Chapter A

Appendix

A.1 Technical results for Theorem 3.12

We next prove the inequality (3.30) under the hypotheses of Theorem 3.12. Let $\eta^1 = (1-\alpha_1)\mu,\ \eta^2 = \alpha_1(1-\alpha_2)\nu,\ \eta^3 = \alpha_1\alpha_2\kappa$, thus

$$\eta := \eta^1 + \eta^2 + \eta^3.$$

Also note that, for every i in I, $\eta_i^1 = (1-\alpha_1)\mu_i,\ \eta_i^2 = \alpha_1(1-\alpha_2)\nu_i$ and $\eta_i^3 = \alpha_1\alpha_2\kappa_i$. Then

$$\eta = (\eta_1, .., \eta_n) = (\eta_1^1 + \eta_1^2 + \eta_1^3, \ldots, \eta_n^1 + \eta_n^2 + \eta_n^3).$$

Since μ^* is a SUP in the set $\mathcal{C}$, $J_1(\mu_1^*, \eta_{-1}) > J_1(\eta_1, \eta_{-1})$. Then using the notation in (2.3) we have the following implications

$$\mathcal{I}_{(\mu_1,\eta_2,\eta_3,\ldots,\eta_n)}U_1 > \mathcal{I}_{(\eta_1,\eta_2,\eta_3,\ldots,\eta_n)}U_1$$

$\Rightarrow$

$$\begin{aligned}
\mathcal{I}_{(\mu_1,\eta_2^1,\eta_3,\ldots,\eta_n)}U_1 \quad + & \quad \mathcal{I}_{(\mu_1,\eta_2^2,\eta_3,\ldots,\eta_n)}U_1 + \mathcal{I}_{(\mu_1,\eta_2^3,\eta_3,\ldots,\eta_n)}U_1 \\
> & \quad \mathcal{I}_{(\eta_1^1,\eta_2^1\eta_3,\ldots,\eta_n)}U_1 + \mathcal{I}_{(\eta_1^1,\eta_2^2,\eta_3,\ldots,\eta_n)}U_1 + \mathcal{I}_{(\eta_1^1,\eta_2^3,\eta_3,\ldots,\eta_n)}U_1 \\
& + \mathcal{I}_{(\eta_1^2,\eta_2^1,\eta_3,\ldots,\eta_n)}U_1 + \mathcal{I}_{(\eta_1^2,\eta_2^2,\eta_3,\ldots,\eta_n)}U_1 + \mathcal{I}_{(\eta_1^2,\eta_2^3,\eta_3,\ldots,\eta_n)}U_1 \\
& + \mathcal{I}_{(\eta_1^3,\eta_2^1,\eta_3,\ldots,\eta_n)}U_1 + \mathcal{I}_{(\eta_1^3,\eta_2^2,\eta_3,\ldots,\eta_n)}U_1 + \mathcal{I}_{(\eta_1^3,\eta_2^3,\eta_3,\ldots,\eta_n)}U_1
\end{aligned}$$

$\Rightarrow$

$$\begin{aligned}
&\mathcal{I}_{(\mu_1,\eta_2^1,\eta_3,\dots,\eta_n)}U_1 + \mathcal{I}_{(\mu_1,\eta_2^2,\eta_3,\dots,\eta_n)}U_1 + \mathcal{I}_{(\mu_1,\eta_2^3,\eta_3,\dots,\eta_n)}U_1 \\
&\quad > \quad (1-\alpha_1)\Big[\mathcal{I}_{(\mu_1,\eta_2^1,\eta_3,\dots,\eta_n)}U_1 + \mathcal{I}_{(\mu_1,\eta_2^2,\eta_3,\dots,\eta_n)}U_1 + \mathcal{I}_{(\mu_1,\eta_2^3,\eta_3,\dots,\eta_n)}U_1\Big] \\
&\qquad + \alpha_1(1-\alpha_2)\Big[\mathcal{I}_{(\nu_1,\eta_2^1,\eta_3,\dots,\eta_n)}U_1 + \mathcal{I}_{(\nu_1,\eta_2^2,\eta_3,\dots,\eta_n)}U_1 + \mathcal{I}_{(\nu_1,\eta_2^3,\eta_3,\dots,\eta_n)}U_1\Big] \\
&\qquad + \alpha_1\alpha_2\Big[\mathcal{I}_{(\kappa_1,\eta_2^1,\eta_3,\dots,\eta_n)}U_1 + \mathcal{I}_{(\kappa_1,\eta_2^2,\eta_3,\dots,\eta_n)}U_1 + \mathcal{I}_{(\kappa_1,\eta_2^3,\eta_3,\dots,\eta_n)}U_1\Big]
\end{aligned}$$

