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Problems for Chapters 19 of
Advanced Mathematics for Applications

Infinite-Dimensional Spaces

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Notation. The following notation is used in the problems that follow:

- $C^k[a,b]$ is the set of functions continuous with their first k derivatives on the closed interval $a \le x \le b$.
- $L^p(a,b)$ is the set of functions u such that $|u|^p$ is integrable (in the sense of Lebesgue) on the interval $a < x < b$.
- ℓ^p is the set of infinite numerical sequences $A = \{a_n\} = (a_1,\, a_2,\, \ldots,\, a_n,\, \ldots)$, real or complex, such that $\sum_{n=0}^{\infty} |a_n|^p < \infty$.

1 General

1. Let x_1, x_2 and x_3 be the component of a vector with respect to a certain orthogonal basis in the ordinary real three-dimensional space $\mathbb{R}^3$. Do vectors of the types

$$\left|\begin{array}{c} x_1 \\ x_2 \\ 2x_2 - x_1 \end{array}\right| , \qquad \left|\begin{array}{c} x_1 \\ x_2 \\ |x_2| \end{array}\right|$$

constitute subspaces of $\mathbb{R}^3$?

2. In a four-dimensional real Cartesian linear space $\mathbb{R}^4$, for what values of the constant a, if any, do sets of the form

$$x_1 + x_2 + x_3 + x_4 \,=\, a$$

constitute a linear vector subspace? Answer the same question for sets of the form

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 \,=\, a^2\,.$$

3. Characterize the subspace S of $\mathbb{R}^4$ spanned by the set $\{1,0,1,0\}$, $\{0,1,0,0\}$, $\{0,0,0,1\}$, i.e. give the general form of the vectors belonging to it.

4. (a) In $\mathbb{R}^4$ find the most general form of the vectors belonging to the subspace S spanned by $\{(1,1,0,0)\,,(1,0,1,1)\}$. Repeat for the subspace spanned by the two vectors $\{(2,-1,3,3)\,,(0,1,-1,-1)\}$. Is this subspace different from the previous one?

5. Show that the set of polynomials which assume the value a at $x = \eta$ is not a linear space if $a = 1$, but is a linear space if $a = 0$.

6. Let $u(x)$ denote a generic element of the space of continuous functions $C[0,1]$. Which, if any, of the following subsets of $C[0,1]$ constitutes a linear subspace?

   (a) All functions such that $u(0) = u(1)$;

(b) All functions such that $u(0) = u(1) = 0$;

(c) All functions such that $u(x_1) = u(x_2)$ for $x_1 + x_2 = 1$;

(d) All functions such that $u(0) = 0$;

(e) All functions such that $u(0) = 1$;

(f) All functions such that $\int_0^1 u(x)\,\mathrm{d}x \ = 0$;

(g) All functions such that $\int_0^1 u(x)\,\mathrm{d}x \ = 1$.

7. Let $\{\mathbb{S}_n\}$ be a set of linear subspaces of a linear vector space $\mathbb{S}$. Show that $\cap_n \mathbb{S}_n$ (namely, the set of elements of $\mathbb{S}$ common to all the $\mathbb{S}_n$'s) is a linear subspace of $\mathbb{S}$. Is the same statement true for $\cup_n \mathbb{S}_n$, the union of the $\mathbb{S}_n$?

8. In the (complex) space $C[-\pi, \pi]$ consider the functions e^{inx}, $n = 0, \pm 1, \pm 2, \ldots$. By differentiating $K - 1$ times a generic linear combinations of K such functions prove that any finite set of them is linearly independent. Argue, on the basis of this result, that $C[-\pi, \pi]$ is infinite-dimensional.

9. Show that a set of elements of a linear vector space is linearly independent if and only if every finite subset of them is linearly independent.

10. Let $f_1(x), f_2(x), \ldots, f_N(x)$ be a set of square-integrable functions over the interval $0 < x < 1$, and consider all the square-integrable functions $u(x)$ such that

$$\int_0^1 f_j(x) u(x)\,\mathrm{d}x \ = \ 0 \qquad j \ = \ 1,\, 2,\, \ldots,\, N\,.$$

Is this a closed linear subspace of the space of square-integrable functions?

11. Consider the set $\mathbb{S}$ of all functions $u(x)$ defined for $0 \le x \le 2\pi$ having the form

$$u(x) \ = \ a \, \cos\left(x + \phi\right),$$

where a and ϕ are real constants. (a) Is this a linear space with the usual definitions of sum of two functions and product of a function by a number? (b) What is its dimension? Give a suitable basis.

2 Normed spaces

1. Consider the set of all *finite* numerical sequences $A \ = \ (a_1,\, a_2,\, \ldots,\, a_n)$ for any finite n. Show that this set is a normed space with the norm

$$\|A\| \ = \ \sup_n |a_n|$$

but it is not a Banach space.

2. Show that the set of infinite convergent numerical sequences $A = \{a_n\} = (a_1,\, a_2,\, \ldots)$ (real or complex) is a normed space with the norm

$$\|A\| \ = \ \sup_n |a_n|\,.$$

3. Consider the set of all infinite bounded numerical sequences $A \ = \ \{a_n\}$ such that $|a_k| < C$ for any k; the constant C depending on the particular sequence considered. Show that this set is a normed space with the norm

$$\|A\| \ = \ \sup_n |a_n|$$

4. Show that the set of infinite numerical sequences $A = \{a_n\}$ (real or complex) such that the series $|\sum_{n=0}^{\infty} n! a_n| < \infty$ is a normed space with the norm

$$\|A\| = \sum_{n=0}^{\infty} n! |a_n| \,.$$

5. Show that the set of infinite numerical sequences $A = \{a_n\}$ (real or complex) such that $a_n \to 0$ is a normed linear space under the norm

$$\|A\| = \sum_{n=0}^{\infty} |a_{n+1} - a_n| \,.$$

6. Show that the space of real functions $u(x)$ belonging to $C^1[0,1]$ is a normed space with the definition

$$\|u\|^2 = u^2(0) + \int_0^1 [u']^2 \,\mathrm{d}x \,.$$

   Can the first term $u^2(0)$ be omitted without compromising the normed nature of the space?

7. Let $A = \{a_n\}$ be an infinite numerical sequence such that $\lim_{n\to\infty} a_n = 0$ and set $f_n = a_n \sin nx$. Is this a Cauchy sequence in the space $C[0, 2\pi]$ equipped with the max norm?

8. In a normed linear space consider a sequence $\{u_n\} \to u$. Let v be an element of the space such that $\|u_n - v\| < a$, where a is a constant, for all n's. Show that, then, $\|u - v\| < a$ as well.

9. Show that, if a Cauchy sequence $\{u_n\}$ has a convergent subsequence, then it is itself convergent.

10. Consider the space C^1 of continuously differentiable functions defined on the interval $-\pi \le x \le \pi$. Do the following definitions

$$\| u \|_1 = \max(|u| + |u'|), \qquad \| u \|_2 = \max|u'|,$$

   give rise to properly defined norms? If not, are they norms in a suitable subspace of C^1? Is the sequence $a_n(x) = (\sin nx)/n$ strongly convergent in the previous norm(s)? Is it convergent according to the usual maximum norm

$$\| u \|_\infty = \max |u| \ ?$$

11. Consider the set $\mathbb{S}$ of all functions $u(x)$ defined for $0 \le x \le 2\pi$ having the form

$$u(x) = a \cos(x + \phi),$$

   where a and ϕ are real constants. Would the relation $\|u\| = |u(t = \pi)|$ define a proper norm in $\mathbb{S}$?

12. Consider over the interval $[0, 1]$ the linear space $\mathbb{S}$ of all polynomials $p(x) \equiv a_0 + a_1 x + a_2 x^2 + \ldots a_N x^N$ of degree not greater than N with complex coefficients. (a) Does the relation

$$\|p\|^2 = \sum_{j=0}^{N} |a_j|^2$$

   define a proper norm over $\mathbb{S}$?

