar{T}_j\right) \\
&= \sum_{k=1}^{m} X_{k,i}X_{k,j},
\end{aligned}$$

thus $C^S = X^{\mathrm{tr}}X$.

(2) As $C^S_{i,j} = C^S_{j,i} \Rightarrow (C^S)^{\mathrm{tr}} = C^S$.

(3) for any $z \in \mathbb{R}^n, z \neq 0$, we have

$$z^{\mathrm{tr}}C^S z = z^{\mathrm{tr}}X^{\mathrm{tr}}Xz = \|Xz\|^2 \geq 0,$$

therefore C^S is semi-definite positive, i.e. the eigenvalues λ_j of C^S are nonnegative.

Exercise 12.1(1) Take each of the examples below, with 16 pixels and 3 bit dynamic range monochrome values and apply a contrast stretch and histogram normalization. What would you expect to see in real images of these types?

$$(A): \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 2 & 1 & 6 \\ 1 & 1 & 6 & 5 \\ 6 & 7 & 7 & 7 \end{bmatrix}, \quad (B): \begin{bmatrix} 7 & 6 & 5 & 4 \\ 6 & 7 & 5 & 5 \\ 5 & 5 & 6 & 7 \\ 4 & 5 & 7 & 6 \end{bmatrix}$$

$$(C): \begin{bmatrix} 2 & 3 & 2 & 3 \\ 3 & 2 & 3 & 5 \\ 3 & 6 & 5 & 6 \\ 6 & 5 & 6 & 5 \end{bmatrix}$$

(2) Suppose that in a binary image you have k white pixels in a total of N pixels. Find the mean and variance of the image.

(3) (a) Show that the mean of an image with dynamic range of n bits is $1/2(2n-1)$ if histogram normalization has been applied to it. (b) Write down a formula for the variance of the images in (a). (c) What is the implication of (a)-(b) for contrast and brightness?

Solution: (1) The results of stretching $(A), (B), (C)$ are

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 2 & 1 & 6 \\ 1 & 1 & 6 & 5 \\ 6 & 7 & 7 & 7 \end{bmatrix}, \quad \begin{bmatrix} 7 & 5 & 2 & 0 \\ 0 & 7 & 1 & 6 \\ 2 & 2 & 5 & 7 \\ 0 & 2 & 7 & 5 \end{bmatrix}, \quad \begin{bmatrix} 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 5 \\ 2 & 7 & 5 & 7 \\ 7 & 5 & 7 & 5 \end{bmatrix}.$$

Nothing has change in (A). The contrast of (B) and (C) has been enhanced.

(2)

$$\text{mean} = \frac{k}{N}, \quad \text{variance} = \frac{1}{N-1}\left(k\left(1-\frac{k}{N}\right)^2 + (N-k)\left(\frac{k}{N}\right)^2\right) = \frac{k(N-k)}{N(N-1)}.$$

(3) Problem??

Exercise 12.2 (a) Consider each of the three images below. Each is the Fourier transform of an image. Roughly sketch the original image

$$(A): \begin{bmatrix} \text{space} & \text{space} & \text{space} \\ \cdot & \cdot & \cdot \\ \text{space} & \text{space} & \text{space} \end{bmatrix}, \quad (B): \begin{bmatrix} \text{space} & \cdot & \text{space} & \text{space} \\ \text{space} & \text{space} & \text{space} & \text{space} \\ \text{space} & \cdot & \text{space} & \text{space} \end{bmatrix}$$

$$(C): \begin{bmatrix} \text{space} & \text{space} & \text{space} & \text{space} & \text{space} \\ \cdot & \text{space} & \text{space} & \cdot & \cdot \\ \text{space} & \text{space} & \text{space} & \text{space} & \text{space} \end{bmatrix}$$

(b) Suppose we take an image which consists of spaces everywhere except on the diagonal where it has a high pixel value. What will its Fourier transform look like? Explain your answer.

