The Appendices

to

Brownian Motion, the Fredholm Determinant, and Time Series Analysis

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The Appendices are composed of Appendix A and Appendix B, and serve as a companion to my book

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Appendix A gives a complete set of solutions to exercises posed at the end of most sections of each chapter in the book. Most of the exercises are concerned with corroborating the results described in the main text, so that the reader can gain a better understanding of the details of the discussions.

Appendix B presents graphs of some probability densities of statistics associated with Brownian motion and fractional Brownian motion dealt with in the main text. These are computed by numerically inverting the characteristic functions derived in the text.

Contents

Appendix A

Solutions to Exercises

Here we intend to present solutions to all exercises embedded in the main text of each chapter. Most of the exercises are concerned with corroborating the results described in the text. It is expected that solving these exercises will give the reader the opportunity to find better solutions than the ones given here.

Chapter 1

1.1.1 Since it holds that

$$\lim_{h\to 0} \mathrm{E}\left[\left(\frac{W(t+h)-W(t)}{h}\right)^2\right] = \lim_{h\to 0}\frac{|h|}{h^2} = \lim_{h\to 0}\frac{1}{|h|} = \infty,$$

$\{W(t)\}$ is not m.s. differentiable.

1.1.2 Since $\mathrm{Var}(X(t)) = \mathrm{E}(X^2(t)) = t$, it holds that $\mathrm{E}(X^2(0)) = 0$, which implies $P(X(0) = 0) = 1$. Increments of $X(t)$ are linear combinations of normal random variables so that they are normal with means 0 and

$$\mathrm{Var}(X(s) - X(t)) = s - 2\min(s,t) + t = |s-t|.$$

Thus $X(t) - X(s) \sim \mathrm{N}(0, |s-t|)$. For $0 \le t_1 < t_2 < t_3 < t_4 \le 1$, we have

$$\mathrm{Cov}\,(X(t_2) - X(t_1),\, X(t_4) - X(t_3)) = t_2 - t_2 - t_1 + t_1 = 0,$$

which implies that increments are independent. Thus $\{X(t)\}$ is Bm since it satisfies all the conditions of the definition of Bm.

1.1.3 Put $\Delta W_k = W(t_k) - W(t_{k-1})$ and $\Delta t_k = t_k - t_{k-1}$. Then we have

$$\sum_{k=1}^{n} \mathrm{E}\left[(\Delta W_k)^2\right] = \sum_{k=1}^{n} \Delta t_k = b - a \le \max_{1\le k\le n} |\Delta W_k| \sum_{k=1}^{n} \mathrm{E}(|\Delta W_k|).$$

The left side is a positive constant, but the first factor on the right side converges to 0. Thus the second factor on the right must diverge, which implies that $W(t)$ is of unbounded variation. We next have

$$\begin{aligned}
\mathrm{E}\left[\left\{\sum_{k=1}^{n}(\Delta W_k)^2 - (b-a)\right\}^2\right] &= \mathrm{E}\left[\left\{\sum_{k=1}^{n}\left((\Delta W_k)^2 - \Delta t_k\right)\right\}^2\right] \\
&= \sum_{k=1}^{n} \mathrm{Var}\left[(\Delta W_k)^2\right] \\
&= 2\sum_{k=1}^{n}(\Delta t_k)^2 \le 2 \max_{1\le k\le n} \Delta t_k (b-a) \ \to \ 0.
\end{aligned}$$

Thus the quadratic variation is finite and bounded away from 0.

1.1.4 It is clear that $\{X(t)\}$ is a Gaussian process with $X(0) = 0$ and $\mathrm{E}(X(t)) = 0$. Using the formula presented in the problem, we also have

$$\begin{aligned}
\mathrm{Cov}(X(s), X(t)) &= \sum_{n=1}^{\infty} \frac{2\sin\left(n-\frac{1}{2}\right)\pi s \ \sin\left(n-\frac{1}{2}\right)\pi t}{\left(n-\frac{1}{2}\right)^2 \pi^2} \\
&= \sum_{n=1}^{\infty} \frac{\cos\left(n-\frac{1}{2}\right)\pi s - \cos\left(n-\frac{1}{2}\right)\pi t}{\left(n-\frac{1}{2}\right)^2 \pi^2} \\
&= \frac{1}{2}\left(1-|s-t|\right) - \frac{1}{2}\left(1-(s+t)\right) = \min(s,t).
\end{aligned}$$

It follows from Exercise 1.1.2 that $\{X(t)\}$ is Bm. Note here that we used the fact that "l.i.m." and "expectation" commute [see (1.6)].

1.1.5 For $s \le t$ we have

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} W(s) \\ W(t) \\ W(1) \end{pmatrix} \sim \mathrm{N}(\mathbf{0}, \Sigma), \qquad \Sigma = \begin{pmatrix} s & s & s \\ s & t & t \\ s & t & 1 \end{pmatrix} = \begin{pmatrix} \Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{yx} & \Sigma_{yy} \end{pmatrix},$$

where $X = (W(s), W(t))'$ and $Y = W(1)$. Then it follows from the property of the multivariate normal distribution that

$$\begin{aligned}
\mathrm{E}(X|Y=0) &= \mathrm{E}(X) + \Sigma_{xy}\Sigma_{yy}^{-1}(0-\mathrm{E}(Y)) = 0, \\
\mathrm{Var}(X|Y=0) &= \Sigma_{xx} - \Sigma_{xy}\Sigma_{yy}^{-1}\Sigma_{yx} = \begin{pmatrix} s-s^2 & s-st \\ s-st & t-t^2 \end{pmatrix}.
\end{aligned}$$

Thus $\mathrm{Cov}(W(s), W(t)|W(1)=0) = \min(s,t) - st$ so that $\{W(t)|W(1) = 0\}$ is the Bb. It is clear that $\tilde{W}(t) = W(t) - tW(1)$ is a Gaussian process with $\mathrm{Cov}(\tilde{W}(s), \tilde{W}(t)) = \min(s,t) - st$. Thus $\tilde{W}(t)$ is the Bb. Since $\mathrm{Cov}(\tilde{W}(t), W(1)) = 0$, $\tilde{W}(t)$ is independent of $W(1)$.

1.1.6 It is clear that $\{X(t)\}$ is a zero-mean Gaussian process with $X(0) = X(1) = 0$ and $\mathrm{E}(X(t)) = 0$. Using the formula presented in the problem, we have

$$\begin{aligned}\mathrm{E}\,(X(s)X(t)) &= \sum_{n=1}^{\infty} \frac{2\sin n\pi s\ \sin n\pi t}{n^2\,\pi^2} \\ &= \sum_{n=1}^{\infty} \frac{1}{n^2\pi^2}\,(\cos n\pi(s-t) - \cos n\pi(s+t)) \\ &= \frac{1}{4}(|s-t|-1)^2 - \frac{1}{4}(|s+t|-1)^2 \\ &= \min(s,t) - st,\end{aligned}$$

which proves that $\{X(t)\}$ is the Bb. Note here that we used the fact that "l.i.m." and "expectation" commute [see (1.6)].

1.2.1 We have

$$\begin{aligned}&\sqrt{\mathrm{E}\,\{(aX_n + bY_n - (aX+bY))^2\}} \\ =\ &\sqrt{\mathrm{E}\,\{(a(X_n - X) + b(Y_n - Y))^2\}} \\ \le\ &\sqrt{\mathrm{E}\,\{(a(X_n - X))^2\}} + \sqrt{\mathrm{E}\,\{(b(Y_n - Y))^2\}} \\ =\ &|a|\sqrt{\mathrm{E}\,\{(X_n - X)^2\}} + |b|\sqrt{\mathrm{E}\,\{(Y_n - Y)^2\}} \\ \to\ &0,\end{aligned}$$

which implies that $\underset{n\to\infty}{\text{l.i.m.}}\,(aX_n + bY_n) = aX + bY$. We also have

$$\begin{aligned}&|\mathrm{E}(X_mY_n) - \mathrm{E}(XY)| \\ =\ &|\mathrm{E}\,\{X(Y_n - Y)\} + \mathrm{E}\,\{(X_m - X)Y\} + \mathrm{E}\,\{(X_m - X)(Y_n - Y)\}\,| \\ \le\ &\mathrm{E}\,\{|X(Y_n - Y)|\} + \mathrm{E}\,\{|(X_m - X)Y|\} + \mathrm{E}\,\{|(X_m - X)(Y_n - Y)|\} \\ \le\ &\sqrt{\mathrm{E}(X^2)\mathrm{E}\,\{(Y_n - Y)^2\}} + \sqrt{\mathrm{E}\,\{(X_m - X)^2\}\,\mathrm{E}(Y^2)} \\ &+\sqrt{\mathrm{E}\,\{(X_m - X)^2\}\,\mathrm{E}\,\{(Y_n - Y)^2\}}\ \to\ 0,\end{aligned}$$

which implies that $\lim_{m,n\to\infty} \mathrm{E}(X_mY_n) = \mathrm{E}(XY)$.

1.2.2 The m.s. integral given by

$$Y = \int_0^1 X(t)\,dt$$

is well defined if $Y_n = \sum_{i=1}^{n} X(t_i')(t_i - t_{i-1})$ converges in the m.s. sense, which is ensured if

$$\lim_{m,n\to\infty} \mathrm{E}(Y_m Y_n) = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} \mathrm{E}[X(s_i')X(t_j'))](s_i - s_{i-1})(t_j - t_{j-1}) \tag{A.1}$$

exists for any sequence of subdivisions $p_m : 0 = s_0 < s_1 < \cdots < s_m = 1$, $q_n : 0 = t_0 < t_1 < \cdots < t_n = 1$ and for any $s_i' \in [s_{i-1}, s_i)$, $t_j' \in [t_{j-1}, t_j)$. If $X(t)$ is m.s. continuous, it follows from Exercise 1.2.1 that $\mathrm{E}(X(s)X(t))$ is continuous so that the right side of (A.1) exists.

1.2.3 Let us consider

$$A_n = \sum_{i=1}^{n} X(t_i')(t_i - t_{i-1}), \quad A = \int_0^1 X(t)\,dt,$$

where it holds that $\underset{n\to\infty}{\mathrm{l.i.m.}}\, A_n = A$. It follows from the m.s. property that

$$\mathrm{E}(A) = \mathrm{E}(\underset{n\to\infty}{\mathrm{l.i.m.}}\, A_n) = \lim_{n\to\infty} \sum_{i=1}^{n} \mathrm{E}(X(t_i'))(t_i - t_{i-1}) = \int_0^1 \mathrm{E}(X(t))\,dt.$$

Noting that $\mathrm{E}(A^2) = \lim_{n\to\infty} \mathrm{E}(A_n^2)$, we obtain

$$\begin{aligned}
\mathrm{E}(A^2) &= \mathrm{E}\left(\int_0^1 \int_0^1 X(s)X(t)\,ds\,dt\right) = \lim_{n\to\infty} \mathrm{E}(A_n^2) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{n} \mathrm{E}(X(t_i')X(t_j'))(t_i - t_{i-1})(t_j - t_{j-1}) \\
&= \int_0^1 \int_0^1 \mathrm{E}(X(s)X(t))\,ds\,dt.
\end{aligned}$$

1.2.4 Since $\{X_n(t)\}$ is a Gaussian process, any finite-dimensional distributions of $X_n(t_i)$ $(i = 1, \ldots, k)$ for each finite k and each collection $t_1 < \cdots < t_k$ are normal for all n. Then, putting $\boldsymbol{Y}_n = \big(X_n(t_1), \ldots, X_n(t_k)\big)'$, it holds that

$$\phi_n(\boldsymbol{\theta}) = \mathrm{E}\left\{\exp(i\boldsymbol{\theta}'\boldsymbol{Y}_n)\right\} = \exp\left\{i\boldsymbol{\theta}'\mathrm{E}(\boldsymbol{Y}_n) - \boldsymbol{\theta}'\mathrm{Var}(\boldsymbol{Y}_n)\boldsymbol{\theta}/2\right\}.$$

Since $\underset{n\to\infty}{\mathrm{l.i.m.}}\, \boldsymbol{Y}_n = \boldsymbol{Y}$, where $\boldsymbol{Y} = \big(X(t_1), \ldots, X(t_k)\big)'$, we have, from Exercise 1.2.1, $\mathrm{E}(\boldsymbol{Y}_n) \to \mathrm{E}(\boldsymbol{Y})$ and $\mathrm{Var}(\boldsymbol{Y}_n) \to \mathrm{Var}(\boldsymbol{Y})$. Thus it follows that

$$\begin{aligned}
\lim_{n\to\infty} \phi_n(\boldsymbol{\theta}) &= \lim_{n\to\infty} \exp\left\{i\boldsymbol{\theta}'\mathrm{E}(\boldsymbol{Y}_n) - \boldsymbol{\theta}'\mathrm{Var}(\boldsymbol{Y}_n)\boldsymbol{\theta}/2\right\} \\
&= \exp\left\{i\boldsymbol{\theta}'\mathrm{E}(\boldsymbol{Y}) - \boldsymbol{\theta}'\mathrm{Var}(\boldsymbol{Y})\boldsymbol{\theta}/2\right\},
\end{aligned}$$

which implies that $\boldsymbol{Y} \sim \mathrm{N}\Big(\mathrm{E}(\boldsymbol{Y}),\, \mathrm{Var}(\boldsymbol{Y})\Big)$.