3 Hilbert spaces

1. Consider the set $\mathbb{S}$ of all functions $u(x)$ defined for $0 \le x \le 2\pi$ having the form

$$u(x) \;=\; a\,\cos\left(x+\phi\right),$$

where a and ϕ are real constants. Would the relation $\|u\| \,=\, |u(t=\pi)|$ define a proper norm in $\mathbb{S}$? Define a suitable scalar product in terms of the constants a_1, ϕ_1, a_2, ϕ_2 defining two elements of the set $\mathbb{S}$.

2. (a). Would the expression

$$(f,g) = \int_a^b \overline{f}'(x)\, g'(x)\, \mathrm{d}x\,.$$

define a proper scalar product in the space $C^1[a,b]$? Would it in the subspace of $C^1[a,b]$ consisting of all the functions such that $u(a)=0$?

3. Consider over the interval $[0,1]$ the linear space $\mathbb{S}$ of all polynomials $p(x) \equiv a_0 + a_1 x + a_2 x^2 + \ldots a_N x^N$ of degree not greater than N with complex coefficients equipped with the norm

$$\|p\|^2 \;=\; \sum_{j=0}^{N} |a_j|^2\,.$$

Can this norm be extended to a scalar product? What is its explicit expression?

4. The sets of either sines or cosines on which the Fourier series is based are a very simple illustration that, in general, an infinity of independent unit vectors does not necessarily constitute a basis. For a less trivial example consider the following construction. Let $\{e_n\}$ be an orthonormal basis for a Hilbert space H and define $h_n = e_1 + e_{n+1}/(n+1)$.
(a) Show that the set $\{h_n\}$ is linearly independent;
(b) Show that the sets $\{h_n\}$ and $\{e_n\}$ have the same closed span. (Note that e_1 is in the closed span of $\{h_n\}$ since $h_n \to e_1$ for $n \to \infty$);
(c) Show that e_1 cannot be expanded in a series in the set $\{h_n\}$. (Hint: Assume that $e_1 = a_1 h_1 + a_2 h_2 + \ldots$, take the inner product with respect to each e_k to arrive at the conclusion).

5. (a) Let the set $\{e_n\}, n = 0,1,2,\ldots$ be orthonormal and complete in a Hilbert space H and define the new set $\{w_n\}$ by $w_n = e_n + e_0, n = 1,2,\ldots$. Is there a non-zero vector $g \in H$ such that g is orthogonal to all the w_n's? Can $\{w_n\}$ be taken as a basis in H? (b) Answer the same question if the definition of the w_n's is changed to $w_n = e_{n-1} + 2e_n$, $n = 1,2,\ldots$. (Hint: Expand g on the basis $\{e_n\}$ and use the coefficients of this expansion.)

6. Let H_1 and H_2 be two Hilbert spaces equipped with the scalar products $(u_1,v_1)_1$ and $(u_2,v_2)_2$ for any pair of vectors $u_1, v_1 \in H_1$ and $u_2, v_2 \in H_2$. Now form the sum space (pp. 541, 555)) H consisting of all the possible ordered pairs of vectors $u = \{u_1, u_2\}$, in which the first element belongs to H_1 and the second one to H_2. Show that the definition

$$(u,v) \;=\; (u_1,v_1)_1 + (u_2,v_2)_2,$$

leads to a good scalar product in the space H, i.e. that the scalar product axioms are satisfied for any pair of elements of $\mathcal{H}$.

7. Find an orthonormal basis for the solution space of the two linear equations

$$x_1 + 3x_2 + x_3 - x_4 = 0, \qquad -2x_1 + 2x_2 - x_3 + x_4 = 0\,.$$

8. Consider the linear space of real or complex solutions of the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \pi^2 y \,=\, 0, \qquad -1 \le x \le 1.$$

Give an orthonormal basis in this space and define a suitable scalar product only expressed in terms of the components of the elements along the basis vectors. Show that the value of this scalar product is independent of the set of basis vectors chosen.

9. It is mentioned on p. 552 that, in a Hilbert space, $u_n \to u$ strongly if and only if $u_n \to u$ weakly and $\|u_n\| \to \|u\|$; prove this statement.

10. Show that in the space ℓ^1 weak convergence implies strong convergence.

11. Give the proper form of the polarization procedure (19.3.13) p. 551 for a real normed space.

12. Show that, in any inner product space (real or complex),

$$(u, v) + (v, u) = \frac{1}{2}(||u + v||^2 - ||u - v||^2),$$

while, for a complex inner product space,

$$(u, v) - (v, u) = \frac{i}{2}(||u - iv||^2 - ||u + iv||^2)\,.$$

Deduce from this the polarization procedure (19.3.13) p. 551 defining the scalar product in terms of suitable norms.

13. Let u and v be non-zero elements of a complex inner-product space. Show that

   (a) $\|u + v\| \;=\; \|u\| + \|v\|$ if and only if v is a positive multiple of u;

   (b) $\|u - v\| \;=\; |\|u\| - \|v\||$ if and only if v is a positive multiple of u;

   (c) the equality $\|u - v\| = \|u - w\| + \|w - v\|$ implies that $w = \lambda u + (1 - \lambda)v$ for some real number λ with $0 < \alpha < 1$.

14. Show that for a sequence $\{u_n\}$ in an inner product space the conditions $\|u_n\| \to \|u\|$ and $(u, u_n) \to \|u\|^2$ imply $\|u - u_n\| \to 0$.

15. Show that any three elements u, v, w of an inner product space satisfy the so-called *Apollonius identity*

$$\|w - u\|^2 + \|w - v\|^2 \;=\; \frac{1}{2}\|u - v\|^2 + 2\|w - \frac{1}{2}(u + v)\|^2\,.$$

16. Show that in an inner product space $u \perp v$ if and only if $\|u + av\| = \|u - av\|$ for all scalars a.

17. Let u and v be elements of an inner-product space and suppose that$\|\lambda u + (1 - \lambda)v\| = \|u\|$ for all λ with $0 \le \lambda \le 1$. Show that, if this is true, then $u = v$. Is this conclusion true also in the absence of an inner product?

18. In an inner product space define the function $f(\lambda) = \|u_1 - \lambda u_2\|$, where u_1 and u_2 are linearly independent. For what λ is $f(\lambda)$ a minimum? Give a geometrical interpretation of the result.

19. Consider the set of all complex functions $u(z)$, $v(z)$, $\ldots$ analytic and square-integrable in the unit disc. (a) Show that

$$\int_{|z|<1} \overline{v}\, u \,\mathrm{d}z$$

defines a proper scalar product. (b) Show that the functions $e_n = \sqrt{n/\pi} z^{n-1}$, for $n = 1, 2, \ldots$, form an orthonormal basis. (c) Compare the coefficients of the expansion of the elements u on this bases with those of the power series expansion.

20. Show that the complex functions $e_n(z) = z^{n-1}/\sqrt{2\pi}$, with $n = 1, 2, 3, \ldots$, form an orthonormal basis in the space of continuous complex functions defined on the unit circle $|z| = 1$ with respect to the inner product

$$(v, u) = \int_C \overline{v}(z)\, u(z)\ \mathrm{d}z\ ,$$

in which the integral is taken around the unit circle.

21. Given the generic vector $\mathbf{x} \equiv (a, b, c, d)$ in $\mathbb{R}^4$, find its orthogonal projection $\mathbf{s}$ onto the subspace $\mathbb{S}$ spanned by $\{(1,1,0,0)\,,(1,0,1,1)\}$ and the vector $\mathbf{y}$ such that $\mathbf{x} = \mathbf{s} + \mathbf{y}$.

22. Let M and N be orthogonal linear subspaces of a Hilbert space H. Is the orthogonal complement of M orthogonal to the orthogonal complement of N?

23. Consider the subspace of $L^2(-1, 1)$ spanned by the functions $u_1(x) = x$ and $u_2(x) = \sin \pi x$. Determine the operator which projects the general element of $L^2(-1, 1)$ onto this subspace.