Solution: (a)

$$(A): |\mathcal{F}^{-1}(*)| = \begin{bmatrix} a & b & c \\ a & b & c \\ a & b & c \end{bmatrix}, \quad (B): |\mathcal{F}^{-1}(*)| = \begin{bmatrix} a & a & a & a \\ b & b & b & b \\ c & c & c & c \end{bmatrix}$$

$$(C): |\mathcal{F}^{-1}(*)| = \begin{bmatrix} a & b & c & d & e \\ a & b & c & d & e \\ a & b & c & d & e \end{bmatrix}$$

(b) The Fourier transform of a squared diagonal graph looks like

$$|\mathcal{F}(*)| = \begin{bmatrix} * & & & & \\ & & & & * \\ & & & * & \\ & & * & & \\ & * & & & \end{bmatrix}$$

which has its dominate values along the -1st anti-diagonal and the upper-left corner. This is because the 2D DFT for squared diagonal graph x is

$$X = WxW,$$

where W is the conjugate of the Fourier transform matrix. If x has large values along its diagonal, then

$$X \approx cW^2,$$

where c is a large positive real number. Since W is symmetric unitary, and its column has the relationship $w_n = e^{2\pi(n-1)i/N} w_1$, the dominate entries of X should be $X_{m,n}$ where

$$\bmod\ (m+n-2, N) = 0.$$

Since $1 \le m, n \le N$, thus $m + n = N + 2$ or $m = n = 1$.

Exercise 12.13 Gray scale morphology operators: (a) Show that the following is true: if f is the intensity function for an image and k is a hemisphere structuring function then

- $fdk = T[U[f]dU[k]]$.
- $fek = T[U[f]eU[k]]$.
- $fok = T[U[f]oU[k]]$.
- $fck = T[U[f]cU[k]]$.

(b) Prove the Umbra Homomorphism Theorem:

- $U[fdg] = U[f]dU[g]$.
- $U[feg] = U[f]eU[g]$.
- $U[fog] = U[feg]dU[g] = U[f]oU[g]$.

Solution: (a) $T[U[f]dU[k]] = T[f]dTU[k] = fdk$.

$T[U[f]eU[k]] = TU[f]eTU[k] = fek$.

$T[U[f]oU[k]] = TU[f]oTU[k] = fok$.

$T[U[f]cU[k]] = TU[f]cTU[k] = fck$.

(b) $U[fdg] = U[T[f]dT[g]] = UT[f]dUT[g] = U[f]dU[g]$.

$U[feg] = U[T[f]eT[g]] = UT[f]eUT[g] = U[f]eU[g]$.

$U[fog] = U[(feg)dg] = U[feg]dU[g] = (U[f]eU[g])dU[g] = U[f]oU[g]$.

Exercise 12.14 Find the opening and closing of the images (A)–(C) in Exercise 254 for suitable structuring elements of your choice.

Solution: Exercise 254?

Exercise 12.15 Prove that if A and B are umbras, then so is AdB.

Solution: if A and B are umbras, we have

$$AdB = UT(A)dUT(B) = U[T(A)dT(B)].$$

Thus, AdB is an umbra.

Exercise 12.16 Consider an image with pixels over $[0, 4]$. The gray scale function for the image intensity is given by $f(x) = x - 1$ for $1 \le x \le 2$, $f(x) = 3 - x$ for $2 \le x \le 3$ and 0 elsewhere. The structuring function is a hemisphere of radius 0.25, i.e.,

$$k(x) = \sqrt{0.0625 - x^2}, \quad -0.25 \le x \le 0.25.$$

Find the opening and closing of f by k and discuss what this should do to the visual aspects of the image.

Solution: The shapes of feg, fdg, fcg, fog are shown in Figure 4 and Figure 5.

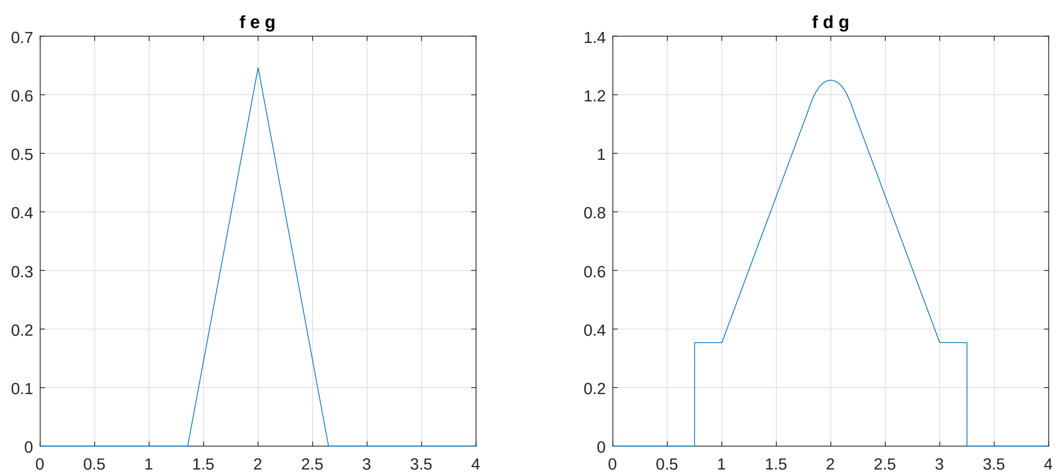


Figure 4: feg and fdg for Exercise 12.16.