1.2.5 It is evident that $\mathrm{E}(I_1) = 0$, whereas we obtain

$$\begin{aligned}\mathrm{E}(I_1^2) &= \mathrm{E}\left(\int_0^1 \int_0^1 W(s)W(t)\,ds\,dt\right) = \int_0^1 \int_0^1 \min(s,t)\,ds\,dt \\ &= \int_0^1 \left(\int_0^t s\,ds + t\int_t^1 ds\right) dt = \frac{1}{3}.\end{aligned}$$

It follows from $W(t) \sim \mathrm{N}(0,t)$ that

$$\begin{aligned}\mathrm{E}(|W(t)|) &= \int_{-\infty}^{\infty} |x| \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}\,dx = \frac{2\sqrt{t}}{\sqrt{2\pi}} \int_0^{\infty} z e^{-z^2/2}\,dz \\ &= \sqrt{\frac{2t}{\pi}} \left[-e^{-z^2/2}\right]_0^{\infty} = \sqrt{\frac{2t}{\pi}}.\end{aligned}$$

Then we have

$$\mathrm{E}(I_2) = \int_0^1 \sqrt{\frac{2t}{\pi}}\,dt = \frac{2}{3}\sqrt{\frac{2}{\pi}}.$$

To compute $\mathrm{E}(|W(s)W(t)|)$, note that $(W(s), W(t))' \sim \mathrm{N}(\mathbf{0}, \Sigma)$, where

$$\Sigma = \begin{pmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2 \end{pmatrix}, \quad \sigma_x^2 = s, \quad \sigma_y^2 = t, \quad \rho = \frac{\min(s,t)}{\sqrt{st}}.$$

Denoting the joint density of $W(s)$ and $W(t)$ as

$$f(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_x^2} - 2\rho\frac{xy}{\sigma_x\sigma_y} + \frac{y^2}{\sigma_y^2}\right)\right\},$$

and putting $x = \sigma_x r\cos\theta$ and $y = \sigma_y r\sin\theta$, $\mathrm{E}(|W(s)W(t)|)$ is equal to

$$\begin{aligned}&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |xy| f(x,y)\,dx\,dy \\ &= \frac{\sigma_x\sigma_y}{2\pi\sqrt{1-\rho^2}} \int_0^{\infty}\int_0^{2\pi} r^3 |\cos\theta\sin\theta| \\ &\qquad\qquad \times \exp\left\{-\frac{r^2}{2(1-\rho^2)}(1 - 2\rho\cos\theta\sin\theta)\right\} dr\,d\theta \\ &= \frac{\sigma_x\sigma_y(1-\rho^2)^{3/2}}{\pi} \int_0^{2\pi} \frac{|\cos\theta\sin\theta|}{(1-2\rho\cos\theta\sin\theta)^2}\,d\theta \int_0^{\infty} u e^{-u}\,du \\ &= \frac{2\sigma_x\sigma_y(1-\rho^2)^{3/2}}{\pi} \int_0^{\pi/2} \left(\frac{\cos\theta\sin\theta}{(1-2\rho\cos\theta\sin\theta)^2} + \frac{\cos\theta\sin\theta}{(1+2\rho\cos\theta\sin\theta)^2}\right) d\theta \\ &= \frac{\sigma_x\sigma_y(1-\rho^2)^{3/2}}{\pi} \frac{d}{d\rho} \int_0^{\pi/2} \left(\frac{1}{1-2\rho\cos\theta\sin\theta} - \frac{1}{1+2\rho\cos\theta\sin\theta}\right) d\theta.\end{aligned}$$

Putting $u = \tan\theta$, it follows that

$$\begin{aligned}
&\mathrm{E}(|W(s)W(t)|) \\
&= \frac{\sigma_x\sigma_y(1-\rho^2)^{3/2}}{\pi}\frac{d}{d\rho}\int_0^\infty \left(\frac{1}{1-2\rho u+u^2} - \frac{1}{1+2\rho u+u^2}\right) du \\
&= \frac{\sigma_x\sigma_y(1-\rho^2)^{3/2}}{\pi}\frac{d}{d\rho}\left[\frac{2}{\sqrt{1-\rho^2}}\tan^{-1}\frac{\rho}{\sqrt{1-\rho^2}}\right] \\
&= \frac{\sigma_x\sigma_y(1-\rho^2)^{3/2}}{\pi}\frac{d}{d\rho}\left[\frac{2}{\sqrt{1-\rho^2}}\sin^{-1}\rho\right] \\
&= \frac{2\sigma_x\sigma_y}{\pi}\left(\sqrt{1-\rho^2}+\rho\,\sin^{-1}\rho\right) \\
&= \frac{2}{\pi}\left(\sqrt{st-(\min(s,t))^2}+\min(s,t)\sin^{-1}\frac{\min(s,t)}{\sqrt{st}}\right).
\end{aligned}$$

Then we have

$$\begin{aligned}
\mathrm{E}(I_2^2) &= \int_0^1\int_0^1 \mathrm{E}(|W(s)W(t)|)\,ds\,dt \\
&= \frac{4}{\pi}\int_0^1\left[\int_0^t\left\{\sqrt{s(t-s)}+s\,\sin^{-1}\sqrt{s/t}\right\}ds\right]dt \\
&= \frac{2}{\pi}\int_0^1\left[\frac{\pi t^2}{8}+\frac{5\pi t^2}{32}+\frac{2\sqrt{t}\,(1-t)^{3/2}}{3}\right. \\
&\qquad \left.+t\,\sin^{-1}\sqrt{t}+\sqrt{1-t}\,t^{3/2}-\frac{\pi t^2}{2}\right]dt = \frac{3}{8}.
\end{aligned}$$

We also have

$$\begin{aligned}
\mathrm{E}(I_3) &= \int_0^1 \mathrm{E}\left(e^{W(t)}\right)dt = \int_0^1 e^{t/2}\,dt = 2(\sqrt{e}-1), \\
\mathrm{E}(I_3^2) &= \int_0^1\left[\int_0^t \mathrm{E}\left(e^{W(s)+W(t)}\right)ds + \int_t^1 \mathrm{E}\left(e^{W(s)+W(t)}\right)ds\right]dt \\
&= 2\int_0^1\left[\int_0^t \exp\left(\frac{3s}{2}+\frac{t}{2}\right)ds\right]dt \\
&= \frac{2e^2}{3}-\frac{8\sqrt{e}}{3}+2.
\end{aligned}$$

1.2.6 Putting $\tilde{W}(t) = W(t) - tW(1)$ and noting that

$$\mathrm{Cov}(W(s), W(t)) = \min(s,t), \quad \mathrm{Cov}(\tilde{W}(s), \tilde{W}(t)) = \min(s,t) - st,$$
$$\mathrm{Cov}(\bar{W}(s), \bar{W}(t)) = \frac{1}{3} - \max(s,t) + \frac{s^2 + t^2}{2}, \quad \bar{W}(t) = W(t) - \int_0^1 W(u)\, du,$$

we have $\mathrm{E}(I_4) = 1/2$ and $\mathrm{E}(I_5) = \mathrm{E}(I_6) = 1/6$. When $(X, Y)' \sim \mathrm{N}(\mathbf{0}, \Sigma)$, where

$$\Sigma = \begin{pmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_y^2 \end{pmatrix},$$

it holds that $\mathrm{E}(X^2Y^2) = \sigma_x^2\sigma_y^2 + 2\sigma_{xy}^2$. Thus

$$\begin{aligned}
\mathrm{E}(I_4^2) &= \int_0^1 \int_0^1 \left[st + 2(\min(s,t))^2 \right] ds\, dt = \frac{7}{12}, \\
\mathrm{E}(I_5^2) &= \int_0^1 \int_0^1 \left[st(1-s)(1-t) + 2(\min(s,t) - st)^2 \right] ds\, dt = \frac{1}{20}, \\
\mathrm{E}(I_6^2) &= \int_0^1 \int_0^1 \left[\left(\frac{1}{3} - s + s^2 \right) \left(\frac{1}{3} - t + t^2 \right) \right. \\
&\qquad \left. + 2 \left(\frac{1}{3} - \max(s,t) + \frac{s^2 + t^2}{2} \right)^2 \right] ds\, dt = \frac{1}{20}.
\end{aligned}$$

1.2.7 Since

$$A = \int_0^1 g(t)\, dW(t) = \underset{n \to \infty}{\mathrm{l.i.m.}} \sum_{i=1}^n g(t_i')(W(t_i) - W(t_{i-1})),$$

and the m.s. limit retains normality, A is normal. Moreover we have

$$\begin{aligned}
\mathrm{E}(A) &= \lim_{n \to \infty} \sum_{i=1}^n g(t_i')\mathrm{E}(W(t_i) - W(t_{i-1})) = 0, \\
\mathrm{Var}(A) &= \lim_{n \to \infty} \sum_{i=1}^n g^2(t_i')\mathrm{Var}(W(t_i) - W(t_{i-1})) \\
&= \lim_{n \to \infty} \sum_{i=1}^n g^2(t_i')(t_i - t_{i-1}) = \int_0^1 g^2(t)\, dt.
\end{aligned}$$

1.3.1 Let us consider

$$X_{m,n} = \sum_{i=1}^m \sum_{j=1}^n K(s_i', t_j')\Delta W(s_i)\Delta W(t_j),$$

where $0 = s_0 < s_1 < \cdots < s_m = 1$, $0 = t_0 < t_1 < \cdots < t_n = 1$, $s'_i \in [s_{i-1}, s_i)$, $t'_j \in [t_{j-1}, t_j)$, and $\Delta W(s_i) = W(s_i) - W(s_{i-1})$. Then

$$\underset{\Delta_{m,n}\to 0}{\text{l.i.m.}}\, X_{m,n} = X = \int_0^1 \int_0^1 K(s,t)\, dW(s)\, dW(t),$$

where $\Delta_{m,n} = \max(s_1 - s_0, \ldots, s_m - s_{m-1}, t_1 - t_0, \ldots, t_n - t_{n-1})$. The m.s. limit X is independent of the partition as well as the choice of s'_i and t'_j. Putting $m = n$, $s_i = t_i$ and $\Delta s_i = s_i - s_{i-1}$, we now have

$$\mathrm{E}(X) = \lim_{n\to\infty} \mathrm{E}(X_{n,n}) = \lim_{n\to\infty} \sum_{i=1}^{n} K(s'_i, s'_i)\Delta s_i = \int_0^1 K(s,s)\, ds,$$

$$\mathrm{E}(X^2) = \lim_{n\to\infty} \mathrm{E}(X_{n,n}^2) = \lim_{n\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} K(s'_i, s'_j)K(s'_k, s'_l)A_{ijkl},$$

where

$$\begin{aligned} A_{ijkl} &= \mathrm{E}\left(\Delta W(s_i)\Delta W(s_j)\Delta W(s_k)\Delta W(s_l)\right) \\ &= \begin{cases} 3(\Delta s_i)^2 & (i = j = k = l) \\ \Delta s_i \Delta s_k & (i = j, k = l, i \neq k) \\ \Delta s_i \Delta s_j & (i = k, j = l, i \neq j) \\ \Delta s_i \Delta s_j & (i = l, j = k, i \neq j). \end{cases} \end{aligned}$$

Therefore we have

$$\begin{aligned} \mathrm{E}\left(X_{n,n}^2\right) &= 3\sum_{i=1}^{n} K^2(s'_i, s'_i)(\Delta s_i)^2 + \sum_{i\neq j} K(s'_i, s'_i)K(s'_j, s'_j)\Delta s_i \Delta s_j \\ &\quad +2\sum_{i\neq j} K^2(s'_i, s'_j)\Delta s_i \Delta s_j \\ &= 2\sum_{i=1}^{n}\sum_{j=1}^{n} K^2(s'_i, s'_j)\Delta s_i \Delta s_j + \left(\sum_{i=1}^{n} K(s'_i, s'_i)\Delta s_i\right)^2 \\ &\to \mathrm{E}(X^2) = 2\int_0^1 \int_0^1 K^2(s,t)\, ds\, dt + \left(\int_0^1 K(s,s)\, ds\right)^2, \end{aligned}$$

which implies that

$$\mathrm{Var}(X) = 2\int_0^1 \int_0^1 K^2(s,t)\, ds\, dt = 4\int_0^1 \int_0^t K^2(s,t)\, ds\, dt,$$

where this last equality holds because $K(s,t)$ is symmetric. Another way for the solution will be given in Theorem 2.4 of Chapter 2.

1.3.2 For $K_2(s,t) = g_1(s)g_1(t) + g_2(s)g_2(t)$ with $g_1(t) = 1$ and $g_2(t) = t$, we have

$$\begin{aligned} \mathrm{E}(X_2) &= \int_0^1 K_2(t,t)\,dt = \int_0^1 (1+t^2)\,dt = \frac{4}{3}, \\ \mathrm{Var}(X_2) &= 2\int_0^1\int_0^1 K_2^2(s,t)\,ds\,dt = 2\int_0^1\int_0^1 (1+st)^2\,ds\,dt = \frac{29}{9}. \end{aligned}$$

Since

$$\begin{pmatrix} \displaystyle\int_0^1 g_1(t)\,dW(t) \\ \displaystyle\int_0^1 g_2(t)\,dW(t) \end{pmatrix} \sim \mathrm{N}(\mathbf{0},\Sigma_2), \quad \Sigma_2 = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{pmatrix},$$

we obtain

$$\mathrm{E}\left(e^{i\theta X_2}\right) = \left|I_2 - 2i\theta\Sigma_2\right|^{-1/2} = \left(1 - \frac{8}{3}\,i\theta - \frac{1}{3}\,\theta^2\right)^{-1/2}.$$

For $K_3(s,t) = g_1(s)g_1(t) + g_2(s)g_2(t) + g_3(s)g_3(t)$ with $g_1(t) = 1$, $g_2(t) = t$ and $g_3(t) = t^2$, we have

$$\begin{aligned} \mathrm{E}(X_3) &= \int_0^1 K_3(t,t)\,dt = \int_0^1 (1+t^2+t^4)\,dt = \frac{23}{15}, \\ \mathrm{Var}(X_3) &= 2\int_0^1\int_0^1 (1+st+s^2t^2)^2\,ds\,dt = \frac{1199}{300}. \end{aligned}$$

Since

$$\begin{pmatrix} \displaystyle\int_0^1 g_1(t)\,dW(t) \\ \displaystyle\int_0^1 g_2(t)\,dW(t) \\ \displaystyle\int_0^1 g_3(t)\,dW(t) \end{pmatrix} \sim \mathrm{N}(\mathbf{0},\Sigma_3), \quad \Sigma_3 = \begin{pmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{pmatrix},$$

we obtain

$$\mathrm{E}\left(e^{i\theta X_3}\right) = \left|I_3 - 2i\theta\Sigma_3\right|^{-1/2} = \left(1 - \frac{46}{15}\,i\theta - \frac{127}{180}\,\theta^2 + \frac{1}{270}\,i\theta^3\right)^{-1/2}.$$

1.3.3 Noting that $W(t) = \int_0^t dW(u)$, we have

$$\begin{aligned}\int_0^1 g(t)W^2(t)\,dt &= \int_0^1 g(t)\left(\int_0^t \int_0^t dW(u)\,dW(v)\right)\,dt \\ &= \int_0^1 \int_0^1 \left(\int_{\max(u,v)}^1 g(t)\,dt\right) dW(u)\,dW(v).\end{aligned}$$

1.3.4 It holds that

$$\begin{aligned}&\int_0^1 \int_0^1 \min(s,t)\,dW(s)\,dW(t) \\ =\ &\int_0^1 \left(\int_0^t s\,dW(s) + t\int_t^1 dW(s)\right) dW(t) \\ =\ &\int_0^1 \left(tW(1) - \int_0^t W(s)\,ds\right) dW(t) \\ =\ &W(1)\left(W(1) - 2\int_0^1 W(t)\,dt\right) + \int_0^1 W^2(t)\,dt \\ =\ &\int_0^1 (W(1) - W(t))^2\,dt.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&\int_0^1 (W(1) - W(t))^2\,dt \\ =\ &W^2(1) - 2W(1)\int_0^1 W(t)\,dt + \int_0^1 W^2(t)\,dt \\ =\ &\int_0^1 \int_0^1 [1 - (1 - s + 1 - t) + 1 - \max(s,t)]\,dW(s)\,dW(t) \\ =\ &\int_0^1 \int_0^1 \min(s,t)\,dW(s)\,dW(t).\end{aligned}$$

1.3.5 Since

$$K_7(s,t) = \sum_{n=1}^{\infty} \frac{f_n(s)f_n(t)}{\lambda_n} = \min(s,t) - st,$$

where $\lambda_n = n^2\pi^2$ and $f_n(t) = \sqrt{2}\sin n\pi t$, it holds that

$$\mathrm{E}(X_7) = \int_0^1 K_7(t,t)\,dt = \int_0^1 t(1-t)\,dt = \frac{1}{6},$$
$$\mathrm{Var}(X_7) = 2\int_0^1\int_0^1 (\min(s,t)-st))^2\,ds\,dt = \frac{1}{45}.$$

We also have

$$\mathrm{E}\left(e^{i\theta X_7}\right) = \mathrm{E}\left[\exp\left\{i\theta\sum_{n=1}^{\infty}\frac{1}{\lambda_n}Z_n^2\right\}\right] = \prod_{n=1}^{\infty}\left(1-\frac{2i\theta}{\lambda_n}\right)^{-1/2} = \left(\frac{\sin\sqrt{2i\theta}}{\sqrt{2i\theta}}\right)^{-1/2}.$$

1.3.6 Noting that $1-\max(s,t)-(1-s)(1-t) = \min(s,t)-st$ and

$$\int_0^1 W^2(t)\,dt = \int_0^1\int_0^1 [1-\max(s,t)]\,dW(s)\,dW(t),$$
$$\int_0^1 W(t)\,dt = \int_0^1 (1-t)\,dW(t),$$

the equality can be proved.

