Solutions to: Geomathematics

Modelling and Solving Mathematical Problems in Geodesy and Geophysics

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Gravitation and Harmonic Functions

Exercises

3.1 In this exercise, the harmonicity of the fundamental solutions

$$\gamma_n(x) := \frac{1}{(n-2)|x|^{n-2}}\,, \quad x \in \mathbb{R}^n \setminus \{0\}, \quad \text{if } n \geq 3\,,$$

and

$$\gamma_2(x) := -\log|x|\,, \qquad x \in \mathbb{R}^2 \setminus \{0\}, \quad \text{if } n = 2\,,$$

is to be proved. For $n \geq 3$, we obtain

$$\begin{aligned}
\gamma_n(x) &= \frac{1}{n-2}\left(\sum_{j=1}^n x_j^2\right)^{(-n+2)/2}\,,\\
\frac{\partial \gamma_n}{\partial x_k}(x) &= \frac{1}{n-2}\,\frac{-n+2}{2}\left(\sum_{j=1}^n x_j^2\right)^{-n/2} 2x_k = -\left(\sum_{j=1}^n x_j^2\right)^{-n/2} x_k\,,\\
\frac{\partial^2 \gamma_n}{\partial x_k^2}(x) &= -\left(-\frac{n}{2}\right)\left(\sum_{j=1}^n x_j^2\right)^{(-n-2)/2} 2x_k^2 - \left(\sum_{j=1}^n x_j^2\right)^{-n/2}.
\end{aligned}$$

Hence, the Laplace operator yields

$$\Delta_x \gamma_n(x) = \sum_{k=1}^n \frac{\partial^2 \gamma_n}{\partial x_k^2}(x) = n|x|^{-n-2} \underbrace{\sum_{k=1}^n x_k^2}_{=|x|^2} - n|x|^{-n} = 0\,.$$

In the case $n = 2$, we obtain

$$\begin{aligned}
\gamma_2(x) &= -\log\left(\sum_{j=1}^2 x_j^2\right)^{1/2}\,,\\
\frac{\partial \gamma_2}{\partial x_k}(x) &= -\left(\sum_{j=1}^2 x_j^2\right)^{-1/2} \frac{1}{2}\left(\sum_{j=1}^2 x_j^2\right)^{-1/2} 2x_k = -\left(\sum_{j=1}^2 x_j^2\right)^{-1} x_k\,,
\end{aligned}$$

$$\frac{\partial^2 \gamma_2}{\partial x_k^2}(x) = \left(\sum_{j=1}^{2} x_j^2\right)^{-2} 2x_k^2 - \left(\sum_{j=1}^{2} x_j^2\right)^{-1}$$

such that

$$\Delta_x \gamma_2(x) = 2|x|^{-4} \underbrace{\sum_{k=1}^{2} x_k^2}_{=|x|^2} - 2|x|^{-2} = 0\,.$$

3.2 The exercise is to show the harmonicity of

$$V(x) := \int_D F(y)\,\frac{1}{|x-y|^{n-2}}\,\mathrm{d}y\,, \quad x \in \mathbb{R}^n\,, \quad \text{if } n \geq 3\,,$$
$$V(x) := \int_D F(y)\log|x-y|\,\mathrm{d}y\,, \quad x \in \mathbb{R}^2, \quad \text{if } n = 2\,,$$

outside $\overline{D}$, where $D \subset \mathbb{R}^n$ is a bounded and open set and $F\colon D \to \mathbb{R}$ is a bounded and Lebesgue-integrable function.

Let $x \in \mathbb{R}^3 \setminus \overline{D}$ be arbitrary but fixed. Then there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subset \mathbb{R}^3 \setminus \overline{D}$. In analogy to the solution of Exercise 3.1, we obtain, for $n \geq 3$,

$$\frac{\partial}{\partial x_k}\frac{1}{|x-y|^{n-2}} = -(n-2)\left(\sum_{j=1}^{n}(x_j-y_j)^2\right)^{-n/2}(x_k-y_k)\,,$$
$$\frac{\partial^2}{\partial x_l \partial x_k}\frac{1}{|x-y|^{n-2}} = (n-2)n\left(\sum_{j=1}^{n}(x_j-y_j)^2\right)^{-(n+2)/2}(x_l-y_l)(x_k-y_k)$$
$$-(n-2)\left(\sum_{j=1}^{n}(x_j-y_j)^2\right)^{-n/2}\delta_{lk}$$

such that

$$\left|\nabla_x \frac{1}{|x-y|^{n-2}}\right| \leq (n-2)\frac{1}{|x-y|^{n-1}}\,,$$
$$\left|\frac{\partial^2}{\partial x_l \partial x_k}\frac{1}{|x-y|^{n-2}}\right| \leq (n-2)(n+1)\frac{1}{|x-y|^{n}}\,,$$

while the case $n = 2$ leads us to

$$\frac{\partial}{\partial x_k}\log|x-y| = \left(\sum_{j=1}^{2}(x_j-y_j)^2\right)^{-1}(x_k-y_k)\,,$$
$$\frac{\partial^2}{\partial x_l \partial x_k}\log|x-y| = -\left(\sum_{j=1}^{2}(x_j-y_j)^2\right)^{-4/2} 2\,(x_l-y_l)(x_k-y_k)$$
$$+\left(\sum_{j=1}^{2}(x_j-y_j)^2\right)^{-2/2}\delta_{kl}$$

with

$$|\nabla_x \log|x-y|| \leq \frac{1}{|x-y|} \,,$$
$$\left|\frac{\partial^2}{\partial x_l \partial x_k} \log|x-y|\right| \leq \frac{3}{|x-y|^2} \,.$$

Hence, there exist constants A_n and B_n such that, for $n \geq 2$, a first-order derivative of the integrand of V with respect to x is bounded by $A_n \varepsilon^{1-n} M$ and any corresponding second-order derivative is bounded by $B_n \varepsilon^{-n}$ for all $y \in D$, where $M := \sup_{y \in D} |F(y)|$. Since D is bounded, the constant bounds are integrable in D and, consequently, we can interchange first- and second-order differentiations with the integral. In view of Exercise 3.1, we get that $\Delta_x V(x) = 0$.

3.3 A possible definition is:

U is regular at infinity (in $\mathbb{R}^n$), if $|U(y)| = \mathcal{O}(|y|^{2-n})$ and $|\nabla U(y)| = \mathcal{O}(|y|^{1-n})$ as $|y| \to \infty$.

Obviously, $y \mapsto |y|^{2-n}$ satisfies the first requirement. From the solution of Exercise 3.1, we see that $\nabla |y|^{2-n} = (2-n)|y|^{-n} y$ and $|\nabla |y|^{2-n}| = (n-2)|y|^{1-n} = \mathcal{O}(|y|^{1-n})$ as $|y| \to \infty$.

3.4 The Kelvin transform is defined by $F^*(x) := |x|^{2-n} F(|x|^{-2}x)$, $x \in R^* \subset \mathbb{R}^n$. The proof of the formula for the application of the Laplace operator is basically analogous to the proof in the case $n = 3$, which is in the book. We obtain

$$\Delta_x F^*(x) = \left(\Delta_x |x|^{2-n}\right) F\left(\frac{x}{|x|^2}\right) + 2\left(\nabla_x |x|^{2-n}\right) \cdot \nabla_x F\left(\frac{x}{|x|^2}\right) + |x|^{2-n} \Delta_x F\left(\frac{x}{|x|^2}\right) \,.$$

According to Exercise 3.1, the first summand on the right-hand side vanishes and $\nabla_x |x|^{2-n} = (2-n)|x|^{-n} x$. The calculation of $\nabla_x F(|x|^{-2}x)$ is independent of the dimension n (see the proof for $n = 3$). Hence, we obtain

$$\begin{aligned} &2\left(\nabla_x |x|^{2-n}\right) \cdot \nabla_x F\left(\frac{x}{|x|^2}\right) \\ &\quad = 4(n-2)\frac{|x|^2}{|x|^{n+4}}\, x \cdot \left(\nabla_y F(y)\right)\Big|_{y=\frac{x}{|x|^2}} - 2\frac{n-2}{|x|^{n+2}}\, x \cdot \left(\nabla_y F(y)\right)\Big|_{y=\frac{x}{|x|^2}} \\ &\quad = \frac{2(n-2)}{|x|^{n+2}}\, x \cdot \left(\nabla_y F(y)\right)\Big|_{y=\frac{x}{|x|^2}} \end{aligned}$$

for all $x \in R^*$. The formula for $\frac{\partial^2}{\partial x_j^2} F(|x|^{-2}x)$ is also entirely independent of the dimension n such that we can take the already derived formula for $n = 3$ here as well. A closer look at the sum over $j = 1, 2, 3$ (also available in the book) reveals that, for general n, only one term changes. We get (the $2n$ in the first line was originally 6 for $n = 3$)

$$\begin{aligned} \Delta_x F\left(\frac{x}{|x|^2}\right) &= \left(\frac{8|x|^2 - 2n|x|^2}{|x|^6} - \frac{4}{|x|^4}\right) x \cdot \left(\nabla_y F(y)\right)\Big|_{y=\frac{x}{|x|^2}} \\ &\quad + \left(\frac{4}{|x|^6} - \frac{2}{|x|^6} - \frac{2}{|x|^6}\right) x^{\mathrm{T}} \left[(\nabla_y \otimes \nabla_y) F(y)\right]\Big|_{y=\frac{x}{|x|^2}} \cdot x \end{aligned}$$

$$+ \frac{1}{|x|^4} (\Delta_y F(y)) \Big|_{y=\frac{x}{|x|^2}} \ .$$

Eventually, the combination of the derivations above leads us to

$$\begin{aligned}
\Delta_x F^*(x) &= \frac{2(n-2)}{|x|^{n+2}} \, x \cdot (\nabla_y F(y))|_{y=\frac{x}{|x|^2}} - \frac{2n-4}{|x|^{n+2}} \, x \cdot (\nabla_y F(y))|_{y=\frac{x}{|x|^2}} \\
&\quad + \frac{1}{|x|^{n+2}} \, (\Delta_y F(y))|_{y=\frac{x}{|x|^2}} \\
&= \frac{1}{|x|^{n+2}} \, (\Delta_y F(y))|_{y=\frac{x}{|x|^2}} \, , \quad x \in R^* \, .
\end{aligned}$$

3.5 A function $U \in \mathrm{C}^{(2)}(D)$, $D \subset \mathbb{R}$, is harmonic, if and only if $U''(x) = 0$ for all $x \in D$. If D is a region, which means in $\mathbb{R}^1$ that D is an open interval, then a harmonic function U on D must have the form $U(x) = ax + b$ for some constants $a, b \in \mathbb{R}$.

Gauß's mean value theorem in $\mathbb{R}^1$ can be postulated as follows: If $U\colon [x_0 - R, x_0 + R] \to \mathbb{R}$ is harmonic on $]x_0 - R, x_0 + R[$ and continuous on $[x_0 - R, x_0 + R]$, then

$$U(x_0) = \frac{1}{2} \left(U\left(x_0 - R\right) + U\left(x_0 + R\right) \right) = \frac{1}{2R} \int_{x_0 - R}^{x_0 + R} U(x) \, \mathrm{d}x \, .$$

The analogy is given in the sense that $[x_0 - R, x_0 + R]$ is a one-dimensional ball with the measure $2R$ and its boundary is given by $\{x_0 - R, x_0 + R\}$, which has 2 elements. The proof is easy. With the considerations above, we get

$$\begin{aligned}
\int_{x_0 - R}^{x_0 + R} U(x) \, \mathrm{d}x &= \int_{x_0 - R}^{x_0 + R} ax + b \, \mathrm{d}x = \left(\frac{a}{2} \, x^2 + bx \right) \Big|_{x_0 - R}^{x_0 + R} \\
&= \frac{a}{2} \left(x_0 + R\right)^2 - \frac{a}{2} \left(x_0 - R\right)^2 + b\left(x_0 + R\right) - b\left(x_0 - R\right) \\
&= \frac{a}{2} \, 4x_0 R + 2bR = 2R(ax_0 + b) = 2RU(x_0)
\end{aligned}$$

and

$$\frac{1}{2} \left(U\left(x_0 - R\right) + U\left(x_0 + R\right) \right) = \frac{1}{2} \left(a\left(x_0 - R\right) + b + a\left(x_0 + R\right) + b \right) = U(x_0) \, .$$

3.6 We use the fundamental theorem on the ball $B_R(x_0)$, namely

$$\begin{aligned}
4\pi U(x_0) &= \int_{S_R(x_0)} \left(\frac{1}{|x - x_0|} \frac{\partial U}{\partial \nu}(x) - U(x) \frac{\partial}{\partial \nu(x)} \frac{1}{|x - x_0|} \right) \mathrm{d}\omega(x) \\
&\quad - \int_{B_R(x_0)} \frac{\Delta U(x)}{|x - x_0|} \, \mathrm{d}x \, , \qquad (3.1)
\end{aligned}$$

and Green's second identity, that is

$$\int_{B_R(x_0)} U(x) \, \underbrace{\Delta_x 1}_{=0} - 1 \, \Delta_x U(x) \, \mathrm{d}x = \int_{S_R(x_0)} U(x) \, \underbrace{\frac{\partial}{\partial \nu(x)} 1}_{=0} - 1 \, \frac{\partial}{\partial \nu(x)} U(x) \, \mathrm{d}\omega(x) \, . \qquad (3.2)$$

From (3.2), we obtain

$$\int_{S_R(x_0)} \frac{1}{|x - x_0|} \frac{\partial U}{\partial \nu}(x) \, \mathrm{d}\omega(x) = \frac{1}{R} \int_{S_R(x_0)} \frac{\partial U}{\partial \nu}(x) \, \mathrm{d}\omega(x) = \frac{1}{R} \int_{B_R(x_0)} \Delta_x U(x) \, \mathrm{d}x \, . \qquad (3.3)$$

Furthermore, by using

$$\nu(x) = \frac{x - x_0}{R}, \quad x \in S_R(x_0)\,,$$

and

$$\nabla_x \frac{1}{|x - x_0|} = -\frac{x - x_0}{|x - x_0|^3}\,,$$

we get

$$\begin{aligned}\int_{S_R(x_0)} U(x)\,\frac{\partial}{\partial\nu(x)}\,\frac{1}{|x - x_0|}\,\mathrm{d}\omega(x) &= \int_{S_R(x_0)} U(x)\,\frac{x - x_0}{R}\cdot\frac{-(x - x_0)}{|x - x_0|^3}\,\mathrm{d}\omega(x)\\ &= -\frac{1}{R^2}\int_{S_R(x_0)} U(x)\,\mathrm{d}\omega(x)\,. \qquad (3.4)\end{aligned}$$

Inserting (3.4) and (3.3) into (3.1), we obtain the desired result.

3.7 Due to the chain rule, we have

$$\nabla|\nabla V| = \frac{1}{2|\nabla V|}\,\nabla|\nabla V|^2\,.$$

Hence,

$$\begin{aligned}\nabla|\nabla V(x)| &= \frac{1}{2|\nabla V(x)|}\left(\frac{\partial}{\partial x_i}\sum_{k=1}^{3}\left(\frac{\partial}{\partial x_k}V(x)\right)^2\right)_{i=1,2,3}\\ &= \frac{1}{|\nabla V(x)|}\left(\sum_{k=1}^{3}\frac{\partial}{\partial x_k}V(x)\,\frac{\partial^2}{\partial x_i\partial x_k}V(x)\right)_{i=1,2,3}\\ &= \frac{1}{|\nabla V(x)|}(\nabla\otimes\nabla V(x))\cdot(\nabla V(x))\\ &= -(\nabla\otimes\nabla V(x))\cdot n(x)\,, \qquad (3.5)\end{aligned}$$

where the dot in the latter and the penultimate line stands for a matrix-vector multiplication.

With the formula $\nabla^* F = \nabla F - (n\cdot\nabla F)n$ for the surface gradient, we can now conclude that

$$\nabla^*|\nabla V| = -(\nabla\otimes\nabla V)n + \left(n^{\mathrm{T}}(\nabla\otimes\nabla V)n\right)n\,. \qquad (3.6)$$

Furthermore, we have, due to the product rule, the identity

$$\begin{aligned}\varphi''(s) &= \frac{\mathrm{d}}{\mathrm{d}s}n(\varphi(s)) = \frac{\mathrm{d}}{\mathrm{d}s}\left(-\frac{\nabla V(\varphi(s))}{|\nabla V(\varphi(s))|}\right) \qquad (3.7)\\ &= \frac{1}{|\nabla V(\varphi(s))|^2}\left(\frac{\mathrm{d}}{\mathrm{d}s}|\nabla V(\varphi(s))|\right)\nabla V(\varphi(s)) - \frac{1}{|\nabla V(\varphi(s))|}\,\frac{\mathrm{d}}{\mathrm{d}s}\nabla V(\varphi(s))\,.\end{aligned}$$

The chain rule also yields

$$\frac{\mathrm{d}}{\mathrm{d}s}\nabla V(\varphi(s)) = (\nabla\otimes\nabla V(\varphi(s)))\cdot\underbrace{\varphi'(s)}_{=n}\,, \qquad (3.8)$$

where the dot again represents the matrix-vector multiplication. Moreover, with (3.5), we arrive at (the dot now represents the Euclidean inner product)

$$\frac{\mathrm{d}}{\mathrm{d}s}|\nabla V(\varphi(s))| = \nabla|\nabla V(\varphi(s))|\cdot\varphi'(s) = -n^{\mathrm{T}}(\nabla\otimes\nabla V)n\big|_{\varphi(s)}\,. \qquad (3.9)$$

Eventually, we combine (3.6), (3.7), (3.8), and (3.9) and we observe that

$$\begin{aligned}\varphi''(s) &= \left\{ \frac{1}{|\nabla V|} \left[\left(n^{\mathrm{T}} (\nabla \otimes \nabla V) n \right) n - (\nabla \otimes \nabla V) n \right] \right\} \Bigg|_{\varphi(s)} \\ &= \left(\frac{1}{|\nabla V|} \nabla^* |\nabla V| \right) \Bigg|_{\varphi(s)} .\end{aligned}$$

3.8 For proving this, we need to assume that $U \in \mathrm{C}^{(2)}(\overline{R})$. We start with the proof of '⇒': for this purpose, we use the stronger version of Gauß's mean value theorem from Exercise 3.6. Moreover, we observe that $|x - x_0| < R$ for all $x \in B_R(x_0)$ and, hence, $|x - x_0|^{-1} - R^{-1} > 0$. Let now $\Delta U \geq 0$ on R and let the ball $B_R(x_0)$ be arbitrarily chosen, with the mere requirement that $\overline{B_R(x_0)} \subset R$. Then we obtain

$$\begin{aligned}U(x_0) &= \frac{1}{4\pi R^2} \int_{S_R(x_0)} U(x) \, \mathrm{d}\omega(x) - \frac{1}{4\pi} \int_{B_R(x_0)} \Delta U(x) \left(\frac{1}{|x - x_0|} - \frac{1}{R} \right) \mathrm{d}x \\ &\leq \frac{1}{4\pi R^2} \int_{S_R(x_0)} U(x) \, \mathrm{d}\omega(x) .\end{aligned}$$

Hence, since a smaller radius of the ball would work as well, we have

$$U(x_0) \leq \frac{1}{4\pi \varrho^2} \int_{S_\varrho(x_0)} U(x) \, \mathrm{d}\omega(x) \quad \text{for all } \varrho \in]0, R] .$$

We easily deduce now that

$$\int_0^R U(x_0) \varrho^2 \, \mathrm{d}\varrho \leq \int_0^R \frac{1}{4\pi \varrho^2} \int_{S_\varrho(x_0)} U(x) \, \mathrm{d}\omega(x) \, \varrho^2 \, \mathrm{d}\varrho .$$

Obviously, this leads us to

$$\frac{R^3}{3} U(x_0) \leq \frac{1}{4\pi} \int_{B_R(x_0)} U(x) \, \mathrm{d}x$$

such that U is subharmonic.

For '⇐', we assume that U is subharmonic and $\overline{B_R(x_0)} \subset R$ is an arbitrary ball.

Then we consider the properties of a function $V \in \mathrm{C}(\overline{B_R(x_0)})$ which is harmonic on $B_R(x_0)$ and satisfies $U \leq V$ on $S_R(x_0)$. Let $W := U - V$. We want to prove that the latter inequality is maintained on $B_R(x_0)$. For this purpose, we assume that there is $y \in B_R(x_0)$ with $U(y) > V(y)$.

Since W is continuous and $\overline{B_R(x_0)}$ is compact, there exists $z \in \overline{B_R(x_0)}$ such that $A := W(z) = \max_{x \in \overline{B_R(x_0)}} W(x)$. Due to the assumption above, A must be positive. However, since W is non-positive on $S_R(x_0)$, it cannot be constant. Therefore, there exists $x_1 \in \overline{B_R(x_0)}$ where $W(x_1) = A$ and each neighbourhood of x_1 contains points x with $W(x) < A$. Thus, the subharmonicity of U and Gauß's mean value theorem for V yield (provided that R is sufficiently small)

$$\begin{aligned}A &= W(x_1) = U(x_1) - V(x_1) \\ &\leq \frac{3}{4\pi R^3} \int_{B_R(x_1)} U(x) \, \mathrm{d}x - \frac{3}{4\pi R^3} \int_{B_R(x_1)} V(x) \, \mathrm{d}x = \frac{3}{4\pi R^3} \int_{B_R(x_1)} W(x) \, \mathrm{d}x \\ &< A , \qquad (3.10)\end{aligned}$$

where the latter inequality holds true due to our previous observation on W in neighbourhoods of x_1 and the continuity of W. However, (3.10) includes the contradiction $A < A$. Hence, our assumption is falsified and $U \le V$ on the whole set $\overline{B_R(x_0)}$.

