# Chapter 11

```
In[1]:= Needs["Graphics`PlotField`"]

In[2]:= Clear[y, β]
```

## à Question 1

(i)

```
In[3]:= sol = Simplify[DSolve[{y'[t] == β f[u] - β (1 - ξ) y[t], y[0] == y0}, y[t], t]]
```

$$Out[3]= \left\{\left\{y[t] \to \frac{e^{t\beta(-1+\xi)}\, y0\,(-1+\xi) + \left(-1+e^{t\beta(-1+\xi)}\right) f[u]}{-1+\xi}\right\}\right\}$$

```
In[4]:= Simplify[Solve[0 == β f[u] - β (1 - ξ) ystar, ystar]]
```

$$Out[4]= \left\{\left\{ystar \to -\frac{f[u]}{-1+\xi}\right\}\right\}$$

```
In[5]:= x[t_] := sol[[1, 1, 2]]

In[6]:= x[t]
```

$$Out[6]= \frac{e^{t\beta(-1+\xi)}\, y0\,(-1+\xi) + \left(-1+e^{t\beta(-1+\xi)}\right) f[u]}{-1+\xi}$$

```
In[7]:= x2[t_] := Collect[x[t], {E^(t β (-1+ξ)), y0}]

In[8]:= x2[t]
```

$$Out[8]= -\frac{f[u]}{-1+\xi} + e^{t\beta(-1+\xi)}\left(y0 + \frac{f[u]}{-1+\xi}\right)$$

Using the algebraic manipulations palette, namely the simplify command, we readily obtain the following expression:

$$In[9]:= x3[t\_] := \frac{f[u]}{1-\xi} + E^{t\beta(-1+\xi)}\left(y0 + \frac{f[u]}{-1+\xi}\right)$$

```
In[10]:= x3[t]
```

$$Out[10]= \frac{f[u]}{1-\xi} + e^{t\beta(-1+\xi)}\left(y0 + \frac{f[u]}{-1+\xi}\right)$$

(ii)

Note that this expression is,

$$y(t) = y^* + E^{-\beta(1-\xi)\,t}(\mathrm{y0} - y^*)$$

where

$$y^* = \frac{f(u)}{1-\xi}$$

and is asymptotically stable so long as $\beta(1\text{-}\xi) < 1$.

## à Question 2

For the system,

$$y_{t-1} = 9 + 0.2\,(5 - p_{t-1})$$

$$\pi_t = \alpha\left(\frac{y_{t-1}-6}{6}\right)$$

we can solve for $p_t$.

```
In[11]:= Simplify[Solve[(p - p1)/p1 == α ((9 + (1/5) (5 - p1)) - 6)/6, p]]
```

```
Out[11]= {{p → p1 + 2 p1 α/3 - p1^2 α/30}}
```

We can solve for $p$ as follows,

```
In[12]:= Solve[p == p + 2 p α/3 - α p^2/30, p]
```

```
Out[12]= {{p → 0}, {p → 20}}
```

which is independent of $\alpha$.

The difference equation in terms of $p$ and $\alpha$ is given by,

$$p_t = f(p_{t-1}) = (1 + \tfrac{2\alpha}{3})\,p_{t-1} - (\tfrac{\alpha}{30})\,p_{t-1}^2$$

whose stability we can investigate by considering,

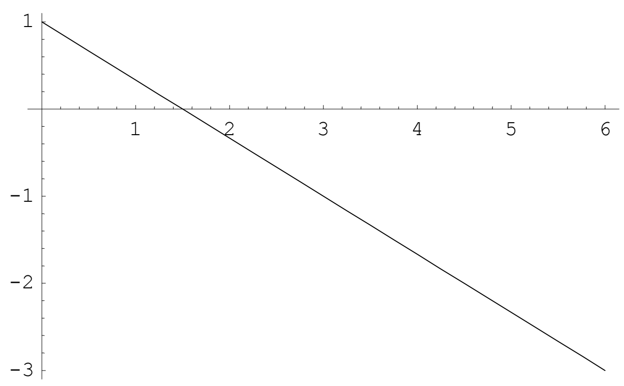
```
In[13]:= slope = D[p1 + 2 p1 α/3 - p1^2 α/30, p1]
```

```
Out[13]= 1 + 2 α/3 - p1 α/15
```

```
In[14]:= eqslope = slope /. p1 -> 20
```

```
Out[14]= 1 - 2 α/3
```

```
In[15]:= Plot[1 - 2 α/3, {α, 0, 6}];
```




```
In[16]:= Solve[1 - 2 α/3 == 0, α]
```

```
Out[16]= {{α → 3/2}}
```

*In[17]:=* $\texttt{Solve}\left[1 - \frac{2\,\alpha}{3} == -1,\ \alpha\right]$

*Out[17]=* $\{\{\alpha \to 3\}\}$

*In[18]:=* $\texttt{Solve}\left[1 - \frac{2\,\alpha}{3} == 1,\ \alpha\right]$

*Out[18]=* $\{\{\alpha \to 0\}\}$

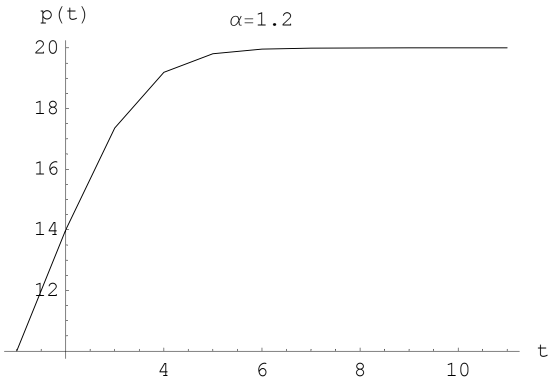
Thus, the system is asymptotically stable if $0 < \alpha < 3$. The system converges steadily on $p^* = 20$, if $0 < \alpha < 3/2$ and oscillates to equilibrium if $3/2 < \alpha < 3$. Beyond $\alpha = 3$ the system gives rise to an explosive oscillation. To see these alternatives, let $\alpha = \{1.2, 2, 3.5\}$.

*In[19]:=* **Clear[f]**

*In[20]:=* $\texttt{f[p\_] := p} + \frac{2\,p\,\alpha}{3} - \frac{\alpha\, p^2}{30}$

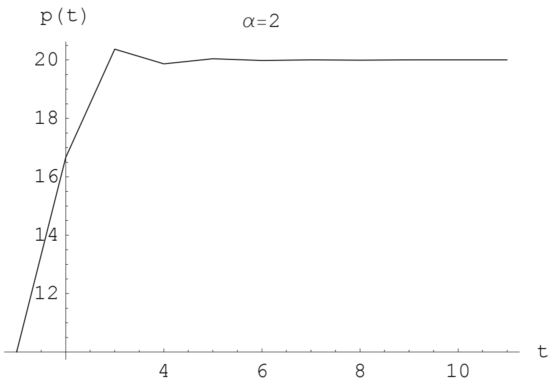
*In[21]:=*
```
pathp1 = NestList[f, 10, 10] /. α -> 1.2;
```

*In[22]:=*
```
ListPlot[pathp1, PlotJoined -> True,
  AxesLabel -> {"t", "p(t)"}, PlotLabel -> "α=1.2"];
```



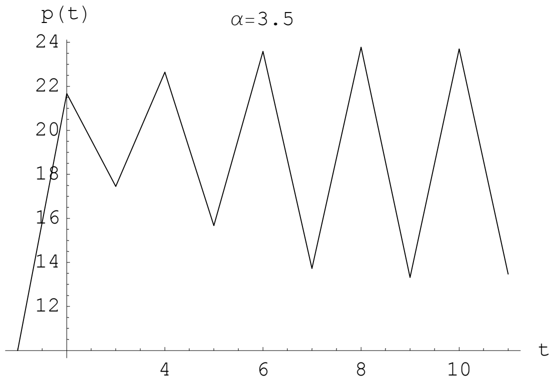

*In[23]:=*
```
pathp2 = N[NestList[f, 10, 10]] /. α -> 2;
```

*In[24]:=*
```
ListPlot[pathp2, PlotJoined -> True,
  AxesLabel -> {"t", "p(t)"}, PlotLabel -> "α=2"];
```




*In[25]:=*
```
pathp3 = N[NestList[f, 10, 10]] /. α -> 3.5;
```

```
In[26]:= ListPlot[pathp3, PlotJoined -> True,
           AxesLabel -> {"t", "p(t)"}, PlotLabel -> "α=3.5"];
```




## à Question 3

Let

```
In[27]:= matrixA = ( -1.85  -10 )
                   (  0.3     0 )

Out[27]= {{-1.85, -10}, {0.3, 0}}

In[28]:= sol = Eigenvalues[matrixA]

Out[28]= {-0.925 + 1.46437 ⅈ, -0.925 - 1.46437 ⅈ}

In[29]:= Re[ sol[[1]] ]

Out[29]= -0.925

In[30]:= Im[ sol[[1]] ]

Out[30]= 1.46437

In[31]:= Re[ sol[[2]] ]

Out[31]= -0.925

In[32]:= Im[ sol[[2]] ]

Out[32]= -1.46437
```

Clear that the real part of the complex conjugate roots is $\alpha = -0.925$.