1.3.7 For $\tilde{W}(t) = W(t) - tW(1)$, we have $\mathrm{Cov}(\tilde{W}(s),\tilde{W}(t)) = \min(s,t) - st$. Since

$$\int_0^1 \tilde{W}^2(t)\,dt = \int_0^1 W^2(t)\,dt - 2W(1)\int_0^1 tW(t)\,dt + \frac{1}{3}W^2(1),$$
$$\int_0^1 tW(t)\,dt = \int_0^1 t\left(\int_0^t dW(u)\right)dt = \frac{1}{2}\int_0^1 (1-u^2)\,dW(u),$$

it holds that $\int_0^1 \tilde{W}^2(t)\,dt$ is equal to

$$\int_0^1\int_0^1\left[1-\max(s,t)-\frac{1}{2}(1-s^2+1-t^2)+\frac{1}{3}\right]dW(s)\,dW(t),$$

which leads to the required equality.

1.4.1 Putting $\Delta W_i = W(s_i) - W(s_{i-1})$ and $\Delta s_i = s_i - s_{i-1}$, we have

$$
\begin{aligned}
&\mathrm{E}\left[\left\{\sum_{i=1}^{n} X(s_{i-1})\left((\Delta W_i)^2 - \Delta s_i\right)\right\}^2\right] \\
&= \mathrm{E}\left[\sum_{i=1}^{n}\sum_{k=1}^{n} X(s_{i-1})X(s_{k-1})\left((\Delta W_i)^2 - \Delta s_i\right)\left((\Delta W_k)^2 - \Delta s_k\right)\right] \\
&= 2\sum_{i=1}^{n} \mathrm{E}\left(X^2(s_{i-1})\right)(\Delta s_i)^2 \le 2\max_i \Delta s_i \sum_{i=1}^{n} \mathrm{E}\left(X^2(s_{i-1})\right)\Delta s_i \\
&\to 0
\end{aligned}
$$

as $n \to \infty$ and $\max_i \Delta s_i \to 0$ so that the relation for $j = 2$ is established. For $j \ge 3$ it holds that

$$
\begin{aligned}
&\mathrm{E}\left[\left\{\sum_{i=1}^{n} X(s_{i-1})(\Delta W_i)^j\right\}^2\right] \\
&\quad = \mathrm{E}\left[\sum_{i=1}^{n}\sum_{k=1}^{n} X(s_{i-1})X(s_{k-1})(\Delta W_i)^j(\Delta W_k)^j\right] \\
&\quad = \sum_{i=1}^{n} \mathrm{E}\left(X^2(s_{i-1})\right)\mathrm{E}((\Delta W_i)^{2j}) \\
&\qquad\quad + \sum_{i \ne k} \mathrm{E}\left(X(s_{i-1})X(s_{k-1})\right)\mathrm{E}((\Delta W_i)^j)\mathrm{E}((\Delta W_k)^j),
\end{aligned}
$$

where $\mathrm{E}((\Delta W_i)^{\ell}) = \text{const.} \times (\Delta s_i)^{\ell/2}$ for even ℓ and 0 for odd ℓ, which leads to the conclusion.

1.4.2 Consider

$$
\begin{aligned}
\sum_{j=1}^{n} y_{j-1}\varepsilon_j &= \sum_{j=1}^{n} y_{j-1}(y_j - y_{j-1}) \\
&= -\frac{1}{2}\left(\sum_{j=1}^{n}(y_j - y_{j-1})^2 - \sum_{j=1}^{n} y_j^2 + \sum_{j=1}^{n} y_{j-1}^2\right) \\
&= \frac{1}{2}y_n^2 - \frac{1}{2}\sum_{j=1}^{n}\varepsilon_j^2,
\end{aligned}
$$

which establishes the required relation by the FCLT and CMT.

1.4.3 Put $\tau_{i-1} = (1-\lambda)t_{i-1} + \lambda t_i$ and $\Delta W_i = W(t_i) - W(t_{i-1})$. Then we have

$$\sum_{i=1}^{m} W(\tau_{i-1})\Delta W_i = \sum_{i=1}^{m} W(t_{i-1})\Delta W_i + \sum_{i=1}^{m} (W(\tau_{i-1}) - W(t_{i-1}))\, \Delta W_i .$$

Here the first term on the right side converges in m.s. to $(W^2(t) - t)/2$, whereas the second term can be rewritten as

$$\sum_{i=1}^{m} (W(\tau_{i-1}) - W(t_{i-1}))^2 + \sum_{i=1}^{m} (W(\tau_{i-1}) - W(t_{i-1}))\,(W(t_i) - W(\tau_{i-1}))\,. \tag{A.2}$$

Since it holds that

$$\mathrm{E}\left[\sum_{i=1}^{m} (W(\tau_{i-1}) - W(t_{i-1}))^2\right] = \sum_{i=1}^{m} (\tau_{i-1} - t_{i-1}) = \lambda \sum_{i=1}^{m} (t_i - t_{i-1}) = \lambda t,$$

$$\mathrm{Var}\left[\sum_{i=1}^{m} (W(\tau_{i-1}) - W(t_{i-1}))^2\right] = 2\sum_{i=1}^{m} (\tau_{i-1} - t_{i-1})^2 \le 2\lambda^2 t \max_i \Delta t_i,$$

the first term in (A.2) converges in m.s. to λt. Noting that $t_{i-1} \le \tau_{i-1} \le t_i$, the second term can be shown to converge in m.s. to 0, which establishes the required relation.

1.4.4 Since it holds that

$$d\left(g(t)W^2(t)\right) = 2g(t)W(t)\, dW(t) + (g'(t)W^2(t) + g(t))\, dt,$$

we obtain

$$\begin{aligned}
&\int_0^1 g(t)W(t)\, dW(t) \\
&= \frac{1}{2}\left[\int_0^1 d(g(t)W^2(t)) - \int_0^1 \left(g'(t)W^2(t) + g(t)\right) dt\right] \\
&= \frac{1}{2}\left[g(1)W^2(1) - \int_0^1 \int_0^1 [g(1) - g(\max(s,t))]\, dW(s)\, dW(t) \right.\\
&\qquad\qquad \left. - \int_0^1 g(t)\, dt\right] \\
&= \frac{1}{2}\int_0^1 \int_0^1 g(\max(s,t))\, dW(s)\, dW(t) - \frac{1}{2}\int_0^1 g(t)\, dt.
\end{aligned}$$

1.4.5 Noting that

$$X(t) = e^{\gamma t} \int_0^t e^{-\gamma u}\, dW(u),$$

it follows that

$$\begin{aligned}\int_0^1 X^2(t)\,dt &= \int_0^1 e^{2\gamma t}\left(\int_0^t\int_0^t e^{-\gamma(u+v)}\,dW(u)\,dW(v)\right)dt\\ &= \int_0^1\int_0^1\left(\int_{\max(u,v)}^1 e^{2\gamma t}\,dt\right)e^{-\gamma(u+v)}\,dW(u)\,dW(v)\\ &= \int_0^1\int_0^1 \frac{e^{2\gamma}-e^{2\gamma\max(u,v)}}{2\gamma}\,e^{-\gamma(u+v)}\,dW(u)\,dW(v)\\ &= \int_0^1\int_0^1 \frac{e^{\gamma(2-s-t)}-e^{\gamma|s-t|}}{2\gamma}\,dW(s)\,dW(t).\end{aligned}$$

We also have

$$\begin{aligned}\mathrm{Cov}(X(s),X(t)) &= \mathrm{E}\left(e^{\gamma(s+t)}\int_0^s\int_0^t e^{-\gamma(u+v)}\,dW(u)\,dW(v)\right)\\ &= e^{\gamma(s+t)}\int_0^{\min(s,t)} e^{-2\gamma u}\,du = \frac{e^{\gamma(s+t)}-e^{\gamma|s-t|}}{2\gamma}.\end{aligned}$$

1.4.6 When H is symmetric, we have, from (1.69),

$$S_1 = \int_0^1 \boldsymbol{W}'(t)H\,d\boldsymbol{W}(t) = \frac{1}{2}\left(\boldsymbol{W}'(1)H\boldsymbol{W}(1) - \mathrm{tr}\,(H)\right). \qquad \text{(A.3)}$$

Then we obtain

$$\begin{aligned}\mathrm{E}\left(e^{i\theta S_1}\right) &= \left|I_q - i\theta H\right|^{-1/2}\exp\left(-i\theta\,\mathrm{tr}\,(H)/2\right)\\ &= \prod_{k=1}^q\left(1-i\theta\lambda_k\right)^{-1/2}\exp\left(-\frac{i\theta}{2}\sum_{k=1}^q\lambda_k\right),\end{aligned}$$

where λ_k $(k = 1, \ldots, q)$ are the eigenvalues of H.

1.4.7 It follows from (A.3) that

$$S_2 = \int_0^1 \boldsymbol{W}'(t)H\,d\boldsymbol{W}(t) = \frac{1}{2}\boldsymbol{W}'(1)H\boldsymbol{W}(1),$$

where

$$H = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}.$$

Since the eigenvalues of H is ± 1, the c.f. of S_2 is $\left(1+\theta^2\right)^{-1/2}$, which is also the c.f. of $\left(W_1^2(1) - W_2^2(1)\right)/2$.

1.4.8 Given $\{W_1(t)\}$, it holds that

$$U = \int_0^1 W_1(t)\,dW_2(t)\bigg|\Big\{W_1(t)\Big\} \sim \mathrm{N}\left(0, \int_0^1 W_1^2(t)\,dt\right).$$

Using (1.29) we have

$$\begin{aligned}\mathrm{E}\left(e^{i\theta U}\right) &= \mathrm{E}\Big[\mathrm{E}\left(e^{i\theta U}\Big|\Big\{W_1(t)\Big\}\right)\Big] = \mathrm{E}\left[\exp\left\{-\frac{\theta^2}{2}\int_0^1 W_1^2(t)\,dt\right\}\right]\\ &= \left(\cos\sqrt{-\theta^2}\right)^{-1/2} = \left(\cosh\theta\right)^{-1/2}.\end{aligned}$$

On the other hand we obtain

$$\mathrm{E}\left[\exp\left\{\frac{i\theta}{2}\sum_{n=1}^{\infty}\frac{Z_{n1}^2 - Z_{n2}^2}{\left(n-\frac{1}{2}\right)\pi}\right\}\right] = \prod_{n=1}^{\infty}\left(1 + \frac{\theta^2}{\left(n-\frac{1}{2}\right)^2\pi^2}\right)^{-1/2} = \left(\cosh\theta\right)^{-1/2},$$

which proves the distributional equivalence.

1.5.1 Putting $F(x;\hat{\xi}) = F(x)$, we obtain

$$\begin{aligned}W_n^2(\hat{\xi}) &= n\int_{-\infty}^{\infty}(F_n(x) - F(x))^2\,dF(x)\\ &= n\left[\sum_{j=1}^{n-1}\int_{X_j'}^{X_{j+1}'}\left(\frac{j}{n} - F(x)\right)^2 dF(x) + \int_{-\infty}^{X_1'} F^2(x)\,dF(x)\right.\\ &\qquad \left. + \int_{X_n'}^{\infty}(1 - F(x))^2\,dF(x)\right].\end{aligned}$$

Using

$$\int F(x)\,dF(x) = \frac{1}{2}F^2(x), \quad \int F^2(x)\,dF(x) = \frac{1}{3}F^3(x),$$

we can arrive at the final expression after some manipulations.

