Exercise: concepts from chapter 10

Reading: Fundamentals of Structural Geology, Ch 10

1) The flow of magma with a viscosity as great as 10^{10} Pa·s, let alone that of rock with a viscosity of 10^{20} Pa·s, is difficult to comprehend because our common experience is with liquids like water at room temperature which has a viscosity of 10^{-3} Pa·s. Recall from (10.7) that the rate of deformation, shear stress, and viscosity for Newton's thought experiment (Figure 10.2) are related by:

$$D_{yx} = \sigma_{yx} / 2\eta \qquad (1)$$

To build intuition investigate Newton's simple shearing configuration using your own weight to shear a 1 m cubic block of liquid (Figure 1).

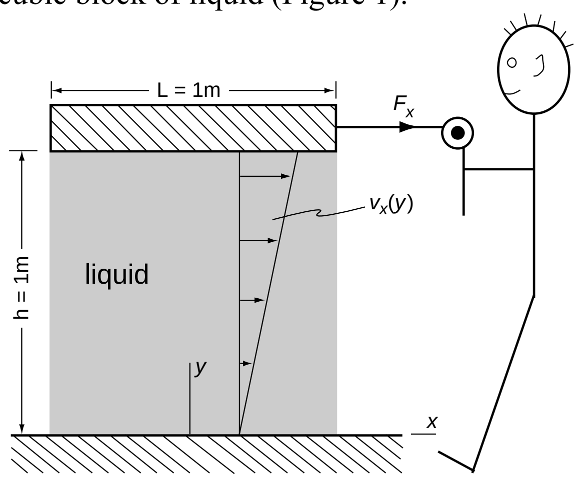

Figure 1. Newton's thought experiment for simple shearing of a viscous liquid. Here the horizontal force is a person's weight transferred to the upper plate through a pulley.

a) Use the kinematic equations to re-write (1) in terms of the velocity and integrate this relationship to find the velocity distribution as a function of vertical position in the cube using the no-slip boundary condition to solve for the constant of integration. Write down the equation for the velocity of the upper plate, *V*, as a function of the shear stress, viscosity, and height of the cube of liquid.

b) Explain how the maximum velocity of the liquid is related to the displacement of a liquid particle in contact with the upper plate. Use this relationship and your result from part a) to write an equation for the displacement, *U*, of the upper plate as a function of shear stress, viscosity, liquid height, and time. Propose an appropriate initial condition to solve for the constant of integration.

c) Use the result from part b) and your own weight to shear the magma and the rock specimens and calculate the displacement, *U*, if you hung on the cable for 1 second, 1 hour, 1 day, and 1 year. Would you think of either the magma or the rock as a liquid?

d) Would the parallel plate apparatus pictured schematically in Figure 1 be a good design for measuring the coefficient of Newtonian viscosity? Discuss the good and the bad aspects of this experimental design.

2) Most viscometers (Figure 2) do not provide a direct measurement of the shear stress or the rate of deformation because these quantities vary throughout the flow in a rather complex way. The quantities that are measured must be related to what is needed for the calculation of viscosity by solving the appropriate boundary-value problem. Because these solutions depend upon assumptions about the flow behavior, the corresponding laboratory tests only provide *apparent viscosities*.