$\Rightarrow$

$$\begin{aligned}
&\mathcal{I}_{(\mu_1,\eta_2^1,\eta_3,\dots,\eta_n)}U_1 + \mathcal{I}_{(\mu_1,\eta_2^2,\eta_3,\dots,\eta_n)}U_1 + \mathcal{I}_{(\mu_1,\eta_2^3,\eta_3,\dots,\eta_n)}U_1 \\
&\quad > \quad (1-\alpha_2)\Big[\mathcal{I}_{(\nu_1,\eta_2^1,\eta_3,\dots,\eta_n)}U_1 + \mathcal{I}_{(\nu_1,\eta_2^2,\eta_3,\dots,\eta_n)}U_1 + \mathcal{I}_{(\nu_1,\eta_2^3,\eta_3,\dots,\eta_n)}U_1\Big] \\
&\qquad + \alpha_2\Big[\mathcal{I}_{(\kappa_1,\eta_2^1,\eta_3,\dots,\eta_n)}U_1 + \mathcal{I}_{(\kappa_1,\eta_2^2,\eta_3,\dots,\eta_n)}U_1 + \mathcal{I}_{(\kappa_1,\eta_2^3,\eta_3,\dots,\eta_n)}U_1\Big]
\end{aligned}$$

$\Rightarrow$

$$\begin{aligned}
&\mathcal{I}_{(\mu_1,\eta_2^1,\eta_3^1,\dots,\eta_n^1)}U_1 + \mathcal{I}_{(\mu_1,\eta_2^1,\eta_3^2,\dots,\eta_n^1)}U_1 + \dots + \mathcal{I}_{(\mu_1,\eta_2^1,\eta_3^3,\dots,\eta_n^3)}U_1 + \dots \\
&+\mathcal{I}_{(\mu_1,\eta_2^2,\eta_3^1,\dots,\eta_n^1)}U_1 + \dots + \mathcal{I}_{(\mu_1,\eta_2^2,\eta_3^2,\dots,\eta_n^2)}U_1 + \dots + \mathcal{I}_{(\mu_1,\eta_2^2,\eta_3^3,\dots,\eta_n^3)}U_1 + \dots \\
&+\mathcal{I}_{(\mu_1,\eta_2^3,\eta_3^1,\dots,\eta_n^1)}U_1 + \dots + \mathcal{I}_{(\mu_1,\eta_2^3,\eta_3^3,\dots,\eta_n^3)}U_1 \\
&\quad > \quad (1-\alpha_2)\Big[\mathcal{I}_{(\nu_1,\eta_2^1,\eta_3^1,\dots,\eta_n^1)}U_1 + \dots + \mathcal{I}_{(\nu_1,\eta_2^1,\eta_3^3,\dots,\eta_n^3)}U_1 \\
&\qquad\qquad + \mathcal{I}_{(\nu_1,\eta_2^2,\eta_3,\dots,\eta_n)}U_1 + \mathcal{I}_{(\nu_1,\eta_2^3,\eta_3,\dots,\eta_n)}U_1\Big] \\
&\qquad + \alpha_2\Big[\mathcal{I}_{(\kappa_1,\eta_2^1,\eta_3^1,\dots,\eta_n^1)}U_1 + \dots + \mathcal{I}_{(\kappa_1,\eta_2^1,\eta_3^3,\dots,\eta_n^3)}U_1 \\
&\qquad\qquad + \mathcal{I}_{(\kappa_1,\eta_2^2,\eta_3,\dots,\eta_n)}U_1 + \mathcal{I}_{(\kappa_1,\eta_2^3,\eta_3,\dots,\eta_n)}U_1\Big]
\end{aligned}$$

$\Rightarrow$

$$\begin{aligned}
&\mathcal{I}_{(\mu_1,\eta_2^1,\eta_3^1,\dots,\eta_n^1)}U_1 + \mathcal{I}_{(\mu_1,\eta_2^2,\eta_3^2,\dots,\eta_n^2)}U_1 + \mathcal{I}_{(\mu_1,\eta_2^3,\eta_3^3,\dots,\eta_n^3)}U_1 \\
&\qquad > \; (1-\alpha_2)\mathcal{I}_{(\nu_1,\eta_2^1,\eta_3^1,\dots,\eta_n^1)}U_1 + \alpha_2\mathcal{I}_{(\kappa_1,\eta_2^1,\eta_3^1,\dots,\eta_n^1)}U_1 + O(\alpha_1)
\end{aligned}$$