1.5.2 Since $F(X_1), \ldots, F(X_n)$ are independent and uniformly distributed over $[0, 1]$, it holds that

$$\begin{aligned}\mathrm{E}[\epsilon\big(t - F(X_j)\big)] &= \mathrm{E}[\epsilon^2\big(t - F(X_j)\big)] = P(X_j < t) = t,\\ \mathrm{Cov}[\epsilon\big(s - F(X_j)\big), \epsilon\big(t - F(X_k)\big)] &= \mathrm{E}[\epsilon\big(s - F(X_j)\big)\,\epsilon\big(t - F(X_k)\big)] - st\\ &= \begin{cases}\min(s,t) - st & (j = k)\\ 0 & (j \neq k).\end{cases}\end{aligned}$$

Thus, putting $H_n(t) = \dfrac{1}{n}\ \sum_{j=1}^{n} \epsilon\Big(t - F(X_j)\Big)$, we have

$$\mathrm{E}(H_n(t)) = \frac{1}{n}\sum_{j=1}^{n} t = t, \quad \mathrm{Cov}(H_n(s), H_n(t)) = \frac{1}{n}\,(\min(s,t) - st).$$

1.5.3 When $F(x;\xi) = R(x - \xi) = t$, we have

$$\frac{\partial F(x;\xi)}{\partial \xi} = \frac{\partial R(x-\xi)}{\partial \xi} = -r(x-\xi), \quad g_\xi(t) = -r(R^{-1}(t)),$$

$$\sigma^{-2}(\xi) = \mathrm{E}\left[\left(\frac{\partial \log f(x;\xi)}{\partial \xi}\right)^2\right] = \mathrm{E}\left[\left(\frac{r'(x-\xi)}{r(x-\xi)}\right)^2\right] = \int_{-\infty}^{\infty} \frac{(r'(x))^2}{r(x)}\,dx.$$

Thus we obtain

$$\sigma(\xi)\,g_\xi(t) = -\left(\int_{-\infty}^{\infty} \frac{(r'(x))^2}{r(x)}\,dx\right)^{-1/2} r(R^{-1}(t)),$$

which does not depend on ξ. When $F(x;\xi) = R(x/\xi) = t$, we have $x = \xi R^{-1}(t)$ and

$$\frac{\partial F(x;\xi)}{\partial \xi} = \frac{\partial R(x/\xi)}{\partial \xi} = -\frac{x}{\xi^2}\,r(x/\xi), \quad g_\xi(t) = -\frac{1}{\xi}\,R^{-1}(t) r(R^{-1}(t)).$$

It follows from $f(x;\xi) = r(x/\xi)/\xi$ that

$$\begin{aligned}
\sigma^{-2}(\xi) = \mathrm{E}\left[\left(\frac{\partial \log f(x;\xi)}{\partial \xi}\right)^2\right] &= \frac{1}{\xi^2}\mathrm{E}\left[\left(1 + \frac{x}{\xi}\frac{r'(x/\xi)}{r(x/\xi)}\right)^2\right] \\
&= \frac{1}{\xi^2}\int_{-\infty}^{\infty} \left(1 + \frac{x}{\xi}\frac{r'(x/\xi)}{r(x/\xi)}\right)^2 \frac{1}{\xi} r(x/\xi)\,dx \\
&= \frac{1}{\xi^2}\int_{-\infty}^{\infty} \left(1 + \frac{xr'(x)}{r(x)}\right)^2 r(x)\,dx \\
&= \frac{1}{\xi^2}\int_{-\infty}^{\infty} \left(r(x) + 2xr'(x) + \frac{(xr'(x))^2}{r(x)}\right) dx \\
&= \frac{1}{\xi^2}\left(\int_{-\infty}^{\infty} \frac{(xr'(x))^2}{r(x)}\,dx - 1\right).
\end{aligned}$$

Thus we obtain

$$\sigma(\xi)\,g_\xi(t) = -\left(\int_{-\infty}^{\infty} \frac{(xr'(x))^2}{r(x)}\,dx - 1\right)^{-1/2} R^{-1}(t)\,r(R^{-1}(t)),$$

which does not depend on ξ.

1.5.4 Since $f(x;\xi) = \xi(R(x))^{\xi-1}r(x)$, we have

$$\sigma^{-2}(\xi) = \mathrm{E}\left[\left(\frac{\partial \log f(x;\xi)}{\partial \xi}\right)^2\right] = \mathrm{E}\left[\left(\frac{1}{\xi} + \log R(x)\right)^2\right]$$
$$= \frac{1}{\xi^2} + A,$$

where

$$A = \mathrm{E}\left[\frac{2}{\xi}\log R(x) + (\log R(x))^2\right]$$
$$= \int_{-\infty}^{\infty}\left(\frac{2}{\xi}\log R(x) + (\log R(x))^2\right)\xi(R(x))^{\xi-1}r(x)\,dx$$
$$= \int_0^1\left(\frac{2}{\xi}\log u + (\log u)^2\right)\xi u^{\xi-1}\,du = 0.$$

Thus $\sigma^2(\xi) = \xi^2$. We now have

$$g_\xi(t) = \left.\frac{\partial F(x;\xi)}{\partial \xi}\right|_{x=x(t;\xi)} = (R(x))^\xi \log R(x)\Big|_{x=x(t;\xi)} = \frac{t}{\xi}\log t,$$

which yields $\sigma(\xi)\,g_\xi(t) = t\,\log t$.

1.5.5 The log-likelihood for $\boldsymbol{y}$ is given by

$$L(\rho,\sigma_\varepsilon^2) = -\frac{n}{2}\log 2\pi\sigma_\varepsilon^2 - \frac{1}{2}\log|I_n + \rho CC'| - \frac{1}{2\sigma_\varepsilon^2}\boldsymbol{y}'(I_n + \rho CC')^{-1}\boldsymbol{y}.$$

Then we have

$$\left.\frac{\partial L(\rho,\sigma_\varepsilon^2)}{\partial \rho}\right|_{\rho=0} = -\frac{1}{2}\,\mathrm{tr}\,(CC') + \frac{1}{2\sigma_\varepsilon^2}\boldsymbol{y}'CC'\boldsymbol{y} = \frac{1}{2}\boldsymbol{Z}'CC'\boldsymbol{Z} - \frac{n(n+1)}{4},$$

where $\boldsymbol{Z} \sim \mathrm{N}(\boldsymbol{0}, I_n)$. Since the (j,k)th element of CC'/n is $\min(j,k)/n$ and its uniform limit in the sense of (1.105) is $K(s,t) = \min(s,t)$, it follows from Theorem 1.11 that

$$\frac{2}{n^2}\frac{\partial L(\rho,\sigma_\varepsilon^2)}{\partial \rho} = \frac{1}{n^2}\boldsymbol{Z}'CC'\boldsymbol{Z} - \frac{n+1}{2n} \Rightarrow \int_0^1\int_0^1 \min(s,t)\,dW(s)\,dW(t) - \frac{1}{2}.$$

1.5.6 Define $P = HF_1$ and $Q = XF_2$, where F_1 is an $(n-p)\times(n-p)$ matrix such that $P'P = I_{n-p}$ and F_2 is a $p\times p$ matrix such that $Q'Q = I_p$. Since $P'Q = F_1'H'XF_2 = 0$, it follows from Lemma 1.12 that, putting $A = \Omega$,

$$P(P'AP)^{-1}P' = H(H'\Omega H)^{-1}H' = N'A^{-1}N = N'\Omega^{-1}N,$$

where $N = I_n - Q(Q'A^{-1}Q)^{-1}Q'A^{-1} = I_n - X(X'\Omega^{-1}X)^{-1}X'\Omega^{-1}$. It also follows from Lemma 1.12 that

$$P(P'AP)^{-2}P' = P(P'AP)^{-1}P'P(P'AP)^{-1}P' = N'A^{-1}A^{-1}N,$$

which leads to $H(H'\Omega H)^{-2}H' = N'\Omega^{-2}N$.

1.5.7 Since $\boldsymbol{y} = (y_1, \ldots, y_n)' \sim \mathrm{N}(\boldsymbol{0}, \sigma_\varepsilon^2\Omega(\alpha))$, the log-likelihood for $\boldsymbol{y}$ is given by

$$L(\alpha, \sigma_\varepsilon^2) = -\frac{n}{2}\log 2\pi\sigma_\varepsilon^2 - \frac{1}{2}\log|\Omega(\alpha)| - \frac{1}{2\sigma_\varepsilon^2}\boldsymbol{y}'\Omega^{-1}(\alpha)\boldsymbol{y}.$$

We have $\partial L(\alpha, \sigma_\varepsilon^2)/\partial\alpha|_{H_0} = 0$ and

$$\left.\frac{\partial^2 L(\alpha, \sigma_\varepsilon^2)}{\partial\alpha^2}\right|_{H_0} = n\frac{\boldsymbol{y}'\Omega^{-2}\boldsymbol{y}}{\boldsymbol{y}'\Omega^{-1}\boldsymbol{y}} - \frac{n(n+5)}{6}, \quad \Omega = \Omega(1).$$

Then we obtain the LBIU statistic $S_n = \boldsymbol{y}'\Omega^{-2}\boldsymbol{y}/(n\boldsymbol{y}'\Omega^{-1}\boldsymbol{y})$. Since $\boldsymbol{\xi} = \Omega^{-1/2}\boldsymbol{y}/\sigma_\varepsilon \sim \mathrm{N}(\boldsymbol{0}, I_n)$ under H_0, it follows that $\boldsymbol{y}'\Omega^{-1}\boldsymbol{y}/(n\sigma_\varepsilon^2) = \boldsymbol{\xi}'\boldsymbol{\xi}/n \to 1$ in probability. Noting that the (j,k)th element of Ω^{-1}/n is given by $\min(j,k)/n - jk/(n(n+1))$, it follows from Theorem 1.11 that

$$\frac{1}{n^2\sigma_\varepsilon^2}\boldsymbol{y}'\Omega^{-2}\boldsymbol{y} = \frac{1}{n}\boldsymbol{\xi}'\left(\frac{1}{n}\Omega^{-1}\right)\boldsymbol{\xi} \;\Rightarrow\; S = \int_0^1\int_0^1 [\min(s,t) - st]\,dW(s)\,dW(t).$$

Thus $S_n \Rightarrow S$ and the c.f. of S is given from (1.35) by $\left(\sin\sqrt{2i\theta}/\sqrt{2i\theta}\right)^{-1/2}$.

1.5.8 Define the partial sum process

$$Y_n(t) = \frac{1}{\sqrt{n}\sigma_\varepsilon}u_{j-1} + n\left(t - \frac{j-1}{n}\right)\frac{u_j - u_{j-1}}{\sqrt{n}\sigma_\varepsilon} \quad \left(\frac{j-1}{n} \le t \le \frac{j}{n}\right).$$

It follows from the FCLT that $\{Y_n(t)\} \Rightarrow \{W(t)\}$. Consider

$$\begin{aligned}\hat{u}_j &= y_j - \hat{\beta}_1 - \hat{\beta}_2 j \\ &= u_j - \frac{\left(\sum_{k=1}^n k^2 - j\sum_{k=1}^n k\right)\sum_{k=1}^n u_k + \left(jn - \sum_{k=1}^n k\right)\sum_{k=1}^n k u_k}{n\sum_{k=1}^n k^2 - \left(\sum_{k=1}^n k\right)^2},\end{aligned}$$

which yields, because of the FCLT and CMT,

$$
\begin{aligned}
&\frac{1}{n^2\sigma_\varepsilon^2}\sum_{j=1}^n \hat{u}_j^2 + o_p(1) \\
&= \frac{1}{n}\sum_{j=1}^n \left(Y_n\left(\frac{j}{n}\right) + \left(\frac{6j}{n}-4\right)\frac{1}{n}\sum_{j=1}^n Y_n\left(\frac{j}{n}\right) - \left(\frac{12j}{n}-6\right)\frac{1}{n}\sum_{j=1}^n \frac{j}{n} Y_n\left(\frac{j}{n}\right)\right)^2 \\
&\Rightarrow \int_0^1 \left(W(t) + (6t-4)\int_0^1 W(s)\,ds - (12t-6)\int_0^1 sW(s)\,ds\right)^2 dt.
\end{aligned}
$$

1.5.9 The (j,k)th element $\Phi_{jk}(\rho)$ of $\Phi(\rho) = C(\rho)C'(\rho)$ is given for $j \le k$ by

$$
\Phi_{jk}(\rho) = \sum_{i=0}^{j-1} \rho^{k-j+2i}.
$$

Thus we obtain, for $j \le k$,

$$
\left.\frac{d\Phi_{jk}(\rho)}{d\rho}\right|_{\rho=1} = \left.\sum_{i=0}^{j-1}(k-j+2i)\,\rho^{k-j+2i-1}\right|_{\rho=1} = jk - j = jk - \min(j,k),
$$

which leads to the required result.

1.5.10 Since $\Phi^{-1}(\rho) = (C'(\rho))^{-1}C^{-1}(\rho)$, where $C(\rho)$ is defined in (1.131), and

$$
C^{-1}(\rho) = \begin{pmatrix} 1 & & & & \\ -\rho & 1 & & 0 & \\ & \cdot & \cdot & & \\ 0 & & \cdot & \cdot & \\ & & & -\rho & 1 \end{pmatrix},
$$

we have

$$
\begin{aligned}
\frac{d\Phi^{-1}(\rho)}{d\rho} &= \frac{d\left\{(C'(\rho))^{-1}\right\}}{d\rho} C^{-1}(\rho) + (C'(\rho))^{-1}\frac{dC^{-1}(\rho)}{d\rho} \\
&= \begin{pmatrix} 2\rho & -1 & & & & \\ -1 & 2\rho & -1 & & 0 & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & 0 & & -1 & 2\rho & -1 \\ & & & & -1 & 0 \end{pmatrix}.
\end{aligned}
$$

Thus we obtain

$$\left.\frac{d^2\Phi^{-1}(\rho)}{d\rho^2}\right|_{\rho=1} = \frac{d}{d\rho}\left(\frac{d\Phi^{-1}(\rho)}{d\rho}\right)\Bigg|_{\rho=1} = 2(I_n - \boldsymbol{e}_n\boldsymbol{e}_n').$$

1.5.11 Note first that $\boldsymbol{y}'N'N\boldsymbol{y} = \boldsymbol{u}'N'N\boldsymbol{u}$ and $\boldsymbol{y}'\tilde{M}\boldsymbol{y} = \boldsymbol{u}'\tilde{M}\boldsymbol{u}$. Consider $N\boldsymbol{u} = (I_n - X(X'(CC')^{-1}X)^{-1}X'(CC')^{-1}\boldsymbol{u}$, where

$$X'(CC')^{-1}X = \begin{pmatrix} 1 & 1 \\ 1 & n \end{pmatrix}, \quad \left(X'(CC')^{-1}X\right)^{-1} = \frac{1}{n-1}\begin{pmatrix} n & -1 \\ -1 & 1 \end{pmatrix}.$$

Then we obtain

$$\begin{aligned} N\boldsymbol{u} &= \boldsymbol{u} - \frac{1}{n-1}(\boldsymbol{e},\boldsymbol{d})\begin{pmatrix} n & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \boldsymbol{e}' \\ \boldsymbol{d}' \end{pmatrix}(C')^{-1}C^{-1}\boldsymbol{u} \\ &= \boldsymbol{u} - \boldsymbol{e}u_1 - \frac{u_n - u_1}{n-1}(\boldsymbol{d} - \boldsymbol{e}), \end{aligned}$$

which leads to the first relation. Since $\boldsymbol{u}'\tilde{M}\boldsymbol{u} = \boldsymbol{u}'N'(C')^{-1}C^{-1}N\boldsymbol{u}$ and it holds that

$$C^{-1}N\boldsymbol{u} = C^{-1}\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 - v_1 \\ \vdots \\ v_n - v_{n-1} \end{pmatrix},$$

where $\boldsymbol{v} = N\boldsymbol{u}$ and v_j is the jth component of $N\boldsymbol{u}$, which is $u_j - u_1 - (j-1)(u_n - u_1)/(n-1)$. Thus the second relation is established.