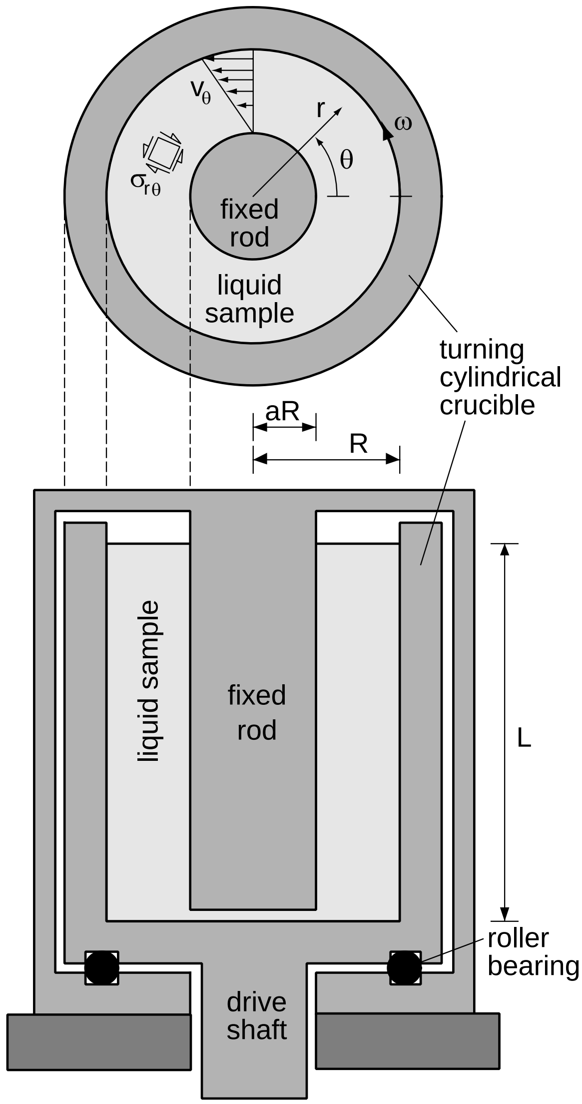

Figure 2. Couette viscometer used to determine the apparent viscosity of liquids.

The Couette viscometer is composed of a cylindrical crucible containing the liquid and a centered cylindrical rod of length, *L*. The annulus between the rod and crucible has an inner radius, a*R*, and an outer radius *R*. The objective of the experiment is to achieve a steady, two-dimensional, and laminar flow within the annulus because these are the conditions assumed for the solution of the boundary value problem. These conditions are most closely approximated if the crucible is turned with angular velocity ω while the rod is held stationary, and if the rod is long compared to the annulus (Bird et al., 1960).

a) A cylindrical coordinate system, (r, θ), is used and the boundary-value problem for the steady state, laminar flow of a Newtonian viscous liquid in an annulus, is solved for the distribution of the circumferential component of velocity:

$$v_\theta = \omega R\left[\frac{(\mathrm{a}R/r)-(r/\mathrm{a}R)}{\mathrm{a}-(1/\mathrm{a})}\right] \tag{2}$$

This is the only non-zero velocity component for this problem. Solve (2) for the velocity at the edge of the rod, $r = \mathrm{a}R$, and at the edge of the crucible, $r = R$. Write the equation relating the angular velocity of the crucible, ω, to the velocity component in the θ direction at the outer wall.

b) Flow in the cylindrical annulus is very similar to flow between parallel plates, except it is wrapped around the axis of the cylinder. One accounts for this different geometry in the relationship between shear stress and velocity gradient as follows:

$$\sigma_{r\theta} = \eta r \frac{d}{dr}\left(\frac{v_\theta}{r}\right) \tag{3}$$

Use the velocity distribution (2) to solve (3) for the shear stress distribution in the annulus. Unlike the parallel plate flow where the shear stress is constant, here the shear stress varies with radial position.

c) Derive the equation relating the net torque, T, on the crucible to the shear stress in the liquid at the edge of the crucible, $r = R$. Substitute for the shear stress to derive the relationship for the apparent viscosity as a function of the applied torque. This is the relationship used to determine the apparent viscosity in the Couette viscometer.

3) Consider the flow of an incompressible, isotropic and linearly viscous liquid down an inclined planar surface that is very long in both the strike and dip directions compared to the thickness, h, of the liquid layer (Figure 3).

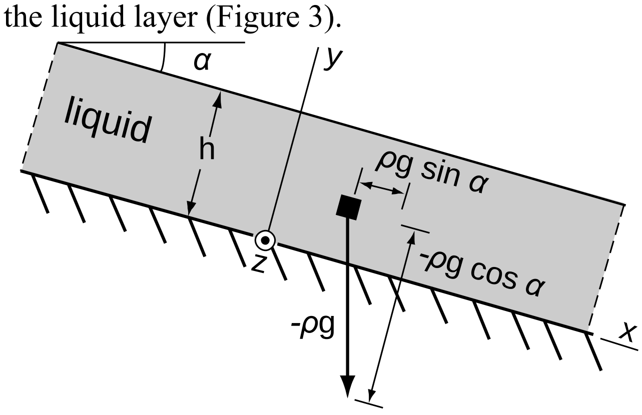

Figure 3. Cross section of a viscous liquid flowing down an inclined plane.