## à Question 4

```
In[33]:= Clear[rhs, p, pe]

In[34]:= pe1 = Solve[pe[t + 1] == pe[t] + (1 - λ) (p[t] - pe[t]), pe[t + 1]]

Out[34]= {{pe[1 + t] → p[t] - λ p[t] + λ pe[t]}}
```

```
In[35]:= newpe1 = Collect[pe1[[1, 1, 2]], p[t]]

Out[35]= (1 - λ) p[t] + λ pe[t]

In[36]:= rhs[i_] := (1 - λ) p[t - i] + λ pe[t - i]

In[37]:= rhs[i]

Out[37]= (1 - λ) p[-i + t] + λ pe[-i + t]

In[38]:= Table[rhs[i], {i, 0, 3}] // TableForm

Out[38]//TableForm=
        (1 - λ) p[t] + λ pe[t]
        (1 - λ) p[-1 + t] + λ pe[-1 + t]
        (1 - λ) p[-2 + t] + λ pe[-2 + t]
        (1 - λ) p[-3 + t] + λ pe[-3 + t]

In[39]:= Collect[
          Simplify[(1 - λ) p[t] + λ ((1 - λ) p[-1 + t] + λ pe[-1 + t])], {p[t], p[t - 1]}]

Out[39]= (1 - λ) λ p[-1 + t] + (1 - λ) p[t] + λ^2 pe[-1 + t]

In[40]:= Collect[
          Simplify[(1 - λ) λ p[-1 + t] + (1 - λ) p[t] + λ^2 ((1 - λ) p[-2 + t] + λ pe[-2 + t])],
          {p[t], p[t - 1], p[t - 2]}]

Out[40]= (1 - λ) λ^2 p[-2 + t] + (1 - λ) λ p[-1 + t] + (1 - λ) p[t] + λ^3 pe[-2 + t]

In[41]:= Collect[ Simplify[(1 - λ) λ^2 p[-2 + t] + (1 - λ) λ p[-1 + t] + (1 - λ) p[t] +
            λ^3 ((1 - λ) p[-3 + t] + λ pe[-3 + t])], {p[t], p[t - 1], p[t - 2], p[t - 3]}]

Out[41]= (1 - λ) λ^3 p[-3 + t] + (1 - λ) λ^2 p[-2 + t] +
          (1 - λ) λ p[-1 + t] + (1 - λ) p[t] + λ^4 pe[-3 + t]
```

This is the same as the expression,

In[42]:= $(1-\lambda)\sum_{k=0}^{3}\lambda^k\, p[t-k] + \lambda^4\, pe[t-3]$

Out[42]= $(1-\lambda)\ (\lambda^3\, p[-3+t] + \lambda^2\, p[-2+t] + \lambda\, p[-1+t] + p[t]) + \lambda^4\, pe[-3+t]$

Taking the series to the limit, we have

$$E_t\, P_{t+1} = (1-\lambda)\sum_{k=0}^{\infty}\lambda^k\, P[t-k]$$

In[43]:= $\text{Solve}\left[b\left(\frac{ep}{p}-1\right) == a - \frac{M}{p},\ p\right]$

Out[43]= $\left\{\left\{p \to \frac{b\ ep + M}{a+b}\right\}\right\}$

In[44]:= $\frac{b\ ep + M}{a+b}$ /. $ep \text{ -> } (1-\lambda)\sum_{k=0}^{\infty}\lambda^k\, p[t-k]$

Out[44]= $\frac{M + b\ (1-\lambda)\ \sum_{k=0}^{\infty}\lambda^k\, p[t-k]}{a+b}$

Therefore,

$$P_t = \frac{M_t}{a+b} + \left(\frac{b}{a+b}\right)(1-\lambda)\sum_{k=0}^{\infty} \lambda^k P_{t-k}$$

## à Question 5

(i)

Setting $y_t^d = y_t^s = y_t$ and solving for $y_t$ and $p_t$ under the assumption of fixed expectations. In what follows we let ep denote the expression $E_{t-1}\, p_t$.

```
In[45]:= solpy = Simplify[Solve[{y == a0 + a1 (m - p), y == yn + b1 (p - ep)}, {y, p}]]
```

$$\text{Out[45]=}\ \left\{\left\{y \to \frac{\text{a0 b1} + \text{a1}\,(\text{b1}\,(-\text{ep} + m) + \text{yn})}{\text{a1} + \text{b1}},\ p \to \frac{\text{a0} + \text{b1 ep} + \text{a1}\, m - \text{yn}}{\text{a1} + \text{b1}}\right\}\right\}$$

Next we impose the money supply following a systematic component $m = \mu 0$.

```
In[46]:= solpy /. {m -> μ0}
```

$$\text{Out[46]=}\ \left\{\left\{y \to \frac{\text{a0 b1} + \text{a1}\,(\text{yn} + \text{b1}\,(-\text{ep} + \mu 0))}{\text{a1} + \text{b1}},\ p \to \frac{\text{a0} + \text{b1 ep} - \text{yn} + \text{a1}\,\mu 0}{\text{a1} + \text{b1}}\right\}\right\}$$

Since,

$$p_t = \frac{\text{a0}-\text{yn}}{\text{a1}+\text{b1}} + \frac{\text{a1}\,\mu 0}{\text{a1}+\text{b1}} + \frac{\text{b1}\,E_{t-1}\,p_t}{\text{a1}+\text{b1}}$$

then

$$E_{t-1}\,p_t = \frac{\text{a0}-\text{yn}}{\text{a1}+\text{b1}} + \frac{\text{a1}\,\mu 0}{\text{a1}+\text{b1}} + \frac{\text{b1}\,E_{t-1}\,p_t}{\text{a1}+\text{b1}}$$

$$\text{In[47]:=}\ \text{Simplify}\left[\text{Solve}\left[\text{ep} == \frac{\text{a0} + \text{b1 ep} - \text{yn} + \text{a1}\,\mu 0}{\text{a1} + \text{b1}},\ \text{ep}\right]\right]$$

$$\text{Out[47]=}\ \left\{\left\{\text{ep} \to \frac{\text{a0} - \text{yn} + \text{a1}\,\mu 0}{\text{a1}}\right\}\right\}$$

or

$$E_{t-1}\,p_t = \left(\frac{\text{a0}-\text{yn}}{\text{a1}}\right) + \mu 0$$

Substituting this into the solution for $y$, we obtain

$$\text{In[48]:=}\ \text{Simplify}\left[\text{Solve}\left[\left\{y == \frac{\text{a0 b1} + \text{a1}\,(\text{yn} + \text{b1}\,(-\text{ep} + \mu 0))}{\text{a1} + \text{b1}},\ \text{ep} == \frac{\text{a0} - \text{yn} + \text{a1}\,\mu 0}{\text{a1}}\right\},\ \{y, \text{ep}\}\right]\right]$$

$$\text{Out[48]=}\ \left\{\left\{y \to \text{yn},\ \text{ep} \to \frac{\text{a0} - \text{yn} + \text{a1}\,\mu 0}{\text{a1}}\right\}\right\}$$

(ii)

```
In[49]:= solpy /. {m -> μ0 + z}
```

$$\text{Out[49]=}\ \left\{\left\{y \to \frac{\text{a0 b1} + \text{a1}\,(\text{yn} + \text{b1}\,(-\text{ep} + z + \mu 0))}{\text{a1} + \text{b1}},\ p \to \frac{\text{a0} + \text{b1 ep} - \text{yn} + \text{a1}\,(z + \mu 0)}{\text{a1} + \text{b1}}\right\}\right\}$$

$E_{t-1}\,p_t$ is as before since $E_{t-1}\,z_t = 0$.

In[50]:= `Simplify[Solve[{y == (a0 b1 + a1 (yn + b1 (-ep + z + μ0)))/(a1 + b1), ep == (a0 - yn + a1 μ0)/a1}, {y, ep}]]`

$$\text{Out[50]= } \left\{\left\{y \to \frac{\text{a1 yn} + \text{b1 yn} + \text{a1 b1 z}}{\text{a1} + \text{b1}},\ \text{ep} \to \frac{\text{a0} - \text{yn} + \text{a1}\,\mu 0}{\text{a1}}\right\}\right\}$$

Or,

$$y_t = \text{yn} + \frac{\text{a1 b1}\, z_t}{\text{a1+b1}}.$$

Income deviates from its natural level only in so far as changes in the money supply are unexpected.