In particular, if V is the (unique) solution of the IDP

$\Delta V = 0$ on $B_R(x_0)$ and $V = U$ on $S_R(x_0)$,

then the latter result implies that $U(x) \le V(x)$ for all $x \in B_R(x_0)$, while Gauß's mean value theorem leads us to

$$U(x_0) \le V(x_0) = \frac{1}{4\pi R^2} \int_{S_R(x_0)} V(x)\,\mathrm{d}\omega(x) = \frac{1}{4\pi R^2} \int_{S_R(x_0)} U(x)\,\mathrm{d}\omega(x)\,.$$

Consequently, the stronger version of Gauss's mean value theorem (Exercise 3.6) yields

$$\frac{1}{4\pi} \int_{B_\varrho(x_0)} \Delta U(x) \underbrace{\left(\frac{1}{|x - x_0|} - \frac{1}{\varrho}\right)}_{>0} \mathrm{d}x \ge 0$$

for all $\varrho \in \,]0, R]$. If we had $\Delta U(x_0) < 0$, then there would exist $\varrho > 0$ with $\Delta U|_{B_\varrho(x_0)} < 0$, which is a contradiction. Due to the arbitrariness of x_0, the proof is finished.

The solution of part b is: U is subharmonic $\Leftrightarrow$ U is convex.

3.9 We observe first that $\mathcal{T}F \in \mathrm{C}(E)$ for all $F \in \mathrm{C}(D)$ and $\mathcal{T}$ is linear due to basic propositions from real analysis. Furthermore, the triangle inequality for integrals and the continuity of the integral kernel K and of F yield

$$|(\mathcal{T}F)(x)| \le \int_D \underbrace{|K(x,y)|}_{\le \|K\|_{\mathrm{C}(E\times D)}} \underbrace{|F(y)|}_{\le \|F\|_{\mathrm{C}(D)}} \,\mathrm{d}y \le \|K\|_{\mathrm{C}(E\times D)} \lambda(D) \|F\|_{\mathrm{C}(D)} \tag{3.11}$$

for all $x \in E$ and all $F \in \mathrm{C}(D)$, where λ is the usual Lebesgue measure. Hence, $\mathcal{T}$ is bounded, because

$$\|\mathcal{T}F\|_{\mathrm{C}(E)} \le \|K\|_{\mathrm{C}(E\times D)} \lambda(D) \|F\|_{\mathrm{C}(D)} \quad \text{for all } F \in \mathrm{C}(D)\,.$$

Let us now prove that $\mathcal{T}$ is compact. For this purpose, we consider the image of the unit sphere $\mathcal{U}$ in $\mathrm{C}(D)$ and show that $\mathcal{T}\mathcal{U}$ is relatively compact by using the Ascoli–Arzelà theorem. This means that we need to prove that $\mathcal{T}\mathcal{U}$ is pointwise bounded and equicontinuous.

From Equation (3.11), we get that $|(\mathcal{T}F)(x)| \le \|K\|_{\mathrm{C}(E\times D)} \lambda(D)$ for all $F \in \mathcal{U}$ and all $x \in E$. Hence, $\mathcal{T}\mathcal{U}$ is pointwise bounded (and, actually, also uniformly bounded).

Furthermore, the Cauchy–Schwarz inequality yields

$$\begin{aligned} \left|(\mathcal{T}F)(x) - (\mathcal{T}F)\left(x'\right)\right|^2 &= \left|\int_D \left(K(x,y) - K\left(x',y\right)\right) F(y)\,\mathrm{d}y\right|^2 \\ &\le \int_D \left(K(x,y) - K\left(x',y\right)\right)^2 \,\mathrm{d}y \int_D (F(y))^2\,\mathrm{d}y\,. \end{aligned} \tag{3.12}$$

Let now $\varepsilon > 0$ be arbitrary. Since $E \times D$ is compact and K is continuous, the kernel K is also uniformly continuous, that is we find a $\delta > 0$ such that the following implication is valid (if $\lambda(D) = 0$, then $\mathcal{T} = 0$ and the compactness is trivial):

$$\left|(x,y) - \left(x',y'\right)\right| < \delta \quad \Rightarrow \quad \left|K(x,y) - K\left(x',y'\right)\right| < \frac{\varepsilon}{\lambda(D)}\,.$$

In particular, we have the implication:

$$\left|x - x'\right| < \delta,\ y \in D \text{ arbitrary} \quad \Rightarrow \quad \left|K(x,y) - K\left(x', y\right)\right| < \frac{\varepsilon}{\lambda(D)}\,.$$

Consequently, if we have $x, x' \in E$ with $|x - x'| < \delta$, then we get, due to (3.12), that

$$\left|(\mathcal{T}F)(x) - (\mathcal{T}F)\left(x'\right)\right|^2 \le \lambda(D)\, \frac{\varepsilon^2}{\lambda(D)^2}\, \|F\|^2_{\mathrm{L}^2(D)}\,. \tag{3.13}$$

Moreover, each $F \in \mathcal{U}$ satisfies

$$\|F\|_{\mathrm{L}^2(D)} = \sqrt{\int_D F(x)^2 \,\mathrm{d}x} \le \|F\|_{\mathrm{C}(D)} \sqrt{\lambda(D)} = \sqrt{\lambda(D)}\,. \tag{3.14}$$

Combining Equations (3.13) and (3.14), we see that all $x, x' \in E$ with $|x - x'| < \delta$ satisfy

$$\left|(\mathcal{T}F)(x) - (\mathcal{T}F)\left(x'\right)\right| \le \varepsilon \quad \text{for each } F \in \mathcal{U}\,.$$

Hence, $\mathcal{T}\mathcal{U}$ is equicontinuous.

In total, we get that $\mathcal{T}\mathcal{U}$ is a relatively compact set and, consequently, $\mathcal{T}$ is a compact operator.

3.10 For proving that $\mathcal{T}F \in \mathrm{L}^2(E)$, we need to show that $\mathcal{T}F$ has a finite norm in this space (because of the definition of $\mathrm{L}^2(E)$). With the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}
\|\mathcal{T}F\|^2_{\mathrm{L}^2(E)} &= \int_E [(\mathcal{T}F)(x)]^2 \,\mathrm{d}x = \int_E \left[\int_D K(x,y)F(y)\,\mathrm{d}y\right]^2 \,\mathrm{d}x \\
&\le \int_E \int_D K(x,y)^2 \,\mathrm{d}y \int_D F(y)^2 \,\mathrm{d}y\,\mathrm{d}x \\
&= \int_{E\times D} K(x,y)^2 \,\mathrm{d}(x,y) \int_D F(y)^2 \,\mathrm{d}y \\
&= \|K\|^2_{\mathrm{L}^2(E\times D)} \|F\|^2_{\mathrm{L}^2(D)} < +\infty\,.
\end{aligned} \tag{3.15}$$

Equation (3.15) also shows that $\mathcal{T}$ is a bounded operator.

Let now $\mathcal{U}$ be the unit sphere in $\mathrm{L}^2(D)$ and let (F_n) be an arbitrary sequence in $\mathcal{U}$. Then (F_n) has a weakly convergent subsequence. Without loss of generality, we assume that (F_n) is already this subsequence. The weak limit is denoted by F. Since $\int_E \int_D K(x,y)^2 \,\mathrm{d}y\,\mathrm{d}x < +\infty$, Fubini's theorem tells us that $\int_D K(x,y)^2 \,\mathrm{d}y < +\infty$ for almost all $x \in E$, that is $K(x,\cdot) \in \mathrm{L}^2(D)$ for almost every $x \in E$.

For every such x, we obtain

$$\begin{aligned}
\lim_{n\to\infty} \left(\mathcal{T}F_n\right)(x) &= \lim_{n\to\infty} \int_D K(x,y)F_n(y)\,\mathrm{d}y \\
&= \lim_{n\to\infty} \langle K(x,\cdot), F_n\rangle_{\mathrm{L}^2(D)} \\
&= \langle K(x,\cdot), F\rangle_{\mathrm{L}^2(D)} \\
&= (\mathcal{T}F)(x)
\end{aligned}$$

and, in analogy to (3.15), also

$$|(\mathcal{T}F_n)(x)|^2 = \left|\int_D K(x,y)F_n(y)\,\mathrm{d}y\right|^2 \leq \underbrace{\int_D K(x,y)^2\,\mathrm{d}y}_{=:G(x)}\,\underbrace{\|F_n\|^2_{\mathrm{L}^2(D)}}_{=1}\,.$$

Hence, $|(\mathcal{T}F_n)(x)|^2 \leq G(x)$ and $[(\mathcal{T}F_n)(x)]^2 \to [(\mathcal{T}F)(x)]^2$ for almost every $x \in E$, where $\int_E G(x)\,\mathrm{d}x < +\infty$. Thus, the dominated convergence theorem allows us to conclude that

$$\lim_{n\to\infty}\int_E [(\mathcal{T}F_n)(x)]^2\,\mathrm{d}x = \int_E [(\mathcal{T}F)(x)]^2\,\mathrm{d}x\,. \tag{3.16}$$

Furthermore, with the weak convergence, we see that every $H \in \mathrm{L}^2(E)$ satisfies

$$\langle \mathcal{T}F_n, H\rangle_{\mathrm{L}^2(E)} = \langle F_n, \mathcal{T}^*H\rangle_{\mathrm{L}^2(D)} \to \langle F, \mathcal{T}^*H\rangle_{\mathrm{L}^2(D)} = \langle \mathcal{T}F, H\rangle_{\mathrm{L}^2(E)}\,.$$

Consequently, $\mathcal{T}F_n \rightharpoonup \mathcal{T}F$, that is we have a weak convergence in $\mathrm{L}^2(E)$. This weak convergence in combination with (3.16) leads us to the *strong* convergence $\mathcal{T}F_n \to \mathcal{T}F$ in $\mathrm{L}^2(E)$. This result means that the sequence $(\mathcal{T}F_n)$ *strongly* converges to a limit in $\overline{\mathcal{T}\mathcal{U}}$. Due to the arbitrariness of the choice of the (sub-)sequence $(F_n) \subset \mathcal{U}$, we obtain that $\mathcal{T}\mathcal{U}$ is relatively compact and, therefore, $\mathcal{T}$ is a compact operator.

3.11 a) We first calculate the single layer potential along the x_3-axis by using the polar coordinates

$$y_1 = r\cos\varphi\,, \qquad y_2 = r\sin\varphi\,, \qquad y_3 = 0 \tag{3.17}$$

with $r \in [0,1]$ and $\varphi \in [0,2\pi]$. We get

$$\begin{aligned} P_{\mathrm{s}}(0,0,x_3) &= \int_D \frac{1}{\sqrt{y_1^2+y_2^2+(x_3-y_3)^2}}\,\mathrm{d}\omega(y) \\ &= \int_0^1\int_0^{2\pi} \frac{r}{\sqrt{r^2+x_3^2}}\,\mathrm{d}\varphi\,\mathrm{d}r = 2\pi\sqrt{r^2+x_3^2}\Big|_{r=0}^{r=1} \\ &= 2\pi\left(\sqrt{1+x_3^2}-|x_3|\right)\,. \end{aligned} \tag{3.18}$$

Note the absolute value in (3.18). The limits of the normal derivatives then become

$$\begin{aligned} \partial_{\nu+}P_{\mathrm{s}}(0) &= \lim_{t\to 0+}\left[\nu(0)\cdot\nabla P_{\mathrm{s}}(0+t\nu(0))\right] = \lim_{t\to 0+}\left(\frac{\partial}{\partial x_3}P(x)\right)\Big|_{x=(0,0,t)} \\ &= \lim_{t\to 0+}\frac{\partial}{\partial t}\left[2\pi\left(\sqrt{1+t^2}-|t|\right)\right] = \lim_{t\to 0+}\left[2\pi\left(\frac{t}{\sqrt{1+t^2}}-1\right)\right] \\ &= -2\pi \end{aligned}$$

and, analogously,

$$\begin{aligned} \partial_{\nu-}P_{\mathrm{s}}(0) &= \lim_{t\to 0+}\frac{\partial}{\partial t}\left[2\pi\left(\sqrt{1+t^2}-(-t)\right)\right] = \lim_{t\to 0+}\left[2\pi\left(\frac{t}{\sqrt{1+t^2}}+1\right)\right] \\ &= 2\pi\,. \end{aligned}$$

Consequently, the jump is

$$\partial_{\nu+}P_{\mathrm{s}}(0) - \partial_{\nu-}P_{\mathrm{s}}(0) = -4\pi = -4\pi F(0)\,.$$

b) By using again the polar coordinates (3.17), we obtain the double layer potential

$$\begin{aligned}
P_{\mathrm{d}}(0,0,x_3) &= \int_D \frac{x_3 - y_3}{|x-y|^3}\,\mathrm{d}\omega(y) \\
&= \int_0^1 r \int_0^{2\pi} \frac{x_3}{\left(r^2 + x_3^2\right)^{3/2}}\,\mathrm{d}\varphi\,\mathrm{d}r = 2\pi x_3 \int_0^1 \frac{r}{\left(r^2+x_3^2\right)^{3/2}}\,\mathrm{d}r \\
&= 2\pi x_3 \left(-\frac{1}{\sqrt{r^2+x_3^2}}\right)\Bigg|_{r=0}^{r=1} = -2\pi x_3 \left(\frac{1}{\sqrt{1+x_3^2}} - \frac{1}{|x_3|}\right) \\
&= -2\pi \left(\frac{x_3}{\sqrt{1+x_3^2}} - \operatorname{sgn} x_3\right).
\end{aligned}$$

With the resulting limits

$$\begin{aligned}
\lim_{t\to 0+} P_{\mathrm{d}}(0,0,t) &= -2\pi(0-1) = 2\pi\,, \\
\lim_{t\to 0-} P_{\mathrm{d}}(0,0,t) &= -2\pi(0+1) = -2\pi\,,
\end{aligned}$$

we obtain the jump

$$(P_{\mathrm{d}})_+(0) - (P_{\mathrm{d}})_-(0) = 4\pi = 4\pi F(0)\,.$$

3.12 For $x \in B_R(0)$ and $y \in S_R(0)$, we have the inequalities

$$|x-y| \le |x| + |y| = |x| + R\,, \tag{3.19}$$

$$|x-y| \ge \big||x| - |y|\big| = |y| - |x| = R - |x|\,. \tag{3.20}$$

Moreover, clearly,

$$R^2 - |x|^2 = (R - |x|)(R + |x|)\,. \tag{3.21}$$

Due to the requirements on U, this function must fulfil the Poisson integral formula (in the version for the IDP-solution). With this formula as well as (3.20), (3.21), and Gauß's mean value theorem, we obtain

$$\begin{aligned}
U(x) &= \int_{S_R(0)} U(y)\,\frac{R^2 - |x|^2}{4\pi R|x-y|^3}\,\mathrm{d}\omega(y) \\
&\le \int_{S_R(0)} U(y)\,\frac{R^2-|x|^2}{(R-|x|)^3}\,\frac{1}{4\pi R}\,\mathrm{d}\omega(y) \\
&= \frac{R+|x|}{(R-|x|)^2}\,\frac{1}{4\pi R}\int_{S_R(0)} U(y)\,\mathrm{d}\omega(y) \\
&= \frac{R+|x|}{(R-|x|)^2}\,R\,U(0)\,.
\end{aligned}$$

Analogously, with (3.19) instead of (3.20), we get

$$\begin{aligned}
U(x) &\ge \frac{R-|x|}{(|x|+R)^2}\,\frac{1}{4\pi R}\int_{S_R(0)} U(y)\,\mathrm{d}\omega(y) \\
&= \frac{R-|x|}{(|x|+R)^2}\,R\,U(0)\,.
\end{aligned}$$

3.13 Let $U\colon \mathbb{R}^3 \to \mathbb{R}$ be harmonic and non-negative. Then U fulfils the conditions for Harnack's inequality in every ball $B_R(0)$. Hence, we get for an arbitrary but fixed $x \in \mathbb{R}^3$ the inequality

$$R\,\frac{R-|x|}{(R+|x|)^2}\,U(0) \le U(x) \le R\,\frac{R+|x|}{(R-|x|)^2}\,U(0)$$

for all $R > |x|$. Cancelling R^2 leads us to

$$\frac{1-\frac{|x|}{R}}{\left(1+\frac{|x|}{R}\right)^2}\,U(0) \le U(x) \le \frac{1+\frac{|x|}{R}}{\left(1-\frac{|x|}{R}\right)^2}\,U(0)$$

for all $R > |x|$. In the limit $R \to \infty$, we get

$$U(0) \le U(x) \le U(0)\,.$$

Since x was arbitrary, we obtain $U(x) = U(0)$ for all $x \in \mathbb{R}^3$.

3.14 Let $\overline{B_r(x_0)} \subset R$ be an arbitrary closed ball inside the region and let V be the solution of the IDP

$$\Delta V = 0 \text{ in } B_r\left(x_0\right)\,, \quad V = U \text{ on } S_r\left(x_0\right)\,.$$

Since V is harmonic, it has Gauß's mean value property on $\overline{B_r(x_0)}$ (and all balls which are subsets of it). Hence, this also holds true for $U - V$.

In the proof of maximum principle I, we only used the following facts: the domain is a region and the function is continuous and has Gauß's mean value property on the domain. Here, $B_r(x_0)$ is the considered domain and we get, in analogy to the proof of the maximum principle, that $U - V$ is either constant or it has neither a maximum nor a minimum in $B_r(x_0)$. However, since $U - V$ vanishes identically on the boundary $S_r(x_0)$, this must also be the case in the interior of the ball and, consequently, $U \equiv V$ on $\overline{B_r(x_0)}$. Hence, U is harmonic on $B_r(x_0)$. Since R is open, every $x_0 \in R$ possesses a ball $B_r(x_0)$ with $\overline{B_r(x_0)} \subset R$. Thus, U is harmonic on the entire set R.

3.15 Note that the standard theorem from real analysis on the interchanging of a limit and a differentiation is not applicable here, because this theorem requires the uniform convergence of the sequence of the *derivatives*. Therefore, we have to choose a different way to prove the proposition.

Let (U_k) be a sequence of harmonic functions on R such that this sequence uniformly converges to a function $V\colon R \to \mathbb{R}$. Due to the harmonicity, each U_k has Gauß's mean value property on every $\overline{B_r(x_0)} \subset R$. Since (U_k) uniformly converges to V, the limit $\lim_{k\to\infty}$ may be interchanged with the integration in Gauß's mean value property. Hence, V also has this property (again in every closed ball contained in R).

Moreover, V is a uniform limit of a sequence of continuous functions and is, therefore, also continuous. Hence, Exercise 3.14 tells us that V must be harmonic.