24. Consider the linear manifold $C[-1, 1]$ of the space $L^2(-1, 1)$. In this manifold, consider all odd functions, i.e., functions such that $f(-x) = -f(x)$. Determine the orthogonal complement of this subset in $L^2(-1, 1)$. What is the unique decomposition predicted by the orthogonal projection theorem?

25. In $L^2(-1, 1)$ consider the linear manifold consisting of functions $u(x)$ such that

$$\int_{-1}^{1} u\, \mathrm{d}x\ = 0\,.$$

Determine the orthogonal complement of this subset in $L^2(-1, 1)$. What is the unique decomposition predicted by the orthogonal projection theorem?

26. Consider the scalar product

$$(f, g) = \int_{-\infty}^{\infty} \exp(-x^2)\, \overline{f}(x)\, g(x)\ \mathrm{d}x\ , \qquad (*)$$

and the norm induced by it. By use of the Gram-Schmidt orthogonalization procedure construct, starting with the family of monomials $\{x^n\}, n = 0, 1, \ldots$, the first few polynomials $\{Q_k\}$ orthogonal according in the sense of the scalar product $(*)$ and having unit norm. The Hermite polynomials $\{H_k\}$ (p. 335) are proportional to the Q_k's and are defined by the normalization condition

$$(H_n, H_m) = 2^n n! \sqrt{\pi} \delta_{n,m}.$$

Write down the H_k's corresponding to the Q_k's that you have calculated. Recall that

$$\int_{-\infty}^{\infty} \exp(-x^2) x^{2n}\, \mathrm{d}x\ = \frac{1 \cdot 3 \cdot \ldots \cdot (2n-1)}{2^n} \sqrt{\pi}\,,$$

while, for $n = 0$, the integral equals $\sqrt{\pi}$.

27. Given the generic vector $\mathbf{w}^T \equiv |a\ b\ c|$ of $\mathbb{R}^3$, find the vector closest to it in the subspace spanned by the two vectors $\mathbf{u}_1^T = |1\ 1\ 1|$ and $\mathbf{u}_2^T = |1\ -1\ 1|$.

28. Obtain the best approximation (in the least-squares sense, p. 556) to the function $\sin(\pi x/2)$ in the interval $-1 \le x \le 1$ using the first 4 Legendre polynomials. Calculate both the L^2 and the max norms of the error (within a 5-10% accuracy is enough). Recall that $P_0 = 1, P_1 = x, P_2 = (3x^2 - 1)/2, P_3 = (5x^3 - 3x)/2$. These are orthogonal, but not orthonormal.

29. Section 19.4.2 p. 556 describes the best approximation procedure (in a least-squares sense) applied to vectors in a Hilbert space. The same technique can be adapted to the problem of finding the "best" approximate solution to an equation. Consider e.g. the problem

$$\mathcal{L}\, u = f$$

in a region Ω, with $u = 0$ on the boundary $\partial\Omega$ of Ω. Here $\mathcal{L}$ is a linear operator and f is given. Suppose that $\{v_j\}, j = 1, 2, \ldots, N$ are a set of functions (not necessarily orthonormal) defined in Ω and vanishing on $\partial\Omega$. Find the best approximation (in the sense of the L^2 - norm) to the solution of the equation in terms of the v_j's. Assume for simplicity that everything is real. After deriving the general formulae, consider in $\Omega \equiv [0, 1]$ the simple case

$$\mathcal{L} = \frac{\mathrm{d}^2}{\mathrm{d}x^2} + k^2,$$

with k a given constant, with, and take $v_1 = \sin \pi x$, $v_2 = \sin 2\pi x$. Give explicit expressions for the coefficients of the linear combination of v_1 and v_2 that best approximates the solution to the problem.

30. In the linear space of the real or complex solutions of the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \pi^2 y \,=\, 0, \qquad -1 \le x \le 1,$$

find the element closest to the solution of

$$\frac{1}{2}x^2\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} - x\frac{\mathrm{d}w}{\mathrm{d}x} + w \,=\, 0, \qquad w(-1) = 0, \quad w(1) = 2,$$

in the sense of the L^2 norm. (Solve the equations first.)