Exercise 12.17 Write a program to implement the 3×3 skeletonization algorithm described above and try it out on some images

Solution:

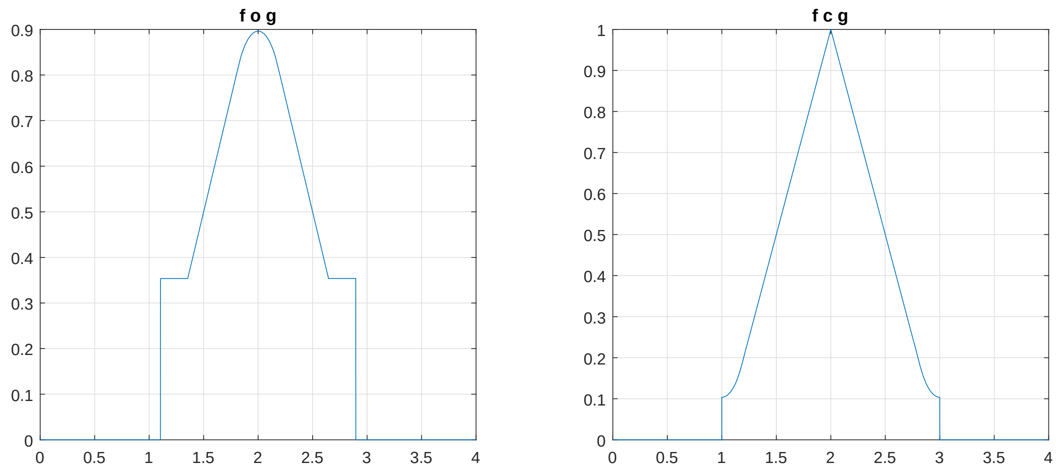


Figure 5: fog and fcg for Exercise 12.16.

```
BW1 = imread('circbw.tif');
BW2 = bwmorph(BW1,'skel',Inf);
figure
subplot(1,2,1)
imshow(BW1)
title('orginal')
subplot(1,2,2)
imshow(BW2)
title('skeletonized')
```

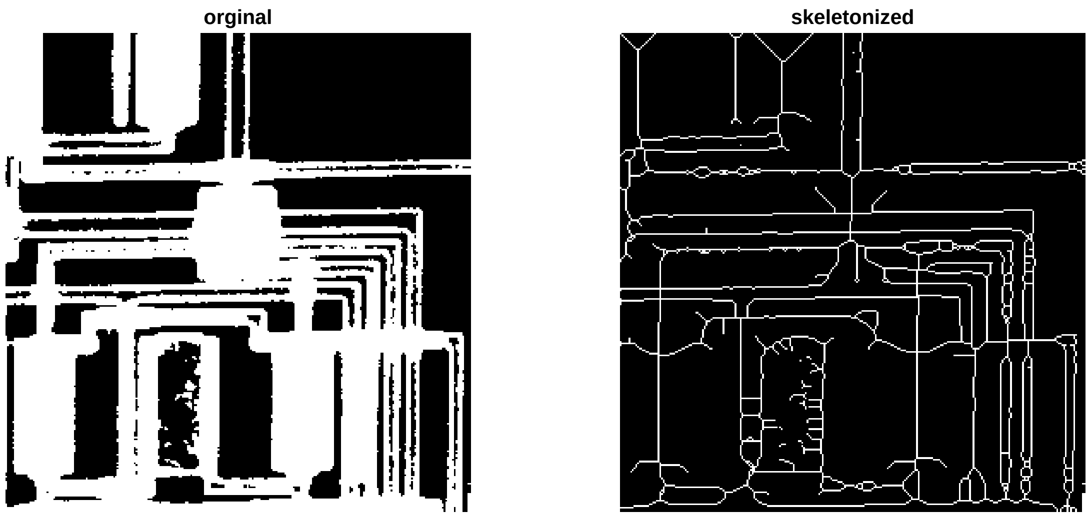


Figure 6: Exercise 12.17.