$\Rightarrow$

$$
\begin{aligned}
&(1-\alpha_1)^{n-1}\mathcal{I}_{(\mu_1,\mu_2,\mu_3,\dots,\mu_n)}U_1\\
&\qquad + \alpha_2^{n-1}(1-\alpha_1)^{n-1}\mathcal{I}_{(\mu_1,\nu_2,\nu_3,\dots,\nu_n)}U_1 + \alpha_2^{n-1}\alpha_1^{n-1}\mathcal{I}_{(\mu_1,\kappa_2,\kappa_3,\dots,\kappa_n^3)}U_1\\
&\qquad\quad > (1-\alpha_2)(1-\alpha_1)^{n-1}\mathcal{I}_{(\nu_1,\mu_2,\mu_3,\dots,\mu_n)}U_1\\
&\qquad\qquad + \alpha_2(1-\alpha_1)^{n-1}\mathcal{I}_{(\kappa_1,\mu_2,\mu_3,\dots,\mu_n)}U_1 + O(\alpha_1)
\end{aligned}
$$

$$\Rightarrow$$

$$
\begin{aligned}
&(1-\alpha_2)(1-\alpha_1)^{n-1}\mathcal{I}_{(\mu_1,\mu_2,\mu_3,\dots,\mu_n)}U_1\\
&\quad + \alpha_2^{n-1}(1-\alpha_1)^{n-1}\mathcal{I}_{(\mu_1,\nu_2,\nu_3,\dots,\nu_n)}U_1 + \alpha_2^{n-1}\alpha_1^{n-1}\mathcal{I}_{(\mu_1,\kappa_2,\kappa_3,\dots,\kappa_n^3)}U_1\\
&\qquad > (1-\alpha_2)(1-\alpha_1)^{n-1}\mathcal{I}_{(\nu_1,\mu_2,\mu_3,\dots,\mu_n)}U_1\\
&\qquad\quad - \alpha_2(1-\alpha_1)^{n-1}\Big[\mathcal{I}_{(\mu_1,\mu_2,\mu_3,\dots,\mu_n)}U_1 - \mathcal{I}_{(\kappa_1,\mu_2,\mu_3,\dots,\mu_n)}U_1\Big] + O(\alpha_1),
\end{aligned}
$$

and (3.30) follows. The inequality (3.32) is obtained similarly. □

A.2 Technical issues for metrics on $\mathbb{P}(A)$

Proposition A.1. *Let (A, r) be a separable metric space. Then the Prokhorov metric r_p and the bounded Lipschitz metric r_{bl} metrize the weak convergence, i.e., for any sequence $\{\mu_n\} \subset \mathbb{P}(A)$, the following statements are equivalent;*

i) *μ_n converges in the weak topology,*

ii) $r_p(\mu_n, \mu) \to 0$,

iii) $r_{bl}(\mu_n, \mu) \to 0$.

Moreover, for any μ and ν in $\mathbb{P}(A)$

$$\frac{1}{3}[r_p(\mu,\nu)]^2 \le r_{bl}(\mu,\nu) \le 2r_p(\mu,\nu).$$

Proof. See Shiryaev (1996) chapter 3. □

Proposition A.2. *Let (A, r) be a Polish space and $1 \le p < \infty$. The L_p-Wasserstein metric r_{w_p} metrizes the weak convergence on $\mathbb{P}_p(A)$, i.e., for any sequence $\{\mu_n\} \subset \mathbb{P}_p(A)$ and $\{\mu\} \subset \mathbb{P}(A)$, the following conditions are equivalent* *i)* μ_n *converges in the weak topology,*

ii) $r_{w_p}(\mu_n, \mu) \to 0$.

Moreover, if A is bounded, then the L_p*-Wasserstein metric* r_{w_p}*, the Prokhorov metric* r_p *, the bounded Lipschitz metric* r_{bl} *and the Kantorovich-Rubinstein metric* r_{kr} *all metrize the weak convergence of probability measures in* $\mathbb{P}(A)$*. Moreover, if* $p = 1$ *then*

$$\frac{1}{3}[r_p(\mu,\nu)]^2 \leq r_{bl}(\mu,\nu) \leq r_{kr}(\mu,\nu) = r_w(\mu,\nu).$$

Proof. See Shiryaev (1996) chapter 3, and Givens and Shortt (1984). □

Proposition A.3. *Let A be a separable metric space. Let* μ *and* ν *in* $\mathbb{P}(A)$*, with* $\nu \ll \mu$*. Then*

$$\|\mu - \nu\| \leq 2[K(\mu,\nu)]^{\frac{1}{2}}.$$

Moreover, if A is a bounded (with diameter $C > 0$*) Polish space, then*

$$r_w(\mu,\nu) \leq C\|\mu - \nu\| \leq 2C[K(\mu,\nu)]^{\frac{1}{2}}.$$

Proof. See Reiss (1989) chapter 3, and Villani (2008) chapter 6. □

Proposition A.4. *Let* (A, r) *a separable metric space and* $1 \leq p < \infty$*. If* μ *and* ν *are in* $\mathbb{P}(A)$*, then*

$$r_{w_p}(\mu,\nu) \leq 2^{\frac{1}{q}} \left[\int_A [r(a,a_0)]^p |\mu - \nu|(dx)\right]^{\frac{1}{p}}, \qquad \frac{1}{p} + \frac{1}{q} = 1.$$

In particular, if A is bounded with diameter $C > 0$*, then*

$$r_w(\mu,\nu) \leq C\|\mu - \nu\|.$$

Proof. Villani (2008) chapter 6. □

A.3 Proof of Lemmas 5.1, 5.2, and 5.9

For the proof of Lemmas 5.2 and 5.9, it is convenient to rewrite (2.1) as in (2.3), that is

$$\mathcal{I}_{(\mu_1,\ldots,\mu_n)} U_i := \int_{A_1} \ldots \int_{A_n} U_i(a_1,\ldots,a_n)\mu_n(da_n)\ldots\mu_1(da_1). \tag{A.1}$$

Hence (2.2) becomes

$$\begin{aligned} J_i(a_i, \mu_{-i}) &= \int_{A_{-i}} U_i(a_i, a_{-i}) \mu_{-i}(da_{-i}) \\ &= \mathcal{I}_{(\mu_1,\ldots,\mu_{i-1},\mu_{i+1},\ldots,\mu_n)} U_i(a_i). \end{aligned} \tag{A.2}$$

A.3.1 Proof of Lemma 5.1

,

We have the following inequalities

$$\begin{aligned} \frac{d\|\mu(t)\|_\infty}{dt} &= \frac{d}{dt} \max_{i\in I} [\|\mu_i(t)\|] \\ &= \lim_{\epsilon\to 0} \frac{1}{\epsilon} \left[\max_{i\in I} [\|\mu_i(t+\epsilon)\|] - \max_{i\in I} [\|\mu_i(t)\|] \right] \\ &\leq \lim_{\epsilon\to 0} \frac{1}{\epsilon} \left[\max_{i\in I} [\|\mu_i(t+\epsilon)\| - \|\mu_i(t)\|] \right] \\ &\leq \lim_{\epsilon\to 0} \frac{1}{\epsilon} \left[\max_{i\in I} [\|\mu_i(t+\epsilon) - \mu_i(t)\|] \right] \\ &= \max_{i\in I} \left[\lim_{\epsilon\to 0} \left\| \frac{\mu_i(t+\epsilon) - \mu_i(t)}{\epsilon} \right\| \right] \\ &= \max_{i\in I} [\|\mu_i'(t)\|] \\ &= \|\mu'(t)\|. \qquad \square \end{aligned}$$

A.3.2 Proof of Lemma 5.2

For any i in I and μ, ν in $\mathbb{P}(A_1) \times \ldots \times \mathbb{P}(A_n)$, using (A.1) we obtain