31. (*a*) In the three-dimensional subspace of $L^2(-1, 1)$ spanned by the monomials $1, x, x^2$ find the best approximation (in the least-squares sense) to the solution of the Fredholm integral equation

$$\int_{-1}^{1} \sin(axy)\, u(y)\; \mathrm{d}y \;+\; \mu u \;=\; f(x),$$

where μ, a are given parameters and f a given function. (Leave indicated the integrals you cannot carry out in closed form). (*b*) What happens if $\mu \to 0$? Why? (*c*) How would you set up a numerical calculation of this type involving many more monomials by computer? (*d*) Can you think of another approximation scheme if $a \ll 1$?

4 Linear functionals

1. Is the functional ℓ on $C[0, 1]$ defined by

$$\ell(u) \;=\; \max_{0 \le x \le 1} u(x)$$

linear? Is it bounded?

2. Equip the space of all infinite bounded sequences $A = (a_0, a_1, \ldots, a_n, \ldots)$ (i.e., such that $|a_k| < C$ for any k, the constant C depending on the particular sequence) with the norm

$$\|A\| \;=\; \sup_n |a_n|$$

Define the functional $\ell(A)$ by $\ell(A) = a_K$, with a fixed K. Is this a linear functional? Is it continuous? Answer the same questions for the functional defined by

$$\ell(A) \;=\; \sup_n a_n\,.$$

3. Let ℓ_1 and ℓ_2 be two continuous linear functionals on a space $\mathbb{S}$. Show that the set of elements $u \in \mathbb{S}$ such that $\ell_1(u) = \ell_2(u)$ is a closed subset of $\mathbb{S}$.

4. For each $f \in L^2(0,1)$ let $u(x)$ be the solution of the equation

$$\frac{\mathrm{d}u}{\mathrm{d}x} + au \,=\, f\,. \qquad u(0) \,=\, 0\,,$$

with $a > 0$ a given constant. Define the linear functional

$$\ell(f) \,=\, \int_0^1 u(x)\,\mathrm{d}x\ .$$

Show that ℓ is bounded and find its representation in the form of a scalar product (λ, u).

5. Consider the linear space of all real polynomials in the variable $x \in [0,1]$ up to and including the degree N. For the generic polynomial $p_k(x)$ in this space define the linear functional

$$\ell_\phi(p_k) \,=\, \int_0^1 \phi(x)\,p_k(x)\ \mathrm{d}x\ ,$$

where ϕ is some fixed continuous function in $0 \le x \le 1$. (a) Set up a Hilbert space structure in this space by selecting a proper scalar product; (b) Show that this functional is bounded in the natural norm induced by the scalar product; (c) Since the functional is bounded and is defined over the entire space, by Riesz's theorem its action can be represented by a scalar product with an element of the space. Find this element.

6. It is pointed out on p. 566 that, in principle, the second step in the derivation of (19.6.9) might be legitimate but the series so obtained might diverge, although this does not happen in a Hilbert space. Prove this fact by considering the scalar product (λ, u), with λ defined in Example 19.6.5 p. 566. Show, by a proper choice of u, that the assumption that λ does not have a finite norm leads to a contradiction.

7. Fill in the missing details in the following sketch of a proof of Riesz's representation theorem (p. 566) in a real linear vector space $\mathbb{S}$. The functional ℓ will take on positive values for elements of $\mathbb{S}$ belonging to a subspace $\mathbb{S}_+$ and negative values for elements of $\mathbb{S}$ belonging to a subspace $\mathbb{S}_-$. The two subspaces must be separated by a linear manifold (or hyperplane) N of co-dimension 1 (p.501; roughly speaking, having one fewer dimension than $\mathbb{S}$; think e.g. of a plane in ordinary three-dimensional space), and ℓ applied to elements of N vanishes. Now decompose any vector u as $u = u_\parallel + u_\perp$, with $u_\parallel \in N$ and $u_\perp$ orthogonal to all the elements of N. Thus $\ell(u) = \ell(u_\perp)$, which is the same as the scalar product (e, u) with e a suitable unit vector perpendicular to N.