## à Question 6

(i)

Since,

$$r\,q = \max_{\lambda}\ \{x(e) - a\,\lambda - w - s\,q + \dot{q}\}$$

and

$$w = b + (r + s + h(q, e))\,(b/\lambda)$$

then

$$r\,q = x(e) - b - s\,q + \dot{q} - \min_{\lambda}\{a\,\lambda + (r + s + h(q, e))\,(b/\lambda)\}$$

Consequently the value of $\lambda$ which maximises $r\,q$ is that which minimises the expression in curly braces. Let this be denoted $z$. Then,

In[51]:= `z = a λ + (r + s + h[q, e]) b/λ`

$$\text{Out[51]= } a\,\lambda + \frac{b\,(r + s + h[q, e])}{\lambda}$$

In[52]:= `D[z, λ]`

$$\text{Out[52]= } a - \frac{b\,(r + s + h[q, e])}{\lambda^2}$$

In[53]:= `D[z, {λ, 2}]`

$$\text{Out[53]= } \frac{2\,b\,(r + s + h[q, e])}{\lambda^3}$$

This last expression is positive. So the result of setting the first derivative to zero gives a minimum.

In[54]:= `Simplify[Solve[D[z, λ] == 0, λ]]`

$$\text{Out[54]= } \left\{\left\{\lambda \to -\frac{\sqrt{b\,(r + s + h[q, e])}}{\sqrt{a}}\right\},\ \left\{\lambda \to \frac{\sqrt{b\,(r + s + h[q, e])}}{\sqrt{a}}\right\}\right\}$$

Ignoring the negative value of $\lambda$, then

$$\lambda = \left(\frac{b}{a}\right)^{\frac{1}{2}} [r + s + h(q, e)]^{\frac{1}{2}}.$$

(ii)

In[55]:= `Simplify[Solve[{w == b + (r + s + h[q, e])` $\frac{b}{\lambda}$, $\lambda ==$ $\sqrt{\frac{b}{a}}\sqrt{r+s+h[q,e]}$`}, {w, λ}]]`

Out[55]= $\left\{\left\{w \to b + a\sqrt{\frac{b}{a}}\sqrt{r+s+h[q,e]},\ \lambda \to \sqrt{\frac{b}{a}}\sqrt{r+s+h[q,e]}\right\}\right\}$

Or,

$$w = b + (a\,b)^{\frac{1}{2}}\,[r + s + h(q,e)]^{\frac{1}{2}}$$

To solve for profit $\pi(q, e)$ we note that,

$$\pi(q,e) = \text{MRP}_L - w = x(e) - a\,\lambda - b - (a\,b)^{\frac{1}{2}}\,[r + s + h(q,e)]^{\frac{1}{2}}$$

In[56]:= `Clear[x]`

In[57]:= `Simplify[x[e] - a` $\left(\sqrt{\frac{b}{a}}\sqrt{r+s+h[q,e]}\right) - b - \sqrt{a\,b}\,\sqrt{r+s+h[q,e]}$ `]`

Out[57]= $-b - \left(a\sqrt{\frac{b}{a}} + \sqrt{a\,b}\right)\sqrt{r+s+h[q,e]} + x[e]$

Hence,

$$\pi(q,e) = x(e) - b - 2\,(a\,b)^{\frac{1}{2}}\,[r + s + h(q,e)]^{\frac{1}{2}}.$$

## à Question 7

(i)

In[58]:= `Clear[m, u, v, e, q, h, c]`

In[59]:= `Solve[{m ==` $\sqrt[4]{u\,v}$, $\frac{m\,q}{v}$ `== c}, {m, v}]`

Out[59]= $\left\{\left\{m \to \left(\frac{q^{4/3}\,u^{4/3}}{c^{4/3}}\right)^{1/4},\ v \to \frac{q^{4/3}\,u^{1/3}}{c^{4/3}}\right\},\right.$
$\left\{m \to \left(-\frac{(-1)^{1/3}\,q^{4/3}\,u^{4/3}}{c^{4/3}}\right)^{1/4},\ v \to -\frac{(-1)^{1/3}\,q^{4/3}\,u^{1/3}}{c^{4/3}}\right\},$
$\left.\left\{m \to \left(\frac{(-1)^{2/3}\,q^{4/3}\,u^{4/3}}{c^{4/3}}\right)^{1/4},\ v \to \frac{(-1)^{2/3}\,q^{4/3}\,u^{1/3}}{c^{4/3}}\right\}\right\}$

Taking only the postive value for $v$, we have

$$v = \left(\tfrac{q}{c}\right)^{\frac{4}{3}}\,u^{\frac{1}{3}}.$$

(ii)

Since,

$$h(q,e) = \tfrac{m(u,v)}{u} = \left(u^{\frac{1}{4}}\,v^{\frac{1}{4}}\right)/u = u^{\frac{-3}{4}}\,v^{\frac{1}{4}}, \qquad v = \left(\tfrac{q}{c}\right)^{\frac{4}{3}}\,u^{\frac{1}{3}},\ \ u = 1 - e$$

then $h(q, e)$ is equal to

In[60]:= `h[q, e] = Simplify[(1 - e)^-3/4 (q / c)^1/3 (1 - e)^1/12]`

Out[60]= $\dfrac{\left(\frac{q}{c}\right)^{1/3}}{(1-e)^{2/3}}$

(iii)

In[61]:= `D[h[q, e], q]`

Out[61]= $\dfrac{1}{3\,c\,(1-e)^{2/3}\left(\frac{q}{c}\right)^{2/3}}$

In[62]:= `D[h[q, e], e]`

Out[62]= $\dfrac{2\left(\frac{q}{c}\right)^{1/3}}{3\,(1-e)^{5/3}}$

which are both positive.

(iv)

Since,

$$\dot{e} = (1-e)\,h(q, e) - s\,e$$

then

In[63]:= `edot = (1 - e) h[q, e] - s e`

Out[63]= $(1-e)^{1/3}\left(\frac{q}{c}\right)^{1/3} - e\,s$

In[64]:= `Simplify[Solve[edot == 0, q]]`

Out[64]= $\left\{\left\{q \to -\frac{c\,e^3\,s^3}{-1+e}\right\}\right\}$

## à Question 8

In[65]:= `Clear[m, u, v, e, c, q]`

(i)

Given,

$$m = \sqrt{u\,v} \qquad \text{and} \qquad \left(\tfrac{m}{u}\right) q = c$$

and noting that $u = 1 - e$, then

In[66]:= `Simplify[Solve[{(√(u v) / u) q == c, u == 1 - e}, {v, u}]]`

Out[66]= $\left\{\left\{v \to -\frac{c^2\,(-1+e)}{q^2},\ u \to 1-e\right\}\right\}$

Hence,

$$v = (1-e)\left(\frac{c}{q}\right)^2$$

(ii)

In[67]:= **Simplify[Solve[{h == $\sqrt{\frac{v}{(1-e)}}$ , v == (1 - e) $\left(\frac{c}{q}\right)^2$}, {h, v}]]**

Out[67]= $\left\{\left\{h \to \sqrt{\frac{c^2}{q^2}},\ v \to -\frac{c^2\ (-1+e)}{q^2}\right\}\right\}$

Thus, $h(q, e) = (c/q)$ and is independent of $e$.

## à Question 9

In[68]:= **Clear[a, s, β, λ, δ, n, x]**

In[69]:= **kdot = s a k^β - (n + δ) k**

```
General::spell1 :
 Possible spelling error: new symbol name "kdot" is similar to existing symbol "edot".
```

Out[69]= $a\ k^{\beta}\ s - k\ (n+\delta)$

In[70]:= **kdot0 = kdot /. {a -> 2, s -> 0.2, β -> 0.25, λ -> 0.05, δ -> 0.03, n -> 0.02}**

Out[70]= $0.4\ k^{0.25} - 0.05\ k$

In[71]:= **NSolve[kdot0 == 0, k]**

Out[71]= {{k → 0.}, {k → 16.}}

Since,

$$x = y\, r^{\frac{-1}{4}}, \qquad y = f(k) = a\, k^{\beta} \qquad f'(k) = \beta\, a\, k^{\beta-1}$$

then

In[72]:= **Solve[x == a k^β r^(-1/4) , r]**

Out[72]= $\left\{\left\{r \to \frac{a^4\ k^{4\beta}}{x^4}\right\}\right\}$

In[73]:= **xdot =** $\left(\beta\ a\ k^{\beta-1} + \lambda - \delta - n - \frac{a^4\ k^{4\beta}}{x^4}\right)$ **x**

```
General::spell :
 Possible spelling error: new symbol name "xdot" is similar to existing symbols {edot, kdot}.
```