Exercise 12.18 Consider the pattern classes $\{(0,0),(0,1)\}$ and $\{(1,0),(1,1)\}$. Use these patterns to train a perceptron algorithm to separate patterns into two classes around these. [Do not forget to augment the patterns and start with zero weights.]

Solution: Let $w^{(0)} = (0,0,0), x_1 = (0,0,1), x_2 = (0,1,1), x_3 = (1,0,1), x_4 = (1,1,1)$; Class 1 requires $(w, -x_i) > 0$ and Class 1 requires $(w, x_i) > 0$.

$(w^{(0)}, -x_1) = 0 \Rightarrow w^{(1)} = w^{(0)} - w_1 = (0,0,-1)$;

$(w^{(1)}, -x_2) = -1 < 0 \Rightarrow w^{(2)} = w^{(1)} - w_2 = (0,-1,-2)$;

$(w^{(2)}, x_3) = -2 < 0 \Rightarrow w^{(3)} = w^{(2)} + w_3 = (1,-1,-1)$;

$(w^{(3)}, x_4) = -1 < 0 \Rightarrow w^{(4)} = w^{(3)} + w_4 = (2,0,0)$;

$(w^{(4)}, -x_1) = 0 \Rightarrow w^{(5)} = w^{(4)} - w_1 = (2,0,-1)$;

$(w^{(5)}, -x_2) = 1 > 0$;

$(w^{(5)}, x_3) = 1 > 0$;

$(w^{(5)}, x_4) = 1 > 0$;

Thus, the hyperplane for separating the two classes is $2x - 1 = 0$.

Exercise 12.19 Suppose we have a collection of pattern classes and in each case the distribution of the patterns around the cluster center is Gaussian. Suppose also that the pattern components are uncorrelated. [This means the covariance matrix is the identity.] Find a formula for the probability that a given pattern vector x belongs to cluster w_j . In this case, given a pattern vector x, deduce that the decision as to which class it belongs to would be made simply on its distance to the nearest cluster center.

Solution: Since the distribution of each class is Gaussian and they are independent, the value of the probability density function (PDF) of each case for each class reveals the chance of that case belonging to the indicated class Thus, we can assign pattern vector x to cluster ω_j with have the maximal PDF with assumed mean (center) among all the clusters. Apparently, higher PDF indicates closer distance of the pattern vector to the center of the cluster. Therefore, the decision of which class a x belonging to would be made on its distance to the nearest cluster center.

Exercise 12.20 Consider the following 20 two-dimensional vectors:

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}:$$

$$(x_1 = 4.1, 2.8, 0.5, 3.1, 4.8, 1.5, 3.8, 2.4, 1.7, 2.0, 0.2, 3.5, 2.1, 0.8, 2.9, 4.4, 3.7, 5, 4, 1.3)$$

$$(x_2 = 9.7, 6.3, 2.4, 7.2, 10.2, 4.4, 9, 5.4, 4.7, 4.8, 1.5, 8, 5.7, 2.5, 6.7, 10, 8.8, 10.5, 9.3, 3.3)$$

Without using a MATLAB program, find its principal components and comment on the results.

Solution: Since

$$x \cdot x^T = \begin{pmatrix} 188.38 & 433.49 \\ 433.39 & 1005..02 \end{pmatrix},$$

its eigen-polynomial is

$$\lambda^2 - 1193.40\lambda + 1412.0875 = 0.$$

thus, its eigenvalues are

$$\lambda_1 = 1192.216, \quad \lambda_2 = 1.1844,$$

and therefore its principle components is the eigenvector

$$v_1 = \begin{pmatrix} 0.396447 \\ 0.918057 \end{pmatrix}.$$

Since $\lambda_1 \gg \lambda_2$, the distribution of $x_i, i = 1, \ldots, 20$ is dominated along v_1 direction.

Exercise 12.21 In this chapter, we have given a self-contained introduction to some techniques dealing with the parsimonious representation of data. MATLAB offers several toolboxes which allow one to implement demos on data. Study the imaging and statistical pattern toolbox demos given in the links:

http://cmp.felk.cvut.cz/cmp/software/stprtool/examples.html

http://www-rohan.sdsu.edu/doc/matlab/toolbox/images/images.html

Solution: