Chapter 2: Integration

Part B: Examples and Applications of Integration

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Table of Contents

Monotone Functions are Integrable

Theorem

Let $f : [a, b] \to \mathbb{R}$ be a monotone function. Then f is integrable on $[a, b]$.

Proof. Suppose f is an increasing function. Let $P = \{x_0, \ldots, x_n\}$ be the partition of $[a, b]$ that cuts it into n equally sized subintervals, each of length $(b - a)/n$.

Define step functions s and t on $[a, b]$ by

$$s(x) = f(x_{i-1}) \quad \text{and} \quad t(x) = f(x_i) \qquad \text{if } x_{i-1} \leq x < x_i,$$

and $s(b) = t(b) = f(b)$. Then we have $s(x) \leq f(x) \leq t(x)$ for every $x \in [a, b]$.

Monotone Functions are Integrable (cont.)

Further,

$$\begin{aligned}\int_a^b t(x)\,dx - \int_a^b s(x)\,dx &= \sum_{i=1}^{n} f(x_i)\frac{b-a}{n} - \sum_{i=1}^{n} f(x_{i-1})\frac{b-a}{n} \\ &= \frac{b-a}{n}\sum_{i=1}^{n}\Big(f(x_i) - f(x_{i-1})\Big) \\ &= \frac{b-a}{n}\Big(f(b) - f(a)\Big).\end{aligned}$$

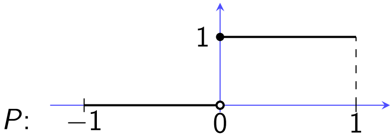
By the Archimedean property, this quantity can be made smaller than any given positive ϵ by taking a large enough n.

Hence f is integrable, by the Riemann condition.

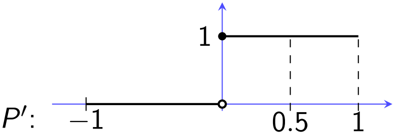
The same approach works for decreasing functions. We just switch the definitions of s and t.

An Example of a Step Function

Consider the Heaviside step function, with the domain restricted to $[-1, 1]$. It is constant on the intervals $(-1, 0)$ and $(0, 1)$, hence the partition $P = \{-1, 0, 1\}$ is adapted to it.



On the other hand, the partition $P' = \{-1, 0.5, 1\}$ is not adapted to it, since the function takes two values on the interval $(-1, 0.5)$.



Integral of a Step Function

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Let $s\colon [a,b] \to \mathbb{R}$ be a step function with adapted partition $P = \{x_0, \ldots, x_n\}$, such that $s(x) = s_i$ if $x_{i-1} < x < x_i$. Then the **integral of** s **from** a **to** b is defined by

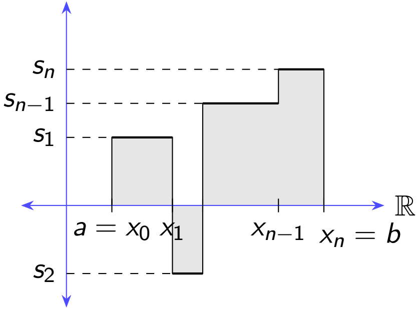
$$\int_a^b s = \int_a^b s(x)\,dx = \sum_{i=1}^{n} s_i(x_i - x_{i-1}).$$

Example: Consider the Heaviside step function H, with the domain as $[-2,3]$, and the adapted partition $P = \{-2,0,3\}$. It takes values $s_1 = 0$ on $(-2,0)$ and $s_2 = 1$ on $(0,3)$. Therefore,

$$\begin{aligned}\int_{-2}^{3} H(x)\,dx &= s_1(x_1 - x_0) + s_2(x_2 - x_1)\\ &= 0 \cdot (0 - (-2)) + 1 \cdot (3 - 0) = 3.\end{aligned}$$

Interpreting the Integral

The integral represents the total **signed area** of the rectangles enclosed by the graph of $s(x)$, the x-axis, and the vertical lines $x = a$ and $x = b$.



The term 'signed area' refers to the area of a rectangle marked by the step function being taken as positive if the rectangle lies above the x-axis and as negative if it lies below the x-axis.

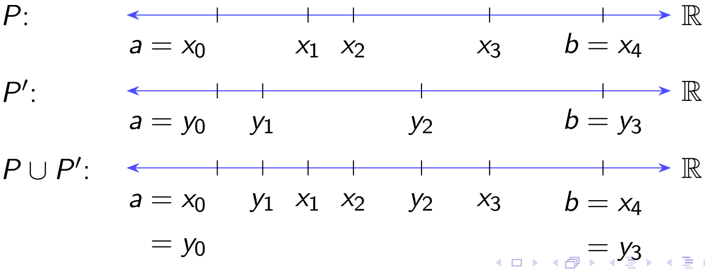
Refinements of Partitions

Let P, P' be partitions of $[a, b]$. We say P' is a **refinement** of P if $P \subseteq P'$.

Task

Suppose a partition P is adapted to a step function s. Show that every refinement of P is also adapted to s.

Note that if P, P' are partitions of $[a, b]$ then $P \cup P'$ is a common refinement for both of them.



Integral is Independent of Choice of Partition

Theorem

Suppose $s\colon [a,b] \to \mathbb{R}$ is a step function and P, Q are partitions of $[a,b]$ which are adapted to s. Then both P and Q lead to the same value of $\int_a^b s(x)\,dx$.

Proof. Let $I(P)$ be the value of the integral of s corresponding to the partition P. We need to prove that $I(P) = I(Q)$. It suffices to prove this when one partition is a refinement of the other.

For, if we have proved this, we'll have $I(P) = I(P \cup Q) = I(Q)$.

Next, it suffices to prove $I(P) = I(Q)$ when Q has just one point more than P. Let $P = \{x_0, \ldots, x_n\}$ and for each i, let s take the value s_i on (x_{i-1}, x_i).

Let $Q = \{x_0, \ldots, x_{k-1}, t, x_k, \ldots, x_n\}$.

Integral is Independent of Choice of Partition (cont.)

Then

$$
\begin{aligned}
I(Q) &= \sum_{i=1}^{k-1} s_i(x_i - x_{i-1}) + s_k(t - x_{k-1}) + s_k(x_k - t) \\
&\quad + \sum_{i=k+1}^{n} s_i(x_i - x_{i-1}) \\
&= \sum_{i=1}^{k-1} s_i(x_i - x_{i-1}) + s_k(x_k - x_{k-1}) + \sum_{i=k+1}^{n} s_i(x_i - x_{i-1}) \\
&= \sum_{i=1}^{n} s_i(x_i - x_{i-1}) = I(P).
\end{aligned}
$$

□

Comparison Theorem

Task

Suppose $s, t\colon [a,b] \to \mathbb{R}$ are step functions. Show there is a partition which is adapted to both s and t.

Theorem (Comparison Theorem)

Let $s, t\colon [a,b] \to \mathbb{R}$ be step functions such that $s(x) \leq t(x)$ for every $x \in [a,b]$. Then

$$\int_a^b s(x)\,dx \leq \int_a^b t(x)\,dx.$$

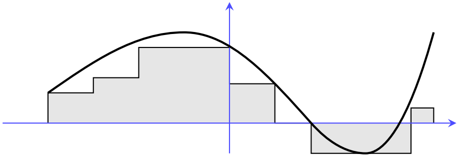
Proof. Let $P = \{x_0, \ldots, x_n\}$ be a partition adapted to both s and t. Let $s(x) = s_i$, $t(x) = t_i$ for $x \in (x_{i-1}, x_i)$. Then $s_i \leq t_i$ and

$$\int_a^b s(x)\,dx = \sum_{i=1}^n s_i(x_i - x_{i-1}) \leq \sum_{i=1}^n t_i(x_i - x_{i-1}) = \int_a^b t(x)\,dx.$$

Table of Contents

Lower and Upper Sums

A function $f\colon [a,b] \to \mathbb{R}$ is **bounded** by a real number M if $-M \le f(x) \le M$ for every $x \in [a,b]$.
Let $f\colon [a,b] \to \mathbb{R}$ be bounded by M. Consider a step function $s\colon [a,b] \to \mathbb{R}$ such that $s(x) \le f(x)$ for every $x \in [a,b]$.



We view $\int_a^b s$ as an underestimate of the 'signed area' under the graph of f and call it a **lower sum** for f.

Similarly, if $t\colon [a,b] \to \mathbb{R}$ is a step function such that $t(x) \ge f(x)$ for every $x \in [a,b]$, then $\int_a^b t$ is viewed as an overestimate of the 'signed area' under the graph of f and called an **upper sum** for f.

Consider the collection of all lowers sums,

$$\mathcal{L}_f = \left\{ \int_a^b s(x)\,dx \;\middle|\; s\colon [a,b] \to \mathbb{R} \text{ is a step function and } s(x) \le f(x) \text{ for every } x \in [a,b] \right\}.$$

$\mathcal{L}_f$ is non-empty because the constant function $-M$ is a step function whose values never exceed those of f.

Similarly, we have the non-empty collection of all upper sums:

$$\mathcal{U}_f = \left\{ \int_a^b t(x)\,dx \;\middle|\; t\colon [a,b] \to \mathbb{R} \text{ is a step function and } t(x) \ge f(x) \text{ for every } x \in [a,b] \right\}.$$

Theorem

Let $f\colon [a,b] \to \mathbb{R}$ be a bounded function. Then $\ell \in \mathcal{L}_f$ and $u \in \mathcal{U}_f$ implies $\ell \leq u$.

Proof. Let $s, t\colon [a,b] \to \mathbb{R}$ be step functions such that

- $s(x) \leq f(x)$ for every $x \in [a,b]$ and $\int_a^b s(x)\,dx = \ell$.
- $t(x) \geq f(x)$ for every $x \in [a,b]$ and $\int_a^b t(x)\,dx = u$.

Then $s(x) \leq f(x) \leq t(x)$ for every $x \in [a,b]$. Hence, by the Comparison Theorem,

$$\ell = \int_a^b s(x)\,dx \leq \int_a^b t(x)\,dx = u. \qquad \square$$

By the Completeness Axiom there is a number I such that $\ell \leq I \leq u$ for every $\ell \in \mathcal{L}_f$ and $u \in \mathcal{U}_f$. This I is our natural candidate for the value of the signed area under the graph of f.

Dirichlet Function

The **Dirichlet function** $D\colon [0,1] \to \mathbb{R}$ is defined by

$$D(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [0,1], \\ 0 & \text{if } x \in \mathbb{Q}^c \cap [0,1]. \end{cases}$$

Let $s\colon [0,1] \to \mathbb{R}$ be a step function such that $s(x) \leq D(x)$ for every $x \in [0,1]$. Each open subinterval of $[0,1]$ contains an irrational number, hence $s(x) \leq 0$ on each open subinterval, and so $\int_0^1 s(x)\,dx \leq 0$.

Similarly, we see that $\int_0^1 t(x)\,dx \geq 1$ if $t\colon [0,1] \to \mathbb{R}$ is a step function such that $t(x) \geq D(x)$ for every $x \in [0,1]$.

Therefore every α between 0 and 1 satisfies $\ell \leq \alpha \leq u$ for every $\ell \in \mathcal{L}_D$ and $u \in \mathcal{U}_D$.

Integrable Functions

For the Dirichlet function, our approach fails to successfully assign a unique 'signed area'.

For the functions where our approach *does* work we have a special term:

A bounded function $f\colon [a,b] \to \mathbb{R}$ is called **integrable** if there is a unique number I such that $\ell \leq I \leq u$ for every $\ell \in \mathcal{L}_f$ and $u \in \mathcal{U}_f$. This unique I is called the **(definite) integral** of f on $[a,b]$ and is denoted by $\int_a^b f(x)\,dx$ or $\int_a^b f$.

Integral of x over $[0, 1]$

Consider $f(x) = x$ on $[0, 1]$. Take the partition $P : x_0 < \cdots < x_n$ which cuts $[0, 1]$ into n equal subintervals. That is, $x_i = i/n$. Define step functions s and t:

$$s(x) = x_{i-1} \quad \text{and} \quad t(x) = x_i \qquad \text{if } x_{i-1} \leq x < x_i,$$

and $s(1) = t(1) = 1$. Then

$$\int_0^1 s(x)\,dx = \frac{1}{n}\sum_{i=1}^{n} x_{i-1} = \frac{1}{n}\sum_{i=1}^{n} \frac{i-1}{n} = \frac{n-1}{2n} = \frac{1}{2} - \frac{1}{2n},$$

$$\int_0^1 t(x)\,dx = \frac{1}{n}\sum_{i=1}^{n} x_i = \cdots = \frac{1}{2} + \frac{1}{2n}.$$

$1/2$ is the only number that fits between all these lower and upper sums, and hence it is the integral of f over $[0, 1]$.

Riemann Condition

Theorem (Riemann Condition)

Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then f is integrable on $[a, b]$ if and only if for each $\epsilon > 0$ there are $\ell \in \mathcal{L}_f$ and $u \in \mathcal{U}_f$ such that $u - \ell \leq \epsilon$.

Proof. First, suppose f is not integrable. Since f is not integrable, there are two numbers I_1, I_2 such that

$$\ell \leq I_1 < I_2 \leq u, \text{ for every } \ell \in \mathcal{L}_f, \; u \in \mathcal{U}_f.$$

Consider $\epsilon = \frac{1}{2}(I_2 - I_1)$. Then $\ell \in \mathcal{L}_f, \; u \in \mathcal{U}_f$ implies $u - \ell > \epsilon$.

Now, suppose f has integral I. Given any $\epsilon > 0$, consider $(I - \epsilon/2, I + \epsilon/2)$. If this does not contain any lower sum of f, then $I - \epsilon/2$ will also meet the conditions for the integral of f. So this interval must contain some $\ell \in \mathcal{L}_f$. Similarly, it must contain some $u \in \mathcal{U}_f$. Then $u - \ell < \epsilon$. □

Riemann Condition (ver. 2)

Theorem

Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then $I \in \mathbb{R}$ is the integral of f over $[a, b]$ if and only if for each $\epsilon > 0$ there are $\ell \in \mathcal{L}_f$ and $u \in \mathcal{U}_f$ such that $I - \ell < \epsilon$ and $u - I < \epsilon$.

Proof. Suppose I is the integral of f over $[a, b]$, and $\epsilon > 0$. If $(I - \epsilon, I] \cap \mathcal{L}_f = \emptyset$ then $I - \epsilon$ also satisfies the definition of integral of f. Hence there is an $\ell \in \mathcal{L}_f$ such that $I - \epsilon < \ell \leq I$. Similarly, there is $u \in \mathcal{U}_f$ such that $I \leq u < I + \epsilon$.

Suppose such ℓ, u exist for every $\epsilon > 0$. By the earlier Riemann Condition, f is integrable. So we only need to show I is between $\mathcal{L}_f$ and $\mathcal{U}_f$. Suppose $\ell \in \mathcal{L}_f$ and $\ell > I$. Then there is $u \in \mathcal{U}_f$ such that $I < u < \ell$, a contradiction. Therefore $\ell \leq I$ for every $\ell \in \mathcal{L}_f$. Similarly, Therefore $I \leq u$ for every $u \in \mathcal{U}_f$. □

Two conventions

- $\int_a^a f(x)\,dx = 0.$
- If $a < b$ then $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$

The first is consistent with line segments having zero area. The second takes into account the direction of travel.

Table of Contents

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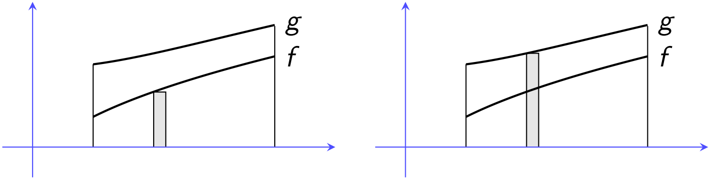
Comparison

We now take up various general properties of integration. We first present the corresponding diagrams. The idea is that the general patterns can be intuited from what happens to rectangles. We begin with the general statement of the Comparison Theorem.

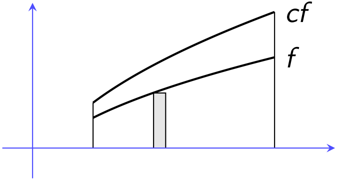
- $f \leq g$ on $[a, b]$ implies $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$.

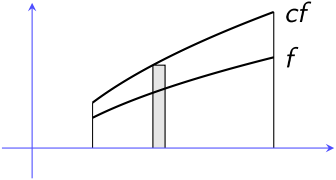


The rectangles for g are higher than those for f.

Homogeneity

- $\int_a^b c\, f(x)\, dx = c \int_a^b f(x)\, dx.$

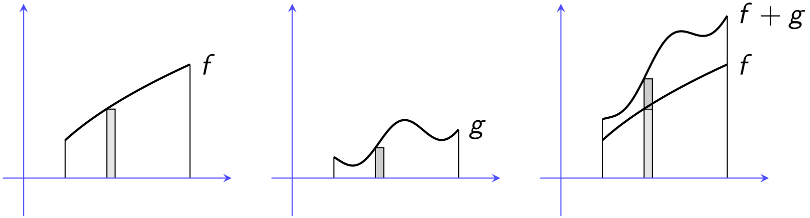




Each rectangle gets scaled vertically by c, hence so does its area.

Additivity

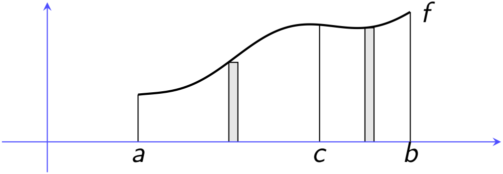
- $$\int_a^b \big(f(x)+g(x)\big)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$



The rectangles for g are placed on top of those for f to get the ones for $f+g$.

Additivity over Intervals

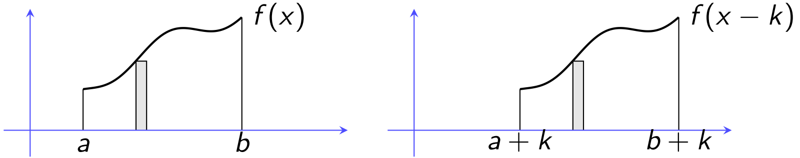
- $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$



The rectangles for $[a, c]$ and $[c, b]$ are pooled to get all the ones for $[a, b]$.

Shift of Interval of Integration

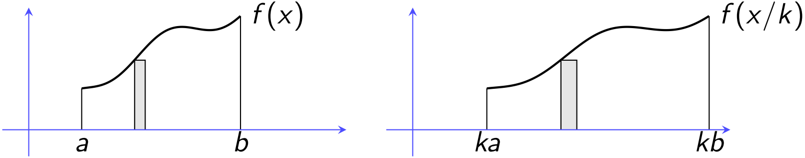
- $\displaystyle\int_{a+k}^{b+k} f(x-k)\,dx = \int_a^b f(x)\,dx.$



Each rectangle shifts without changing its dimensions.

Scaling of Interval of Integration

- $k > 0 \implies \int_{ka}^{kb} f(x/k)\,dx = k \int_a^b f(x)\,dx.$



The width of each rectangle is scaled by k, hence so is its area.

Comparison Theorem

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Theorem

Suppose $f, g\colon [a, b] \to \mathbb{R}$ *are integrable functions such that* $f(x) \le g(x)$ *for every* $x \in [a, b]$. *Then*

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$$

Proof. Suppose that $\int_a^b f > \int_a^b g$. Let $s, t\colon [a, b] \to \mathbb{R}$ be step functions such that $s(x) \le f(x) \le t(x)$ for every $x \in [a, b]$.

We have $s(x) \le f(x) \le g(x)$ for every $x \in [a, b]$, and hence $\int_a^b s \le \int_a^b g$. We also have $\int_a^b g < \int_a^b f \le \int_a^b t$. Thus the integral of g satisfies the defining properties of the integral of f and so must be equal to it. But this contradicts the assumed inequality. □

Homogeneity

Theorem

Let $f : [a, b] \to \mathbb{R}$ be an integrable function and $c \in \mathbb{R}$. Then cf is integrable and

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx.$$

Proof. First, we prove the result for a step function s. Let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a, b]$, such that $s(x) = s_i$ if $x_{i-1} < x < x_i$. Then $c \cdot s(x) = c \cdot s_i$ if $x_{i-1} < x < x_i$. Hence

$$\int_a^b cs = \sum_{i=1}^n (cs_i) \cdot (x_i - x_{i-1}) = c\sum_{i=1}^n s_i \cdot (x_i - x_{i-1}) = c\int_a^b s.$$

Homogeneity (cont.)

Second, consider an arbitrary integrable function f and $c > 0$. Let $\epsilon > 0$. By the Riemann Condition there are step functions s, t such that $s(x) \leq f(x) \leq t(x)$ for every $x \in [a, b]$ and

$$\int_a^b f - \int_a^b s < \frac{\epsilon}{c}, \qquad \int_a^b t - \int_a^b f < \frac{\epsilon}{c}.$$

It follows that cs, ct are step functions such that $cs(x) \leq cf(x) \leq ct(x)$ for every $x \in [a, b]$ and

$$c\int_a^b f - \int_a^b cs < \epsilon, \qquad \int_a^b ct - c\int_a^b f < \epsilon.$$

By Riemann Condition again, cf is integrable and $\int_a^b cf = c\int_a^b f$.

The $c < 0$ case can be done in a similar fashion. The $c = 0$ case is trivial.

Additivity

Theorem

Let $f, g\colon [a,b] \to \mathbb{R}$ be integrable functions. Then $f+g$ is an integrable function and

$$\int_a^b \Big(f(x)+g(x)\Big)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

Proof. We first prove this for step functions. Suppose s, t are step functions on $[a,b]$. If partitions P_s and P_t are adapted to s and t respectively, then $P = P_s \cup P_t$ is adapted to all of s, t and $s+t$. Let $P = \{x_0 \ldots, x_n\}$, and

$$s(x) = s_i \quad \text{if} \quad x_{i-1} < x < x_i, \qquad t(x) = t_i \quad \text{if} \quad x_{i-1} < x < x_i.$$

Then $s(x) + t(x) = s_i + t_i$ if $x_{i-1} < x < x_i$.

Additivity (cont.)

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$$\int_a^b \left(s+t\right) = \sum_{i=1}^n (s_i + t_i)(x_i - x_{i-1})$$
$$= \sum_{i=1}^n s_i(x_i - x_{i-1}) + \sum_{i=1}^n t_i(x_i - x_{i-1}) = \int_a^b s + \int_a^b t.$$

Now consider any integrable functions $f, g\colon [a,b] \to \mathbb{R}$. Let $\epsilon > 0$. We have step functions s_f, s_g, t_f, t_g such that the following hold:

$$s_f(x) \leq f(x) \leq t_f(x) \text{ and } s_g(x) \leq g(x) \leq t_g(x),$$
$$\int_a^b f - \int_a^b s_f < \frac{\epsilon}{2} \text{ and } \int_a^b g - \int_a^b s_g < \frac{\epsilon}{2},$$
$$\int_a^b t_f - \int_a^b f < \frac{\epsilon}{2} \text{ and } \int_a^b t_g - \int_a^b g < \frac{\epsilon}{2}.$$

Additivity (cont.)

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Then $s_f + s_g$ and $t_f + t_g$ are step functions such that

$$s_f(x) + s_g(x) \leq f(x) + g(x) \leq t_f(x) + t_g(x),$$

$$\int_a^b f + \int_a^b g - \int_a^b \left(s_f + s_g\right) < \epsilon,$$

$$\int_a^b \left(t_f + t_g\right) - \int_a^b f - \int_a^b g < \epsilon.$$

Now apply the Riemann Condition. □

Additivity over Intervals

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Theorem

Let $a < c < b$ and suppose $f : [a, b] \to \mathbb{R}$ is a bounded function. Then

1. *f is integrable on $[a, b]$ if and only if f is integrable on both $[a, c]$ and $[c, b]$.*
2. *If f is integrable on $[a, b]$ then*
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Proof. Exercise. □

The equality $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ holds for *any* ordering of a, b, c. For example, suppose $a < b < c$. Then,

$$\int_a^c f + \int_c^b f = \int_a^b f + \int_b^c f - \int_b^c f = \int_a^b f.$$

Shift of Interval of Integration

Theorem

Let $f : [a, b] \to \mathbb{R}$ *be integrable and let* $k \in \mathbb{R}$*. Then* $f(x - k)$ *is integrable on* $[a + k, b + k]$ *and*

$$\int_{a+k}^{b+k} f(x - k)\, dx = \int_a^b f(x)\, dx.$$

Proof. Hint: If $s(x)$ is a step function with domain $[a, b]$ then $s(x - k)$ is a step function with domain $[a + k, b + k]$. □

Scaling of Interval of Integration

Theorem

Let $f(x)$ be integrable on $[a, b]$ and let $k > 0$. Then $f(x/k)$ is integrable on $[ka, kb]$ and

$$\int_{ka}^{kb} f(x/k)\,dx = k\int_{a}^{b} f(x)\,dx.$$

Proof. Hint: If $s(x)$ is a step function with domain $[a, b]$ then $s(x/k)$ is a step function with domain $[ka, kb]$. □