Both the viscosity and the density are uniform throughout the layer. Furthermore, the flow is steady and the flow regime is laminar (low Reynolds number). This could be considered the most elementary model for the flow of lava down the slope of a volcano, or the flow of ice in a very broad glacier.

a) In general the velocity vector is a function of the three Cartesian coordinates and time: $\mathbf{v} = \mathbf{v}(x, y, z, t)$. However, more restricted dependence is indicated for the flow regime depicted in Figure 1. Describe which of the four independent variables each component of the velocity vector depends upon and consider whether the thickness is constant. Justify your choices based upon the geometry, the liquid properties, and the flow conditions. Given these choices, describe how the components of the rate of deformation tensor are related to the gradients in velocity.

b) Describe both qualitatively and mathematically the boundary conditions at the interface between the bottom of the liquid and underlying solid substrate, and at the interface between the top of the liquid and the overlying atmosphere. Indicate how the components of the velocity vector and stress tensor at the two interfaces are constrained by these boundary conditions.

c) The velocity distribution for flow down the inclined plane is given in (10.25) as:

$$v_x(y) = -(\rho g / \eta)\sin(\alpha)\left(\tfrac{1}{2}y^2 - \mathrm{h}y\right) \tag{4}$$

Here the density, ρ, the acceleration of gravity, g, and the Newtonian viscosity, η, are uniform and constant. Derive the equation for the maximum velocity and use this to rewrite (4) in dimensionless form. Plot the dimensionless velocity profile and describe it.

d) Estimate the range of thicknesses of a lava flow that you could just outrun (say for 100m) on slopes ranging from $\alpha = 1$ to 10 degrees and summarize your results graphically. Use values for the constants given by Macdonald (1955) for basaltic lava in Hawaii where flows typically range from 1 to 5 m thick:

density, $\rho = 2.65 \times 10^3$ kg/m^3
acceleration of gravity, g = 9.8 m/s^2
apparent Newtonian viscosity, η = 3000 Pa·s

This viscosity is approximately the median for the values quoted by Macdonald which range from 1,900 to 3,800 Pa·s.

4) George Stokes considered a constitutive law for an isotropic viscous fluid (7.160) in which the stress components are related linearly to the thermodynamic pressure and the rate of deformation components.

a) Write down this constitutive law in three dimensions using indicial notation and treating the stress components as the dependent variables. Describe each physical quantity. Rewrite this equation for this fluid at rest (or in uniform motion) and in doing so indicate the relationships between the thermodynamic pressure, p, the mean normal pressure, $\bar{p}$, and the static pressure, $\bar{p}_o$.

b) For the flowing fluid considered by Stokes align the Cartesian coordinates with the directions of principal stresses:

$$\sigma_{11} = \sigma_1,\ \sigma_{22} = \sigma_2,\ \sigma_{33} = \sigma_3, \text{ so } \sigma_{ij} = 0,\ i \neq j \tag{5}$$

Note that the shear stress components are zero. What are the magnitudes of the maximum shear stresses and what are the orientations of the planes on which these shear stresses act? Use the condition described in (5) to define the mean normal pressure in terms of the thermodynamic pressure, the two material constants, and the rate of volume change, D_{kk}. Use this relationship, in turn, to define the bulk viscosity, κ, and describe what the bulk viscosity measures.

c) Describe the two special conditions for the flowing fluid with this constitutive law in non-uniform motion under which the thermodynamic pressure is exactly equal to the mean normal pressure. Rewrite the constitutive law to be consistent with each of these conditions. In this context describe what is meant by an incompressible fluid by referring to the equation of continuity (7.81):

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\rho\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) \tag{6}$$

Rearrange the constitutive law for the incompressible fluid treating the rate of deformation components as the dependent variables and the stress components as the independent variables.