3.16 A corresponding figure can be found in the book for comparison.

3.17 With the chain rule, we obtain

$$\begin{pmatrix} \frac{\partial U}{\partial r} \\ \frac{\partial U}{\partial \varphi} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \varphi} & \frac{\partial y}{\partial \varphi} \end{pmatrix} \begin{pmatrix} \frac{\partial U}{\partial x} \\ \frac{\partial U}{\partial y} \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -r\sin\varphi & r\cos\varphi \end{pmatrix} \begin{pmatrix} \frac{\partial U}{\partial x} \\ \frac{\partial U}{\partial y} \end{pmatrix}\,.$$

By inverting the matrix, we get

$$\begin{pmatrix} \frac{\partial U}{\partial x} \\ \frac{\partial U}{\partial y} \end{pmatrix} = \frac{1}{r} \begin{pmatrix} r\cos\varphi & -\sin\varphi \\ r\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} \frac{\partial U}{\partial r} \\ \frac{\partial U}{\partial \varphi} \end{pmatrix}$$

such that

$$\frac{\partial U}{\partial x} = \cos\varphi \frac{\partial U}{\partial r} - \frac{1}{r} \sin\varphi \frac{\partial U}{\partial \varphi}, \qquad \frac{\partial U}{\partial y} = \sin\varphi \frac{\partial U}{\partial r} + \frac{1}{r} \cos\varphi \frac{\partial U}{\partial \varphi}.$$

For the second-order derivatives we use the results from above and again the chain rule and we arrive at

$$\begin{aligned} \frac{\partial^2 U}{\partial x^2} &= \cos\varphi \frac{\partial}{\partial r}\frac{\partial U}{\partial x} - \frac{1}{r} \sin\varphi \frac{\partial}{\partial \varphi}\frac{\partial U}{\partial x} \\ &= \cos\varphi \left(\cos\varphi \frac{\partial^2 U}{\partial r^2} + \frac{1}{r^2} \sin\varphi \frac{\partial U}{\partial \varphi} - \frac{1}{r} \sin\varphi \frac{\partial^2 U}{\partial r \partial \varphi} \right) \\ &\quad - \frac{1}{r} \sin\varphi \left(-\sin\varphi \frac{\partial U}{\partial r} + \cos\varphi \frac{\partial^2 U}{\partial \varphi \partial r} - \frac{1}{r} \cos\varphi \frac{\partial U}{\partial \varphi} - \frac{1}{r} \sin\varphi \frac{\partial^2 U}{\partial \varphi^2} \right) \end{aligned} \tag{3.22}$$

and

$$\begin{aligned} \frac{\partial^2 U}{\partial y^2} &= \sin\varphi \frac{\partial}{\partial r}\frac{\partial U}{\partial y} + \frac{1}{r} \cos\varphi \frac{\partial}{\partial \varphi}\frac{\partial U}{\partial y} \\ &= \sin\varphi \left(\sin\varphi \frac{\partial^2 U}{\partial r^2} - \frac{1}{r^2} \cos\varphi \frac{\partial U}{\partial \varphi} + \frac{1}{r} \cos\varphi \frac{\partial^2 U}{\partial r \partial \varphi} \right) \\ &\quad + \frac{1}{r} \cos\varphi \left(\cos\varphi \frac{\partial U}{\partial r} + \sin\varphi \frac{\partial^2 U}{\partial \varphi \partial r} - \frac{1}{r} \sin\varphi \frac{\partial U}{\partial \varphi} + \frac{1}{r} \cos\varphi \frac{\partial^2 U}{\partial \varphi^2} \right). \end{aligned} \tag{3.23}$$

By summing up (3.22) and (3.23), we obtain the Laplacian of U:

$$\Delta U = \frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \varphi^2}.$$

3.18 With Exercise 3.17, we can write the 2D Laplace equation in polar coordinates as follows (for $(x, y) \neq (0, 0)$):

$$\begin{aligned} \Delta U = 0 \Leftrightarrow \quad & F''(r)Y(\varphi) + \frac{1}{r} F'(r)Y(\varphi) + \frac{1}{r^2} F(r)Y''(\varphi) = 0 \\ \Leftrightarrow \quad & r^2 F''(r)Y(\varphi) + rF'(r)Y(\varphi) = -F(r)Y''(\varphi) \\ \Leftrightarrow \quad & r^2 \frac{F''(r)}{F(r)} + r\frac{F'(r)}{F(r)} = -\frac{Y''(\varphi)}{Y(\varphi)}, \end{aligned}$$

where the latter identity holds true outside zeros of U. In the latter identity, we see that the left-hand side only depends on the radial coordinate and the right-hand side only depends on the angular coordinate. Hence, both sides must be constant. Thus, there is a constant $C \in \mathbb{R}$ such that

$$r^2 F''(r) + rF'(r) = CF(r), \tag{3.24}$$

$$Y''(\varphi) = -CY(\varphi) \tag{3.25}$$

for all $r \in [0, R]$ and all $\varphi \in [0, 2\pi]$ — the missing points (r, φ) can be included by a continuous extension. For (3.25), we use (in analogy to the 3D-case) the argument that U and, consequently, Y need to be 2π-periodic in φ such that only solutions of the type

$$Y(\varphi) = a_j \cos(j\varphi) + b_j \sin(j\varphi)\,, \quad \varphi \in [0, 2\pi]\,,$$

for arbitrary constants $a_j, b_j \in \mathbb{R}$ and $j \in \mathbb{N}_0$ (with $C = j^2$) are admissible. The particular case $C = 0$ refers to a constant solution. For (3.24) with a fixed $C = j^2$, we use the power series *ansatz* $F(r) = \sum_{n=0}^{\infty} c_n r^n$, which yields

$$\begin{aligned}
&\sum_{n=0}^{\infty} c_n n(n-1) r^n + \sum_{n=0}^{\infty} c_n n r^n = j^2 \sum_{n=0}^{\infty} c_n r^n \\
\Leftrightarrow \quad & c_n \left(n^2 - n + n\right) = j^2 c_n \quad \text{for all } n \in \mathbb{N}_0 \\
\Leftrightarrow \quad & c_n = 0 \quad \text{for all } n \in \mathbb{N}_0 \setminus \{j\} \text{ with arbitrary } c_j\,.
\end{aligned}$$

We get $F(r) = c_j r^j$. Since (3.24) is of order 2, we need a second solution which is linearly independent to the known solution. It is easy to see that $F(r) = \log r$ works for $j = 0$ and $F(r) = r^{-j}$ works for $j > 0$. However, each one does not exist in $r = 0$.

Hence, $F(r) = cr^j$ with arbitrary $c \in \mathbb{R}$ is the general solution. It suffices here to choose $c = 1$. Thus, we obtain the general solution of the 2D Laplace equation as follows:

$$U(x, y)|_{(x,y)=(x(r,\varphi),y(r,\varphi))} = \sum_{j=0}^{\infty} a_j r^j \cos(j\varphi) + \sum_{j=1}^{\infty} b_j r^j \sin(j\varphi)$$

with arbitrary constants $a_j, b_j \in \mathbb{R}$.

4

Basis Functions

Exercises

4.1 We start with $\nabla^* \cdot \nabla^*$. For the calculation, we basically need the product rule and the formulae for the derivatives of the local orthonormal basis ε^r, ε^φ, ε^t. In detail, we get

$$
\begin{aligned}
&\nabla^* \cdot \nabla^* F \\
&\quad = \left(\varepsilon^\varphi \frac{1}{\sqrt{1-t^2}} \frac{\partial}{\partial\varphi} + \varepsilon^t \sqrt{1-t^2} \frac{\partial}{\partial t}\right) \cdot \left(\varepsilon^\varphi \frac{1}{\sqrt{1-t^2}} \frac{\partial}{\partial\varphi} + \varepsilon^t \sqrt{1-t^2} \frac{\partial}{\partial t}\right) F \\
&\quad = \frac{1}{\sqrt{1-t^2}} \varepsilon^\varphi \cdot \left(\frac{\partial}{\partial\varphi}\varepsilon^\varphi\right) \frac{1}{\sqrt{1-t^2}} \frac{\partial}{\partial\varphi} F + \frac{1}{\sqrt{1-t^2}} \varepsilon^\varphi \cdot \varepsilon^\varphi \frac{1}{\sqrt{1-t^2}} \frac{\partial^2}{\partial\varphi^2} F \\
&\qquad + \frac{1}{\sqrt{1-t^2}} \varepsilon^\varphi \cdot \left(\frac{\partial}{\partial\varphi}\varepsilon^t\right) \sqrt{1-t^2} \frac{\partial}{\partial t} F + \frac{1}{\sqrt{1-t^2}} \varepsilon^\varphi \cdot \varepsilon^t \sqrt{1-t^2} \frac{\partial^2}{\partial\varphi \partial t} F \\
&\qquad + \sqrt{1-t^2}\, \varepsilon^t \cdot \left(\frac{\partial}{\partial t}\varepsilon^\varphi\right) \frac{1}{\sqrt{1-t^2}} \frac{\partial}{\partial\varphi} F + \sqrt{1-t^2}\, \varepsilon^t \cdot \varepsilon^\varphi \frac{\partial}{\partial t} \left(\frac{1}{\sqrt{1-t^2}} \frac{\partial}{\partial\varphi} F\right) \\
&\qquad + \sqrt{1-t^2}\, \varepsilon^t \cdot \left(\frac{\partial}{\partial t}\varepsilon^t\right) \sqrt{1-t^2} \frac{\partial}{\partial t} F + \sqrt{1-t^2}\, \varepsilon^t \cdot \varepsilon^t \frac{\partial}{\partial t} \left(\sqrt{1-t^2} \frac{\partial}{\partial t} F\right) \\
&\quad = \frac{1}{1-t^2} \frac{\partial^2}{\partial\varphi^2} F - t \frac{\partial}{\partial t} F + \sqrt{1-t^2} \frac{\partial}{\partial t} \left(\sqrt{1-t^2} \frac{\partial}{\partial t} F\right) \\
&\quad = \frac{1}{1-t^2} \frac{\partial^2}{\partial\varphi^2} F - t \frac{\partial}{\partial t} F + \sqrt{1-t^2} \frac{-t}{\sqrt{1-t^2}} \frac{\partial}{\partial t} F + \left(\sqrt{1-t^2}\right)^2 \frac{\partial^2}{\partial t^2} F \\
&\quad = \frac{1}{1-t^2} \frac{\partial^2}{\partial\varphi^2} F - 2t \frac{\partial}{\partial t} F + \left(1-t^2\right) \frac{\partial^2}{\partial t^2} F \\
&\quad = \frac{1}{1-t^2} \frac{\partial^2}{\partial\varphi^2} F + \frac{\partial}{\partial t} \left[\left(1-t^2\right) \frac{\partial}{\partial t} F\right] \\
&\quad = \Delta^* F\,.
\end{aligned}
$$

The calculation of $\mathrm{L}^* \cdot \mathrm{L}^*$ is essentially analogous to the calculations above. We get

$$
\begin{aligned}
&\mathrm{L}^* \cdot \mathrm{L}^* F \\
&\quad = \left(-\varepsilon^\varphi \sqrt{1-t^2} \frac{\partial}{\partial t} + \varepsilon^t \frac{1}{\sqrt{1-t^2}} \frac{\partial}{\partial\varphi}\right) \cdot \left(-\varepsilon^\varphi \sqrt{1-t^2} \frac{\partial}{\partial t} + \varepsilon^t \frac{1}{\sqrt{1-t^2}} \frac{\partial}{\partial\varphi}\right) F \\
&\quad = \sqrt{1-t^2}\, \varepsilon^\varphi \cdot \varepsilon^\varphi \frac{\partial}{\partial t} \left(\sqrt{1-t^2} \frac{\partial}{\partial t} F\right)
\end{aligned}
$$

$$
\begin{aligned}
&- \sqrt{1-t^2}\,\varepsilon^\varphi \cdot \frac{\partial}{\partial t}\varepsilon^t \frac{1}{\sqrt{1-t^2}} \frac{\partial}{\partial \varphi}F - \sqrt{1-t^2}\,\varepsilon^\varphi \cdot \varepsilon^t \frac{\partial}{\partial t}\left(\frac{1}{\sqrt{1-t^2}}\frac{\partial}{\partial \varphi}F\right) \\
&- \frac{1}{\sqrt{1-t^2}}\,\varepsilon^t \cdot \frac{\partial}{\partial \varphi}\varepsilon^\varphi \sqrt{1-t^2}\,\frac{\partial}{\partial t}F - \frac{1}{\sqrt{1-t^2}}\,\varepsilon^t \cdot \varepsilon^\varphi \sqrt{1-t^2}\,\frac{\partial^2}{\partial \varphi \partial t}F \\
&+ \frac{1}{\sqrt{1-t^2}}\,\varepsilon^t \cdot \frac{\partial}{\partial \varphi}\varepsilon^t \frac{1}{\sqrt{1-t^2}} \frac{\partial}{\partial \varphi}F + \frac{1}{\sqrt{1-t^2}}\,\varepsilon^t \cdot \varepsilon^t \frac{1}{\sqrt{1-t^2}} \frac{\partial^2}{\partial \varphi^2}F \\
&= \sqrt{1-t^2}\left(\frac{-t}{\sqrt{1-t^2}}\frac{\partial}{\partial t}F + \sqrt{1-t^2}\,\frac{\partial^2}{\partial t^2}F\right) - \frac{t}{\sqrt{1-t^2}}\sqrt{1-t^2}\,\frac{\partial}{\partial t}F \\
&+ \frac{1}{\sqrt{1-t^2}}\,\frac{1}{\sqrt{1-t^2}}\frac{\partial^2}{\partial \varphi^2}F \\
&= -t\frac{\partial}{\partial t}F + \left(1-t^2\right)\frac{\partial^2}{\partial t^2}F - t\,\frac{\partial}{\partial t}F + \frac{1}{1-t^2}\,\frac{\partial^2}{\partial \varphi^2}F \\
&= \frac{1}{1-t^2}\,\frac{\partial^2}{\partial \varphi^2}F + \frac{\partial}{\partial t}\left[\left(1-t^2\right)\frac{\partial}{\partial t}F\right] \\
&= \Delta^* F\,.
\end{aligned}
$$

4.2 We derive the formulae for the fully normalized spherical harmonics. We have

$$
Y_{n,j}(\xi(\varphi,t)) = c_{n,j}\,P_{n,|j|}(t)\,G_j(\varphi)\,,
$$

where

$$
\begin{aligned}
c_{n,j} &:= \sqrt{\frac{(2n+1)(n-|j|)!(2-\delta_{j0})}{4\pi(n+|j|)!}}\,, \\
P_{n,j}(t) &= \frac{1}{2^n n!}\left(1-t^2\right)^{j/2}\frac{\mathrm{d}^{n+j}}{\mathrm{d}t^{n+j}}\left(t^2-1\right)^n = \left(1-t^2\right)^{j/2}\frac{\mathrm{d}^j}{\mathrm{d}t^j}P_n(t)\,,
\end{aligned}
$$

and

$$
G_j(\varphi) := \begin{cases} \cos(j\varphi), & \text{for } j=-n,\dots,0\,, \\ \sin(j\varphi), & \text{for } j=1,\dots,n\,. \end{cases}
$$

The degree 0 case is easy:

$$
Y_{0,0} \equiv \frac{1}{\sqrt{4\pi}}\,.
$$

The scalar case of degree 1 corresponds to the functions

$$
\begin{aligned}
Y_{1,0}(\xi(\varphi,t)) &= \sqrt{\frac{3}{4\pi}}\,t = \sqrt{\frac{3}{4\pi}}\,\xi_3\,, \\
Y_{1,-1}(\xi(\varphi,t)) &= \sqrt{\frac{3\cdot 2}{4\pi\cdot 2}}\sqrt{1-t^2}\,\cdot 1\cdot\cos\varphi = \sqrt{\frac{3}{4\pi}\,(1-t^2)}\,\cos\varphi = \sqrt{\frac{3}{4\pi}}\,\xi_1 \\
Y_{1,1}(\xi(\varphi,t)) &= \sqrt{\frac{3}{4\pi}\,(1-t^2)}\,\sin\varphi = \sqrt{\frac{3}{4\pi}}\,\xi_2\,.
\end{aligned}
$$

Amongst the vector spherical harmonics, there is only one type which belongs to degree 0:

$$
y^{(1)}_{0,0}(\xi(\varphi,t)) = \xi\,Y_{0,0} = \frac{1}{\sqrt{4\pi}}\begin{pmatrix}\xi_1\\ \xi_2\\ \xi_3\end{pmatrix} = \frac{1}{\sqrt{4\pi}}\begin{pmatrix}\sqrt{1-t^2}\,\cos\varphi\\ \sqrt{1-t^2}\,\sin\varphi\\ t\end{pmatrix}\,.
$$

We continue with degree 1 and start with order 0. We get (the arguments of some functions on the right-hand side are omitted for reasons of readability and brevity)

$$y^{(1)}_{1,0}(\xi(\varphi,t)) = \xi\, Y_{1,0} = \sqrt{\frac{3}{4\pi}} \begin{pmatrix} t\sqrt{1-t^2}\,\cos\varphi \\ t\sqrt{1-t^2}\,\sin\varphi \\ t^2 \end{pmatrix} = \sqrt{\frac{3}{4\pi}} \begin{pmatrix} \xi_1\xi_3 \\ \xi_2\xi_3 \\ \xi_3^2 \end{pmatrix} ,$$

$$\begin{aligned} y^{(2)}_{1,0}(\xi(\varphi,t)) &= \frac{1}{\sqrt{2}}\,\nabla^* Y_{1,0} = \frac{1}{\sqrt{2}}\,\frac{1}{\sqrt{1-t^2}}\left(\frac{\partial}{\partial\varphi}Y_{1,0}\right)\varepsilon^\varphi + \sqrt{\frac{1-t^2}{2}}\left(\frac{\partial}{\partial t}Y_{1,0}\right)\varepsilon^t \\ &= \sqrt{\frac{1-t^2}{2}}\sqrt{\frac{3}{4\pi}} \begin{pmatrix} -t\cos\varphi \\ -t\sin\varphi \\ \sqrt{1-t^2} \end{pmatrix} = \sqrt{\frac{3}{8\pi}} \begin{pmatrix} -t\sqrt{1-t^2}\,\cos\varphi \\ -t\sqrt{1-t^2}\,\sin\varphi \\ 1-t^2 \end{pmatrix} \\ &= \sqrt{\frac{3}{8\pi}} \begin{pmatrix} -\xi_1\xi_3 \\ -\xi_2\xi_3 \\ 1-\xi_3^2 \end{pmatrix} , \end{aligned}$$

$$\begin{aligned} y^{(3)}_{1,0}(\xi(\varphi,t)) &= \xi \times y^{(2)}_{1,0} = \sqrt{\frac{3}{8\pi}} \begin{pmatrix} \xi_2 - \xi_2\xi_3^2 + \xi_2\xi_3^2 \\ -\xi_1\xi_3^2 - \xi_1 + \xi_1\xi_3^2 \\ -\xi_1\xi_2\xi_3 + \xi_1\xi_2\xi_3 \end{pmatrix} \\ &= \sqrt{\frac{3}{8\pi}} \begin{pmatrix} \xi_2 \\ -\xi_1 \\ 0 \end{pmatrix} = \sqrt{\frac{3}{8\pi}} \begin{pmatrix} \sqrt{1-t^2}\,\sin\varphi \\ -\sqrt{1-t^2}\,\cos\varphi \\ 0 \end{pmatrix} . \end{aligned}$$

For the orders ±1, we obtain

$$y^{(1)}_{1,1}(\xi(\varphi,t)) = \xi\, Y_{1,1} = \sqrt{\frac{3}{4\pi}} \begin{pmatrix} \xi_1\xi_2 \\ \xi_2^2 \\ \xi_2\xi_3 \end{pmatrix} = \sqrt{\frac{3}{4\pi}} \begin{pmatrix} (1-t^2)\cos\varphi\,\sin\varphi \\ (1-t^2)\sin^2\varphi \\ t\sqrt{1-t^2}\,\sin\varphi \end{pmatrix} ,$$

$$y^{(1)}_{1,-1}(\xi(\varphi,t)) = \xi\, Y_{1,-1} = \sqrt{\frac{3}{4\pi}} \begin{pmatrix} \xi_1^2 \\ \xi_1\xi_2 \\ \xi_1\xi_3 \end{pmatrix} = \sqrt{\frac{3}{4\pi}} \begin{pmatrix} (1-t^2)\cos^2\varphi \\ (1-t^2)\cos\varphi\,\sin\varphi \\ t\sqrt{1-t^2}\,\cos\varphi \end{pmatrix}$$

for the normal vector fields, whereas the tangential vector fields of type 2 are

$$\begin{aligned} y^{(2)}_{1,1}(\xi(\varphi,t)) &= \frac{1}{\sqrt{2}}\,\nabla^* Y_{1,1} = \frac{1}{\sqrt{2}}\,\frac{1}{\sqrt{1-t^2}}\left(\frac{\partial}{\partial\varphi}Y_{1,1}\right)\varepsilon^\varphi + \sqrt{\frac{1-t^2}{2}}\left(\frac{\partial}{\partial t}Y_{1,1}\right)\varepsilon^t \\ &= \frac{1}{\sqrt{2}}\,\frac{1}{\sqrt{1-t^2}}\sqrt{\frac{3}{4\pi}\,(1-t^2)}\,\cos\varphi\,\varepsilon^\varphi \\ &\quad + \sqrt{\frac{1-t^2}{2}}\sqrt{\frac{3}{4\pi}}\,\frac{-t}{\sqrt{1-t^2}}\,\sin\varphi\,\varepsilon^t \\ &= \sqrt{\frac{3}{8\pi}}\left(\cos\varphi\,\varepsilon^\varphi - t\sin\varphi\,\varepsilon^t\right) \\ &= \sqrt{\frac{3}{8\pi}} \begin{pmatrix} -\sin\varphi\cos\varphi + t^2\sin\varphi\cos\varphi \\ \cos^2\varphi + t^2\sin^2\varphi \\ 0 - t\sqrt{1-t^2}\,\sin\varphi \end{pmatrix} \\ &= \sqrt{\frac{3}{8\pi}} \begin{pmatrix} (t^2-1)\sin\varphi\cos\varphi \\ \cos^2\varphi + t^2\sin^2\varphi \\ -t\sqrt{1-t^2}\,\sin\varphi \end{pmatrix} = \sqrt{\frac{3}{8\pi}} \begin{pmatrix} -\xi_1\xi_2 \\ 1-\xi_2^2 \\ -\xi_2\xi_3 \end{pmatrix} , \end{aligned}$$

$$y^{(2)}_{1,-1}(\xi(\varphi,t)) = \frac{1}{\sqrt{2}}\,\nabla^* Y_{1,-1} = \frac{1}{\sqrt{2}}\,\frac{1}{\sqrt{1-t^2}}\left(\frac{\partial}{\partial\varphi}Y_{1,-1}\right)\varepsilon^\varphi + \sqrt{\frac{1-t^2}{2}}\left(\frac{\partial}{\partial t}Y_{1,-1}\right)\varepsilon^t$$
$$= -\frac{1}{\sqrt{2}}\,\frac{1}{\sqrt{1-t^2}}\sqrt{\frac{3}{4\pi}}\,(1-t^2)\,\sin\varphi\,\varepsilon^\varphi$$
$$+\sqrt{\frac{1-t^2}{2}}\sqrt{\frac{3}{4\pi}}\,\frac{-t}{\sqrt{1-t^2}}\,\cos\varphi\,\varepsilon^t$$
$$= \sqrt{\frac{3}{8\pi}}\left(-\sin\varphi\,\varepsilon^\varphi - t\cos\varphi\,\varepsilon^t\right),$$
$$= \sqrt{\frac{3}{8\pi}}\begin{pmatrix}\sin^2\varphi + t^2\cos^2\varphi\\ -\sin\varphi\cos\varphi + t^2\sin\varphi\cos\varphi\\ 0 - t\sqrt{1-t^2}\,\cos\varphi\end{pmatrix}$$
$$= \sqrt{\frac{3}{8\pi}}\begin{pmatrix}\sin^2\varphi + t^2\cos^2\varphi\\ (t^2-1)\sin\varphi\cos\varphi\\ -t\sqrt{1-t^2}\,\cos\varphi\end{pmatrix} = \sqrt{\frac{3}{8\pi}}\begin{pmatrix}1-\xi_1^2\\ -\xi_1\xi_2\\ -\xi_1\xi_3\end{pmatrix},$$

because $\cos^2\varphi + t^2\sin^2\varphi = 1 - \sin^2\varphi + t^2\sin^2\varphi = 1-(1-t^2)\sin^2\varphi$ (and analogously for exchanged sin and cos). Eventually, the missing functions of type 3 are

$$y^{(3)}_{1,1}(\xi(\varphi,t)) = \xi\times y^{(2)}_{1,1} = \sqrt{\frac{3}{8\pi}}\begin{pmatrix}-\xi_2^2\xi_3 - \xi_3 + \xi_2^2\xi_3\\ -\xi_1\xi_2\xi_3 + \xi_1\xi_2\xi_3\\ \xi_1 - \xi_1\xi_2^2 + \xi_1\xi_2^2\end{pmatrix}$$
$$= \sqrt{\frac{3}{8\pi}}\begin{pmatrix}-\xi_3\\ 0\\ \xi_1\end{pmatrix} = \sqrt{\frac{3}{8\pi}}\begin{pmatrix}-t\\ 0\\ \sqrt{1-t^2}\,\cos\varphi\end{pmatrix},$$
$$y^{(3)}_{1,-1}(\xi(\varphi,t)) = \xi\times y^{(2)}_{1,-1} = \sqrt{\frac{3}{8\pi}}\begin{pmatrix}-\xi_1\xi_2\xi_3 + \xi_1\xi_2\xi_3\\ \xi_3 - \xi_1^2\xi_3 + \xi_1^2\xi_3\\ -\xi_1^2\xi_2 - \xi_2 + \xi_1^2\xi_2\end{pmatrix}$$
$$= \sqrt{\frac{3}{8\pi}}\begin{pmatrix}0\\ \xi_3\\ -\xi_2\end{pmatrix} = \sqrt{\frac{3}{8\pi}}\begin{pmatrix}0\\ t\\ -\sqrt{1-t^2}\,\sin\varphi\end{pmatrix}.$$

4.3 The term from the addition theorem for spherical harmonics is the searched reproducing kernel: for all $Y_n\in\mathrm{Harm}_n(\Omega)$ and $\xi\in\Omega$, we obtain

$$\int_\Omega \frac{2n+1}{4\pi}\,P_n(\xi\cdot\eta)\,Y_n(\eta)\,\mathrm{d}\omega(\eta) = \sum_{j=-n}^{n} Y_{n,j}(\xi)\int_\Omega Y_{n,j}(\eta)\,Y_n(\eta)\,\mathrm{d}\omega(\eta)$$
$$= \sum_{j=-n}^{n} Y_{n,j}(\xi)\,\langle Y_{n,j}, Y_n\rangle_{\mathrm{L}^2(\Omega)}$$
$$= Y_n(\xi),$$

because $\{Y_{n,j}\}_{j=-n,\ldots,n}$ is an orthonormal basis of $(\mathrm{Harm}_n(\Omega), \langle\cdot,\cdot\rangle_{\mathrm{L}^2(\Omega)})$.

4.4 For arbitrary but fixed $\xi,\eta,\zeta\in\Omega$ and each degree $n\in\mathbb{N}_0$, we find a $\tau\in[-1,1]$, by using the mean value theorem of differentiation, such that

$$|P_n(\xi\cdot\zeta) - P_n(\eta\cdot\zeta)| = \left|P_n'(\tau)\right|\,|\xi\cdot\zeta - \eta\cdot\zeta|$$

$$\leq \left\| P_n' \right\|_{\mathrm{C}[-1,1]} \left| (\xi - \eta) \cdot \zeta \right|$$
$$\leq \left\| P_n' \right\|_{\mathrm{C}[-1,1]} \left| \xi - \eta \right| \left| \zeta \right| . \tag{4.1}$$

It remains to estimate the maximum norm of P_n'. We already know that $P_n'(1) = n(n+1)/2$. We show that this is (just like for the 0-th derivative) also the maximum norm on $[-1,1]$. From one of the recurrence formulae of the Legendre polynomials, we see that

$$P_{n+1}'(x) = (2n+1)P_n(x) + P_{n-1}'(x) \tag{4.2}$$

for all $n \in \mathbb{N}$ and $x \in [-1,1]$. Obviously, $P_0' \equiv 0 = 0(0+1)/2$ and $P_1' \equiv 1 = 1(1+1)/2$ such that a simple induction applied to (4.2) yields

$$\begin{aligned} \left| P_{n+1}'(x) \right| &\leq (2n+1) \left| P_n(x) \right| + \left| P_{n-1}'(x) \right| \\ &\leq 2n + 1 + \frac{(n-1)n}{2} = \frac{4n+2+n^2-n}{2} \\ &= \frac{(n+1)(n+2)}{2} \end{aligned}$$

and, therefore,

$$\left| P_n'(x) \right| \leq \frac{n(n+1)}{2} \quad \text{for all } x \in [-1,1] \text{ and } n \in \mathbb{N}_0 . \tag{4.3}$$

Hence, the combination of (4.1) and (4.3) leads us to

$$\left| P_n(\xi \cdot \zeta) - P_n(\eta \cdot \zeta) \right| \leq \frac{n(n+1)}{2} \left| \xi - \eta \right| .$$

4.5 From the Sobolev lemma, we know that functions in $\mathcal{H}_s(\Omega)$, $s > 1$, have uniformly convergent Fourier series in the $Y_{n,j}$-functions. For arbitrary $F \in \mathcal{H}_s(\Omega)$, $s > 2$, and $\xi, \eta \in \Omega$, we obtain then, by using the Cauchy–Schwarz inequality, the definition of the $\| \cdot \|_{\mathcal{H}_s(\Omega)}$-norm, the addition theorem for spherical harmonics, and Exercise 4.4, that

$$\begin{aligned} |F(\xi) - F(\eta)|^2 &= \left| \sum_{n=0}^{\infty} \sum_{j=-n}^{n} F^\wedge(n,j) \left(Y_{n,j}(\xi) - Y_{n,j}(\eta) \right) \right|^2 \\ &= \left| \sum_{n=0}^{\infty} \sum_{j=-n}^{n} \left(n + \frac{1}{2} \right)^s \left(n + \frac{1}{2} \right)^{-s} F^\wedge(n,j) \left(Y_{n,j}(\xi) - Y_{n,j}(\eta) \right) \right|^2 \\ &\leq \left(\sum_{n=0}^{\infty} \sum_{j=-n}^{n} \left(n + \frac{1}{2} \right)^{2s} \left(F^\wedge(n,j) \right)^2 \right) \\ &\quad \times \left(\sum_{n=0}^{\infty} \sum_{j=-n}^{n} \left(n + \frac{1}{2} \right)^{-2s} \left(Y_{n,j}(\xi) - Y_{n,j}(\eta) \right)^2 \right) \\ &= \|F\|_{\mathcal{H}_s(\Omega)}^2 \sum_{n=0}^{\infty} \left(n + \frac{1}{2} \right)^{-2s} \sum_{j=-n}^{n} \left(\left(Y_{n,j}(\xi) \right)^2 - 2Y_{n,j}(\xi) Y_{n,j}(\eta) + \left(Y_{n,j}(\eta) \right)^2 \right) \\ &= \|F\|_{\mathcal{H}_s(\Omega)}^2 \sum_{n=0}^{\infty} \left(n + \frac{1}{2} \right)^{-2s} \frac{2n+1}{4\pi} \, 2 \left(P_n(1) - P_n(\xi \cdot \eta) \right) \end{aligned}$$

$$\le \|F\|^2_{\mathcal{H}_s(\Omega)} \sum_{n=0}^{\infty} \left(n+\frac{1}{2}\right)^{-2s} \frac{2n+1}{4\pi}\, 2\, P_n'(1)\, (1-\xi\cdot\eta)$$

$$= \|F\|^2_{\mathcal{H}_s(\Omega)} \sum_{n=0}^{\infty} \left(n+\frac{1}{2}\right)^{-2s} \frac{2n+1}{4\pi}\, 2\, \frac{n(n+1)}{2}\, \frac{1}{2}\, |\xi-\eta|^2\,.$$

Hence,

$$|F(\xi)-F(\eta)| \le \|F\|_{\mathcal{H}_s(\Omega)} \left(\frac{1}{2}\sum_{n=1}^{\infty} \frac{2n+1}{4\pi}\, \frac{n(n+1)}{(n+1/2)^{2s}}\right)^{1/2} |\xi-\eta|\,.$$

The summand for $n = 0$ vanishes. The Lipschitz constant is finite, because its summands are of the order $\mathcal{O}(n^{1+2-2s}) = \mathcal{O}(n^{3-2s})$, where $s > 2$ and, therefore, $3-2s < -1$.

4.6 Previously, we had already derived a formula for $\nabla^*\otimes\nabla^*H + \mathrm{L}^*\otimes\mathrm{L}^*H$. If we follow this derivation and replace '+' at the appropriate places by '−', then we obtain

$$\begin{aligned}
&\nabla^*\otimes\nabla^*H - \mathrm{L}^*\otimes\mathrm{L}^*H\\
&\quad= \varepsilon^\varphi\otimes\varepsilon^r \left(-\frac{1}{\sqrt{1-t^2}}\,\frac{\partial H}{\partial\varphi} - \frac{1}{\sqrt{1-t^2}}\,\frac{\partial H}{\partial\varphi}\right)\\
&\qquad+ \varepsilon^\varphi\otimes\varepsilon^t \left(\frac{t}{1-t^2}\,\frac{\partial H}{\partial\varphi} + \frac{\partial^2 H}{\partial\varphi\partial t} + \frac{t}{1-t^2}\,\frac{\partial H}{\partial\varphi} + \frac{\partial^2 H}{\partial t\partial\varphi}\right)\\
&\qquad+ \varepsilon^\varphi\otimes\varepsilon^\varphi \left(\frac{1}{1-t^2}\,\frac{\partial^2 H}{\partial\varphi^2} - t\frac{\partial H}{\partial t} + t\frac{\partial H}{\partial t} - \left(1-t^2\right)\frac{\partial^2 H}{\partial t^2}\right)\\
&\qquad+ \varepsilon^t\otimes\varepsilon^\varphi \left(\frac{t}{1-t^2}\,\frac{\partial H}{\partial\varphi} + \frac{\partial^2 H}{\partial t\partial\varphi} + \frac{t}{1-t^2}\,\frac{\partial H}{\partial\varphi} + \frac{\partial^2 H}{\partial\varphi\partial t}\right)\\
&\qquad+ \varepsilon^t\otimes\varepsilon^r \left(-\sqrt{1-t^2}\,\frac{\partial H}{\partial t} - \sqrt{1-t^2}\,\frac{\partial H}{\partial t}\right)\\
&\qquad+ \varepsilon^t\otimes\varepsilon^t \left(-t\,\frac{\partial H}{\partial t} + \left(1-t^2\right)\frac{\partial^2 H}{\partial t^2} - \frac{1}{1-t^2}\,\frac{\partial^2 H}{\partial\varphi^2} + t\frac{\partial H}{\partial t}\right).
\end{aligned}$$

Hence, the subtraction of the transposed tensor leads us to

$$\begin{aligned}
&(\nabla^*\otimes\nabla^*H - \mathrm{L}^*\otimes\mathrm{L}^*H) - (\nabla^*\otimes\nabla^*H - \mathrm{L}^*\otimes\mathrm{L}^*H)^{\mathrm{T}}\\
&\quad= (\varepsilon^\varphi\otimes\varepsilon^r - \varepsilon^r\otimes\varepsilon^\varphi)\left(-\frac{2}{\sqrt{1-t^2}}\,\frac{\partial H}{\partial\varphi}\right)\\
&\qquad+ \left(\varepsilon^t\otimes\varepsilon^r - \varepsilon^r\otimes\varepsilon^t\right)\left(-2\sqrt{1-t^2}\,\frac{\partial H}{\partial t}\right)\\
&\quad= 2\left[\varepsilon^r\otimes\nabla^*H - (\nabla^*H)\otimes\varepsilon^r\right].
\end{aligned}$$

5
Inverse Problems

Exercises

5.1 The equation $\Delta_x(F(x)|x|^{-p}) = 0$ in $B_R(0)$ is solved, if and only if the function $B_R(0) \ni x \mapsto F(x)|x|^{-p}$ can be expanded in inner harmonics. This, however, is equivalent to the fact that F is expandable in the functions $B_R(0) \ni x \mapsto |x|^{n+p} Y_{n,j}(x/|x|)$. In the notation of the *ansatz* which we used for the inverse gravimetric problem, this means that $F_{n,j}(r) = f_{n,j} r^{n+p+1}$ for arbitrary constants $f_{n,j}$. By inserting this into the spectral formula which we derived, we arrive at

$$\frac{4\pi G}{2n+1} f_{n,j} \int_0^R r^{2n+p+2}\, \mathrm{d}r = V_{n,j}(R+\varepsilon)^n \quad \text{for all } n, j\,.$$

This holds true, if and only if

$$\frac{4\pi G}{2n+1} f_{n,j} \frac{R^{2n+p+3}}{2n+p+3} = V_{n,j}(R+\varepsilon)^n \quad \text{for all } n, j\,. \tag{5.1}$$

By resolving (5.1) for $f_{n,j}$, we obtain the unique solution

$$F(x) = \sum_{n=0}^{\infty} (2n+p+3) \frac{(R+\varepsilon)^n}{R^{2n+p+3}} \frac{2n+1}{4\pi G} |x|^{n+p} \sum_{j=-n}^{n} V_{n,j} Y_{n,j}\left(\frac{x}{|x|}\right).$$

5.2 Again we transfer the scenario to the notation of the *ansatz* which we used for the inverse gravimetric problem. We get $F_{n,j}(r) = F^{\mathrm{L}}_{n,j} r$ for $r \in [\tau, \tau+\delta]$ and $F_{n,j}(r) = 0$ else. The insertion into the spectral formula yields here

$$\begin{aligned} &\frac{4\pi G}{2n+1} \int_\tau^{\tau+\delta} r^{n+2}\, \mathrm{d}r\, F^{\mathrm{L}}_{n,j} = V_{n,j}(R+\varepsilon)^n \quad \text{for all } n, j \\ \Leftrightarrow\quad &\frac{4\pi G}{2n+1} \frac{(\tau+\delta)^{n+3} - \tau^{n+3}}{n+3} F^{\mathrm{L}}_{n,j} = V_{n,j}(R+\varepsilon)^n \quad \text{for all } n, j\,. \end{aligned}$$

Hence,

$$F^{\mathrm{L}}_{n,j} = V_{n,j} \frac{2n+1}{4\pi G} (n+3) \frac{(R+\varepsilon)^n}{(\tau+\delta)^{n+3} - \tau^{n+3}} \quad \text{for all } n, j\,.$$

5.3 Let us start with the properties of an inner product. For the positive definiteness, we see that

$$\left\langle \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle_{\mathcal{X}\times\mathcal{Y}} = \langle x, x\rangle_{\mathcal{X}} + \langle y, y\rangle_{\mathcal{Y}} = \|x\|^2_{\mathcal{X}} + \|y\|^2_{\mathcal{Y}} \geq 0\,, \tag{5.2}$$

because $\langle\cdot,\cdot\rangle_{\mathcal{X}}$ and $\langle\cdot,\cdot\rangle_{\mathcal{Y}}$ are inner products. For the same reason, the term in (5.2) can only vanish if both summands $\|x\|_{\mathcal{X}}^2$ and $\|y\|_{\mathcal{Y}}^2$ vanish, which is true if and only if $x = 0$ and $y = 0$.

Regarding the symmetry, we observe that

$$\begin{aligned}\left\langle \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right\rangle_{\mathcal{X}\times\mathcal{Y}} &= \langle x_1, x_2\rangle_{\mathcal{X}} + \langle y_1, y_2\rangle_{\mathcal{Y}} \\ &= \langle x_2, x_1\rangle_{\mathcal{X}} + \langle y_2, y_1\rangle_{\mathcal{Y}} = \left\langle \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \right\rangle_{\mathcal{X}\times\mathcal{Y}},\end{aligned}$$

because $\langle\cdot,\cdot\rangle_{\mathcal{X}}$ and $\langle\cdot,\cdot\rangle_{\mathcal{Y}}$ are inner products.

For the bilinearity, we use again that the new mapping is composed out of inner products. We get

$$\begin{aligned}&\left\langle r\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + s\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} \right\rangle_{\mathcal{X}\times\mathcal{Y}} = \left\langle \begin{pmatrix} rx_1 + sx_2 \\ ry_1 + sy_2 \end{pmatrix}, \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} \right\rangle_{\mathcal{X}\times\mathcal{Y}} \\ &= \langle rx_1 + sx_2, x_3\rangle_{\mathcal{X}} + \langle ry_1 + sy_2, y_3\rangle_{\mathcal{Y}} \\ &= r\langle x_1, x_3\rangle_{\mathcal{X}} + s\langle x_2, x_3\rangle_{\mathcal{X}} + r\langle y_1, y_3\rangle_{\mathcal{Y}} + s\langle y_2, y_3\rangle_{\mathcal{Y}} \\ &= r\left(\langle x_1, x_3\rangle_{\mathcal{X}} + \langle y_1, y_3\rangle_{\mathcal{Y}}\right) + s\left(\langle x_2, x_3\rangle_{\mathcal{X}} + \langle y_2, y_3\rangle_{\mathcal{Y}}\right) \\ &= r\left\langle \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} \right\rangle_{\mathcal{X}\times\mathcal{Y}} + s\left\langle \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} \right\rangle_{\mathcal{X}\times\mathcal{Y}}.\end{aligned}$$

Finally, we need to show that the Cartesian product space is complete. For this purpose, let $((x_n, y_n)^{\mathrm{T}})_n$ be an arbitrary Cauchy sequence in $\mathcal{X} \times \mathcal{Y}$. This means (where $\forall$ and $\exists$ represent, as usual, 'for all' and 'there exists'):

$$\forall \varepsilon > 0\, \exists n_0\, \forall n, m \geq n_0\colon \left\| \begin{pmatrix} x_n \\ y_n \end{pmatrix} - \begin{pmatrix} x_m \\ y_m \end{pmatrix} \right\|^2 < \varepsilon^2,$$

that is

$$\forall \varepsilon > 0\, \exists n_0\, \forall n, m \geq n_0\colon\ \|x_n - x_m\|_{\mathcal{X}}^2 + \|y_n - y_m\|_{\mathcal{Y}}^2 < \varepsilon^2.$$

This implies, in particular, that

$$\forall \varepsilon > 0\, \exists n_0\, \forall n, m \geq n_0\colon\ \|x_n - x_m\|_{\mathcal{X}} < \varepsilon \text{ and } \|y_n - y_m\|_{\mathcal{Y}} < \varepsilon.$$

Consequently, $(x_n)_n$ is a Cauchy sequence in $\mathcal{X}$ and $(y_n)_n$ is a Cauchy sequence in $\mathcal{Y}$. Since both spaces were assumed to be complete, there exist (strong) limits: $x_n \to \xi \in \mathcal{X}$ and $y_n \to \eta \in \mathcal{Y}$. In other words,

$$\forall \varepsilon > 0\, \exists n_1, n_2\, \forall n \geq n_1,\, n' \geq n_2\colon\ \|x_n - \xi\|_{\mathcal{X}} < \varepsilon \text{ and } \|y_{n'} - \eta\|_{\mathcal{Y}} < \varepsilon.$$

For each $\varepsilon > 0$, we set $n_3(\varepsilon) := \max(n_1(\varepsilon), n_2(\varepsilon))$. Then we get, for all $n \geq n_3(\varepsilon)$:

$$\left\| \begin{pmatrix} x_n \\ y_n \end{pmatrix} - \begin{pmatrix} \xi \\ \eta \end{pmatrix} \right\|_{\mathcal{X}\times\mathcal{Y}} = \sqrt{\|x_n - \xi\|_{\mathcal{X}}^2 + \|y_n - \eta\|_{\mathcal{Y}}^2} \leq \sqrt{2\varepsilon^2} = \sqrt{2}\,\varepsilon.$$

Hence,

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} \longrightarrow \begin{pmatrix} \xi \\ \eta \end{pmatrix} \quad \text{in } \mathcal{X} \times \mathcal{Y}$$

and $(\mathcal{X} \times \mathcal{Y}, \langle\cdot,\cdot\rangle_{\mathcal{X}\times\mathcal{Y}})$ is complete.