1.5.12 Let the left side be S_n, for which

$$S_n = \frac{1}{n^2}\sum_{j=1}^{n}\left(u_j - \frac{j}{n}u_n\right)^2 + o_p(1) = \frac{1}{n}\sum_{j=1}^{n}\left(Y_n\left(\frac{j}{n}\right) - \frac{j}{n}Y_n(1)\right)^2 + o_p(1),$$

where $\{Y_n(t)\}$ is the partial sum process for $\{u_j\}$. It follows from the FCLT that $\{Y_n(t)\} \Rightarrow \{W(t)\}$. Then we arrive at the conclusion from the CMT. Alternatively it follows from Exercise 1.5.11 that $\boldsymbol{u}'N'N\boldsymbol{u} \stackrel{\mathcal{D}}{=} \boldsymbol{\varepsilon}'NCC'N'\boldsymbol{\varepsilon}$, where $\boldsymbol{\varepsilon} \sim \mathrm{N}(\boldsymbol{0}, \sigma_\varepsilon^2 I_n)$ and

$$\begin{aligned} NCC'N' &= CC' - X(X'(CC')^{-1}X)^{-1}X' \\ &= CC' - \frac{1}{n-1}(\boldsymbol{e},\boldsymbol{d})\begin{pmatrix} n & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \boldsymbol{e}' \\ \boldsymbol{d}' \end{pmatrix} \\ &= CC' - \frac{1}{n-1}\left(n\boldsymbol{e}\boldsymbol{e}' - \boldsymbol{d}\boldsymbol{e}' - \boldsymbol{e}\boldsymbol{d}' + \boldsymbol{d}\boldsymbol{d}'\right). \end{aligned}$$

Since the (j,k)th element of $NCC'N'/n$ is given by

$$B_n(j,k) = \frac{\min(j,k)}{n} - \frac{n-j-k+jk}{n(n-1)},$$

it follows from Theorem 1.11 and (1.33) that

$$\begin{aligned}\frac{1}{n^2}\,\boldsymbol{\varepsilon}'NCC'N'\boldsymbol{\varepsilon} &\Rightarrow \int_0^1\int_0^1 [\min(s,t) - st]\,dW(s)\,dW(t)\\ &\stackrel{\mathcal{D}}{=} \int_0^1 (W(t) - tW(1))^2\;dt.\end{aligned}$$

Chapter 2

2.1.1 Following Hochstadt (1973), consider

$$\frac{d}{d\lambda}\,D_N(\lambda) = (\partial_1 + \partial_2 + \cdots + \partial_N)D_N(\lambda),$$

where ∂_j is a differentiation operator with respect to λ, acting only on the terms in the jth column. Then it holds that

$$\begin{aligned}a_n(N) &= \left(\frac{d}{d\,\lambda}\right)^n D_N(\lambda)\Big|_{\lambda=0} = (\partial_1 + \partial_2 + \cdots + \partial_N)^n D_N(\lambda)\Big|_{\lambda=0}\\ &= \sum_{\alpha_1+\cdots+\alpha_N=n} \frac{n!}{\alpha_1!\cdots\alpha_N!}\,\partial_1^{\alpha_1}\cdots\partial_N^{\alpha_N} D_N(\lambda)\Bigg|_{\lambda=0}.\end{aligned}$$

Since each column of $D_N(\lambda)$ is linear in λ, every term with $\alpha_j \geq 2$ vanishes. Thus we have

$$\begin{aligned}a_n(N) &= \sum_{\sum\alpha_j=n;\,\alpha_j(1-\alpha_j)=0} n!\;\partial_1^{\alpha_1}\cdots\partial_N^{\alpha_N}\;D_N(\lambda)\Bigg|_{\lambda=0}\\ &= \sum_{\pi}\partial_{j_1}\partial_{j_2}\cdots\partial_{j_n}\partial^0_{j_{n+1}}\partial^0_{j_{n+2}}\cdots\partial^0_{j_N}\;D_N(\lambda)\Bigg|_{\lambda=0}, \qquad \text{(A.4)}\end{aligned}$$

where this last sum $\sum_{\pi}$ runs over all permutations of the subscripts $1,2,\ldots,N$.

Observing that $\partial_1 \cdots \partial_n \, D_N(\lambda)|_{\lambda=0}$ is equal to

$$\begin{vmatrix} -\frac{1}{N}K\left(\frac{1}{N},\frac{1}{N}\right) & -\frac{1}{N}K\left(\frac{1}{N},\frac{2}{N}\right) & \cdots & -\frac{1}{N}K\left(\frac{1}{N},\frac{n}{N}\right) & 0\cdots 0 \\ \vdots & \vdots & & \vdots & \vdots \\ -\frac{1}{N}K\left(\frac{n}{N},\frac{1}{N}\right) & -\frac{1}{N}K\left(\frac{n}{N},\frac{2}{N}\right) & \cdots & -\frac{1}{N}K\left(\frac{n}{N},\frac{n}{N}\right) & 0\cdots 0 \\ -\frac{1}{N}K\left(\frac{n+1}{N},\frac{1}{N}\right) & -\frac{1}{N}K\left(\frac{n+1}{N},\frac{2}{N}\right) & \cdots & -\frac{1}{N}K\left(\frac{n+1}{N},\frac{n}{N}\right) & 1\cdots 0 \\ \vdots & \vdots & & \vdots & \ddots \\ -\frac{1}{N}K\left(\frac{N}{N},\frac{1}{N}\right) & -\frac{1}{N}K\left(\frac{N}{N},\frac{2}{N}\right) & \cdots & -\frac{1}{N}K\left(\frac{N}{N},\frac{n}{N}\right) & 0\cdots 1 \end{vmatrix},$$

and similarly for the other terms in (A.4). Then we can arrive at the required expression.

2.1.2 Putting $M = \max |K(s,t)|$ and using the Hadamard inequality, we have

$$\left| K\begin{pmatrix} t_1 & \cdot & \cdot & \cdot & t_n \\ t_1 & \cdot & \cdot & \cdot & t_n \end{pmatrix} \right| \le M^n n^{n/2} \quad (n = 1, 2, \dots)$$

and

$$|D(\lambda)| \le 1 + \sum_{n=1}^{\infty} \frac{\left(|\lambda| M \sqrt{n}\right)^n}{n!} = 1 + \sum_{n=1}^{\infty} b_n.$$

The ratio test gives us

$$\lim_{n\to\infty} \frac{b_{n+1}}{b_n} = \lim_{n\to\infty} \frac{\left(|\lambda| M \sqrt{n+1}\right)^{n+1} / (n+1)!}{\left(|\lambda| M \sqrt{n}\right)^n / n!} = \lim_{n\to\infty} \frac{|\lambda| M (1 + 1/n)^{n/2}}{\sqrt{n+1}} = 0,$$

which ensures that $D(\lambda)$ as a function of λ converges for all λ.

2.1.3 For $K(s,t) = 1 + s + t$, we have $d_1 = 2$,

$$d_2 = \int_0^1 \int_0^1 [(1+2s)(1+2t) - (1+s+t)^2] \, ds \, dt = -\frac{1}{6},$$

and $d_n = 0$ for $n \ge 3$. Thus $D(\lambda) = 1 - 2\lambda - \lambda^2/12$. The function $K(s,t) = 1+s+t+st = (1+s)(1+t)$ is the form given in (2.19). We have $d_1 = 7/3$ and $d_n = 0$ for $n \ge 2$, which gives $D(\lambda) = 1 - 7\lambda/3$. For $K(s,t) = 1+s+t+s^2t^2$, we have $d_1 = 11/5$,

$$d_2 = \int_0^1 \int_0^1 \begin{vmatrix} 1 + 2s + s^4 & 1 + s + t + s^2t^2 \\ 1 + s + t + s^2t^2 & 1 + 2t + t^4 \end{vmatrix} ds \, dt = \frac{7}{90},$$

$$d_3 = \int_0^1 \int_0^1 \int_0^1 \begin{vmatrix} K(s,s) & K(s,t) & K(s,u) \\ K(t,s) & K(t,t) & K(t,u) \\ K(u,s) & K(u,t) & K(u,u) \end{vmatrix} ds \, dt \, du = -\frac{1}{360},$$

and $d_n = 0$ for $n \geq 4$. Thus $D(\lambda) = 1 - 11\lambda/5 + 7\lambda^2/180 + \lambda^3/2160$.

2.1.4 Since

$$\sigma_{jk} = \mathrm{Cov}\left(\int_0^1 \cos j\pi t\, dW(t), \int_0^1 \cos k\pi t\, dW(t)\right) = \frac{1}{2}\,\delta_{jk},$$

we obtain, as the FD of $K_1(s,t)$,

$$D_1(\lambda) = \left|I_n - \lambda\Sigma\right| = \left(1 - \frac{\lambda}{2}\right)^n .$$

Similarly, we obtain $D_2(\lambda) = (1 - \lambda/2)^n$ as the FD of $K_2(s,t)$. Consider

$$\begin{aligned} S_3 &= \int_0^1 \int_0^1 \sum_{j=1}^n \cos j\pi(s-t)\, dW(s)\, dW(t) \\ &= \sum_{j=1}^n \left[\left(\int_0^1 \cos j\pi t\, dW(t)\right)^2 + \left(\int_0^1 \sin j\pi t\, dW(t)\right)^2\right] = \sum_{j=1}^n \left(X_j^2 + Y_j^2\right), \end{aligned}$$

where

$$X_j = \int_0^1 \cos j\pi t\, dW(t), \quad Y_j = \int_0^1 \sin j\pi t\, dW(t).$$

Since $(X_j, Y_j)' \sim \mathrm{NID}(\mathbf{0}, I_2/2)$, we have $\mathrm{E}\left(e^{i\theta S_3}\right) = (1 - i\theta)^{-n}$ so that the FD of $K_3(s,t)$ is given by $D_3(\lambda) = (1 - \lambda/2)^{2n}$. Similarly, consider

$$\begin{aligned} S_4 &= \int_0^1 \int_0^1 \sum_{j=1}^n \sin j\pi(s+t)\, dW(s)\, dW(t) \\ &= 2\sum_{j=1}^n \int_0^1 \cos j\pi t\, dW(t) \int_0^1 \sin j\pi t\, dW(t) = 2\sum_{j=1}^n X_j\, Y_j. \end{aligned}$$

Then we obtain $\mathrm{E}\left(e^{2i\theta X_j Y_j}\right) = \left(1 + 4\theta^2\right)^{-1/2}$ so that $\mathrm{E}\left(e^{i\theta S_4}\right) = \left(1 + 4\theta^2\right)^{-n/2}$. Thus we obtain $D_4(\lambda) = \left(1 - \lambda^2\right)^n$ as the FD of $K_4(s,t)$.

2.1.5 We have, for $0 \leq t_1 \leq \cdots \leq t_n \leq 1$,

$$\begin{aligned} & K\begin{pmatrix} t_1 & \cdot & \cdot & \cdot & t_n \\ t_1 & \cdot & \cdot & \cdot & t_n \end{pmatrix} \\ &= \begin{vmatrix} t_1(1-t_1) & t_1(1-t_2) & \cdot & \cdot & \cdot & t_1(1-t_n) \\ t_1(1-t_2) & t_2(1-t_2) & \cdot & \cdot & \cdot & t_2(1-t_n) \\ \vdots & \vdots & & & \vdots & \vdots \\ t_1(1-t_n) & t_2(1-t_n) & \cdot & \cdot & & t_n(1-t_n) \end{vmatrix} \end{aligned}$$

$$= (1-t_n)\begin{vmatrix} t_1(1-t_1) & t_1(1-t_2) & \cdot & \cdot & \cdot & t_1(1-t_{n-1}) & t_1 \\ t_1(1-t_2) & t_2(1-t_2) & \cdot & \cdot & \cdot & t_2(1-t_{n-1}) & t_2 \\ \vdots & \vdots & & & \vdots & \vdots & \vdots \\ t_1(1-t_n) & t_2(1-t_n) & \cdot & \cdot & \cdot & t_{n-1}(1-t_n) & t_n \end{vmatrix}$$

$$= (1-t_n)\begin{vmatrix} 0 & 0 & 0 & \cdot & \cdot & 0 & t_1 \\ t_1-t_2 & 0 & 0 & \cdot & \cdot & 0 & t_2 \\ & t_2-t_3 & 0 & \cdot & \cdot & 0 & t_3 \\ & & \cdot & 0 & & \vdots & \vdots \\ & * & & \cdot & & 0 & t_{n-1} \\ & & & & & t_{n-1}-t_n & t_n \end{vmatrix}$$

$$= t_1(t_2-t_1)(t_3-t_2)\cdots(t_n-t_{n-1})(1-t_n),$$

$$\int_0^1 \cdots \int_0^1 K\begin{pmatrix} t_1 & \cdot & \cdot & \cdot & t_n \\ t_1 & \cdot & \cdot & \cdot & t_n \end{pmatrix} dt_1\cdots dt_n$$
$$= n! \int \cdots \int_A t_1(t_2-t_1)\cdots(t_n-t_{n-1})(1-t_n)\, dt_1\cdots dt_n = \frac{n!}{(2n+1)!},$$

where $A = \{0 \le t_1 \le \cdots \le t_n \le 1\}$. Then the FD of $K(s,t) = \min(s,t) - st$ is given by

$$D(\lambda) = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} \frac{n!}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{(2n+1)!} = \frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}}.$$

where the series expansion for the sin function is employed.

2.2.1 Mercer's theorem gives the relation for $j = 1$. Suppose that the relation holds for $j = k$. Then we have

$$\begin{aligned} K^{(k+1)}(s,t) &= \int_0^1 K^{(k)}(s,u)K(u,t)\,du \\ &= \int_0^1 \sum_{m=1}^{\infty} \frac{1}{\lambda_m^k} f_m(s)f_m(u) \sum_{n=1}^{\infty} \frac{1}{\lambda_n} f_n(u)f_n(t)\,du \\ &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{\lambda_m^k\lambda_n} f_m(s)f_n(t) \int_0^1 f_m(u)f_n(u)\,du \\ &= \sum_{n=1}^{\infty} \frac{f_n(s)f_n(t)}{\lambda_n^{k+1}}, \end{aligned}$$

which proves the required relation.

2.2.2 Using Theorem 2.8, we compute

$$B_2 = \sqrt{\int_0^1 \int_0^1 K_3^2(s,t)\, ds\, dt} = \frac{1}{6}\sqrt{2\int_0^1 \left(\int_0^t s^4(3t-s)^2\, ds\right) dt}$$
$$= \frac{1}{6}\sqrt{\frac{33}{140}} = 0.08092.$$

Thus $1/\lambda_1 = 0.08089 < B_2 = 0.08092$.

2.2.3 It follows from

$$L_m(s,t) = \sum_{j=1}^{m} \alpha_j K^{(j)}(s,t) = \sum_{n=1}^{\infty} \left(\sum_{j=1}^{m} \frac{\alpha_j}{\lambda_n^j}\right) f_n(s) f_n(t)$$

that

$$\int_0^1 L_m(s,t)\, f_n(s)\, ds = \left(\sum_{j=1}^{m} \frac{\alpha_j}{\lambda_n^j}\right) f_n(t).$$

Thus the eigenvalues and eigenfunctions of $L_m(s,t)$ are given by

$$\left\{\left(1 \middle/ \sum_{j=1}^{m} \frac{\alpha_j}{\lambda_n^j}\ ,\ f_n\right) :\ n = 1,2,\ldots\right\}.$$

2.2.4 It follows from Theorem 2.10 that the c.f. of S_2 is given by

$$\mathrm{E}\left(e^{i\theta S_2}\right) = \left[D(\beta_1(2i\theta))\, D(\beta_2(2i\theta))\right]^{-1/2}$$
$$= \left[D\left(i\theta + \sqrt{-\theta^2 + 2ic^2\theta}\right)\, D\left(i\theta - \sqrt{-\theta^2 + 2ic^2\theta}\right)\right]^{-1/2},$$

where

$$\beta_1(\lambda) = \frac{\lambda + \sqrt{\lambda^2 + 4c^2\lambda}}{2}, \quad \beta_2(\lambda) = \frac{\lambda - \sqrt{\lambda^2 + 4c^2\lambda}}{2},$$

and $D(\lambda)$ is the FD of $K(s,t)$ given in (1.79) as

$$D(\lambda) = \frac{12}{\lambda}\left(\frac{2(1-\cos\sqrt{\lambda})}{\lambda} - \frac{\sin\sqrt{\lambda}}{\lambda}\right).$$

2.2.5 The distributional equivalence in

$$\int_0^1 \int_0^1 K(s,t)\, dW(s)\, dW(t) \overset{\mathcal{D}}{=} \sum_{n=1}^{\infty} \frac{1}{\lambda_n} Z_n^2$$

is ensured due to Mercer's theorem [Theorem 2.2]. On the other hand, it follows from the Karhunen-Loève expansion in (2.36) that

$$\begin{aligned}\int_0^1 Y^2(t)\,dt &= \int_0^1 \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{f_m(t)}{\sqrt{\lambda_m}}\frac{f_n(t)}{\sqrt{\lambda_n}}\,Z_m Z_n\,dt \\ &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{\sqrt{\lambda_m\lambda_n}}\,Z_m Z_n \int_0^1 f_m(t) f_n(t)\,dt = \sum_{n=1}^{\infty} \frac{1}{\lambda_n}\,Z_n^2,\end{aligned}$$

which establishes the required relations.

2.3.1 Consider first the case where $z = (2m + 1/2)\pi + \beta i$ with $|\beta| \le (2m + 1/2)\pi$. Then

$$\begin{aligned}\cos z &= \frac{e^{iz} + e^{-iz}}{2} &&= \frac{1}{2}\left(e^{i\pi/2-\beta} + e^{-i\pi/2+\beta}\right) = \frac{i}{2}\left(e^{-\beta} - e^{\beta}\right), \\ \sin z &= \frac{e^{iz} - e^{-iz}}{2i} &&= \frac{1}{2i}\left(e^{i\pi/2-\beta} - e^{-i\pi/2+\beta}\right) = \frac{1}{2}\left(e^{-\beta} + e^{\beta}\right).\end{aligned}$$

On the other hand, when $z = \alpha + i(2m + 1/2)\pi$ with $|\alpha| \le (2m + 1/2)\pi$, we have

$$\begin{aligned}\cos z &= \frac{1}{2}\left(e^{i\alpha-(2m+1/2)\pi} + e^{-i\alpha+(2m+1/2)\pi}\right), \\ \sin z &= \frac{1}{2i}\left(e^{i\alpha-(2m+1/2)\pi} - e^{-i\alpha+(2m+1/2)\pi}\right).\end{aligned}$$

It can be checked that $\cos z/\sin z$ is bounded on the two sides of C_m. Since $\cos z/\sin z$ is an odd function, $\cos z/\sin z$ is bounded on each side of C_m.

2.3.2 It can be shown easily that the integral equation leads to the differential equation with five conditions. We show the reverse. Denote by R the right side of the integral equation to be proved. Then, using the conditions in (2.62), R is equal to

$$\begin{aligned}&\int_0^1 [\min(s,t) - st](-f''(s) - \lambda a\psi''(s))\,ds \\ &\qquad -\psi(t)\int_0^1 \psi(s)(-f''(s) - \lambda a\psi''(s))\,ds \\ &= -\int_0^t s(1-t)(f''(s) + \lambda a\psi''(s))\,ds - \int_t^1 t(1-s)(f''(s) + \lambda a\psi''(s))\,ds \\ &\qquad +\psi(t)\left[\left[\psi(s)(f'(s) + \lambda a\psi'(s))\right]_0^1 - \int_0^1 \psi'(s)(f'(s) + \lambda a\psi'(s))\,ds\right],\end{aligned}$$

which can be shown, after some manipulations, to be equal to $f(t)$.

2.3.3 The expression (2.74) gives the integral equation

$$f(t) = \lambda \left[\int_0^t L_1(s,t) f(s)\, ds + \int_t^1 L_2(s,t) f(s)\, ds \right],$$

where

$$L_1(s,t) = \frac{1}{(g!)^2} \int_0^s ((s-u)(t-u))^g \; du,$$
$$L_2(s,t) = \frac{1}{(g!)^2} \int_0^t ((s-u)(t-u))^g \; du.$$

Let us define, for $j = 0, 1, \ldots, g$,

$$A_j(s,t) = \frac{\partial^j L_1(s,t)}{\partial t^j} = \frac{g(g-1)\cdots(g-j+1)}{(g!)^2} \int_0^s (s-u)^g (t-u)^{g-j}\, du,$$
$$B_j(s,t) = \frac{\partial^j L_2(s,t)}{\partial t^j} = \frac{g(g-1)\cdots(g-j+1)}{(g!)^2} \int_0^t (s-u)^g (t-u)^{g-j}\, du,$$

where $A_j(t,t) = B_j(t,t)$ and $A_j(0,t) = B_j(s,0) = 0$. Then it holds that

$$f^{(j)}(t) = \lambda \left[\int_0^t A_j(s,t) f(s)\, ds + \int_t^1 B_j(s,t) f(s)\, ds \right], \quad f^{(j)}(0) = 0.$$

Since

$$A_g(s,t) = \frac{\partial^g L_1(s,t)}{\partial t^g} = \frac{1}{g!} \int_0^s (s-u)^g\, du = \frac{s^{g+1}}{(g+1)!},$$
$$B_g(s,t) = \frac{\partial^g L_2(s,t)}{\partial t^g} = \frac{1}{g!} \int_0^t (s-u)^g\, du = \frac{-1}{(g+1)!} \left((s-t)^{g+1} - s^{g+1} \right),$$

it holds that $A_j(s,t) = 0$ for $j > g$ and, for $k = 1, \ldots, g+1$,

$$f^{(g)}(t) = -\frac{\lambda}{(g+1)!} \left[\int_t^1 (s-t)^{g+1} f(s)\, ds - \int_0^1 s^{g+1} f(s)\, ds \right],$$
$$f^{(g+k)}(t) = \frac{\lambda(-1)^{k-1}}{(g+1-k)!} \int_t^1 (s-t)^{(g+1-k)} f(s)\, ds, \quad f^{(g+k)}(1) = 0,$$
$$f^{(2g+1)}(t) = \lambda(-1)^g \int_t^1 f(s)\, ds,$$

so that we obtain the homogeneous differential equation of order $2g+2$

$$f^{(2g+2)}(t) + (-1)^g \lambda f(t) = 0$$

with $2g + 2$ boundary conditions

$$f^{(j)}(0) = 0, \quad f^{(g+1+j)}(1) = 0 \qquad (j = 0, 1, \ldots, g).$$

2.3.4 Define

$$C_j(s,t) = \frac{\partial^j L_1(s,t)}{\partial s^j} = \frac{g(g-1)\cdots(g-j+1)}{(g!)^2} \int_0^s (s-u)^{g-j}(t-u)^g \, du,$$
$$D_j(s,t) = \frac{\partial^j L_2(s,t)}{\partial s^j} = \frac{g(g-1)\cdots(g-j+1)}{(g!)^2} \int_0^t (s-u)^{g-j}(t-u)^g \, du,$$

so that

$$C_g(s,t) = \frac{1}{g!} \int_0^s (t-u)^g \, du = \frac{-1}{(g+1)!} \left((t-s)^{g+1} - t^{g+1} \right),$$
$$D_g(s,t) = \frac{1}{g!} \int_0^t (t-u)^g \, du = \frac{t^{g+1}}{(g+1)!}.$$

It holds that $C_j(t,t) = D_j(t,t)$ and $C_j(0,t) = D_j(s,0) = D_{g+1+j}(s,t) = 0$ for $j = 0, 1, \ldots, g$, whereas

$$C_{g+1+j}(t,t) = 0 \quad (j = 0, 1, \ldots, g-1), \quad C_{2g+1}(s,t) = (-1)^g.$$

Using integration by parts, $\lambda \int_0^1 L(s,t) f(s) \, ds$ is equal to

$$\begin{aligned}
& (-1)^{g+1} \int_0^1 L(s,t) f^{(2g+2)}(s) \, ds \\
= \ & (-1)^{g+1} \left[\left[L(s,t) f^{(2g+1)}(s) \right]_0^1 - \int_0^1 \frac{\partial L(s,t)}{\partial s} f^{(2g+1)}(s) \, ds \right] \\
= \ & (-1)^{g+1} \Big[L(1,t) f^{(2g+1)}(1) - L(0,t) f^{(2g+1)}(0) \\
& \qquad - \int_0^t C_1(s,t) f^{(2g+1)}(s) \, ds - \int_t^1 D_1(s,t) f^{(2g+1)}(s) \, ds \Big] \\
= \ & (-1)^{g+2} \left[\int_0^t C_1(s,t) f^{(2g+1)}(s) \, ds + \int_t^1 D_1(s,t) f^{(2g+1)}(s) \, ds \right],
\end{aligned}$$

where we have used $f^{(2g+1)}(1) = 0$ and $L(0,t) = 0$. Proceeding further by integration by parts and using the boundary conditions (2.76), we obtain

$$\lambda \int_0^1 L(s,t) f(s) \, ds = - \left[\int_0^t C_g(s,t) f^{(g+2)}(s) \, ds + \int_t^1 D_g(s,t) f^{(g+2)}(s) \, ds \right]$$

$$
\begin{aligned}
&= \int_0^t C_{g+1}(s,t) f^{(g+1)}(s)\, ds = -\int_0^t C_{g+2}(s,t) f^{(g)}(s)\, ds \\
&= (-1)^g \int_0^t C_{2g+1}(s,t) f'(s)\, ds = (-1)^{2g} \int_0^t f'(s)\, ds \\
&= \int_0^t f'(s)\, ds = f(t) - f(0) = f(t),
\end{aligned}
$$

which gives the integral equation (2.75).

2.3.5 Since the solutions to the characteristic equation of the differential equation (2.72) are $x_1, \ldots, x_{2g+2}$ and it holds that $x_{g+1+k} = -x_k$ for $k = 1, \ldots, g+1$, the general solution is given by

$$
\begin{aligned}
f(t) &= \sum_{j=1}^{2g+2} b_j e^{x_j t} = \sum_{k=1}^{g+1} \left(b_k e^{i(-ix_k t)} + b_{g+1+k} e^{i(ix_k t)} \right) \\
&= \sum_{k=1}^{g+1} \left[b_k \left(\cos ix_k t - i \sin ix_k t \right) + b_{g+1+k} \left(\cos ix_k t + i \sin ix_k t \right) \right] \\
&= \sum_{k=1}^{g+1} \left[(b_k + b_{g+1+k}) \cos ix_k t + i(b_{g+1+k} - b_k) \sin ix_k t \right],
\end{aligned}
$$

which gives the expression in (2.78).

2.3.6 It follows from the relation in (2.84) that $P(\lambda) - 1$ is equal to

$$
\begin{aligned}
&\frac{\lambda}{\pi^2} \int_0^1 \Bigg[\int_0^t \psi'(s) \Bigg\{ \frac{1}{2\lambda/\pi^2} - \frac{\pi}{2} \frac{\cos \sqrt{\lambda}(1-s-t)}{\sqrt{\lambda} \sin \sqrt{\lambda}/\pi} \\
&\qquad\qquad + \frac{1}{2\lambda/\pi^2} - \frac{\pi}{2} \frac{\cos \sqrt{\lambda}(1+s-t)}{\sqrt{\lambda} \sin \sqrt{\lambda}/\pi} \Bigg\}\, ds \\
&\quad + \int_t^1 \psi'(s) \Bigg\{ \frac{1}{2\lambda/\pi^2} - \frac{\pi}{2} \frac{\cos \sqrt{\lambda}(1-s-t)}{\sqrt{\lambda} \sin \sqrt{\lambda}/\pi} \\
&\qquad\qquad + \frac{1}{2\lambda/\pi^2} - \frac{\pi}{2} \frac{\cos \sqrt{\lambda}(1-s+t)}{\sqrt{\lambda} \sin \sqrt{\lambda}/\pi} \Bigg\}\, ds \Bigg] \psi'(t)\, dt \\
= &-\frac{\sqrt{\lambda}}{\sin \sqrt{\lambda}} \int_0^1 \int_0^1 \psi'(s) \psi'(t) L_1(s,t)\, ds\, dt.
\end{aligned}
$$

2.3.7 It follows from the relation in (2.88) that $P(\lambda) - 1$ is equal to

$$\frac{2\lambda}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{\left(n-\frac{1}{2}\right)^2-\frac{\lambda}{\pi^2}}\left(\int_0^1 \psi'(t)\sin\left(n-\frac{1}{2}\right)\pi t\,dt\right)^2$$

$$= \frac{\sqrt{\lambda}}{2\cos\sqrt{\lambda}}\int_0^1\left[\int_0^t \psi'(s)\left\{\sin\sqrt{\lambda}(1+s-t)-\sin\sqrt{\lambda}(1-s-t)\right\}ds\right.$$

$$\left.+\int_t^1 \psi'(s)\left\{\sin\sqrt{\lambda}(1-s+t)-\sin\sqrt{\lambda}(1-s-t)\right\}ds\right]\psi'(t)\,dt$$

$$= \frac{\sqrt{\lambda}}{\cos\sqrt{\lambda}}\int_0^1\int_0^1 \psi'(s)\psi'(t)L_2(s,t)\,ds\,dt.$$

2.3.8 Since $f_n(t) = \sqrt{2}\cos(n-1/2)\pi t$ and

$$b_{jn} = \int_0^1 \psi_j(t)f_n(t)\,dt = -\frac{\sqrt{2}}{(n-1/2)\pi}\int_0^1 \psi_j'(t)\sin\left(n-\frac{1}{2}\right)\pi t\,dt,$$

we obtain $P_{jj}(\lambda)$ from Theorem 2.18. On the other hand, putting $\lambda_n = (n-1/2)^2\pi^2$, we have

$$\begin{aligned}P_{jk}(\lambda) &= \lambda\sum_{n=1}^{\infty}\frac{b_{jn}b_{kn}}{1-\lambda/\lambda_n}\\ &= \frac{2\lambda}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(n-1/2)^2-\lambda/\pi^2}\int_0^1\int_0^1 \psi_j'(s)\psi_k'(t)\sin(n-1/2)\pi s\\ &\qquad\times\sin(n-1/2)\pi t\,ds\,dt\\ &= \frac{\lambda}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(n-1/2)^2-\lambda/\pi^2}\int_0^1\int_0^1 \psi_j'(s)\psi_k'(t)\\ &\qquad\times\left\{\cos(n-1/2)\pi(s-t)-\cos(n-1/2)\pi(s+t)\right\}ds\,dt,\end{aligned}$$

which leads to $P_{jk}(\lambda)$ in the theorem because of Theorem 2.18. Thus the theorem is established.