$$
\begin{aligned}
\left|\int_A U_i(a)\eta(da) - \int_A U_i(a)\nu(da)\right| & \\
\leq\ & |\mathcal{I}_{(\eta_1,\eta_2,\ldots,\eta_n)}U_i - \mathcal{I}_{(\nu_1,\eta_2,\ldots,\eta_n)}U_i| \\
& +|\mathcal{I}_{(\nu_1,\eta_2,\eta_3,\ldots,\eta_n)}U_i - \mathcal{I}_{(\nu_1,\nu_2,\eta_3,\ldots,\eta_n)}U_i| \\
& + \ldots \\
& +|\mathcal{I}_{(\nu_1,\ldots,\nu_{n-2},\eta_{n-1},\eta_n)}U_i - \mathcal{I}_{(\nu_1,\ldots,\nu_{n-2},\nu_{n-1},\eta_n)}U_i| \\
& +|\mathcal{I}_{(\nu_1,\ldots,\nu_{n-1},\eta_n)}U_i - \mathcal{I}_{(\nu_1,\ldots,\nu_{n-1},\nu_n)}U_i| \\
\leq\ & \|U_i\|\|\eta_2 \times \ldots. \times \eta_n\|\|\eta_1 - \nu_1\| \\
& +\|U_i\|\|\nu_1 \times \eta_3 \times \ldots \times \eta_n\|\|\eta_2 - \nu_2\| \\
& + \ldots \\
& +\|U_i\|\|\nu_1 \times \ldots \times \nu_{n-2} \times \eta_n\|\|\eta_{n-1} - \nu_{n-1}\| \\
& +\|U_i\|\|\nu_1 \times \ldots. \times \nu_{n-1}\|\|\eta_n - \nu_n\| \\
\leq\ & n\|U_i\| \max_{j\in I}\|\eta_j - \nu_j\|. \qquad (A.3)
\end{aligned}
$$

Similarly, using (A.2),

$$|J_i(a_i,\mu_{-i}) - J_i(a_i,\nu_{-i})| \leq (n-1)\|U_i\|\|\mu - \nu\|_\infty. \qquad (A.4)$$

Using (A.3) and (A.4) we have

$$
\begin{aligned}
\|F_i(\mu) - F_i(\nu)\|_\infty &= \sup_{\|f\|\leq 1}\int_{A_i} f(a_i)[F_i(\mu) - F_i(\nu)](da_i) \\
&\leq \sup_{\|f\|\leq 1}\int_{A_i} f(a_i)|J_i(a_i,\mu_{-i})|[\mu_i - \nu_i](da) \\
&\quad + \sup_{\|f\|\leq 1}\int_{A_i} f(a_i)|J_i(a_i,\mu_{-i}) - J_i(a_i,\nu_{-i})|\nu_i(da) \\
&\quad + \sup_{\|f\|\leq 1}\int_{A} f(a_i)|J_i(\mu_i,\mu_{-i})|[\nu_i - \mu_i](da) \\
&\quad + \sup_{\|f\|\leq 1}\int_{A} f(a_i)|J_i(\nu_i,\nu_{-i}) - J(\mu_i,\mu_{-i})|\nu_i(da) \\
&\leq \|U_i\|\|\mu_i - \nu_i\| + (n-1)\|U_i\|\|\mu - \nu\|_\infty\|\nu_i\| \\
&\quad + \|U_i\|\|\mu_i - \nu_i\| + n\|U_i\|\|\mu - \nu\|_\infty\|\nu_i\| \\
&\leq H\|\mu - \nu\|_\infty + (n-1)H\|\mu - \nu\|_\infty + H\|\mu - \nu\|_\infty + nH\|\mu - \nu\|_\infty \\
&= (2n+1)H\|\mu - \nu\|_\infty,
\end{aligned}
$$

where $H := \max_{i\in I}\|U_i\|$. □

A.3.3 Proof of Lemma 5.9

For any i and j in I and a_{-j} in A_{-j} let

$$\|U_i(\cdot, a_{-j})\|_L := \sup_{a_j, b_j \in A_j} \frac{|U_i(a_j, a_{-j}) - U_i(b_j, a_{-j})|}{\vartheta^*((a_j, a_{-j}), (b_j, a_{-j}))} \leq \|U_i\|_L, \quad \text{and}$$

$$U_i^j := \frac{U_i(a_j, a_{-j})}{\|U_i(\cdot, a_{-j})\|_L}$$

Then for any i in I and μ, ν in $\mathbb{P}(A_1) \times ... \times \mathbb{P}(A_1)$, using (A.1) we see that