5.4 We have the following equivalences (note that $z_k, \xi, \mathcal{P}\mathcal{T}\xi, \mathcal{P}y \in \mathcal{H}_n$ and $\mathcal{P}$ is self-adjoint):

$$\begin{aligned} &\sum_{j=1}^{n} \alpha_j \langle \mathcal{T} z_j, z_k \rangle = \langle y, z_k \rangle \quad \text{for all } k = 1, \dots, n \\ \Leftrightarrow \quad & \langle \mathcal{T}\xi, z_k \rangle = \langle y, z_k \rangle \quad \text{for all } k = 1, \dots, n \\ \Leftrightarrow \quad & \langle \mathcal{T}\xi, \mathcal{P} z_k \rangle = \langle y, \mathcal{P} z_k \rangle \quad \text{for all } k = 1, \dots, n \\ \Leftrightarrow \quad & \langle \mathcal{P}\mathcal{T}\xi, z_k \rangle = \langle \mathcal{P} y, z_k \rangle \quad \text{for all } k = 1, \dots, n \\ \Leftrightarrow \quad & \mathcal{P}\mathcal{T}\xi = \mathcal{P} y \\ \Leftrightarrow \quad & \mathcal{P}\mathcal{T}\mathcal{P}\xi = \mathcal{P} y \,. \end{aligned}$$

5.5 We first show that the Moore–Penrose inverse $\mathcal{T}^+$ fulfils the axioms. We already know that $\mathcal{T}\mathcal{T}^+ = \mathcal{P}_{\overline{\mathcal{T}(\mathcal{X})}}$. Moreover, with the decomposition of an arbitrary $f \in \mathcal{X}$ into $f = f_{\mathrm{k}} + f_{\mathrm{k}^\perp}$, $f_{\mathrm{k}} \in \ker \mathcal{T}$, $f_{\mathrm{k}^\perp} \in \ker(\mathcal{T})^\perp$, we obtain analogously that $\mathcal{T}^+\mathcal{T} f = \mathcal{T}^+\mathcal{T} f_{\mathrm{k}^\perp}$ is the minimum-norm solution of

$$\mathcal{T}^* \underbrace{\mathcal{T}\left(\mathcal{T}^+\mathcal{T} f\right)}_{\in \overline{\mathcal{T}(\mathcal{X})} = \ker(\mathcal{T}^*)^\perp} = \mathcal{T}^*\mathcal{T} f \,.$$

Hence,

$$\mathcal{T} \underbrace{\mathcal{T}^+\mathcal{T} f}_{\in \mathcal{T}^+(\mathcal{D}(\mathcal{T}^+)) = \ker(\mathcal{T})^\perp} = \mathcal{T} f \,.$$

Consequently, we have

$$\mathcal{T}^+\mathcal{T} f = \left(\mathcal{T}|_{\ker(\mathcal{T})^\perp}\right)^{-1} \mathcal{T} f = \left(\mathcal{T}|_{\ker(\mathcal{T})^\perp}\right)^{-1} \mathcal{T} f_{\mathrm{k}^\perp} = f_{\mathrm{k}^\perp} \,.$$

In total, we get

$$\mathcal{T}^+\mathcal{T} = \mathcal{P}_{\ker(\mathcal{T})^\perp} = \mathcal{P}_{\mathcal{T}^+(\mathcal{D}(\mathcal{T}^+))} = \mathcal{P}_{\overline{\mathcal{T}^*(\mathcal{Y})}} \,.$$

Furthermore, for arbitrary $f \in \mathcal{X}$ and $g \in \mathcal{D}(\mathcal{T}^+)$, we obtain

$$\mathcal{T}\mathcal{T}^+\mathcal{T} f = \mathcal{P}_{\overline{\mathcal{T}(\mathcal{X})}}(\mathcal{T} f) = \mathcal{T} f$$

and

$$\mathcal{T}^+\mathcal{T}\mathcal{T}^+ g = \mathcal{P}_{\mathcal{T}^+(\mathcal{D}(\mathcal{T}^+))}\left(\mathcal{T}^+ g\right) = \mathcal{T}^+ g$$

such that

$$\mathcal{T}\mathcal{T}^+\mathcal{T} = \mathcal{T} \quad \text{and} \quad \mathcal{T}^+\mathcal{T}\mathcal{T}^+ = \mathcal{T}^+ \,.$$

Let now $\mathcal{S}$ be an arbitrary operator which fulfils the Moore–Penrose axioms. If $g \in \mathcal{D}(\mathcal{T}^+)$ and $h \in \mathcal{X}$ are arbitrary, then the fourth axiom yields

$$\begin{aligned} \langle \mathcal{T}^*\mathcal{T}\mathcal{S} g - \mathcal{T}^* g, h \rangle_{\mathcal{X}} &= \langle \mathcal{T}\mathcal{S} g - g, \mathcal{T} h \rangle_{\mathcal{Y}} = \left\langle \mathcal{P}_{\overline{\mathcal{T}(\mathcal{X})}} g - g, \mathcal{T} h \right\rangle_{\mathcal{Y}} \\ &= \left\langle -\mathcal{P}_{\overline{\mathcal{T}(\mathcal{X})}^\perp} g, \mathcal{T} h \right\rangle_{\mathcal{Y}} = 0 \,. \end{aligned}$$

Thus, $f := \mathcal{S} g$ solves the normal equation $\mathcal{T}^*\mathcal{T} f = \mathcal{T}^* g$.

We decompose, like above, f into $f = f_{\mathrm{k}} + f_{\mathrm{k}^\perp}$, $f_{\mathrm{k}} \in \ker \mathcal{T}$, $f_{\mathrm{k}^\perp} \in \ker(\mathcal{T})^\perp$. Since the second axiom yields $\mathcal{STS} = \mathcal{S}$, we must have $\mathcal{STS}g = \mathcal{S}g$ and, therefore, the following chain of implications (where we use the third axiom):

$$\begin{aligned}
&\Rightarrow \quad \mathcal{ST}\,(f_{\mathrm{k}} + f_{\mathrm{k}^\perp}) = f_{\mathrm{k}} + f_{\mathrm{k}^\perp} &&\Rightarrow \quad \mathcal{ST} f_{\mathrm{k}^\perp} = f_{\mathrm{k}} + f_{\mathrm{k}^\perp}\\
&\Rightarrow \quad \underbrace{\mathcal{P}_{\overline{\mathcal{T}^*(\mathcal{Y})}} f_{\mathrm{k}^\perp}}_{\in \ker(\mathcal{T})^\perp} = f_{\mathrm{k}} + f_{\mathrm{k}^\perp} &&\Rightarrow \quad f_{\mathrm{k}} = 0\\
&\Rightarrow \quad \mathcal{S}g \in \ker(\mathcal{T})^\perp &&\Rightarrow \quad \mathcal{S}g = \mathcal{T}^+ g\,.
\end{aligned}$$

Hence, $\mathcal{S} = \mathcal{T}^+$.

5.6 a) Since $\mathcal{A}$ is injective, we have $\ker \mathcal{A} = \{0\}$. On the other hand, we also have $\ker \mathcal{A} = \mathcal{A}^*(\mathcal{X})^\perp = \mathcal{A}(\mathcal{X})^\perp$, because $\mathcal{A} = \mathcal{A}^*$. Hence, $\overline{\mathcal{A}(\mathcal{X})} = \mathcal{X}$.
b) With the Cauchy–Schwarz inequality and the assumption on $\mathcal{A}$, we have

$$\|\mathcal{A}x\|\,\|x\| \geq \langle \mathcal{A}x, x\rangle \geq \gamma \|x\|^2 \quad \text{for all } x \in \mathcal{X}$$

such that

$$\|\mathcal{A}x\| \geq \gamma \|x\| \quad \text{for all } x \in \mathcal{X}\,. \tag{5.3}$$

Since $\mathcal{A}$ is injective, there exists an inverse $\mathcal{A}^{-1}\colon \mathcal{A}(\mathcal{X}) \to \mathcal{X}$. Due to (5.3), it is continuous and we have

$$\left\|\mathcal{A}^{-1}y\right\| \leq \frac{1}{\gamma}\,\left\|\mathcal{A}\mathcal{A}^{-1}y\right\| = \frac{1}{\gamma}\,\|y\| \quad \text{for all } y \in \mathcal{A}(\mathcal{X})$$

such that $\|\mathcal{A}^{-1}\| \leq \gamma^{-1}$. We have already seen that injective, linear, and continuous mappings between Hilbert spaces are continuously invertible on their image if and only if the image is closed. In combination with part a, we see that this closed image must be the whole space $\mathcal{X}$. Thus, $\mathcal{A}$ is also surjective and we get the continuous inverse $\mathcal{A}^{-1}\colon \mathcal{X} \to \mathcal{X}$.

5.7 Clearly, due to Fubini's theorem, the adjoint operator is given by

$$(\mathcal{T}^* F)\,(x) = \int_D \overline{k(y,x)}\, F(y)\,\mathrm{d}y\,, \quad x \in D\,.$$

a) From Exercise 3.10, we know that the operator $\mathcal{T}$ is compact. Thus, Fredholm's theorem tells us that $\sigma_{\mathrm{p}}(\mathcal{T})$ is finite or countable. Let now $\mu \neq 0$ be an arbitrary eigenvalue of $\mathcal{T}$, that is $\ker(\mathcal{T} - \mu\mathcal{I}) \supsetneqq \{0\}$.

Let us assume that the image of $\mathcal{T} - \mu\mathcal{I}$ is given by $\mathrm{L}^2(D) =: \mathcal{H}$ (i.e. we assume that $\mathcal{T} - \mu\mathcal{I}$ is surjective). Then whichever element $x_0 \in [\ker(\mathcal{T} - \mu\mathcal{I})] \setminus \{0\}$ we choose, it is also an element of this image. Hence, there is $x_1 \in \mathcal{H}$ with $(\mathcal{T} - \mu\mathcal{I})x_1 = x_0$. However, due to the assumed surjectivity, this initiates a never-ending recursion: there is $x_2 \in \mathcal{H}$ with $(\mathcal{T} - \mu\mathcal{I})x_2 = x_1$ and so on. We obtain, consequently, a sequence $(x_n)_n \subset \mathcal{H}$ with $(\mathcal{T} - \mu\mathcal{I})x_n = x_{n-1}$ for all n.

We continue with another assumption: let us consider the case that the first l elements, that is $x_0, \ldots, x_{l-1}$, are linearly independent while the first $l+1$ elements are not, that is there exist $\alpha_0, \ldots, \alpha_{l-1}$ such that $x_l = \sum_{k=0}^{l-1} \alpha_k x_k$. However, then we get (by using the linearity of $\mathcal{T}$ and the construction of x_0) that

$$x_{l-1} = (\mathcal{T} - \mu\mathcal{I})x_l = \sum_{k=0}^{l-1} \alpha_k (\mathcal{T} - \mu\mathcal{I})x_k = \sum_{k=1}^{l-1} \alpha_k x_{k-1} = \sum_{k=0}^{l-2} \alpha_{k+1} x_k\,. \tag{5.4}$$

The result of (5.4) contradicts the linear independence of $x_0, \ldots, x_{l-1}$. Hence, amongst our nested assumptions, the inner one is false (for every l). Thus, all x_n together are still linearly independent (i.e. each finite subsystem is linearly independent).

We can now use the Gram–Schmidt orthonormalization to get orthonormal vectors

$$e_k = \sum_{j=0}^{k} \beta_{k,j} x_j \quad \text{for all } k\,.$$

The application of $\mathcal{T}$ yields

$$\begin{aligned}\mathcal{T} e_k &= \sum_{j=0}^{k} \beta_{k,j} (\mathcal{T} - \mu\mathcal{I} + \mu\mathcal{I}) x_j \\ &= \beta_{k,0}\mu\, x_0 + \sum_{j=1}^{k} \beta_{k,j}\, (x_{j-1} + \mu x_j) \\ &= \mu e_k + \sum_{j=0}^{k-1} \beta_{k,j+1} x_j\,,\end{aligned}$$

where $\sum_{j=0}^{k-1} \beta_{k,j+1} x_j \in \operatorname{span}\{x_0, \ldots, x_{k-1}\} = \operatorname{span}\{e_0, \ldots, e_{k-1}\}$. Hence, there exist coefficients $\gamma_{k,j}$ such that $\mathcal{T} e_k = \sum_{j=0}^{k} \gamma_{k,j} e_j$ for each k, while $\gamma_{k,k} = \mu$.

Let now $k, l \in \mathbb{N}_0$ where, without loss of generality, $k > l$. Then the orthonormality of the e_j yields

$$\|\mathcal{T} e_k - \mathcal{T} e_l\|^2 = \sum_{j=0}^{l} |\gamma_{k,j} - \gamma_{l,j}|^2 + \sum_{j=l+1}^{k} |\gamma_{k,j}|^2 \geq |\gamma_{k,k}|^2\,. \tag{5.5}$$

In our nested hierarchy of assumptions, the outer one now also obtains its contradiction: since $\mathcal{T}$ is compact, the sequence $(\mathcal{T} e_k)_k$ must have a convergent subsequence (which is, then, also a Cauchy subsequence). Hence, for each $\varepsilon > 0$, there exists k_0 such that, for all $k, l \geq k_0$, we have $\|\mathcal{T} e_k - \mathcal{T} e_l\| < \varepsilon$. In combination with (5.5) and the fact that $\gamma_{k,k} = \mu$ for all k, we see that $\mu = 0$ would have to hold true, which was excluded.

Thus, we have $(\mathcal{T} - \mu\mathcal{I})(\mathcal{H}) \subsetneqq \mathcal{H}$ and, since this image is closed according to Fredholm's theorem, there is a non-trivial orthogonal complement. Let, therefore, $x^* \in [(\mathcal{T} - \mu\mathcal{I})(\mathcal{H})]^\perp \setminus \{0\}$. We get:

$$\begin{aligned} & & \langle (\mathcal{T} - \mu\mathcal{I}) x, x^* \rangle &= 0 \quad \text{for all } x \in \mathcal{H} \\ &\Leftrightarrow & \langle x, (\mathcal{T}^* - \overline{\mu}\mathcal{I})\, x^* \rangle &= 0 \quad \text{for all } x \in \mathcal{H} \\ &\Leftrightarrow & (\mathcal{T}^* - \overline{\mu}\mathcal{I})\, x^* &= 0\,,\end{aligned}$$

which means that $\overline{\mu}$ is an eigenvalue of $\mathcal{T}^*$.

In the same way, we see that, if $\overline{\mu} \neq 0$ is an eigenvalue of $\mathcal{T}^*$, then $\overline{\overline{\mu}} = \mu$ is an eigenvalue of $\mathcal{T}^{**} = \mathcal{T}$.

b) From Fredholm's theorem, we know that the null spaces $\ker(\mathcal{T} - \lambda\mathcal{I})$ and $\ker(\mathcal{T}^* - \overline{\lambda}\mathcal{I})$ are finite-dimensional with equal dimensions for $\lambda \neq 0$. This actually concludes the proof of this proposition.

If we want to show the identity of the dimensions without using the corresponding proposition from Fredholm's theorem, then we can proceed as follows: let $\{x_1, \ldots, x_n\}$

be an orthonormal basis (onb) of $\ker(\mathcal{T} - \lambda\mathcal{I})$ and let $\{y_1, \ldots, y_m\}$ be an onb of $\ker(\mathcal{T}^* - \overline{\lambda}\mathcal{I})$. Without loss of generality, we assume that $n \leq m$ (otherwise, exchange $\mathcal{T}$ and $\mathcal{T}^*$). Let now $\mathcal{S}\colon \mathcal{H} \to \mathcal{H}$ be defined by

$$\mathcal{S}x := \mathcal{T}x - \sum_{j=1}^{n} \langle x, x_j\rangle\, y_j, \quad x \in \mathcal{H}\,. \tag{5.6}$$

Since $\mathcal{T}$ is compact and the operator corresponding to the summation in (5.6) has a finite-dimensional image and is, therefore, also compact, the operator $\mathcal{S}$ is compact as well. Let now $x \in \ker(\mathcal{S} - \lambda\mathcal{I})$. Then

$$(\mathcal{T} - \lambda\mathcal{I})x = (\mathcal{T} - \mathcal{S} + \mathcal{S} - \lambda\mathcal{I})x = \sum_{j=1}^{n} \langle x, x_j\rangle\, y_j \tag{5.7}$$

and, consequently, due to the orthonormality, (5.7), and the construction of the y_k,

$$\begin{aligned}\left\langle \sum_{j=1}^{n} \langle x, x_j\rangle\, y_j, y_k \right\rangle &= \langle x, x_k\rangle = \langle (\mathcal{T} - \lambda\mathcal{I})x, y_k\rangle \\ &= \left\langle x, \left(\mathcal{T}^* - \overline{\lambda}\mathcal{I}\right) y_k\right\rangle = 0 \quad \text{for all } k = 1, \ldots, n\,.\end{aligned}$$

Hence, $(\mathcal{T} - \lambda\mathcal{I})x = 0$ and $x \in \ker(\mathcal{T} - \lambda\mathcal{I})$. With the orthonormal basis, we see that

$$x = \sum_{j=1}^{n} \langle x, x_j\rangle\, x_j = 0\,.$$

Consequently, $\ker(\mathcal{S} - \lambda\mathcal{I}) = \{0\}$ such that λ is not an eigenvalue of $\mathcal{S}$. Since $\lambda \neq 0$ and $\mathcal{S}$ is compact (which implies that non-vanishing elements of the spectrum must be eigenvalues), λ cannot be an element of the spectrum of $\mathcal{T}$. Hence, in particular, $(\mathcal{S} - \lambda\mathcal{I})(\mathcal{H}) = \mathcal{H}$.

Let us assume now that $n < m$, that is there exists y_{n+1} as an onb element. Due to the considerations above, there exists $z \in \mathcal{H}$ with $y_{n+1} = (\mathcal{S} - \lambda\mathcal{I})z$. This implies (by using again the orthonormality)

$$\begin{aligned}\|y_{n+1}\|^2 &= \langle (\mathcal{S} - \lambda\mathcal{I})z, y_{n+1}\rangle \\ &= \left\langle (\mathcal{T} - \lambda\mathcal{I})z - \sum_{j=1}^{n} \langle z, x_j\rangle\, y_j, y_{n+1} \right\rangle \\ &= \left\langle z, \left(\mathcal{T}^* - \overline{\lambda}\mathcal{I}\right) y_{n+1}\right\rangle \\ &= 0\,.\end{aligned}$$

However, $y_{n+1} = 0$ cannot be a part of an onb, which is a contradiction. Consequently, $n = m$.

c) The equation is solvable. $\Leftrightarrow G \in (\mathcal{T} - \lambda\mathcal{I})(\mathcal{H}) = \ker(\mathcal{T}^* - \overline{\lambda}\mathcal{I})^{\perp} \Leftrightarrow G \perp \ker(\mathcal{T}^* - \overline{\lambda}\mathcal{I})$. Remember that, according to Fredholm's theorem, $(\mathcal{T} - \lambda\mathcal{I})(\mathcal{H})$ is closed.

d) The equation is solvable. $\Leftrightarrow G \in (\mathcal{T}^* - \overline{\lambda}\mathcal{I})(\mathcal{H}) = \ker(\mathcal{T} - \lambda\mathcal{I})^{\perp} \Leftrightarrow G \perp \ker(\mathcal{T} - \lambda\mathcal{I})$.