2.4.1 Denote by R the right side of the integral equation to be proved. Then

$$\begin{aligned}R &= \lambda\int_0^1 [\min(s,t)-st]\,f(s)\,ds\\ &= -\left[\int_0^t sf''(s)\,ds + t\int_t^1 f''(s)\,ds - t\int_0^1 sf''(s)\,ds\right]\end{aligned}$$

$$= -\left[sf'(s)\right]_0^t + \int_0^t f'(s)\,ds - t(f'(1) - f'(t))$$
$$+t\left[sf'(s)\right]_0^1 - t\int_0^1 f'(s)\,ds = f(t),$$

which is the left side of the integral equation to be proved.

2.4.2 We have

$$R = \lambda\left[-t\int_0^t f(s)\,ds - \int_t^1 sf(s)\,ds + \int_0^1 \left(\frac{s^2+t^2}{2} + \frac{1}{3}\right) f(s)\,ds\right]$$
$$= -t\int_0^t (-f''(s) + \lambda a)\,ds - \int_t^1 s(-f''(s) + \lambda a)\,ds$$
$$+\int_0^1 \left(\frac{s^2+t^2}{2} + \frac{1}{3}\right)(-f''(s) + \lambda a)\,ds$$
$$= t(f'(t) - f'(0)) - \lambda a t^2 + \left[sf'(s)\right]_t^1 - \int_t^1 f'(s)\,ds - \frac{\lambda a}{2}(1 - t^2)$$
$$-\left[\left(\frac{s^2+t^2}{2} + \frac{1}{3}\right) f'(s)\right]_0^1 + \int_0^1 sf'(s)\,ds + \frac{\lambda a}{2}(1 + t^2)$$
$$= f(t) - \int_0^1 f(s)\,ds = f(t) - a,$$

which shows that R leads to the left side of the integral equation only when $a = 0$.

2.4.3 It is clear that the integral equation with the present kernel leads to the differential equation with the two boundary conditions given in the problem. Conversely, consider

$$R = -\int_0^1 [1 - \max(s,t) + b]\; f''(s)\,ds$$
$$= t\int_0^t f''(s)\,ds + \int_t^1 sf''(s)\,ds - (1+b)\int_0^1 f''(s)\,ds$$
$$= -f(1) - bf'(1) + f(t),$$

where it holds that $f(1) = -bf'(1)$ since

$$f(1) = \lambda b\int_0^1 f(s)\,ds = -b\int_0^1 f''(s)\,ds = -bf'(1).$$

Thus R reduces to $f(t)$, which is the left side of the integral equation to be proved.

2.4.4 When $b = 0$, it is clear that all eigenvalues are positive. Assume that $b \neq 0$. The zeros of $h(z) = \cos z - bz \sin z$ ($z \neq 0$) satisfy $\tan z = 1/(bz)$, where $z = \sqrt{\lambda}$ is real for $\lambda > 0$ or purely imaginary for $\lambda < 0$. Suppose first that b is positive. Then no purely imaginary number $z = ix$ satisfies $\tan ix = 1/(bix) \Leftrightarrow \tanh x = -1/(bx)$, as can be seen from the graphs of $\tanh x$ and $-1/(bx)$ with $b > 0$. Thus the eigenvalues are all positive when b is positive. When b is negative, the graphs of $\tanh x$ and $-1/(bx)$ cross at two points, $\pm a$, say, which implies that there is only one negative eigenvalue.

2.4.5 We prove that

$$h(z) = (1 + b_2^2 z^2)\frac{\sin z}{z} + 2(b_1 + b_2 + b_2^2)(\cos z - 1)$$

has an infinite product expansion described in Theorem 2.13. Consider

$$\frac{h'(z)}{h(z)} = \frac{p_1(z)\cos z + p_2(z)\sin z}{q_0(z) + q_1(z)\cos z + q_2(z)\sin z},$$

where

$$p_1(z) = z + b_2^2 z^3, \quad p_2(z) = -1 - (b_2^2 + 2b_1 + 2b_2)z^2,$$
$$q_0(z) = -2(b_1 + b_2 + b_2^2)z^2, \quad q_1(z) = -q_0(z), \quad q_2(z) = p_1(z).$$

Then it can be shown that $\cos z/\sin z$ is bounded on each side of squares C_m in the complex plane with vertices $(2m + 1/2)\,\pi(\pm 1 \pm i)$ ($m = 1, 2, \ldots$). Noting that $p_2(z)$ is a lower order polynomial than $p_1(z) = q_2(z)$, and that $q_0(z)$ and $q_1(z)$ are lower order polynomials than $q_2(z)$, it is ensured that $h'(z)/h(z)$ is bounded on each side of C_m. Since $h'(0) = 0$ and $h(z)$ is symmetric with simple zeros of $h(z)$, Theorem 2.13 ensures that $h(z)$ admits an infinite product expansion and we conclude that $D(\lambda)$ is the FD of the present kernel.

2.4.6 Since

$$X(t) = e^{\gamma t} \int_0^t e^{-\gamma s}\, dW(s),$$

we have

$$\int_0^1 X^2(t)\, dt = \int_0^1 e^{2\gamma t} \left(\int_0^t \int_0^t e^{-\gamma(u+v)}\, dW(u)\, dW(v) \right) dt$$
$$= \int_0^1 \int_0^1 \left(\int_{\max(u,v)}^1 e^{2\gamma t}\, dt \right) e^{-\gamma(u+v)}\, dW(u)\, dW(v)$$

$$= \frac{1}{2\gamma}\int_0^1\int_0^1 \left(e^{2\gamma} - e^{2\gamma\max(u,v)}\right) e^{-\gamma(u+v)}\, dW(u)\, dW(v)$$
$$= \frac{1}{2\gamma}\int_0^1\int_0^1 \left(e^{\gamma(2-s-t)} - e^{\gamma|s-t|}\right) dW(s)\, dW(t).$$

We also have

$$\mathrm{Cov}(X(s), X(t)) = e^{\gamma(s+t)}\int_0^{\min(s,t)} e^{-2\gamma u}\, du = \frac{1}{2\gamma}\left(e^{\gamma(s+t)} - e^{\gamma|s-t|}\right),$$

which gives the required distributional relation.

2.4.7 We have

$$\begin{aligned}
& f(t) \\
= \ & \frac{\lambda}{2\gamma}\left[-e^{\gamma t}\int_0^t e^{-\gamma s} f(s)\, ds - e^{-\gamma t}\int_t^1 e^{\gamma s} f(s)\, ds + e^{\gamma t}\int_0^1 e^{\gamma s} f(s)\, ds\right], \\
& f'(t) \\
= \ & \frac{\lambda}{2}\left[-e^{\gamma t}\int_0^t e^{-\gamma s} f(s)\, ds + e^{-\gamma t}\int_t^1 e^{\gamma s} f(s)\, ds + e^{\gamma t}\int_0^1 e^{\gamma s} f(s)\, ds\right], \\
& f''(t) \\
= \ & \frac{\gamma\lambda}{2}\left[-e^{\gamma t}\int_0^t e^{-\gamma s} f(s)\, ds - e^{-\gamma t}\int_t^1 e^{\gamma s} f(s)\, ds + e^{\gamma t}\int_0^1 e^{\gamma s} f(s)\, ds\right] \\
& \quad -\lambda f(t) \\
= \ & (\gamma^2 - \lambda) f(t), \quad f(0) = 0, \quad f'(1) = \gamma f(1).
\end{aligned}$$

Conversely, assume the differential equation with two boundary conditions. Then

$$(\lambda - \gamma^2)\int_0^t e^{-\gamma s} f(s)\, ds = -\int_0^t e^{-\gamma s} f''(s)\, ds$$
$$= -e^{-\gamma t} f'(t) + f'(0) - \gamma e^{-\gamma t} f(t) + \gamma f(0) - \gamma^2\int_0^t e^{-\gamma s} f(s)\, ds,$$

which gives

$$\lambda\int_0^t e^{-\gamma s} f(s)\, ds = -e^{-\gamma t} f'(t) + f'(0) - \gamma e^{-\gamma t} f(t) + \gamma f(0).$$

Consider next

$$(\lambda - \gamma^2)\int_t^1 e^{\gamma s} f(s)\, ds = -\int_t^1 e^{\gamma s} f''(s)\, ds$$
$$= -e^{\gamma} f'(1) + e^{\gamma t} f'(t) + \gamma e^{\gamma} f(1) - \gamma e^{\gamma t} f(t) - \gamma^2\int_t^1 e^{\gamma s} f(s)\, ds,$$

which gives

$$\lambda \int_t^1 e^{\gamma s} f(s)\, ds = e^{\gamma t} f'(t) - \gamma e^{\gamma t} f(t), \quad \lambda \int_0^1 e^{\gamma s} f(s)\, ds = f'(0) - \gamma f(0).$$

Denote by R the right side of the integral equation to be proved. Then we have

$$\begin{aligned} R &= \frac{1}{2\gamma} \left[-e^{\gamma t} \left(-e^{-\gamma t} f'(t) + f'(0) - \gamma e^{-\gamma t} f(t) + \gamma f(0) \right) \right. \\ &\qquad \left. -e^{-\gamma t} \left(e^{\gamma t} f'(t) - \gamma e^{\gamma t} f(t) \right) + e^{\gamma t} \left(f'(0) - \gamma f(0) \right) \right] \\ &= f(t), \end{aligned}$$

which is the left side of the integral equation to be proved.

2.4.8 Consider

$$h(\mu) = \cos\mu - \gamma \frac{\sin\mu}{\mu}, \quad \frac{h'(\mu)}{h(\mu)} = \frac{(\gamma - \mu^2)\sin\mu - \gamma\mu\cos\mu}{\mu^2\cos\mu - \gamma\mu\sin\mu},$$

where $h(0) = 1 - \gamma$ and $h'(0) = 0$. Since $\sin\mu/\cos\mu$ is bounded on squares C_m with vertices $2m\pi(\pm 1 \pm i)$, $h'(\mu)/h(\mu)$ is also bounded on C_m as $m \to \infty$. Noting that $h(\mu)$ is symmetric with simple zeros at $\pm a_1, \pm a_2, \ldots$, it follows from Theorem 2.13 that

$$\begin{aligned} h(\mu) &= (1 - \gamma) \prod_{n=1}^{\infty} \left(1 - \frac{\mu^2}{a_n^2} \right) = (1 - \gamma) \prod_{n=1}^{\infty} \left(1 - \frac{\lambda - \gamma^2}{\lambda_n - \gamma^2} \right) \\ &= (1 - \gamma) \prod_{n=1}^{\infty} \frac{\lambda_n}{\lambda_n - \gamma^2} \left(1 - \frac{\lambda}{\lambda_n} \right). \end{aligned}$$

Since

$$h(\mu)\,|_{\lambda=0} = e^{-\gamma} = (1 - \gamma) \prod_{n=1}^{\infty} \frac{\lambda_n}{\lambda_n - \gamma^2},$$

we have

$$h(\mu) = e^{-\gamma} \prod_{n=1}^{\infty} \left(1 - \frac{\lambda}{\lambda_n} \right).$$

Thus $D(\lambda) = e^{\gamma} h(\mu)$ admits an infinite product expansion.

2.4.9 We prove that

$$h(\mu) = \cos\mu - (\gamma + b(\mu^2 + \gamma^2)) \frac{\sin\mu}{\mu}$$

admits an infinite product expansion, where $h(0) = 1-\gamma-b\gamma^2$ and $h'(0) = 0$. Consider

$$\frac{h'(\mu)}{h(\mu)} = \frac{-(b\mu^3 + (\gamma + b\gamma^2)\mu)\cos\mu + (-(b+1)\mu^2 + \gamma + b\gamma^2)\sin\mu}{-(b\mu^3 + (\gamma + b\gamma^2)\mu)\sin\mu + \mu^2\cos\mu}.$$

Since $\cos\mu/\sin\mu$ is bounded on squares C_m with vertices $(2m+1/2)\pi(\pm1\pm i)$, $h'(\mu)/h(\mu)$ is also bounded on C_m as $m \to \infty$. Noting that $h(\mu)$ is symmetric with simple zeros at $\pm a_1, \pm a_2, \ldots$, it follows from Theorem 2.13 that

$$\begin{aligned} h(\mu) &= (1-\gamma-b\gamma^2)\prod_{n=1}^{\infty}\left(1-\frac{\mu^2}{a_n^2}\right) = (1-\gamma-b\gamma^2)\prod_{n=1}^{\infty}\left(1-\frac{\lambda-\gamma^2}{\lambda_n-\gamma^2}\right) \\ &= (1-\gamma-b\gamma^2)\prod_{n=1}^{\infty}\frac{\lambda_n}{\lambda_n-\gamma^2}\left(1-\frac{\lambda}{\lambda_n}\right). \end{aligned}$$

Since

$$h(\mu)\,|_{\lambda=0} = e^{-\gamma} = (1-\gamma-b\gamma^2)\prod_{n=1}^{\infty}\frac{\lambda_n}{\lambda_n-\gamma^2},$$

we have

$$h(\mu) = e^{-\gamma}\prod_{n=1}^{\infty}\left(1-\frac{\lambda}{\lambda_n}\right).$$

Thus $D(\lambda) = e^{\gamma}h(\mu)$ admits an infinite product expansion.

2.4.10 Define

$$G(t) = \int_0^t g(s)\,ds, \qquad G(0) = 0,$$

where $g(t)$ is any continuous function. Then we have

$$\begin{aligned} I &= \frac{1}{4}\int_0^1\int_0^1 [1-2|s-t|]\,g(s)g(t)\,ds\,dt \\ &= \frac{1}{4}G^2(1) - \frac{1}{2}\int_0^1\left(\int_0^t (t-s)g(s)\,ds + \int_t^1 (s-t)g(s)\,ds\right)g(t)\,dt \\ &= \frac{1}{4}G^2(1) - \frac{1}{2}\int_0^1\left(\int_0^t G(s)\,ds + (1-t)G(1) - \int_t^1 G(s)\,ds\right)g(t)\,dt \\ &= \frac{1}{4}\int_0^1 (G(1)-2G(t))^2\,dt \ge 0. \end{aligned}$$

We next prove that the differential equation with two boundary conditions leads to the integral equation with $K(s,t)$. Denote by R the right side of the integral equation to be proved. Then R is equal to

$$
\begin{aligned}
&\frac{-1}{4}\int_0^1 [1-2\,|s-t|]\ f''(s)\,ds \\
=\ &\frac{-1}{4}\left[f'(1)-f'(0)-2\left\{\int_0^t (t-s)f''(s)\,ds+\int_t^1 (s-t)f''(s)\,ds\right\}\right] \\
=\ &f(t),
\end{aligned}
$$

which is the left side of the integral equation to be proved.