Out[73]= $x\left(-n - \frac{a^4\ k^{4\beta}}{x^4} + a\ k^{-1+\beta}\ \beta - \delta + \lambda\right)$

In[74]:= **xdot0 = xdot /. {a -> 2, s -> 0.2, β -> 0.25, λ -> 0.05, δ -> 0.03, n -> 0.02}**

```
General::spell1 :
 Possible spelling error: new symbol name "xdot0" is similar to existing symbol "kdot0".
```

Out[74]= $\left(0. + \frac{0.5}{k^{0.75}} - \frac{16\ k^{1.}}{x^4}\right)\ x$

In[75]:= **NSolve[** $\frac{0.5}{k^{0.75}} - \frac{16\ k^{1.}}{x^4}$ **== 0, x]**

Out[75]= $\{\{x \to -2.37841\ k^{7/16}\},\ \{x \to (0. - 2.37841\ i)\ k^{7/16}\},$
$\{x \to (0. + 2.37841\ i)\ k^{7/16}\},\ \{x \to 2.37841\ k^{7/16}\}\}$

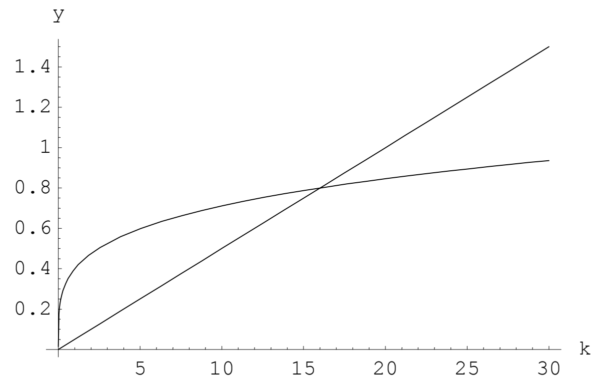
The only positive real solution for $x$ is $x = 2.37841\, k^{7/16} = 2.37841\, k^{0.4375}$.

```
In[76]:= NSolve[{x == 2.37841 k^0.4375, k == 16}, {k, x}]
```

```
Out[76]= {{k → 16., x → 7.99999}}
```

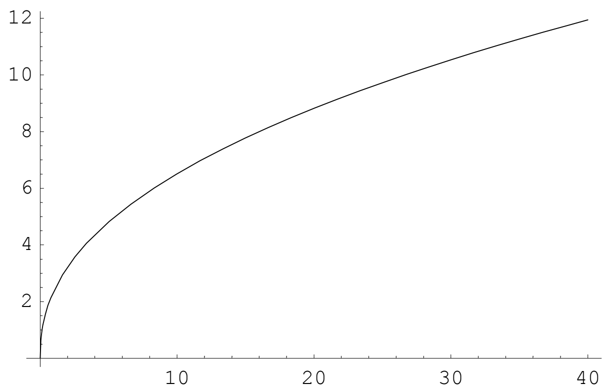
Equilibrium $k$ is shown by the intersection of $s\, f(k)$ and $(n + \delta)\, k$.

```
In[77]:= initdiag = Plot[{0.05 k, 0.4 k^0.25}, {k, 0, 30}, AxesLabel -> {"k", "y"}];
```



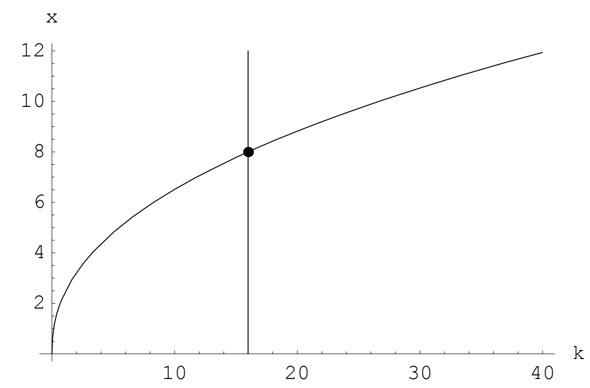

The dynamics are shown by the direction-field diagram. In plotting this we first derive the isoclines and plot these along with the equilibrium point. These are then superimposed on the direction-field diagram. It is important when displaying the direction field to include the option ScaleFunction->(1&). What this does is make all vectors of equal length.

```
In[78]:= xdot0 = Plot[2.37841 k^0.4375, {k, 0, 40}];
```

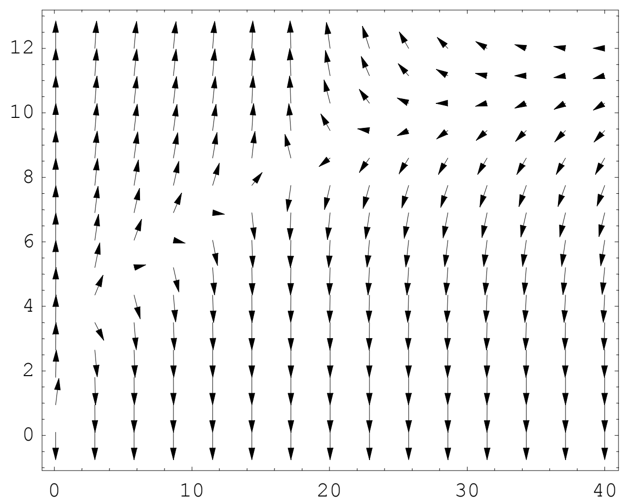



```
In[79]:= linept0 =
           Graphics[{Line[{{16, 0}, {16, 12}}], PointSize[0.02], Point[{16, 8}]}];
```

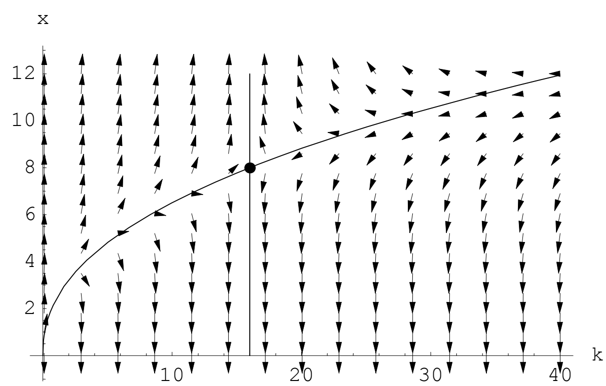
```
In[80]:= isoclines0 = Show[linept0, xdot0, Axes -> True, AxesLabel -> {"k", "x"}];
```




```
In[81]:= field0 = PlotVectorField[{0.4 k^0.25 - 0.05 k, (0.5/k^0.75 - 16 k/x^4) x}, {k, 0.1, 40},
           {x, 0.1, 12}, Frame -> True, ScaleFunction -> (1 &), AspectRatio -> .8];
```




```
In[82]:= Show[isoclines0, field0];
```




### (i) Rise in *s* from 0.2 to 0.3

```
In[83]:= kdot1 = kdot /. {a -> 2, s -> 0.3, β -> 0.25, λ -> 0.05, δ -> 0.03, n -> 0.02}

Out[83]= 0.6 k^0.25 - 0.05 k

In[84]:= NSolve[kdot1 == 0, k]

Out[84]= {{k → 0.}, {k → 27.4731}}
```

In[85]:= **xdot1 = xdot /. {a -> 2, s -> 0.3, β -> 0.25, λ -> 0.05, δ -> 0.03, n -> 0.02}**

General::spell1 :
Possible spelling error: new symbol name "xdot1" is similar to existing symbol "kdot1".

Out[85]= $\left(0. + \frac{0.5}{k^{0.75}} - \frac{16\,k^{1.}}{x^4}\right) x$

In[86]:= **NSolve**$\left[\frac{0.5}{k^{0.75}} - \frac{16\,k^{1.}}{x^4} == 0, x\right]$

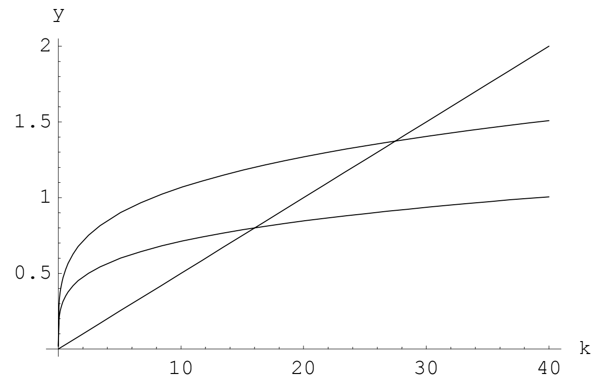
Out[86]= $\{\{x \to -2.37841\,k^{7/16}\}, \{x \to (0. - 2.37841\,i)\,k^{7/16}\}, \{x \to (0. + 2.37841\,i)\,k^{7/16}\}, \{x \to 2.37841\,k^{7/16}\}\}$

Which is the same isocline for $\dot{x} = 0$.