5.8 Since $\mathcal{T}$ is compact, it must have a singular-value decomposition

$$\mathcal{T}x = \sum_{n} \sigma_n \langle x, u_n\rangle_{\mathcal{X}} v_n\,, \quad x \in \mathcal{X}\,. \tag{5.8}$$

We set now $p \coloneqq (\alpha+\beta)/\beta$ and $q \coloneqq (\alpha+\beta)/\alpha$. Then, obviously, $1/p + 1/q = (\beta+\alpha)/(\alpha+\beta) = 1$ and we may apply the Hölder inequality as follows:

$$\begin{aligned}
\left\| |\mathcal{T}|^\beta x \right\|_{\mathcal{X}}^2 &= \sum_n \sigma_n^{2\beta} \left|\langle x, u_n\rangle_{\mathcal{X}}\right|^2 \\
&= \sum_n \left(\sigma_n^{2\beta} \left|\langle x, u_n\rangle_{\mathcal{X}}\right|^{2/p}\right) \left|\langle x, u_n\rangle_{\mathcal{X}}\right|^{2/q} \\
&\le \left[\sum_n \left(\sigma_n^{2\beta} \left|\langle x, u_n\rangle_{\mathcal{X}}\right|^{2/p}\right)^p\right]^{1/p} \left(\sum_n \left|\langle x, u_n\rangle_{\mathcal{X}}\right|^{2q/q}\right)^{1/q} \\
&= \left[\sum_n \sigma_n^{2(\alpha+\beta)} \left|\langle x, u_n\rangle_{\mathcal{X}}\right|^2\right]^{\beta/(\alpha+\beta)} \left(\sum_n \left|\langle x, u_n\rangle_{\mathcal{X}}\right|^2\right)^{\alpha/(\alpha+\beta)} .
\end{aligned}$$

Hence, by using the Bessel inequality in the last summation term, we get

$$\left\| |\mathcal{T}|^\beta x \right\|_{\mathcal{X}} \le \left\| |\mathcal{T}|^{\alpha+\beta} x \right\|_{\mathcal{X}}^{\beta/(\alpha+\beta)} \|x\|_{\mathcal{X}}^{\alpha/(\alpha+\beta)} .$$

5.9 The derivations are here rather simple. According to the assumptions, the singular-value decomposition of $\mathcal{T}$ is given by (5.8).

a) Since the singular values of a compact operator tend to zero (or are only a finite set), we get, for all $x \in \mathcal{X}$, the inequality

$$\|\mathcal{T}x\|_{\mathcal{Y}}^2 = \sum_n \sigma_n^2 \left|\langle x, u_n\rangle_{\mathcal{X}}\right|^2 \le \underbrace{\max_n \sigma_n^2}_{=\sigma_1^2} \underbrace{\sum_n \left|\langle x, u_n\rangle_{\mathcal{X}}\right|^2}_{\le \|x\|_{\mathcal{X}}^2} ,$$

while $\|\mathcal{T}u_1\|_{\mathcal{Y}} = \|\sigma_1 v_1\|_{\mathcal{Y}} = \sigma_1$.

b) We start with the following expansions:

$$\mathcal{T}^* y = \sum_n \sigma_n \langle y, v_n\rangle_{\mathcal{Y}} u_n \in \mathcal{T}^*(\mathcal{Y}) \subset \ker(\mathcal{T})^\perp , \tag{5.9}$$

$$\varphi\left(\mathcal{T}^*\mathcal{T}\right) x = \sum_n \varphi\left(\sigma_n^2\right) \langle x, u_n\rangle_{\mathcal{X}} u_n + \varphi(0) \mathcal{P}_{\ker(\mathcal{T})} x ,$$

$$\Rightarrow \quad \varphi\left(\mathcal{T}^*\mathcal{T}\right) \mathcal{T}^* y = \sum_n \varphi\left(\sigma_n^2\right) \sigma_n \langle y, v_n\rangle_{\mathcal{Y}} u_n + 0 \tag{5.10}$$

for all $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. Since (5.9) is the singular-value decomposition of $\mathcal{T}^*$, we get, for all $y \in \mathcal{Y}$, the identity

$$\begin{aligned}
\varphi\left(\mathcal{T}\mathcal{T}^*\right) y &= \varphi\left(\mathcal{T}^{**}\mathcal{T}^*\right) y \\
&= \sum_n \varphi\left(\sigma_n^2\right) \langle y, v_n\rangle_{\mathcal{Y}} v_n + \varphi(0) \mathcal{P}_{\ker(\mathcal{T}^*)} y .
\end{aligned}$$

Hence, since $v_n \in \ker(\mathcal{T}^*)^\perp$ for all n, we obtain with (5.9) and (5.10) the identity

$$\begin{aligned}
\mathcal{T}^* \varphi\left(\mathcal{T}\mathcal{T}^*\right) y &= \sum_n \sigma_n \left\langle \sum_k \varphi\left(\sigma_k^2\right) \langle y, v_k\rangle_{\mathcal{Y}} v_k + \varphi(0) \mathcal{P}_{\ker(\mathcal{T}^*)} y, v_n \right\rangle_{\mathcal{Y}} u_n \\
&= \sum_n \sigma_n \varphi\left(\sigma_n^2\right) \langle y, v_n\rangle_{\mathcal{Y}} u_n \\
&= \varphi\left(\mathcal{T}^*\mathcal{T}\right) \mathcal{T}^* y \quad \text{for all } y \in \mathcal{Y} .
\end{aligned}$$

c) Since we have

$$\psi\left(\mathcal{T}^*\mathcal{T}\right)x = \sum_n \psi\left(\sigma_n^2\right)\langle x, u_n\rangle_{\mathcal{X}} u_n + \psi(0)\mathcal{P}_{\ker(\mathcal{T})}x$$

for all $x \in \mathcal{X}$, we also get

$$\varphi\left(\mathcal{T}^*\mathcal{T}\right)\psi\left(\mathcal{T}^*\mathcal{T}\right)x = \sum_n \varphi\left(\sigma_n^2\right)\psi\left(\sigma_n^2\right)\langle x, u_n\rangle_{\mathcal{X}} u_n + \varphi(0)\psi(0)\mathcal{P}_{\ker(\mathcal{T})}^2 x\,,$$

where the projection certainly fulfils $\mathcal{P}_{\ker(\mathcal{T})}^2 = \mathcal{P}_{\ker(\mathcal{T})}$. In analogy to part a, where we complement $\{u_n\}_n$ in order to obtain an orthonormal basis for $\mathcal{X}$, we obtain, by using $\sigma_n \to 0$ and the requirements on φ, ψ,

$$\begin{aligned}\left\|\varphi\left(\mathcal{T}^*\mathcal{T}\right)\psi\left(\mathcal{T}^*\mathcal{T}\right)\right\| &= \max\left(\left\{\left|\varphi\left(\sigma_n^2\right)\psi\left(\sigma_n^2\right)\right|\right\}_n \cup \{|\varphi(0)\psi(0)|\}\right)\\ &= \sup\left\{\left|\varphi\left(\sigma_n^2\right)\psi\left(\sigma_n^2\right)\right|\right\}_n\\ &\le \sup_{\lambda\in[0,\|\mathcal{T}\|^2]} |\varphi(\lambda)\psi(\lambda)|\,.\end{aligned}$$

d) We use (5.10) from part b and proceed in analogy to part a. Then we obtain

$$\left\|\varphi\left(\mathcal{T}^*\mathcal{T}\right)\mathcal{T}^*\right\| = \sup_n\left\{\left|\varphi\left(\sigma_n^2\right)\right|\sigma_n\right\}_n = \sup_{\lambda\in[0,\|\mathcal{T}\|^2]}\left(|\varphi(\lambda)|\sqrt{\lambda}\right)\,.$$

5.10 Let $\{(\sigma_n, u_n, v_n)\}_n$ be the singular system of $\mathcal{T}$ and assume that the singular values are arranged in a monotonically decreasing order.

a) For $\vartheta = 1$, the inequality is trivial. So, let $\vartheta < 1$. With the Hölder inequality for $p := \vartheta^{-1}$ and $q := (1-\vartheta)^{-1}$ ($\Rightarrow p^{-1} + q^{-1} = \vartheta + (1-\vartheta) = 1$), we obtain

$$\begin{aligned}\|x\|_{\vartheta\nu+(1-\vartheta)\mu}^2 &= \sum_n \sigma_n^{-2[\vartheta\nu+(1-\vartheta)\mu]}\left|\langle x, u_n\rangle_{\mathcal{X}}\right|^2\\ &= \sum_n \left(\sigma_n^{-2\nu}\left|\langle x, u_n\rangle_{\mathcal{X}}\right|^2\right)^{\vartheta}\left(\sigma_n^{-2\mu}\left|\langle x, u_n\rangle_{\mathcal{X}}\right|^2\right)^{1-\vartheta}\\ &\le \left(\sum_n \sigma_n^{-2\nu}\left|\langle x, u_n\rangle_{\mathcal{X}}\right|^2\right)^{\vartheta}\left(\sum_n \sigma_n^{-2\mu}\left|\langle x, u_n\rangle_{\mathcal{X}}\right|^2\right)^{1-\vartheta}\\ &= \|x\|_\nu^{2\vartheta}\|x\|_\mu^{2(1-\vartheta)}\,.\end{aligned}$$

b) We use Exercise 5.9 to obtain (by utilizing $\sigma_n/\sigma_1 \in\,]0,1]$ and $\nu \ge \mu$)

$$\begin{aligned}\|x\|_\mu^2 &= \sum_n \sigma_n^{-2\mu}\left|\langle x, u_n\rangle_{\mathcal{X}}\right|^2\\ &= \sigma_1^{-2\mu}\sum_n\left(\frac{\sigma_n}{\sigma_1}\right)^{-2\mu}\left|\langle x, u_n\rangle_{\mathcal{X}}\right|^2\\ &\le \sigma_1^{-2\mu}\sum_n\left(\frac{\sigma_n}{\sigma_1}\right)^{-2\nu}\left|\langle x, u_n\rangle_{\mathcal{X}}\right|^2\\ &= \sigma_1^{2(\nu-\mu)}\sum_n \sigma_n^{-2\nu}\left|\langle x, u_n\rangle_{\mathcal{X}}\right|^2\\ &= \|\mathcal{T}\|^{2(\nu-\mu)}\|x\|_\nu^2\,.\end{aligned}$$

5.11 We have

$$f_1 - f_2 \in \mathcal{X}_\nu\,,$$

$$\|\mathcal{T}(f_1 - f_2)\|_{\mathcal{Y}} \le \|\mathcal{T}f_1 - g\|_{\mathcal{Y}} + \|g - \mathcal{T}f_2\|_{\mathcal{Y}} \le 2\varepsilon\,,$$
$$\|f_1 - f_2\|_\nu \le \|f_1\|_\nu + \|f_2\|_\nu \le 2\max\{\|f_1\|_\nu\,, \|f_2\|_\nu\}\ .$$

Hence, with the corresponding theorem on the best-possible worst-case error, we obtain

$$\|f_1 - f_2\|_{\mathcal{X}} \le \mathrm{e}_\nu\left(2\varepsilon, 2\max\{\|f_1\|_\nu\,, \|f_2\|_\nu\}\right)\ .$$

5.12 As usual, the singular system of $\mathcal{T}$ is denoted by $\{(\sigma_n, u_n, v_n)\}_n$. Let $f^+ \in \mathcal{X}_\mu$ with $\|f^+\|_\mu \le \varrho$, that is there exists $w \in \mathcal{X}$ such that $f^+ = |\mathcal{T}|^\mu w$ and $\|f^+\|_\mu = \|w\|_{\mathcal{X}} \le \varrho$. Moreover, let $g := \mathcal{T}f^+$. By using propositions which we had before in the book as well as part c of Exercise 5.9, we obtain

$$\begin{aligned}
\left\|f^+ - \mathcal{R}_t g\right\|_{\mathcal{X}} &= \left\|\sum_n \left(\sigma_n^{-1} - F_t\left(\sigma_n^2\right)\sigma_n\right)\langle g, v_n\rangle_{\mathcal{Y}} u_n\right\|_{\mathcal{X}} \\
&= \left\|\sum_n \left(1 - \sigma_n^2 F_t\left(\sigma_n^2\right)\right)\sigma_n^{-1}\langle g, v_n\rangle_{\mathcal{Y}} u_n\right\|_{\mathcal{X}} \\
&= \left\|\sum_n p_t\left(\sigma_n^2\right)\left\langle f^+, u_n\right\rangle_{\mathcal{X}} u_n\right\|_{\mathcal{X}} \\
&= \left\|\sum_n p_t\left(\sigma_n^2\right)\sigma_n^\mu\langle w, u_n\rangle_{\mathcal{X}} u_n\right\|_{\mathcal{X}} \\
&= \left\|p_t\left(\mathcal{T}^*\mathcal{T}\right)\left(\mathcal{T}^*\mathcal{T}\right)^{\mu/2} w\right\|_{\mathcal{X}} \\
&\le \sup_{\sigma\in\left[0,\|\mathcal{T}\|_{\mathcal{L}}^2\right]}\left(|p_t(\sigma)|\,\sigma^{\mu/2}\right)\|w\|_{\mathcal{X}} \\
&\le \omega_\mu(t)\,\varrho
\end{aligned} \tag{5.11}$$

for all $t \in\,]0, t_0]$. Let now $g^\varepsilon \in \overline{B_\varepsilon(g)}$. Then, with (5.11) and derivations from earlier in the book, we see that there exist constants $C_3, C_4 \in \mathbb{R}^+$ such that, for sufficiently small ε (guaranteeing that $\gamma(\varepsilon) \le t_0$),

$$\begin{aligned}
&\left\|f^+ - \mathcal{R}_{\gamma(\varepsilon)} g^\varepsilon\right\|_{\mathcal{X}} \le \left\|f^+ - \mathcal{R}_{\gamma(\varepsilon)} g\right\|_{\mathcal{X}} + \left\|\mathcal{R}_{\gamma(\varepsilon)} g - \mathcal{R}_{\gamma(\varepsilon)} g^\varepsilon\right\|_{\mathcal{X}} \\
&\quad\le \omega_\mu(\gamma(\varepsilon))\,\varrho + \varepsilon\sqrt{C_F M(\gamma(\varepsilon))} \\
&\quad\le C_3\,\gamma(\varepsilon)^{\mu/2}\varrho + \varepsilon\sqrt{C_F C_4}\,\gamma(\varepsilon)^{-1/2} \\
&\quad\le C_3\left[C_2\left(\frac{\varepsilon}{\varrho}\right)^{2/(\mu+1)}\right]^{\mu/2}\varrho + \varepsilon\sqrt{C_F C_4}\left[C_1\left(\frac{\varepsilon}{\varrho}\right)^{2/(\mu+1)}\right]^{-1/2} \\
&\quad= C_3 C_2^{\mu/2}\varepsilon^{\mu/(\mu+1)}\varrho^{-\mu/(\mu+1)+1} + \varepsilon\sqrt{C_F C_4}\,C_1^{-1/2}\varrho^{1/(\mu+1)}\varepsilon^{-1/(\mu+1)} \\
&\quad= \widetilde{C}\varrho^{1/(\mu+1)}\varepsilon^{\mu/(\mu+1)}\,,
\end{aligned}$$

where $C_F := \sup_{\sigma\in[0,\|\mathcal{T}\|_{\mathcal{L}}^2]}(\sigma|F_t(\sigma)|)$ and $\widetilde{C} := C_3 C_2^{\mu/2} + \sqrt{C_F C_4}\,C_1^{-1/2}$.

6

The Magnetic Field

Exercises

6.1 f is curl-free and defined on the whole $\mathbb{R}^3$ and is, therefore, a gradient field. It is easy to see that $f(x) = \operatorname{grad}(|x|^2/2) = \operatorname{grad}(|x|^2/2) + \operatorname{curl} 0$.

g is divergence-free on the whole $\mathbb{R}^3$. Therefore, g is a curl field, that is $g = \operatorname{grad} 0 + \operatorname{curl} A$. For finding the corresponding vector potential A, we observe that $g = \operatorname{curl} A$ is equivalent to the system

$$\begin{aligned} \frac{\partial}{\partial x_2} A_3 - \frac{\partial}{\partial x_3} A_2 &= x_2\,, \\ \frac{\partial}{\partial x_3} A_1 - \frac{\partial}{\partial x_1} A_3 &= x_3\,, \\ \frac{\partial}{\partial x_1} A_2 - \frac{\partial}{\partial x_2} A_1 &= x_1\,. \end{aligned}$$

A closer look reveals that

$$A(x) = \frac{1}{2} \begin{pmatrix} x_3^2 \\ x_1^2 \\ x_2^2 \end{pmatrix}$$

is one possible solution.

Note that the Helmholtz decomposition is not unique.

6.2 For $f(r\xi) := \varepsilon^\varphi(\varphi)$, $r \in \,]\alpha, \beta[$, $\xi \in \Omega$, we obtain

$$\begin{aligned} \nabla \cdot f(r\xi) &= \frac{1}{r} \left(\varepsilon^\varphi \frac{1}{\sqrt{1-t^2}} \frac{\partial}{\partial \varphi} + \varepsilon^t \sqrt{1-t^2}\, \frac{\partial}{\partial t} \right) \cdot \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix} \\ &= \frac{1}{r\sqrt{1-t^2}}\, \varepsilon^\varphi \cdot \begin{pmatrix} -\cos\varphi \\ -\sin\varphi \\ 0 \end{pmatrix} \\ &= 0\,. \end{aligned}$$

Hence, f is divergence-free. Moreover, for each $\varrho \in \,]\alpha, \beta[$, we get

$$\int_{S_\varrho(0)} f(x) \cdot \frac{x}{|x|}\, \mathrm{d}\omega(x) = \int_{S_\varrho(0)} \varepsilon^\varphi \cdot \varepsilon^r \, \mathrm{d}\omega = 0\,.$$

Consequently, f is solenoidal due to one of the criteria which are listed in the book.

Since f is obviously also tangential (to each $S_\varrho(0)$), it must also be toroidal (due to another criterion in the book). Since only the zero function is toroidal and poloidal, f can, therefore, not be poloidal.

The Mie representation of the toroidal function f is, thus,

$$f(x) = \nabla \times (x \times \nabla 0) + x \times \nabla Q(x)\,.$$

With sufficient experience, one can guess a toroidal scalar Q here. If one does not succeed with that, it is also possible to make some conclusions on requirements on Q for finally obtaining a possible Q.

With the known properties of the local orthonormal basis and the cross product, we deduce that

$$\begin{aligned}1 &= \varepsilon^\varphi \cdot \varepsilon^\varphi = (x \times \nabla Q) \cdot \varepsilon^\varphi = ((r\varepsilon^r) \times \nabla Q) \cdot \varepsilon^\varphi = (\varepsilon^\varphi \times (r\varepsilon^r)) \cdot (\nabla Q)\\ &= -r\varepsilon^t \cdot (\nabla Q)\end{aligned}$$

and

$$\begin{aligned}0 &= \varepsilon^\varphi \cdot \varepsilon^t = (x \times \nabla Q) \cdot \varepsilon^t = ((r\varepsilon^r) \times \nabla Q) \cdot \varepsilon^t = \left(\varepsilon^t \times (r\varepsilon^r)\right) \cdot (\nabla Q)\\ &= r\varepsilon^\varphi \cdot (\nabla Q)\,.\end{aligned}$$

It would not be helpful to use $0 = \varepsilon^\varphi \cdot \varepsilon^r$ as well, because ε^r is anyway orthogonal to the cross product $x \times \nabla Q$. We obtain now

$$\nabla Q(x) = -\frac{1}{r}\,\varepsilon^t + F\varepsilon^r$$

for an unknown F. Since gradient fields are curl-free, we also get

$$\begin{aligned}0 &= \nabla \times (\nabla Q)\\
&= \frac{1}{r}\left(\varepsilon^\varphi \frac{1}{\sqrt{1-t^2}}\frac{\partial}{\partial\varphi} + \varepsilon^t \sqrt{1-t^2}\,\frac{\partial}{\partial t}\right) \times \left(-\frac{1}{r}\,\varepsilon^t + F\varepsilon^r\right)\\
&\quad + \varepsilon^r \times \left(\frac{1}{r^2}\,\varepsilon^t\right) + \varepsilon^r \times \left(\frac{\partial F}{\partial r}\,\varepsilon^r\right)\\
&= \frac{-1}{r^2\sqrt{1-t^2}}\,\varepsilon^\varphi \times (-t\varepsilon^\varphi) + \frac{F}{r\sqrt{1-t^2}}\,\varepsilon^\varphi \times \left(\sqrt{1-t^2}\,\varepsilon^\varphi\right)\\
&\quad - \frac{\sqrt{1-t^2}}{r^2}\,\varepsilon^t \times \left(\frac{-1}{\sqrt{1-t^2}}\,\varepsilon^r\right) + F\,\frac{\sqrt{1-t^2}}{r}\,\varepsilon^t \times \left(\frac{1}{\sqrt{1-t^2}}\,\varepsilon^t\right)\\
&\quad + \frac{1}{r}\left(\frac{1}{\sqrt{1-t^2}}\frac{\partial F}{\partial\varphi}\,\varepsilon^\varphi \times \varepsilon^r + \sqrt{1-t^2}\,\frac{\partial F}{\partial t}\,\varepsilon^t \times \varepsilon^r\right) - \frac{1}{r^2}\,\varepsilon^\varphi\\
&= \frac{1}{r^2}\varepsilon^\varphi + \frac{1}{r}\left(-\frac{1}{\sqrt{1-t^2}}\frac{\partial F}{\partial\varphi}\,\varepsilon^t + \sqrt{1-t^2}\,\frac{\partial F}{\partial t}\,\varepsilon^\varphi\right) - \frac{1}{r^2}\,\varepsilon^\varphi\,.\end{aligned}$$

This is only possible if

$$\frac{\partial F}{\partial\varphi} = 0 \quad \text{and} \quad \frac{\partial F}{\partial t} = 0\,.$$

Since merely radially dependent summands are irrelevant for toroidal scalars, we try $\nabla Q(r\xi) = -r^{-1}\varepsilon^t$, that is

$$\left[\varepsilon^r\,\frac{\partial}{\partial r} + \frac{1}{r}\left(\varepsilon^\varphi\,\frac{1}{\sqrt{1-t^2}}\frac{\partial}{\partial\varphi} + \varepsilon^t\,\sqrt{1-t^2}\,\frac{\partial}{\partial t}\right)\right] Q = -\frac{1}{r}\,\varepsilon^t\,.$$

Hence, $\frac{\partial Q}{\partial r} = 0 = \frac{\partial Q}{\partial \varphi}$ and we need

$$\frac{1}{r}\sqrt{1-t^2}\,\frac{\partial Q}{\partial t} = -\frac{1}{r}\,.$$

This leads us to

$$\frac{\partial Q}{\partial t} = -\frac{1}{\sqrt{1-t^2}}\,.$$

Therefore, one possible choice as a toroidal scalar of f is

$$Q(x) = -\arcsin t = -\arcsin \xi_3 = -\arcsin \frac{x_3}{|x|}\,.$$

Note that $\arccos y = \pi/2 - \arcsin y$ such that $Q(x) = \arccos(x_3/|x|)$ is another possibility.

6.3 The divergence of $f := \varepsilon^t$ is given by

$$\begin{aligned}\nabla\cdot f &= \frac{1}{r}\left(\varepsilon^\varphi \frac{1}{\sqrt{1-t^2}}\frac{\partial}{\partial\varphi} + \varepsilon^t\sqrt{1-t^2}\,\frac{\partial}{\partial t}\right)\cdot\varepsilon^t\\ &= \frac{1}{r\sqrt{1-t^2}}\varepsilon^\varphi\cdot(-t\varepsilon^\varphi) + \frac{\sqrt{1-t^2}}{r}\,\varepsilon^t\cdot\left(-\frac{1}{\sqrt{1-t^2}}\,\varepsilon^r\right)\\ &= -\frac{t}{r\sqrt{1-t}}\,.\end{aligned}$$

Since this vector field is not divergence-free, ε^t is neither solenoidal, nor poloidal, nor toroidal.

For $f := \varepsilon^r$, we obtain

$$\nabla\cdot f = \nabla\cdot\left(\frac{x}{|x|}\right) = \sum_{j=1}^{3}\frac{1\cdot|x| - x_j\,\frac{1}{2|x|}2x_j}{|x|^2} = \frac{3}{|x|} - \frac{|x|^2}{|x|^3} = \frac{2}{|x|}\,.$$

Thus, ε^r is also neither solenoidal, nor poloidal, nor toroidal.

6.4 For the normal basis vector field, it is clear that

$$\varepsilon^r = \xi\cdot 1 + \nabla^* 0 + \mathrm{L}^* 0\,.$$

For the tangential basis vector fields, we remember that $\varepsilon^t = \varepsilon^r\times\varepsilon^\varphi$ and $\varepsilon^\varphi = \varepsilon^r\times(-\varepsilon^t)$. For this reason, we try the *ansatz* $\varepsilon^t = \nabla^* G$, because this would automatically yield $\varepsilon^\varphi = \mathrm{L}^*(-G)$ (actually, $\varepsilon^\varphi = \nabla^* G$ would not work, because we know from Exercise 6.2 that $0 = \nabla^*\cdot\varepsilon^\varphi$ such that we would get $0 = \Delta^* G$, which is only true for constant G, which, however, yield $\nabla^* G = 0$).

If $\varepsilon^t = \nabla^* G$, then we need

$$\varepsilon^t = \left(\varepsilon^\varphi\frac{1}{\sqrt{1-t^2}}\frac{\partial}{\partial\varphi} + \varepsilon^t\sqrt{1-t^2}\,\frac{\partial}{\partial t}\right)G\,.$$

This motivates the *ansatz* $\frac{\partial G}{\partial\varphi} = 0$ and

$$\sqrt{1-t^2}\,\frac{\partial G}{\partial t} = 1\,.$$

The latter equation leads us to

$$\frac{\partial G}{\partial t} = \frac{1}{\sqrt{1-t^2}}\,,$$

which is solved, for example, by $G(\xi(\varphi,t)) = \arcsin t = \arcsin \xi_3$. Hence,

$$\varepsilon^t = \xi \cdot 0 + \nabla^* \arcsin \frac{x_3}{|x|} + \mathrm{L}^* 0$$

and

$$\varepsilon^\varphi = \xi \cdot 0 + \nabla^* 0 + \mathrm{L}^* \left(- \arcsin \frac{x_3}{|x|} \right) .$$

Note that $\arccos y = \pi/2 - \arcsin y$ such that arcsin may, for instance, be replaced by $-\arccos$ above (since constant summands are irrelevant).

6.5 The solution basically requires the rules for the use of the Levi–Cività alternating symbol as well as Schwarz's theorem (all summations are over $1, 2, 3$):

$$\begin{aligned}
\operatorname{curl}\operatorname{curl}\mathrm{L}P &= \left(\sum_{j,k} \varepsilon_{ijk} \frac{\partial}{\partial x_j} (\operatorname{curl}\mathrm{L}P)_k \right)_{i=1,2,3} \\
&= \left(\sum_{j,k,l,m} \varepsilon_{ijk}\varepsilon_{klm} \frac{\partial}{\partial x_j} \left[\frac{\partial}{\partial x_l} (\mathrm{L}P)_m \right] \right)_{i=1,2,3} \\
&= \left(\sum_{j,k,l,m,n,p} \varepsilon_{kij}\varepsilon_{klm}\varepsilon_{mnp} \frac{\partial}{\partial x_j} \left[\frac{\partial}{\partial x_l} \left(x_n \frac{\partial P}{\partial x_p} \right) \right] \right)_{i=1,2,3} \\
&= \left(\sum_{j,l,m,n,p} (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\, \varepsilon_{mnp} \frac{\partial}{\partial x_j} \left[\delta_{ln} \frac{\partial P}{\partial x_p} + x_n \frac{\partial^2 P}{\partial x_l \partial x_p} \right] \right)_{i=1,2,3} \\
&= \left(\sum_{j,l,m,n,p} (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\, \varepsilon_{mnp} \right. \\
&\qquad \left. \times \left[\delta_{ln} \frac{\partial^2 P}{\partial x_j \partial x_p} + \delta_{jn} \frac{\partial^2 P}{\partial x_l \partial x_p} + x_n \frac{\partial^3 P}{\partial x_j \partial x_l \partial x_p} \right] \right)_{i=1,2,3} \\
&= \left(\underbrace{\sum_{j,p} \left(\varepsilon_{jip} \frac{\partial^2 P}{\partial x_j \partial x_p} + \underbrace{\varepsilon_{jjp}}_{=0} \frac{\partial^2 P}{\partial x_i \partial x_p} \right)}_{-(\nabla \times \nabla P)_i = 0} + \sum_{j,n,p} \varepsilon_{jnp} x_n \frac{\partial^3 P}{\partial x_j \partial x_i \partial x_p} \right)_{i=1,2,3} \\
&\quad - \left(\underbrace{\sum_{j,p} \left(\varepsilon_{ijp} \frac{\partial^2 P}{\partial x_j \partial x_p} + \varepsilon_{ijp} \frac{\partial^2 P}{\partial x_j \partial x_p} \right)}_{=2(\nabla \times \nabla P)_i = 0} + \sum_{j,n,p} \varepsilon_{inp} x_n \frac{\partial^3 P}{\partial x_j^2 \partial x_p} \right)_{i=1,2,3} \\
&= \underbrace{\left(\sum_j \underbrace{\sum_{n,p} \varepsilon_{jnp} x_n \frac{\partial}{\partial x_p}}_{=\mathrm{L}_j} \frac{\partial}{\partial x_j} \frac{\partial P}{\partial x_i} \right)_{i=1,2,3}}_{=\mathrm{L}\cdot\nabla(\nabla P) = \mathrm{L}^* \cdot \left(\xi \frac{\partial}{\partial r} + \frac{1}{r} \nabla^* \right) \nabla P = 0} - \underbrace{\left(\sum_{n,p} \varepsilon_{inp} x_n \frac{\partial}{\partial x_p} \sum_j \frac{\partial^2 P}{\partial x_j^2} \right)_{i=1,2,3}}_{=x \times \nabla(\Delta P)} .
\end{aligned}$$

Hence, since toroidal fields are divergence-free, we obtain

$$\Delta \mathrm{L}P = \operatorname{grad}\operatorname{div}\mathrm{L}P - \operatorname{curl}\operatorname{curl}\mathrm{L}P = -\operatorname{curl}\operatorname{curl}\mathrm{L}P = \mathrm{L}\Delta P\,.$$

Alternatively and a bit shorter, one can also proceed as follows:

$$\begin{aligned}
\Delta \mathrm{L}P &= \sum_j \frac{\partial^2}{\partial x_j^2}(x \times \nabla P) \\
&= \left(\sum_j \frac{\partial^2}{\partial x_j^2} \sum_{k,l} \varepsilon_{ikl} x_k \frac{\partial P}{\partial x_l} \right)_{i=1,2,3} \\
&= \left(\sum_{j,k,l} \varepsilon_{ikl} \left(2\delta_{jk} \frac{\partial^2 P}{\partial x_j \partial x_l} + x_k \frac{\partial^3 P}{\partial x_j^2 \partial x_l} \right) \right)_{i=1,2,3} \\
&= \left(2 \sum_{k,l} \varepsilon_{ikl} \frac{\partial^2 P}{\partial x_k \partial x_l} + \sum_{j,k,l} \varepsilon_{ikl} x_k \frac{\partial^3 P}{\partial x_l \partial x_j^2} \right)_{i=1,2,3} \\
&= 2 \underbrace{\nabla \times \nabla P}_{=0} + \left(\sum_{k,l} \varepsilon_{ikl} x_k \frac{\partial}{\partial x_l} \sum_j \frac{\partial^2 P}{\partial x_j^2} \right)_{i=1,2,3} \\
&= x \times \nabla \Delta P \\
&= \mathrm{L}\Delta P\,.
\end{aligned}$$

7

Mathematical Models in Seismology

Exercises

7.1 While we have

$$\phi(X,0) = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = X ,$$

$B_1(0)$ is a simple body, and ϕ is obviously twice continuously differentiable, we observe, however, that the deformation gradient is

$$F(X,t)$$

$$= \begin{pmatrix} -2X_1 t \mathrm{e}^{-\left(X_1^2+X_2^2\right)t} X_1 + \mathrm{e}^{-\left(X_1^2+X_2^2\right)t} & -2X_2 t \mathrm{e}^{-\left(X_1^2+X_2^2\right)t} X_1 & 0 \\ -2X_1 t \mathrm{e}^{-\left(X_1^2+X_2^2\right)t} X_2 & -2X_2 t \mathrm{e}^{-\left(X_1^2+X_2^2\right)t} X_2 + \mathrm{e}^{-\left(X_1^2+X_2^2\right)t} & 0 \\ -2X_1 t \mathrm{e}^{-\left(X_1^2+X_2^2\right)t} (t+1) X_3 & -2X_2 t \mathrm{e}^{-\left(X_1^2+X_2^2\right)t} (t+1) X_3 & \mathrm{e}^{-\left(X_1^2+X_2^2\right)t} (t+1) \end{pmatrix}$$

such that the Jacobian

$$\begin{aligned} J(X,t) &= \mathrm{e}^{-\left(X_1^2+X_2^2\right)t}(t+1) \Big(4X_1^2 X_2^2 t^2 \mathrm{e}^{-2\left(X_1^2+X_2^2\right)t} - 2X_1^2 t \mathrm{e}^{-2\left(X_1^2+X_2^2\right)t} \\ &\qquad - 2X_2^2 t \mathrm{e}^{-2\left(X_1^2+X_2^2\right)t} + \mathrm{e}^{-2\left(X_1^2+X_2^2\right)t} \\ &\qquad - 4X_1^2 X_2^2 t^2 \mathrm{e}^{-2\left(X_1^2+X_2^2\right)t} \Big) \\ &= \mathrm{e}^{-3\left(X_1^2+X_2^2\right)t}(t+1) \left[4X_1^2 X_2^2 t^2 - 2\left(X_1^2+X_2^2\right)t + 1 - 4X_1^2 X_2^2 t^2 \right] \\ &= \mathrm{e}^{-3\left(X_1^2+X_2^2\right)t}(t+1) \left[1 - 2\left(X_1^2+X_2^2\right)t \right] \end{aligned}$$

vanishes for all X with $X_1^2 + X_2^2 = (2t)^{-1}$. Hence, the general assumption on motions is violated.

The material velocity and acceleration are determined as follows:

$$\begin{aligned} V(X,t) &= \begin{pmatrix} -\left(X_1^2+X_2^2\right) \mathrm{e}^{-\left(X_1^2+X_2^2\right)t} X_1 \\ -\left(X_1^2+X_2^2\right) \mathrm{e}^{-\left(X_1^2+X_2^2\right)t} X_2 \\ -\left(X_1^2+X_2^2\right) \mathrm{e}^{-\left(X_1^2+X_2^2\right)t} (t+1) X_3 + \mathrm{e}^{-\left(X_1^2+X_2^2\right)t} X_3 \end{pmatrix} \\ &= \mathrm{e}^{-\left(X_1^2+X_2^2\right)t} \left[-\left(X_1^2+X_2^2\right) \begin{pmatrix} X_1 \\ X_2 \\ (t+1)X_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ X_3 \end{pmatrix} \right] \end{aligned}$$

and

$$A(X,t) = -\left(X_1^2 + X_2^2\right) \mathrm{e}^{-\left(X_1^2+X_2^2\right)t} \left[-\left(X_1^2 + X_2^2\right) \begin{pmatrix} X_1 \\ X_2 \\ (t+1)X_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ X_3 \end{pmatrix} \right]$$
$$- \mathrm{e}^{-\left(X_1^2+X_2^2\right)t} \left(X_1^2 + X_2^2\right) \begin{pmatrix} 0 \\ 0 \\ X_3 \end{pmatrix}$$
$$= \mathrm{e}^{-\left(X_1^2+X_2^2\right)t} \left[\left(X_1^2 + X_2^2\right)^2 \begin{pmatrix} X_1 \\ X_2 \\ (t+1)X_3 \end{pmatrix} - 2\left(X_1^2 + X_2^2\right) \begin{pmatrix} 0 \\ 0 \\ X_3 \end{pmatrix} \right].$$

7.2 According to the Laplace expansion of a determinant, we have

$$\det A = \sum_{k=1}^{n} a_{jk} \det A_{j,k} \, (-1)^{j+k} \tag{7.1}$$

for all $j = 1, \ldots, n$. Note that each $a_{j,k}$ does not occur in $A_{j,k}$. Moreover, with the chain rule, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}(\det A) = \sum_{j,k=1}^{n} \frac{\partial \det A}{\partial a_{j,k}} \frac{\mathrm{d}a_{j,k}}{\mathrm{d}t}. \tag{7.2}$$

By combining (7.1) with (7.2), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(\det A) = \sum_{j,k=1}^{n} \det A_{j,k} \, (-1)^{j+k} \frac{\mathrm{d}a_{j,k}}{\mathrm{d}t},$$

which completes the proof.

7.3 It is clear that, if $F \equiv G$, then the integrals of F and G must coincide on every arbitrary domain.

Let now $\int_{\mathcal{U}} F(Y)\,\mathrm{d}Y = \int_{\mathcal{U}} G(Y)\,\mathrm{d}Y$ for every nice region $\mathcal{U} \subset \mathcal{B}$. Moreover, let $X \in \mathcal{B}$ be an arbitrary point. We assume that $F(X) \neq G(X)$. Without loss of generality, let $F(X) - G(X) > 0$. Since F and G are continuous and $\mathcal{B}$ is open, there exists a ball $B_r(X) \subset \mathcal{B}$ such that $F - G > 0$ on $B_r(X)$. Obviously, $B_r(X)$ is a nice region. Hence,

$$0 = \int_{B_r(X)} F(Y)\,\mathrm{d}Y - \int_{B_r(X)} G(Y)\,\mathrm{d}Y = \int_{B_r(X)} F(Y) - G(Y)\,\mathrm{d}Y,$$

which contradicts the positivity of $F - G$ on this ball. Since X was arbitrary, we have $F \equiv G$ on $\mathcal{B}$.

7.4 Clearly, $\phi_0(X) = X$ for all $X \in B_1(0)$, the ball $B_1(0)$ is a simple body, and ϕ is twice continuously differentiable. Moreover, the equation

$$\phi_t(X) = \begin{pmatrix} X_1(1+t) \\ X_2\left(1+t^2\right) \\ X_3(1+2t) \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

is easily solvable for each $t \in \mathbb{R}_0^+$:

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \phi_t^{-1}(x) = \begin{pmatrix} \frac{1}{1+t}\, x_1 \\ \frac{1}{1+t^2}\, x_2 \\ \frac{1}{1+2t}\, x_3 \end{pmatrix}, \tag{7.3}$$

which means that ϕ_t is invertible. The image $\phi_t(B_1(0))$ is given by all $x \in \mathbb{R}^3$ for which

$$|x_1| < 1 + t$$

and

$$|x_2| < \sqrt{1 - \left(\frac{x_1}{1+t}\right)^2}\ (1 + t^2)$$

and

$$|x_3| < \sqrt{1 - \left(\frac{x_1}{1+t}\right)^2 - \left(\frac{x_2}{1+t^2}\right)^2}\ (1 + 2t)$$

and is, therefore, an open set (more precisely, an open ellipsoid). Hence, ϕ is a $\mathrm{C}^{(2)}$-regular motion. Eventually, we obtain the deformation gradient

$$F(X,t) = \begin{pmatrix} 1+t & 0 & 0 \\ 0 & 1+t^2 & 0 \\ 0 & 0 & 1+2t \end{pmatrix}$$

with its determinant

$$J(X,t) = (1+t)\left(1+t^2\right)(1+2t) > 0 \quad \text{for all } t \geq 0\,. \tag{7.4}$$

Hence, the general assumptions on a motion are satisfied.

Regarding the equation which has to be validated, we calculate

$$\frac{\partial J}{\partial t}(X,t) = 1 \cdot \left(1+t^2\right)(1+2t) + (1+t)2t(1+2t) + (1+t)\left(1+t^2\right)2 \tag{7.5}$$

and

$$V(X,t) = \begin{pmatrix} X_1 \\ 2tX_2 \\ 2X_3 \end{pmatrix}. \tag{7.6}$$

We use (7.3) to transfer (7.6) into spatial coordinates which leads us to the spatial velocity

$$v(x,t) = \begin{pmatrix} \frac{1}{1+t}\, x_1 \\ \frac{2t}{1+t^2}\, x_2 \\ \frac{2}{1+2t}\, x_3 \end{pmatrix}.$$

Hence,

$$\operatorname{div}_x v(x,t) = \frac{1}{1+t} + \frac{2t}{1+t^2} + \frac{2}{1+2t}\,. \tag{7.7}$$

By combining (7.4), (7.7), and (7.5), we finally get

$$\begin{aligned}(\operatorname{div}_x v\,(\phi_t(X),t))\,J(X,t) &= \left(1+t^2\right)(1+2t) + (1+t)2t(1+2t) + 2(1+t)\left(1+t^2\right)\\ &= \frac{\partial J}{\partial t}(X,t)\,.\end{aligned}$$

7.5 Since ϕ is a $\mathrm{C}^{(2)}$-function, we can apply Schwarz's theorem such that we get, due to the chain rule, the identity

$$\begin{aligned}\frac{\partial}{\partial t}F(X,t) &= \frac{\partial}{\partial t}\frac{\partial}{\partial X}\phi(X,t) = \frac{\partial}{\partial X}\frac{\partial}{\partial t}\phi(X,t) = \frac{\partial}{\partial X}V(X,t)\\ &= \frac{\partial}{\partial X}V\left(\phi_t^{-1}\left(\phi_t(X)\right),t\right) = \frac{\partial}{\partial X}v\left(\phi_t(X),t\right)\\ &= \frac{\partial}{\partial x}v\left(\phi_t(X),t\right)\cdot\frac{\partial\phi_t(X)}{\partial X}\\ &= \frac{\partial}{\partial x}v\left(\phi_t(X),t\right)\cdot F(X,t)\,.\end{aligned}$$

7.6 The requirement on a perfect fluid means that every non-trivial vector is an eigenvector of the Cauchy stress tensor σ. In particular, there exist scalars $\lambda_1,\lambda_2,\lambda_3\in\mathbb{R}$ such that $\sigma\varepsilon^j=\lambda_j\varepsilon^j$ for all $j=1,2,3$, where ε^j is the usual j-th standard orthonormal vector. Since $\sigma\varepsilon^j$ yields the j-th column of σ, we see already here that

$$\sigma = \begin{pmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{pmatrix}.$$

However, the vector $(1,1,1)^{\mathrm{T}}$ is also an eigenvector, that is there is $\lambda_4\in\mathbb{R}$ such that

$$\sigma\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}\lambda_4\\ \lambda_4\\ \lambda_4\end{pmatrix}, \quad \text{while} \quad \sigma\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{pmatrix}.$$

Hence, there is $\lambda(=\lambda_j\ \forall j)$ such that

$$\sigma = \begin{pmatrix}\lambda & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \lambda\end{pmatrix}.$$

We set $p := -\lambda$.

For the divergence of σ, we calculate (with I representing the identity tensor)

$$\begin{aligned}\operatorname{div}_x\sigma = \operatorname{div}_x(-pI) = -\operatorname{div}_x\begin{pmatrix}p & 0 & 0\\ 0 & p & 0\\ 0 & 0 & p\end{pmatrix} &= -\begin{pmatrix}\operatorname{div}_x(p,0,0)\\ \operatorname{div}_x(0,p,0)\\ \operatorname{div}_x(0,0,p)\end{pmatrix} = -\begin{pmatrix}\frac{\partial p}{\partial x_1}\\ \frac{\partial p}{\partial x_2}\\ \frac{\partial p}{\partial x_3}\end{pmatrix}\\ &= -\nabla_x p\,.\end{aligned}$$

7.7 For verifying that $\mathcal{R}(U) = (1/2)(\operatorname{curl}_X U)\times I$, we calculate the right-hand side as follows (a single index attached to a tensor refers to its corresponding row):

$$(\operatorname{curl}_X U)\times I = \left(\sum_{n,m=1}^{3}\varepsilon_{lnm}(\operatorname{curl}U)_n I_m\right)_{l=1,2,3}$$

$$\begin{aligned}
&= \left(\sum_{n,m=1}^{3} \varepsilon_{lnm} \left(\sum_{j,k=1}^{3} \varepsilon_{njk} \frac{\partial U_k}{\partial X_j} \right) I_m \right)_{l=1,2,3} \\
&= \left(\sum_{n,m,j,k=1}^{3} \varepsilon_{lnm} \varepsilon_{njk} \frac{\partial U_k}{\partial X_j} I_m \right)_{l=1,2,3} \\
&= \left(\sum_{n,m,j,k=1}^{3} (-\varepsilon_{nlm} \varepsilon_{njk}) \frac{\partial U_k}{\partial X_j} I_m \right)_{l=1,2,3} \\
&= -\left(\sum_{m,j,k=1}^{3} (\delta_{lj}\delta_{mk} - \delta_{lk}\delta_{jm}) \frac{\partial U_k}{\partial X_j} I_m \right)_{l=1,2,3} \\
&= -\left(\sum_{k=1}^{3} \frac{\partial U_k}{\partial X_l} I_k - \sum_{j=1}^{3} \frac{\partial U_l}{\partial X_j} I_j \right)_{l=1,2,3} \\
&= -\left(\sum_{k=1}^{3} \frac{\partial U_k}{\partial X_l} I_k \right)_{l=1,2,3} + \left(\sum_{j=1}^{3} \frac{\partial U_l}{\partial X_j} I_j \right)_{l=1,2,3} \\
&= -\begin{pmatrix} \frac{\partial U_1}{\partial X_1} & \frac{\partial U_2}{\partial X_1} & \frac{\partial U_3}{\partial X_1} \\ \frac{\partial U_1}{\partial X_2} & \frac{\partial U_2}{\partial X_2} & \frac{\partial U_3}{\partial X_2} \\ \frac{\partial U_1}{\partial X_3} & \frac{\partial U_2}{\partial X_3} & \frac{\partial U_3}{\partial X_3} \end{pmatrix} + \begin{pmatrix} \frac{\partial U_1}{\partial X_1} & \frac{\partial U_1}{\partial X_2} & \frac{\partial U_1}{\partial X_3} \\ \frac{\partial U_2}{\partial X_1} & \frac{\partial U_2}{\partial X_2} & \frac{\partial U_2}{\partial X_3} \\ \frac{\partial U_3}{\partial X_1} & \frac{\partial U_3}{\partial X_2} & \frac{\partial U_3}{\partial X_3} \end{pmatrix} \\
&= -(\nabla_X U)^{\mathrm{T}} + \nabla_X U\,.
\end{aligned}$$

Moreover, in the context of the linearization, we wrote (actually, without any loss of accuracy) $F = I + \nabla_X U$. Hence,

$$\begin{aligned}
F &= I + \nabla_X U = I + \frac{1}{2}\left(\nabla_X U + (\nabla_X U)^{\mathrm{T}}\right) + \frac{1}{2}\left(\nabla_X U - (\nabla_X U)^{\mathrm{T}}\right) \\
&= I + \mathcal{E} + \mathcal{R}\,.
\end{aligned}$$

7.8 We have $S \simeq 2\Xi : \mathcal{E}$ due to Hooke's law. With the formula for an isotropic Ξ, we get

$$\begin{aligned}
S_{ij} &\simeq 2 \sum_{k,l=1}^{3} \left[\lambda \delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)\right] \mathcal{E}_{kl} \\
&= 2\lambda \delta_{ij} \sum_{k=1}^{3} \mathcal{E}_{kk} + 2\mu\left(\mathcal{E}_{ij} + \mathcal{E}_{ji}\right) \\
&= \lambda \delta_{ij} \sum_{k=1}^{3} \left(\frac{\partial U_k}{\partial X_k} + \frac{\partial U_k}{\partial X_k} \right) + \mu\left(\frac{\partial U_i}{\partial X_j} + \frac{\partial U_j}{\partial X_i} + \frac{\partial U_j}{\partial X_i} + \frac{\partial U_i}{\partial X_j} \right) \\
&= 2\lambda \delta_{ij} \operatorname{div}_X U + 2\mu \left(\frac{\partial U_i}{\partial X_j} + \frac{\partial U_j}{\partial X_i} \right)
\end{aligned}$$

such that

$$S \simeq 2\lambda \operatorname{div}_X U\, I + 4\mu \mathcal{E} \tag{7.8}$$

and

$$\mathrm{tr} S = \sum_{j=1}^{3} S_{jj} \simeq 6\lambda\, \mathrm{div}_X\, U + 2\mu \sum_{j=1}^{3} \left(\frac{\partial U_j}{\partial X_j} + \frac{\partial U_j}{\partial X_j} \right)$$
$$= (6\lambda + 4\mu)\, \mathrm{div}_X\, U\,.$$

Moreover, with (7.8), we also obtain

$$\mathcal{E} \simeq \frac{1}{4\mu} (S - 2\lambda\, \mathrm{div}_X\, U\, I) = \frac{1}{4\mu} \left(S - \frac{\lambda}{3\lambda + 2\mu}\, \mathrm{tr} S\, I \right)\,.$$

7.9 We have from Exercise 7.8 the identity

$$\mathcal{E} \simeq \frac{1}{4\mu} \left(S - \frac{\lambda}{3\lambda + 2\mu}\, \mathrm{tr} S\, I \right) = \frac{1}{4\mu} \left(S_{11} \varepsilon^1 \otimes \varepsilon^1 - \frac{\lambda}{3\lambda + 2\mu}\, S_{11}\, I \right)\,.$$

Hence, $\mathcal{E}_{ij} = 0$ whenever $i \neq j$. Moreover,

$$\mathcal{E}_{11} \simeq S_{11}\, \frac{1}{4\mu} \left(1 - \frac{\lambda}{3\lambda + 2\mu} \right) = S_{11}\, \frac{\lambda + \mu}{2\mu(3\lambda + 2\mu)}$$

and

$$\mathcal{E}_{22} = \mathcal{E}_{33} \simeq -S_{11}\, \frac{\lambda}{4\mu(3\lambda + 2\mu)}\,.$$

Therefore, Young's modulus is given by

$$\frac{S_{11}}{\mathcal{E}_{11}} \simeq \frac{2\mu}{\lambda + \mu}\, (3\lambda + 2\mu)$$

and Poisson's ratio is

$$-\frac{\mathcal{E}_{22}}{\mathcal{E}_{11}} = -\frac{\mathcal{E}_{33}}{\mathcal{E}_{11}} \simeq \frac{\lambda}{4\mu(3\lambda + 2\mu)} \frac{2\mu(3\lambda + 2\mu)}{\lambda + \mu} = \frac{1}{2} \frac{\lambda}{\lambda + \mu}\,.$$

7.10 With the assumption of isotropy, we get

$$\sigma = \Gamma : (\nabla_x u) = \lambda (\mathrm{div}_x\, u) I + \mu \left[\nabla_x u + (\nabla_x u)^{\mathrm{T}} \right]\,.$$

We have already seen that the *ansatz* $u(x, t) = \alpha \varphi(x \cdot k - ct)$ for a plane progressive wave leads us to

$$\frac{\partial u_i}{\partial x_j} = \alpha_i k_j \varphi'(x \cdot k - ct)$$

such that

$$\mathrm{div}_x\, u = \alpha \cdot k \varphi'(x \cdot k - ct)$$

and

$$\nabla_x u + (\nabla_x u)^{\mathrm{T}} = \varphi'(x \cdot k - ct)(\alpha \otimes k + k \otimes \alpha)\,.$$

Hence, we get the stress tensor

$$\sigma(x, t) = \varphi'(x \cdot k - ct) \left[\lambda\, \alpha \cdot k\, I + \mu\, (\alpha \otimes k + k \otimes \alpha) \right]\,.$$

For P-waves, where $\alpha \parallel k$, we see that this turns out to be

$$\sigma_{\mathrm{P}}(x, t) = \varphi'(x \cdot k - ct) \left(\lambda\, \alpha \cdot k\, I + 2\mu \alpha \otimes k \right)\,,$$

while we obtain for S-waves, which correspond to $\alpha \perp k$, the stress tensor

$$\sigma_S(x,t) = \varphi'(x \cdot k - ct)\mu\,(\alpha \otimes k + k \otimes \alpha)\ .$$

7.11 From Exercise 3.17, we have the following representation for the Laplace operator in 2D-polar coordinates:

$$\Delta U = \frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \varphi^2}\,.$$

Hence, the separation *ansatz* $U(x) = F(r)G(\varphi)$ for the Helmholtz equation $\Delta U + \omega^2 U = 0$ leads us to

$$F''(r)G(\varphi) + \frac{1}{r}\,F'(r)G(\varphi) + \frac{1}{r^2}\,F(r)G''(\varphi) + \omega^2 F(r)G(\varphi) = 0\,. \tag{7.9}$$

Dividing (7.9) by FG outside the zeros and multiplying with r^2, we arrive at

$$\left(F''(r) + \frac{1}{r}\,F'(r) + \omega^2 F(r)\right)\frac{r^2}{F(r)} = -\frac{G''(\varphi)}{G(\varphi)}\,. \tag{7.10}$$

Since we completely separated here the dependencies on the two polar coordinates, both sides of (7.10) need to be constant, that is there is a constant $C \in \mathbb{R}$ such that

$$r^2F''(r) + rF'(r) + \omega^2 r^2 F(r) = CF(r)\,, \tag{7.11}$$

$$G''(\varphi) = -CG(\varphi)\,. \tag{7.12}$$

Due to the nature of the polar coordinates, we need $G(0) = G(2\pi)$ and $G'(0) = G'(2\pi)$. Note that such identities for higher-order derivatives are then implied by (7.12). We distinguish three cases:

$C < 0$: Then the general solution of (7.12) is

$$G(\varphi) = C_1 \exp\left(-\sqrt{-C}\,\varphi\right) + C_2 \exp\left(\sqrt{-C}\,\varphi\right)\ .$$

However, the 2π-periodicity requirement can only be fulfilled by this solution if $C_1 = C_2$, which can easily be seen by solving the corresponding 2×2 system of linear equations.

$C = 0$: Then the general solution of (7.12) is

$$G(\varphi) = C_3 + C_4\varphi\,.$$

Obviously, the 2π-periodicity is given if and only if $C_4 = 0$.

$C > 0$: The general solution of (7.12) is now

$$G(\varphi) = C_5 \sin\left(\sqrt{C}\,\varphi\right) + C_6 \cos\left(\sqrt{C}\,\varphi\right)\ .$$

The 2π-periodicity requirement leads us to

$$C_6 = C_5 \sin\left(2\pi\sqrt{C}\right) + C_6 \cos\left(2\pi\sqrt{C}\right)\ ,$$

$$C_5\sqrt{C} = C_5\sqrt{C}\cos\left(2\pi\sqrt{C}\right) - C_6\sqrt{C}\sin\left(2\pi\sqrt{C}\right)\ .$$

This system of linear equations, namely

$$0 = C_5 \sin\left(2\pi\sqrt{C}\right) + C_6\left[\cos\left(2\pi\sqrt{C}\right) - 1\right]\ ,$$

$$0 = C_5\left[\cos\left(2\pi\sqrt{C}\right) - 1\right] - C_6 \sin\left(2\pi\sqrt{C}\right)\ ,$$

is non-trivially solvable if and only if it has a vanishing determinant, that is

$$-\sin^2\left(2\pi\sqrt{C}\right)-\left[\cos\left(2\pi\sqrt{C}\right)-1\right]^2=0\,. \tag{7.13}$$

Equation (7.13) is equivalent to

$$\cos\left(2\pi\sqrt{C}\right)=1\,.$$

As a consequence, we get non-trivial solutions if and only if $\sqrt{C}\in\mathbb{N}$. Therefore, the general solution of (7.12) with 2π-periodicity is representable as

$$G(\varphi)=\sum_{k=0}^{\infty}\left(A_k\sin(k\varphi)+B_k\cos(k\varphi)\right)\,,\quad \varphi\in[0,2\pi]\,.$$

Note that $k=0$ corresponds to the case $C=0$ and $A_0:=0$.

With the knowledge that $C=k^2$, $k\in\mathbb{N}_0$, we can write (7.11) as

$$r^2F''(r)+rF'(r)+\left(\omega^2r^2-k^2\right)F(r)=0\,.$$

We substitute now $\varrho:=\omega r$ and $\widetilde{F}(\varrho):=F(r)$ and obtain

$$\varrho^2\widetilde{F}''(\varrho)+\varrho\widetilde{F}'(\varrho)+\left(\varrho^2-k^2\right)\widetilde{F}(\varrho)=0\,. \tag{7.14}$$

Hence, we see that every multiple of the Bessel function J_k solves (7.14) and, since the solution needs to exist for $\varrho=0$, these are the only solutions. Consequently, all solutions of the Helmholtz equation on the 2D-disc are given by

$$U(x(r,\varphi))=\sum_{k=0}^{\infty}J_k(\omega r)\left(A_k\sin(k\varphi)+B_k\cos(k\varphi)\right)\,.$$

7.12 We have already seen that $\widehat{F}_{n,j}$ solves

$$\widehat{F}''_{n,j}(r)+\frac{2}{r}\,\widehat{F}'_{n,j}(r)+\left(\frac{\omega^2}{c^2}-\frac{n(n+1)}{r^2}\right)\widehat{F}_{n,j}(r)=0$$

(with a finite limit $r\to 0+$) if and only if $\widehat{F}_{n,j}(r)=C\widetilde{j}_n(\omega c^{-1}r)$ for an arbitrary constant C. Hence, we have

$$\frac{c^2}{\omega^2}\,\widehat{F}''_{n,j}(r)+\frac{c}{\omega}\,\frac{2}{r}\,\frac{c}{\omega}\,\widehat{F}'_{n,j}(r)+\left(1-\frac{n(n+1)}{r^2}\,\frac{c^2}{\omega^2}\right)\widehat{F}_{n,j}(r)=0\,,$$

which is equivalent to

$$\widetilde{j}''_n(x)+\frac{2}{x}\,\widetilde{j}'_n(x)+\left(1-\frac{n(n+1)}{x^2}\right)\widetilde{j}_n(x)=0\,. \tag{7.15}$$

Let now $G(x):=(\sin x)/x$. Then we have

$$\begin{aligned}G'(x)&=\frac{\cos x}{x}-\frac{\sin x}{x^2}\,,\\ G''(x)&=-\frac{\sin x}{x}-2\,\frac{\cos x}{x^2}+2\,\frac{\sin x}{x^3}\,.\end{aligned}$$

Hence, we obtain

$$G''(x)+\frac{2}{x}\,G'(x)=-\frac{\sin x}{x}-2\,\frac{\cos x}{x^2}+2\,\frac{\sin x}{x^3}+2\,\frac{\cos x}{x^2}-2\,\frac{\sin x}{x^3}=-\frac{\sin x}{x}$$

$$= -\left(1 - \frac{0(0+1)}{x^2}\right) G(x)\,.$$

Hence, G solves (7.15) for $n = 0$.

Now let $H(x) := x^{-1}(x^{-1}\sin x - \cos x)$. Then we obtain

$$H'(x) = -\frac{1}{x^2}\,(G(x) - \cos x) + \frac{1}{x}\,\bigl(G'(x) + \sin x\bigr)$$

and

$$H''(x) = 2\,\frac{1}{x^3}\,(G(x) - \cos x) - 2\,\frac{1}{x^2}\,\bigl(G'(x) + \sin x\bigr) + \frac{1}{x}\,\bigl(G''(x) + \cos x\bigr)$$

such that

$$\begin{aligned} H''(x) + \frac{2}{x}\,H'(x) &= \frac{1}{x}\,\bigl(G''(x) + \cos x\bigr) \\ &= \frac{1}{x^2}\left(-\sin x - 2\,\frac{\cos x}{x} + 2\,\frac{\sin x}{x^2}\right) + \frac{1}{x}\,\cos x \\ &= -\left[\frac{\sin x}{x^2} - \frac{\cos x}{x} - \frac{2}{x^2}\left(\frac{\sin x}{x^2} - \frac{\cos x}{x}\right)\right] \\ &= -\left(1 - \frac{1(1+1)}{x^2}\right) H(x)\,. \end{aligned}$$

7.13 For $n = 1$, the toroidal frequency equation $\chi\tilde{j}_{n+1}(\chi) = (n-1)\tilde{j}_n(\chi)$ becomes the equation

$$\chi\tilde{j}_2(\chi) = 0\,. \tag{7.16}$$

With the formula for $\tilde{j}_2$, this yields

$$\left(\frac{3}{\chi^2} - 1\right)\sin\chi - \frac{3}{\chi}\,\cos\chi = 0\,. \tag{7.17}$$

If $\cos\chi \neq 0$, then (7.17) is equivalent to

$$\tan\chi = \frac{\frac{3}{\chi}}{\frac{3}{\chi^2} - 1}\,,$$

which can also be written as

$$\frac{\tan\chi}{\chi} = \frac{3}{3 - \chi^2}\,, \tag{7.18}$$

because $\chi \neq 0$ is clear.

In the remaining case of a positive root $\chi = (2k+1)\pi/2$, $k \in \mathbb{N}_0$, of the cosine function, the spherical Bessel function would take the value

$$\begin{aligned} \tilde{j}_2\left(\frac{2k+1}{2}\,\pi\right) &= \left(\frac{3\cdot 4}{(2k+1)^2\pi^2} - 1\right)\frac{(-1)^k}{\frac{2k+1}{2}\,\pi} - 0 \\ &= \underbrace{\left(\underbrace{\frac{12}{(2k+1)^2\pi^2}}_{\notin\mathbb{Q}} - 1\right)}_{\neq 0}\underbrace{(-1)^k\,\frac{2}{(2k+1)\pi}}_{\neq 0} \\ &\neq 0\,. \end{aligned}$$

Hence, roots of the cosine function do not solve the frequency equation (7.16) and, therefore, all solutions must fulfil (7.18).

For solving Equation (7.18) numerically, we write the equation equivalently as

$$\tan \chi = \frac{3\chi}{3 - \chi^2} \, .$$

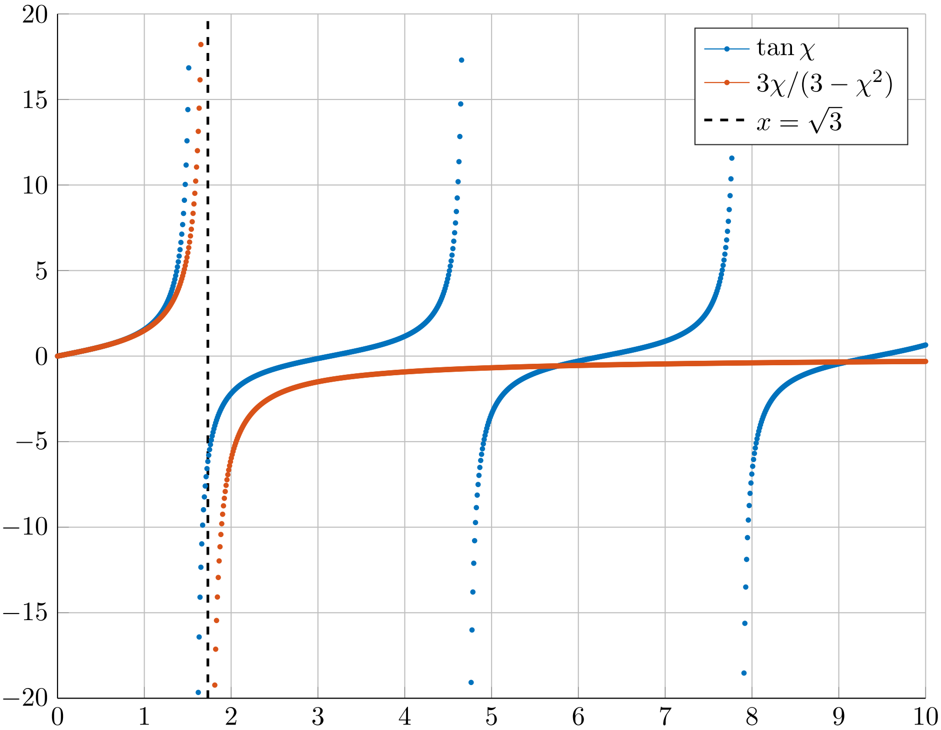

Figure 7.1 The crossing points of the coloured graphs are solutions of the frequency equation for $n = 2$. Note the singularity of the rational term at $\sqrt{3}$, which makes clear that there is no missed crossing point in this area.

Figure 7.1 shows that the smallest positive solution can be expected at approximately $\chi \approx 6$. We, therefore, start a Newton iteration for

$$f(\chi) := \tan \chi - \frac{3\chi}{3 - \chi^2} \quad \Rightarrow \quad f'(\chi) = \frac{1}{\cos^2 \chi} - \frac{9 + 3\chi^2}{(3 - \chi^2)^2} \, .$$

With $\chi_0 := 6$, the iteration

$$\chi_{k+1} := \chi_k - \frac{f(\chi_k)}{f'(\chi_k)}$$

converges to $\chi = 5.76345919689455\ldots$

7.14 We already know that

$$\frac{\mathrm{d}}{\mathrm{d}s} (S\mathfrak{p}) (X(s)) = \nabla_x S(X(s)) \, .$$

In the case of a constant velocity field, the right-hand side must vanish. Hence, with the definition of the unit slowness vector $\mathfrak{p}$, we get

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\nabla_x T(X(s))\right) = 0\,.$$

In other words, $\nabla_x T$ is constant along each ray. However, by definition of the rays,

$$\frac{\mathrm{d}}{\mathrm{d}s}X(s) = \frac{\nabla_x T(X(s))}{|\nabla_x T(X(s))|}$$

is the unit tangential vector along the ray $s \mapsto X(s)$. Therefore, this (arbitrarily chosen) ray must be a straight line.

8

Final comment

Note that the references of the book are also valid as a bibliography regarding exercises. In particular, some of the problems solved in these exercises here can certainly also be found together with derivations in other literature such as those publications which are listed in the references of the book.