$$\begin{aligned}
&\left| \int_A U_i(a)\eta(da) - \int_A U_i(a)\nu(da) \right| \\
&\leq \|U_i(\cdot, a_{-1})\|_L |\mathcal{I}_{(\eta_1,\eta_2,...,\eta_n)} U_i^1 - \mathcal{I}_{(\nu_1,\eta_2,...,\eta_n)} U_i^1| \\
&\quad + \|U_i(\cdot, a_{-2})\|_L |\mathcal{I}_{(\nu_1,\eta_2,\eta_3,...,\eta_n)} U_i^2 - \mathcal{I}_{(\nu_1,\nu_2,\eta_3,...,\eta_n)} U_i^2| \\
&\quad + ... \\
&\quad + \|U_i(\cdot, a_{-(n-1)})\|_L |\mathcal{I}_{(\nu_1,...,\nu_{n-2},\eta_{n-1},\eta_n)} U_i^{n-1} - \mathcal{I}_{(\nu_1,...,\nu_{n-2},\nu_{n-1},\eta_n)} U_i^{n-1}| \\
&\quad + \|U_i(\cdot, a_{-n})\|_L |\mathcal{I}_{(\nu_1,...,\nu_{n-1},\eta_n)} U_i^n - \mathcal{I}_{(\nu_1,...,\nu_{n-1},\nu_n)} U_i^n| \\
&\leq \|U_i\|_L \|\eta_2 \times .... \times \eta_n\| \|\eta_1 - \nu_1\|_{kr} \\
&\quad + \|U_i\|_L \|\nu_1 \times \eta_3 \times ... \times \eta_n\| \|\eta_2 - \nu_2\|_{kr} \\
&\quad + ... \\
&\quad + \|U_i\|_L \|\nu_1 \times ... \times \nu_{n-2} \times \eta_n\| \|\eta_{n-1} - \nu_{n-1}\|_{kr} \\
&\quad + \|U_i\|_L \|\nu_1 \times .... \times \nu_{n-1}\| \|\eta_n - \nu_n\|_{kr} \\
&\leq n\|U_i\|_L \|\eta_j - \nu_j\|_\infty^{kr}. \qquad \text{(A.5)}
\end{aligned}$$

Similary, using (A.2),

$$|J_i(a_i, \mu_{-i}) - J_i(a_i, \nu_{-i})| \leq (n-1)\|U_i\|_L \|\mu - \nu\|_\infty^{kr}. \qquad \text{(A.6)}$$

Using (A.5) and (A.6) we have

$$\|F_i(\mu) - F_i(\nu)\|_{kr}$$

$$
\begin{aligned}
&= \sup_{\substack{\|f\|_L \le 1 \\ f(a_0)=0}} \int_{A_i} f(a_i)[F_i(\mu) - F_i(\nu)](da_i) \\
&\le \sup_{\substack{\|f\|_L \le 1 \\ f(a_0)=0}} \int_{A_i} f(a_i)|J_i(a_i, \mu_{-i})|[\mu_i - \nu_i](da) \\
&\quad + \sup_{\substack{\|f\|_L \le 1 \\ f(a_0)=0}} \int_{A_i} f(a_i)|J_i(a_i, \mu_{-i}) - J_i(a_i, \nu_{-i})|\nu_i(da) \\
&\quad + \sup_{\substack{\|f\|_L \le 1 \\ f(a_0)=0}} \int_{A} f(a_i)|J_i(\mu_i, \mu_{-i})|[\nu_i - \mu_i](da) \\
&\quad + \sup_{\substack{\|f\|_L \le 1 \\ f(a_0)=0}} \int_{A} f(a_i)|J_i(\nu_i, \nu_{-i}) - J(\mu_i, \mu_{-i})|\nu_i(da) \\
&\le \|U_i\|\|\mu_i - \nu_i\|_{kr} + (n-1)\|U_i\|_L\|\mu - \nu\|_\infty^{kr} \sup_{\substack{\|f\|_L \le 1 \\ f(a_0)=0}} \int_{A_i} f(a_i)\nu_i(da_i) \\
&\quad + \|U_i\|\|\mu_i - \nu_i\|_{kr} + n\|U_i\|_L\|\mu - \nu\|_\infty^{kr} \sup_{\substack{\|f\|_L \le 1 \\ f(a_0)=0}} \int_{A_i} f(a_i)\nu_i(da_i) \\
&\le 2H\|\mu - \nu\|_\infty^{kr} + (2n-1)H_L\|\mu - \nu\|_\infty^{kr} C_i \\
&= [2H + (2n-1)CH_L]\|\mu - \nu\|_\infty.
\end{aligned}
$$

where $H := \max_{i \in I} \|U_i\|$, $H_L := \max_{i \in I} \|U_i\|_L$, and $C := \max_{i \in I} C_i$. □