In[87]:= **NSolve[{x == 2.37841 k**$^{0.4375}$**, k == 27.4731}, {k, x}]**

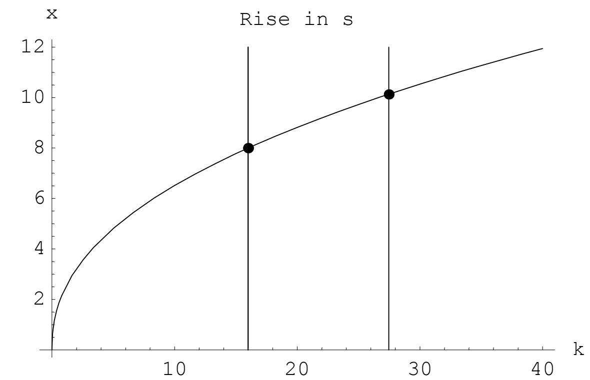
Out[87]= {{k → 27.4731, x → 10.1347}}

In[88]:= **diag1 = Plot[{0.05 k, 0.4 k**$^{0.25}$**, 0.6 k**$^{0.25}$**}, {k, 0, 40}, AxesLabel -> {"k", "y"}];**

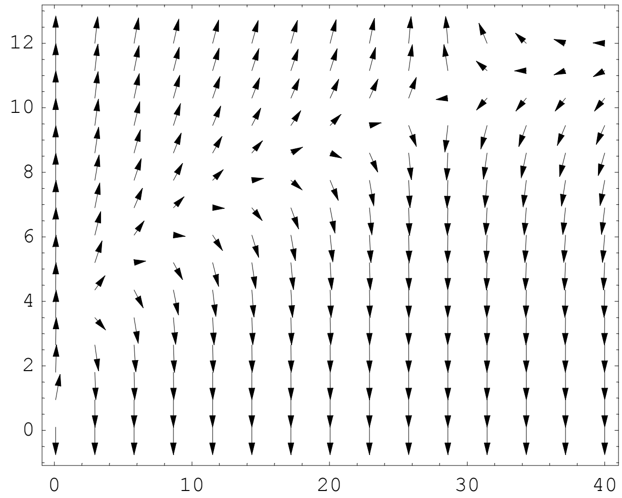



In[89]:= **linept1 =**
**Graphics[{Line[{{16, 0}, {16, 12}}], Line[{{27.4731, 0}, {27.4731, 12}}],**
**PointSize[0.02], Point[{16, 8}], Point[{27.4731, 10.1347}]}];**

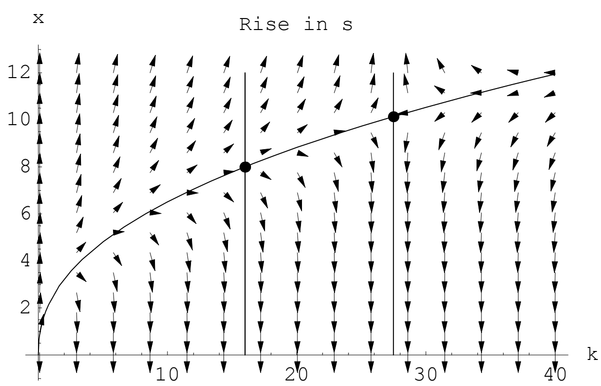
In[90]:= **isoclines1 = Show[linept0, linept1, xdot0,**
**Axes -> True, AxesLabel -> {"k", "x"}, PlotLabel -> "Rise in s"];**



```
In[91]:= field1 = PlotVectorField[{0.6 k^0.25 - 0.05 k, (0.5/k^0.75 - 16 k/x^4) x}, {k, 0.1, 40},
           {x, 0.1, 12}, Frame -> True, ScaleFunction -> (1 &), AspectRatio -> .8];
```




```
In[92]:= Show[isoclines1, field1];
```




Note that a rise in $s$ shifts only $s\,f(k)$ in $(k,y)$-space and only the $\dot{k} = 0$ isocline in $(k,x)$-space.

### (ii) A rise in *n* from 0.02 to 0.03

```
In[93]:= kdot2 = kdot /. {a -> 2, s -> 0.2, β -> 0.25, λ -> 0.05, δ -> 0.03, n -> 0.03}

Out[93]= 0.4 k^0.25 - 0.06 k

In[94]:= NSolve[kdot2 == 0, k]

Out[94]= {{k → 0.}, {k → 12.5471}}

In[95]:= xdot2 = xdot /. {a -> 2, s -> 0.2, β -> 0.25, λ -> 0.05, δ -> 0.03, n -> 0.03}

         General::spell1 :
          Possible spelling error: new symbol name "xdot2" is similar to existing symbol "kdot2".

Out[95]= (-0.01 + 0.5/k^0.75 - 16 k^1./x^4) x
```

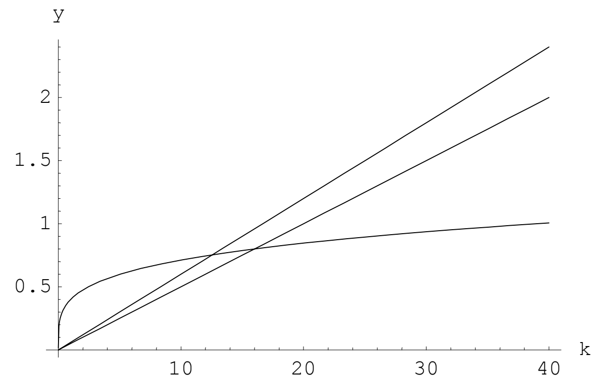
In[96]:= $\text{NSolve}\left[\left(-0.01+\frac{0.5}{k^{0.75}}-\frac{16\,k^{1.}}{x^4}\right)x==0,\ x\right]$

Out[96]= $\left\{\left\{x\to-\frac{(4.47214+4.47214\,i)\,k^{7/16}}{(-50.+1.\,k^{3/4})^{1/4}}\right\},\left\{x\to-\frac{(4.47214-4.47214\,i)\,k^{7/16}}{(-50.+1.\,k^{3/4})^{1/4}}\right\},\left\{x\to\frac{(4.47214-4.47214\,i)\,k^{7/16}}{(-50.+1.\,k^{3/4})^{1/4}}\right\},\left\{x\to\frac{(4.47214+4.47214\,i)\,k^{7/16}}{(-50.+1.\,k^{3/4})^{1/4}}\right\}\right\}$

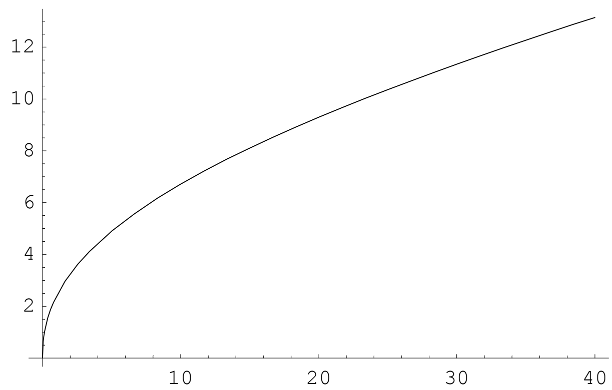
In[97]:= $\text{NSolve}\left[\left\{x==\frac{(4.47214+4.47214\,I)\,k^{7/16}}{(-50+k^{3/4})^{1/4}},\ k==12.5471\right\},\ \{k,x\}\right]$

Out[97]= $\text{NSolve}\left[\left\{x==\frac{(4.47214+4.47214\,i)\,k^{7/16}}{(-50+k^{3/4})^{1/4}},\ k==12.5471\right\},\ \{k,x\}\right]$

In[98]:= `diag2 = Plot[{0.05 k, 0.06 k, 0.4 k^0.25}, {k, 0, 40}, AxesLabel -> {"k", "y"}];`

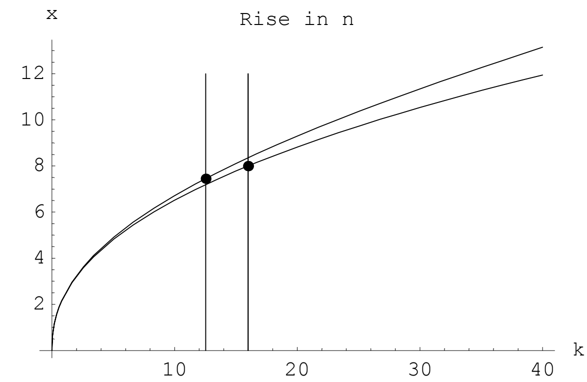



In[99]:= $\text{xdot2 = Plot}\left[\frac{(4.47214+4.47214\,I)\,k^{0.4375}}{(-50+k^{0.75})^{0.25}},\ \{k,0,40\}\right];$

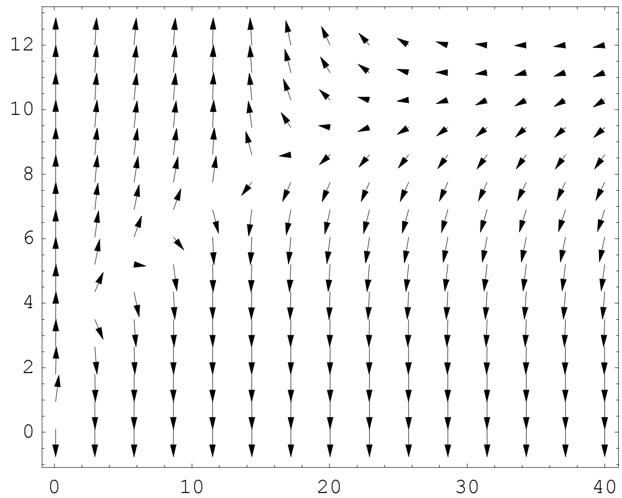



In[100]:=
```
linept2 = Graphics[{
            Line[{{16, 0}, {16, 12}}],
      Line[{{12.5471, 0}, {12.5471, 12}}], PointSize[0.02],
      Point[{16, 8}], Point[{12.5471, 7.45483}]
          }];
```

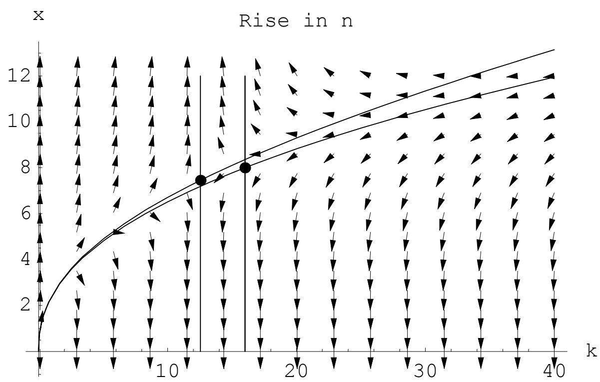
In[101]:=
```
isoclines2 = Show[linept0, linept2, xdot0, xdot2,
   Axes -> True, AxesLabel -> {"k", "x"}, PlotLabel -> "Rise in n"];
```




In[102]:=
```
field2 =
  PlotVectorField[{0.4 k^0.25 - 0.06 k, (-0.01 + 0.5/k^0.75 - 16 k/x^4) x}, {k, 0.1, 40},
   {x, 0.1, 12}, Frame -> True, ScaleFunction -> (1 &), AspectRatio -> 0.8];
```




In[103]:=
```
Show[isoclines2, field2];
```




Note that a rise in $n$ shifts only $(n + \delta)\, k$ in $(k,y)$-space but shifts both isoclines in $(k,x)$-space.

(iii) **A rise in technology (a rise in *a* from 2 to 5)**

*In[104]:=* **kdot3 = kdot /. {a -> 5, s -> 0.2, β -> 0.25, λ -> 0.05, δ -> 0.03, n -> 0.02}**

*Out[104]=* $1.\ k^{0.25} - 0.05\ k$

*In[105]:=* **NSolve[kdot3 == 0, k]**

*Out[105]=* $\{\{k \to 0.\}, \{k \to 54.2884\}\}$

*In[106]:=* **xdot3 = xdot /. {a -> 5, s -> 0.2, β -> 0.25, λ -> 0.05, δ -> 0.03, n -> 0.02}**

```
General::spell1 :
 Possible spelling error: new symbol name "xdot3" is similar to existing symbol "kdot3".
```

*Out[106]=* $\left(0. + \frac{1.25}{k^{0.75}} - \frac{625\ k^{1.}}{x^4}\right) x$

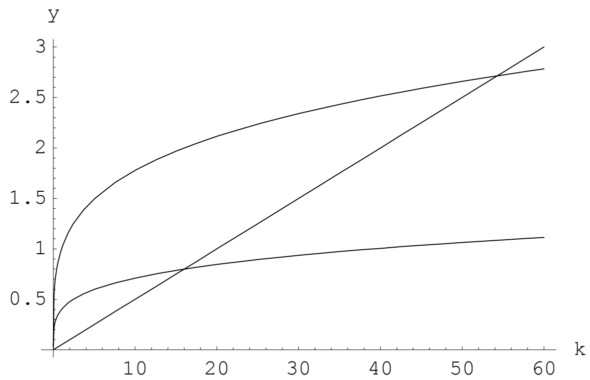
*In[107]:=* **NSolve**$\left[\frac{1.25}{k^{0.75}} - \frac{625\ k}{x^4} == 0, x\right]$

*Out[107]=* $\{\{x \to -4.72871\ k^{7/16}\}, \{x \to (0. - 4.72871\ i)\ k^{7/16}\},$
$\{x \to (0. + 4.72871\ i)\ k^{7/16}\}, \{x \to 4.72871\ k^{7/16}\}\}$

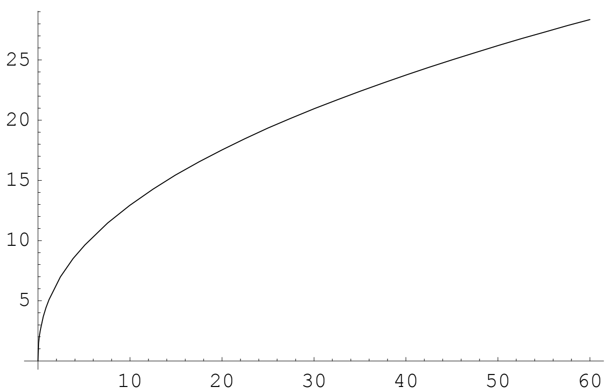
*In[108]:=* **NSolve[{x == 4.72871** $k^{7/16}$**, k == 54.2884}, {k, x}]**

*Out[108]=* NSolve[{x == 4.72871 $k^{7/16}$, k == 54.2884}, {k, x}]

*In[109]:=* **diag3 = Plot[{0.05 k, 0.4** $k^{0.25}$**,** $k^{0.25}$**}, {k, 0, 60}, AxesLabel -> {"k", "y"}];**

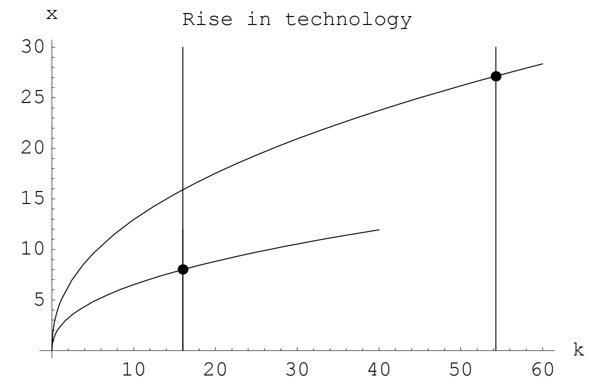



*In[110]:=* **xdot3 = Plot[4.72871** $k^{0.4375}$**, {k, 0, 60}];**

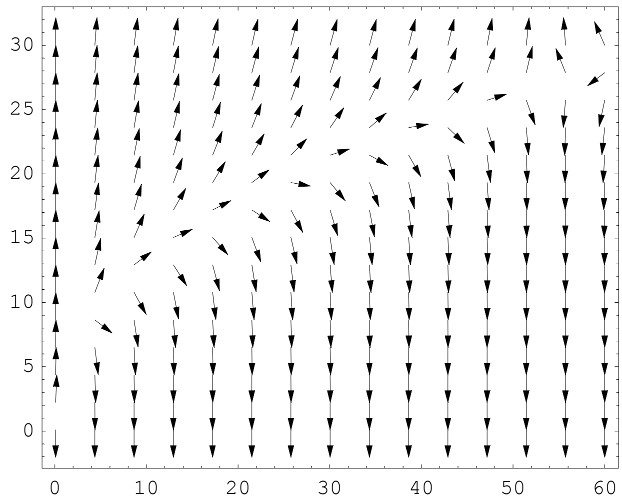



*In[111]:=*
```
linept3 = Graphics[{
              Line[{{16, 0}, {16, 30}}],
      Line[{{54.2884, 0}, {54.2884, 30}}], PointSize[0.02],
      Point[{16, 8}], Point[{54.2884, 27.1442}]
          }];
```

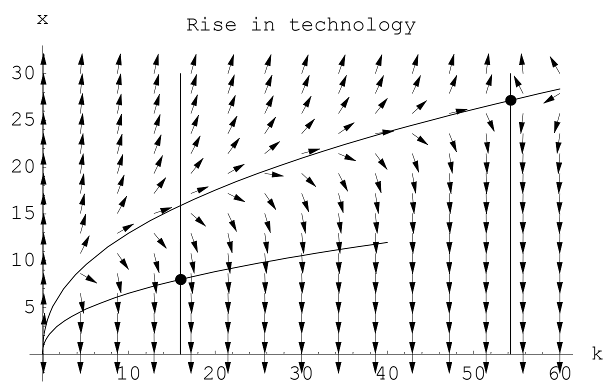
In[112]:=
```
isoclines3 = Show[linept0, linept3, xdot0, xdot3, Axes -> True,
    AxesLabel -> {"k", "x"}, PlotLabel -> "Rise in technology"];
```




In[113]:=
```
field3 = PlotVectorField[{k^0.25 - 0.05 k, (1.25/k^0.75 - 625 k/x^4) x}, {k, 0.1, 60},
    {x, 0.1, 30}, Frame -> True, ScaleFunction -> (1 &), AspectRatio -> 0.8];
```




In[114]:=
```
Show[isoclines3, field3];
```




Note that a rise in technology shifts only $s\,f(k)$ in $(k,y)$-space but shifts both isoclines in $(k,x)$-space.

## à Question 10

This is a continuation of the previous question and we shall identify all plots with the number 4.

*In[115]:=* `kdot4 = kdot /. {a -> 2, s -> 0.2, β -> 0.25, λ -> 0.06, δ -> 0.03, n -> 0.02}`

*Out[115]=* $0.4\, k^{0.25} - 0.05\, k$

*In[116]:=* `NSolve[kdot4 == 0, k]`

*Out[116]=* $\{\{k \to 0.\}, \{k \to 16.\}\}$

*In[117]:=* `xdot4 = xdot /. {a -> 2, s -> 0.2, β -> 0.25, λ -> 0.06, δ -> 0.03, n -> 0.02}`

```
General::spell1 :
 Possible spelling error: new symbol name "xdot4" is similar to existing symbol "kdot4".
```

*Out[117]=* $\left(0.01 + \frac{0.5}{k^{0.75}} - \frac{16\, k^{1.}}{x^4}\right) x$

*In[118]:=* $\mathbf{NSolve}\left[0.01 + \frac{0.5}{k^{0.75}} - \frac{16\, k}{x^4} == 0,\ x\right]$

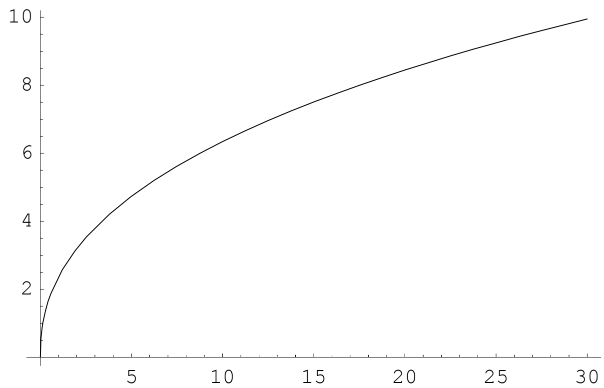
*Out[118]=* $\left\{\left\{x \to -\frac{6.32456\, k^{7/16}}{(50. + k^{3/4})^{1/4}}\right\}, \left\{x \to -\frac{(0. + 6.32456\, i)\, k^{7/16}}{(50. + k^{3/4})^{1/4}}\right\},\right.$
$\left.\left\{x \to \frac{(0. + 6.32456\, i)\, k^{7/16}}{(50. + k^{3/4})^{1/4}}\right\}, \left\{x \to \frac{6.32456\, k^{7/16}}{(50. + k^{3/4})^{1/4}}\right\}\right\}$

*In[119]:=* $\mathbf{NSolve}\left[\left\{x == \frac{6.32456\, k^{0.4375}}{(50 + k^{0.75})^{0.25}},\ k == 16\right\},\ \{k, x\}\right]$

*Out[119]=* $\{\{k \to 16.,\ x \to 7.70861\}\}$

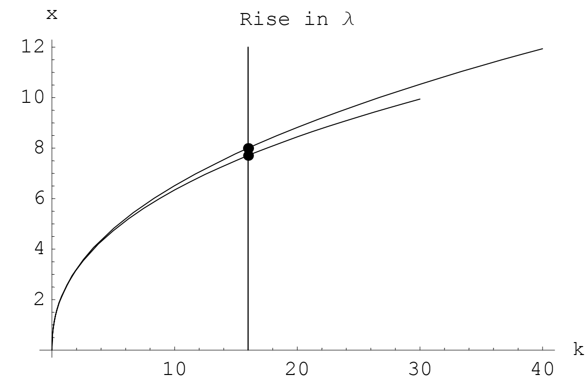
Since a change in $\lambda$ affects neither $(n + \delta)\, k$ nor $s\, f(k)$, then there is no change to equilibrium $k$, and the diagram in $(k,y)$-space remains unaffected. However, we do note that the $\dot{x} = 0$ isocline shifts.

*In[120]:=* $\mathbf{xdot4 = Plot}\left[\frac{6.32456\, k^{0.4375}}{(50 + k^{0.75})^{0.25}},\ \{k, 0, 30\}\right];$

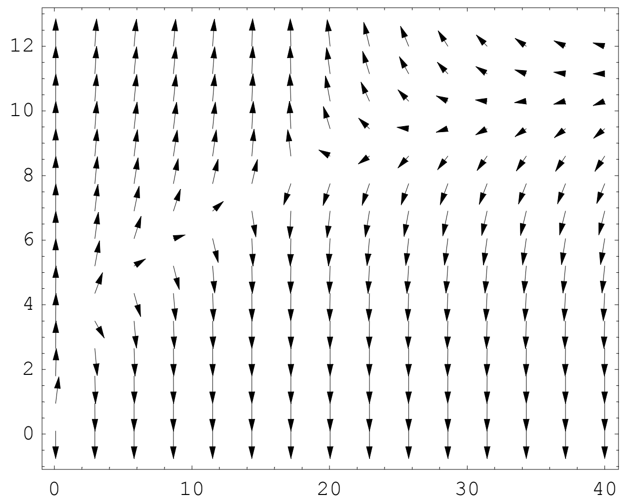



*In[121]:=*
```
lineupt4 = Graphics[{
               Line[{{16, 0}, {16, 12}}],
      PointSize[0.02], Point[{16, 8}], Point[{16, 7.70861}]
         }];
```

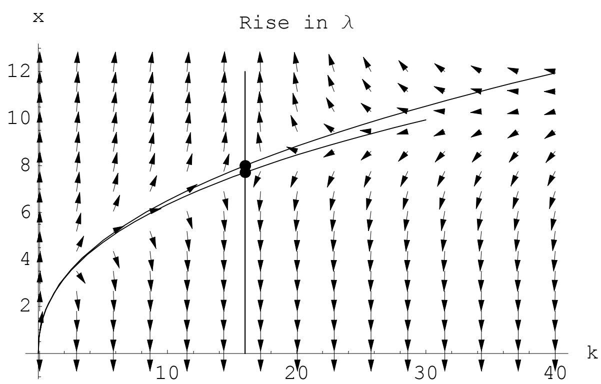
```
In[122]:= isoclines4 = Show[linept0, linept4, xdot0, xdot4,
            Axes -> True, AxesLabel -> {"k", "x"}, PlotLabel -> "Rise in λ"];
```




```
In[123]:= field4 =
           PlotVectorField[{0.4 k^0.25 - 0.05 k, (0.01 + 0.5/k^0.75 - 16 k/x^4) x}, {k, 0.1, 40},
            {x, 0.1, 12}, Frame -> True, ScaleFunction -> (1 &), AspectRatio -> 0.8];
```




```
In[124]:= Show[isoclines4, field4];
```




The rise in $\lambda$ shifts the $\dot{x} = 0$ isocline down, equilibrium $k$ remains at 16 while equilibrium $x$ falls from 8 to 7.7. The economy's trajectory, therefore, is vertically down from one equilibrium to the other.

# à Question 11

In[125]:= `Clear[AA, BB, CC, DD, EE, FF, G, H, J]`

## ■ (i) and (ii)

In[126]:= $AA = \frac{-\alpha\ (a + i0 + g)}{1 - b\ (1 - t) + (k\ h / u)} + \alpha\ yn$

Out[126]= $-\frac{(a + g + i0)\ \alpha}{1 - b\ (1 - t) + \frac{h\ k}{u}} + yn\ \alpha$

In[127]:= $BB = \frac{-\alpha\ (h / u)}{1 - b\ (1 - t) + (k\ h / u)}$

Out[127]= $-\frac{h\ \alpha}{(1 - b\ (1 - t) + \frac{h\ k}{u})\ u}$

In[128]:= $CC = \frac{-\alpha\ h}{1 - b\ (1 - t) + (k\ h / u)} - 1$

Out[128]= $-1 - \frac{h\ \alpha}{1 - b\ (1 - t) + \frac{h\ k}{u}}$

In[129]:= parameters = {a → 100, b → 0.8, i0 → 600, yn → 3000,
g → 525, t → 0.25, k → 0.25, h → 2.5, u → 5, α → 0.2, β → 0.05}

Out[129]= {a → 100, b → 0.8, i0 → 600, yn → 3000, g → 525,
t → 0.25, k → 0.25, h → 2.5, u → 5, α → 0.2, β → 0.05}

In[130]:= `AA /. parameters`

Out[130]= 133.333

In[131]:= `BB /. parameters`

Out[131]= -0.190476

In[132]:= `CC /. parameters`

Out[132]= -1.95238

In[133]:= `Simplify[Solve[0 == 133.333 - 0.190476 ms - 1.95238 pie, pie]]`

Out[133]= {{pie → 68.2925 - 0.0975609 ms}}

In[134]:= $DD = \frac{\alpha\ \beta\ (a + i0 + g)}{1 - b\ (1 - t) + (k\ h / u)} - \alpha\ \beta\ yn$

Out[134]= $\frac{(a + g + i0)\ \alpha\ \beta}{1 - b\ (1 - t) + \frac{h\ k}{u}} - yn\ \alpha\ \beta$

In[135]:= $EE = \frac{\alpha\ \beta\ (h / u)}{1 - b\ (1 - t) + (k\ h / u)}$

Out[135]= $\frac{h\ \alpha\ \beta}{(1 - b\ (1 - t) + \frac{h\ k}{u})\ u}$

In[136]:= $FF = \frac{\alpha\,\beta\,h}{1 - b\,(1 - t) + (k\,h / u)}$

Out[136]= $\frac{h\,\alpha\,\beta}{1 - b\,(1 - t) + \frac{h\,k}{u}}$

In[137]:= **DD /. parameters**

Out[137]= -6.66667

In[138]:= **EE /. parameters**

Out[138]= 0.00952381

In[139]:= **FF /. parameters**

Out[139]= 0.047619

In[140]:= **Simplify[Solve[0 == -6.66667 + 0.00952381 ms + 0.047619 pie, pie]]**

Out[140]= {{pie → 140. - 0.2 ms}}

In[141]:= **Solve[{0 == 133.333 - 0.190476 ms - 1.95238 pie,**
**0 == -6.666670 + 0.00952381 ms + 0.047619 pie}, {ms, pie}]**

Out[141]= {{ms → 700.002, pie → -0.00026}}

i.e., equilibrium values are (ms*, $\pi^{e*}$) = (700, 0).

In[142]:= **mA = {{-0.190476, -1.95238}, {0.00952381, 0.047619}}**

Out[142]= {{-0.190476, -1.95238}, {0.00952381, 0.047619}}

In[143]:= **Tr[mA]**

Out[143]= -0.142857

In[144]:= **Det[mA]**

Out[144]= 0.00952382

In[145]:= **Eigenvalues[mA]**

Out[145]= {-0.0714285 + 0.0664965 i, -0.0714285 - 0.0664965 i}

In[146]:= **Eigenvectors[mA]**

Out[146]= {{0.99757 + 0. i, -0.0608274 - 0.0339764 i},
{0.99757 + 0. i, -0.0608274 + 0.0339764 i}}

## ■ (iii)

In[147]:= $G = \frac{(k / u)\,(a + i0 + g)}{1 - b\,(1 - t) + (k\,h / u)}$

Out[147]= $\frac{(a + g + i0)\,k}{\left(1 - b\,(1 - t) + \frac{h\,k}{u}\right)\,u}$

*In[148]:=* $H = \frac{(h/u)\ (k/u)}{1 - b\ (1 - t) + (k\,h/u)} - \frac{1}{u}$

*Out[148]=* $\frac{h\,k}{\left(1 - b\ (1 - t) + \frac{h\,k}{u}\right) u^2} - \frac{1}{u}$

*In[149]:=* $J = \frac{(k\,h/u)}{1 - b\ (1 - t) + (k\,h/u)}$

*Out[149]=* $\frac{h\,k}{\left(1 - b\ (1 - t) + \frac{h\,k}{u}\right) u}$

*In[150]:=*
```
G /. parameters
```

*Out[150]=* 116.667

*In[151]:=*
```
H /. parameters
```

*Out[151]=* -0.152381

*In[152]:=*
```
J /. parameters
```

*Out[152]=* 0.238095

*In[153]:=*
```
Simplify[Solve[0 == 116.667 - 0.152381 ms + 0.238095 pie, pie]]
```

*Out[153]=*
```
{{pie → -490.002 + 0.640001 ms}}
```

*In[154]:=*
```
dfield = PlotVectorField[{133.333 - 0.190476 ms - 1.95238 pie,
    -6.66667 + 0.00952381 ms + 0.047619 pie}, {ms, 600, 800},
   {pie, -5, 5}, Frame → True, PlotPoints → 20, AspectRatio → 1,
   DisplayFunction → Identity, ScaleFunction → (1 &)];
```

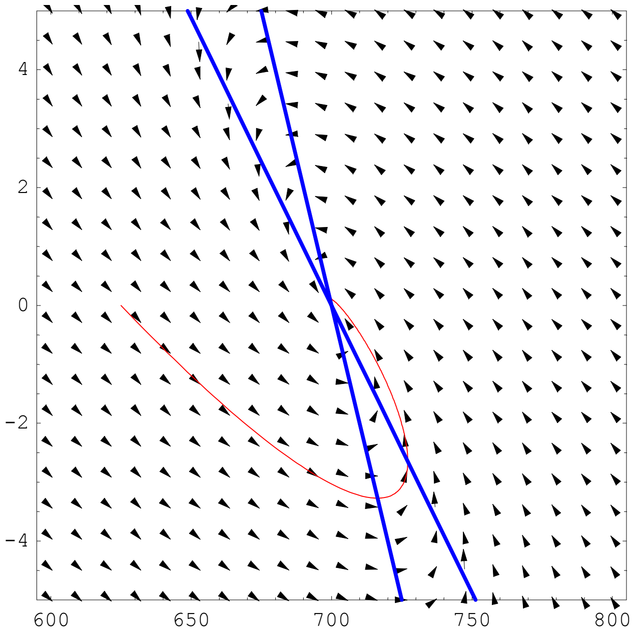
*In[155]:=*
```
sol = NDSolve[{ms'[t] == 133.333 - 0.190476 ms[t] - 1.95238 pie[t],
   pie'[t] == -6.66667 + 0.00952381 ms[t] + 0.047619 pie[t],
   ms[0] == 625, pie[0] == 0}, {ms, pie}, {t, 0, 100}]
```

*Out[155]=*
```
{{ms → InterpolatingFunction[{{0., 100.}}, <>],
  pie → InterpolatingFunction[{{0., 100.}}, <>]}}
```

*In[156]:=*
```
traj = ParametricPlot[Evaluate[{ms[t], pie[t] } /. sol,
    {t, 0, 100}, PlotPoints → 500], DisplayFunction → Identity,
   PlotStyle → {RGBColor[1, 0, 0], Thickness[0.01]}];
```

*In[157]:=*
```
lines = Plot[{68.2925 - 0.0975609 ms, 140 - 0.2 ms},
   {ms, 600, 800}, PlotRange → {-5, 5}, DisplayFunction → Identity,
   PlotStyle → {{RGBColor[0, 0, 1], Thickness[0.007]},
     {RGBColor[0, 0, 1], Thickness[0.007]}}];
```

```
In[158]:= Show[dfield, traj, lines, AxesLabel → {"ms", "πᵉ"},
            PlotRange → {-5, 5}, DisplayFunction → $DisplayFunction];
```




## à Question 12

### ■ (i)

Since

$S = \frac{\Delta M}{P} = \frac{\Delta M}{M}\ \frac{M}{P} = \lambda\ \frac{M}{P}$ then

$\text{Ln}(S) = \text{Ln}(\lambda) + m - p$

$\quad = \text{Ln}(\lambda) - \alpha\,\pi$

since $\pi^e = \pi$

Hence

$\text{Ln}(S) = \text{Ln}(\lambda) - \alpha\,\lambda$

### ■ (ii)

```
In[159]:= Solve[D[Log[λ] - α λ, λ] == 0, λ]
```

Out[159]= $\{\{\lambda \to \frac{1}{\alpha}\}\}$