2.4.11 Noting that $\lambda_n = (2n-1)^2\pi^2$, eigenfunctions are given by

$$f_n(t) = c_1 \cos\sqrt{\lambda_n}\,t + c_2 \sin\sqrt{\lambda_n}\,t = c_1\cos(2n-1)\pi t + c_2\sin(2n-1)\pi t,$$

where c_1 and c_2 are arbitrary constants. Since the multiplicity of each λ_n is two, there are two linealy independet eigenfunctions $c_1\cos(2n-1)\pi t$ and $c_2\sin(2n-1)\pi t$. Then we have two linearly independent orthonormal eigenfunctions by putting $c_1 = c_2 = \sqrt{2}$.

2.4.12 We have

$$
\begin{aligned}
\mathrm{Cov}(\tilde{X}(s),\tilde{X}(t)) &= \mathrm{Cov}\left(\tilde{W}(s)-\int_0^1 \tilde{W}(u)\,du,\ \tilde{W}(t)-\int_0^1 \tilde{W}(v)\,dv\right) \\
&= \min(s,t)-st-\int_0^1 [\min(s,v)-sv]\,dv \\
&\quad -\int_0^1 [\min(u,t)-ut]\,du+\int_0^1\int_0^1 [\min(u,v)-uv]\,du\,dv \\
&= \min(s,t)-st-\frac{1}{2}(s-s^2+t-t^2)+\int_0^1 \frac{1}{2}(v-v^2)\,dv \\
&= K(s,t),
\end{aligned}
$$

which proves the first equality. We also have

$$\int_0^1 \tilde{X}^2(t)\,dt = \int_0^1 \tilde{W}^2(t)\,dt - \left(\int_0^1 \tilde{W}(t)\,dt\right)^2,$$

where

$$
\begin{aligned}
\int_0^1 \tilde{W}^2(t)\,dt &= \int_0^1 (W(t)-tW(1))^2\,dt \\
&= \int_0^1\int_0^1 \left[1-\max(s,t)-\left(1-\frac{s^2+t^2}{2}\right)+\frac{1}{3}\right] dW(s)\,dW(t),
\end{aligned}
$$

$$\int_0^1 \tilde{W}(t)\,dt = \int_0^1 (1-t)\,dW(t) - \frac{1}{2}W(1) = \int_0^1 \left(\frac{1}{2} - t\right) dW(t),$$

which proves the second equality.

2.4.13 It is clear that the integral equation with the present kernel leads to (2.129). Suppose that (2.129) holds. Then, using the two boundary conditions and noting that

$$\left(\frac{f'(t)}{t^{2m}}\right)' + \lambda f(t) = 0, \quad t^{2m} f(t) = -\frac{1}{\lambda}\left(f''(t) - \frac{2m}{t} f'(t)\right),$$

we have

$$\begin{aligned}
&\lambda \int_0^1 \left[1 - (\max(s,t))^{2m+1}\right] f(s)\,ds \\
&= -\int_0^1 \left(\frac{f'(s)}{s^{2m}}\right)' ds + t^{2m+1}\int_0^t \left(\frac{f'(s)}{s^{2m}}\right)' ds + \int_t^1 \left(s f''(s) - 2m f'(s)\right) ds \\
&= -f'(1) + t^{2m+1}\frac{f'(t)}{t^{2m}} + f'(1) - t f'(t) - (2m+1)(f(1) - f(t)) \\
&= (2m+1) f(t),
\end{aligned}$$

which establishes the required relation.

2.4.14 It is clear that the integral equation with the present kernel is equivalent to (2.135). The general solution to the differential equation is given by

$$\begin{aligned}
h(t) &= \sqrt{t}\left\{c_1 J_{1/2(m+1)}\left(\frac{\sqrt{\lambda}}{m+1} t^{m+1}\right) + c_2 J_{-1/2(m+1)}\left(\frac{\sqrt{\lambda}}{m+1} t^{m+1}\right)\right\} \\
&= c_1 \left(\frac{\sqrt{\lambda}}{2(m+1)}\right)^{1/2(m+1)} t \left[\frac{1}{\Gamma(\nu+1)} - \frac{\lambda t^{2(m+1)/4(m+1)^2}}{\Gamma(\nu+2)} + \cdots\right] \\
&\quad + c_2 \left(\frac{\sqrt{\lambda}}{2(m+1)}\right)^{-1/2(m+1)} \left[\frac{1}{\Gamma(-\nu+1)} - \frac{\lambda t^{2(m+1)/4(m+1)^2}}{\Gamma(-\nu+2)} + \cdots\right],
\end{aligned}$$

where $\nu = 1/2(m+1)$. The boundary condition $h(0) = 0$ implies $c_2 = 0$ and the other condition $h(1) = 0$ yields $J_\nu(\sqrt{\lambda}/(m+1)) = 0$. Then, using (2.125), we can obtain the FD given in (2.136).

2.4.15 It follows from

$$\cos z = \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{(n-1/2)^2\pi^2}\right)$$

that

$$\cos\frac{\pi\sqrt{1+4z}}{2} = \prod_{n=1}^{\infty}\left(1-\frac{\pi^2(1+4z)/4}{(n-1/2)^2\pi^2}\right) = -4z\prod_{n=2}^{\infty}\left(1-\frac{(1+4z)/4}{(n-1/2)^2}\right)$$
$$= -4z\prod_{n=2}^{\infty}\left(1-\frac{z}{n(n-1)}\right)\prod_{n=2}^{\infty}\left(1-\frac{1/4}{(n-1/2)^2}\right)$$
$$= -4z\prod_{n=1}^{\infty}\left(1-\frac{z}{n(n+1)}\right)\prod_{n=1}^{\infty}\left(1-\frac{1/4}{(n+1/2)^2}\right)$$
$$= -\pi z\prod_{n=1}^{\infty}\left(1-\frac{z}{n(n+1)}\right),$$

where we have used the fact that

$$\prod_{n=1}^{\infty}\left(1-\frac{1/4}{(n+1/2)^2}\right) = \frac{\pi}{4}.$$

Alternatively, we can use the product expansion formula:

$$\prod_{n=1}^{\infty}\left(1+\frac{x(1-x)}{n(n+1)}\right) = \frac{\sin\pi x}{\pi x(1-x)},$$

which leads to $D(\lambda)$ by putting $x(1-x) = -\lambda$.

2.4.16 Putting $f(t) = t^{1/(2m+1)}h(t)$, the integral equation with the present kernel is equivalent to the differential equation

$$h''(t) + \frac{2m+2}{2m+1}\frac{h'(t)}{t} + \frac{\lambda}{(2m+1)^2}\,t^{-2m/(2m+1)}\,h(t) = 0 \qquad \text{(A.5)}$$

with the boundary conditions

$$h(1) = 0, \quad \lim_{t\to 0} t^{1/(2m+1)}h(t) = 0.$$

The differential equation (A.5) is a special case of Bessel's equation in (2.123) with

$$\alpha = \frac{-1}{2(2m+1)}, \quad \beta = \frac{\sqrt{\lambda}}{m+1}, \quad \gamma = \frac{m+1}{2m+1}, \quad \nu = \frac{1}{2(m+1)},$$

so that the general solution is

$$h(t) = t^{-1/2(2m+1)}\left\{c_1 J_\nu\left(\frac{\sqrt{\lambda}}{m+1}t^{(m+1)/(2m+1)}\right) + c_2 J_{-\nu}\left(\frac{\sqrt{\lambda}}{m+1}t^{(m+1)/(2m+1)}\right)\right\}.$$

Then the boundary condition $t^{1/(2m+1)}h(t) \to 0$ as $t \to 0$ implies $c_2 = 0$ so that we obtain the same FD as that in Exercise 2.4.14 from $h(1) = 0$.

2.4.17 S is equal to

$$\begin{aligned}
&\int_0^1 t^{2m} W^2(t)\,dt - (2m+1)\left(\int_0^1 s^{2m} W(s)\,ds\right)^2 \\
&= \int_0^1 \int_0^1 \left(\int_{\max(u,v)}^1 t^{2m}\,dt\right) dW(u)\,dW(v) \\
&\qquad -(2m+1)\left(\int_0^1 \left(\int_u^1 t^{2m}\,dt\right) dW(u)\right)^2 \\
&= \int_0^1 \int_0^1 \frac{1}{2m+1}\left[(\min(s,t))^{2m+1} - (st)^{2m+1}\right] dW(s)\,dW(t).
\end{aligned}$$

2.4.18 The integral equation with the present kernel gives

$$f'(t)t^{-2m} = \lambda\left[\int_t^1 f(s)\,ds - \int_0^1 s^{2m+1} f(s)\,ds\right],$$

from which it follows that

$$f''(t) - \frac{2m}{t} f'(t) + \lambda t^{2m} f(t) = 0, \qquad f(0) = f(1) = 0.$$

The differential equation is the same as (2.129) whose general solution is given in (2.130) as

$$f(t) = t^{(2m+1)/2}\left\{c_1 J_\nu\left(\frac{\sqrt{\lambda}}{m+1} t^{m+1}\right) + c_2 J_{-\nu}\left(\frac{\sqrt{\lambda}}{m+1} t^{m+1}\right)\right\},$$

where $\nu = (2m+1)/2(m+1)$. The boundary condition $f(0) = 0$ implies $c_2 = 0$ and it follows from $f(1) = 0$ that a necessary and sufficient condition for $\lambda \neq 0$ to be an eigenvalue is $J_\nu(\sqrt{\lambda}/(m+1)) = 0$. Then we obtain the FD given in (2.141) using (2.125).

2.4.19 Since

$$J_{-\nu-1}(\beta) = \left(\frac{\beta}{2}\right)^{-\nu-1}\left(\frac{1}{\Gamma(-\nu)} + \frac{\lambda m/2(m+1)^2}{\Gamma(-\nu+1)} + \sum_{k=2}^{\infty} \frac{(\lambda m/2(m+1)^2)^k}{k!\,\Gamma(-\nu+k)}\right),$$

where $\nu = m/(m+1)$ and $\beta = \sqrt{-2\lambda m}/(m+1)$, it holds that

$$\frac{J_{-\nu-1}(\beta)\Gamma(-\nu)}{(\beta/2)^{-\nu-1}} = 1 + \frac{\Gamma(-\nu)\lambda m/2(m+1)^2}{\Gamma(-\nu+1)} + \Gamma(-\nu)\sum_{k=2}^{\infty} \frac{(\lambda m/2(m+1)^2)^k}{k!\,\Gamma(-\nu+k)}$$

$$= 1 + \frac{\lambda m/2(m+1)^2}{-\nu} + \sum_{k=2}^{\infty} \frac{(\lambda/2(m+1)^2)^k m^k}{k!\,(-\nu+k-1)\cdots(-\nu+1)(-\nu)}$$
$$\to 1 - \frac{\lambda}{2} \quad (m \to 0).$$

2.4.20 It follows from the text that we have only to prove that $g(z) = 720\,h(z)/z^8$ admits an infinite product expansion of the form in (2.39), where

$$h(z) = 4z^2 + 24 + 8(z^2-3)\cos z + z(z^2-24)\sin z, \quad z = \sqrt{\lambda}.$$

It can be checked that every zero ($\neq 0$) of $h(z)$ is simple by showing that there exists no solution common to $h(z) = 0$ and $h'(z) = 0$. It also holds that $\mathrm{rank}(M(\lambda_n)) = 3$ for each eigenvalue λ_n. Thus the multiplicity is unity for each λ_n. The function $g(z) = 720\,h(z)/z^8$ is even and analytic for all z with $g(0) = 1$ and $g'(0) = 0$. The zeros of $g(z)$ can be denoted by $\pm a_n$ ($n = 1,2,\dots$). Furthermore, $g'(z)/g(z)$ is bounded on the squares C_m ($m = 1,2,\dots$) with the vertices at $(2m+1/2)\pi(\pm 1 \pm i)$. It follows from Theorem 2.13 that $g(z)$ has an infinite product expansion (2.39) with $\lambda_n = a_n^2$ and $\ell_n = 1$, which ensures that the FD of the present kernel is $g(\sqrt{\lambda})$.

2.4.21 Theorem 2.20 gives us

$$P_{jj}(\lambda) = 1 - \frac{2\sqrt{\lambda}}{\sin\sqrt{\lambda}} \int_0^1 \left(\int_0^t \psi_j'(s) \cos\sqrt{\lambda}s\, ds \right) \psi_j'(t) \cos\sqrt{\lambda}(1-t)\, dt,$$
$$P_{jk}(\lambda) = -\frac{\sqrt{\lambda}}{\sin\sqrt{\lambda}} \int_0^1 \left(\int_0^t \psi_j'(s) \cos\sqrt{\lambda}s \cos\sqrt{\lambda}(1-t)\, ds + \int_t^1 \psi_j'(s) \cos\sqrt{\lambda}t \cos\sqrt{\lambda}(1-s)\, ds \right) \psi_k'(t)\, dt,$$

where $j \neq k$, from which we obtain, using computerized algebra,

$$P_{11}(\lambda) = -5 - \frac{72}{\lambda} - \frac{144\left(\cos\sqrt{\lambda} - 1\right)}{\lambda\sqrt{\lambda}\sin\sqrt{\lambda}},$$
$$P_{22}(\lambda) = 9 + \frac{120}{\lambda} - \frac{360}{\lambda^2} + \frac{360\cos\sqrt{\lambda}}{\lambda\sqrt{\lambda}\sin\sqrt{\lambda}},$$
$$P_{33}(\lambda) = -3 - \frac{120}{\lambda} + \frac{1080}{\lambda^2} - \frac{120\left(5\cos\sqrt{\lambda} + 4\right)}{\lambda\sqrt{\lambda}\sin\sqrt{\lambda}},$$

$$P_{12}(\lambda) = 3\sqrt{5}i\left[1 + \frac{12}{\lambda} + \frac{24\left(\cos\sqrt{\lambda} - 1\right)}{\lambda\sqrt{\lambda}\sin\sqrt{\lambda}}\right],$$

$$P_{13}(\lambda) = \frac{i}{\sqrt{3}}P_{12}(\lambda),$$

$$P_{23}(\lambda) = 3\sqrt{3}i\left[1 + \frac{20}{\lambda} - \frac{120}{\lambda} + \frac{40\left(2\cos\sqrt{\lambda} + 1\right)}{\lambda\sqrt{\lambda}\sin\sqrt{\lambda}}\right],$$

which yields

$$|P(\lambda)| = \frac{720}{\lambda^3\sqrt{\lambda}\sin\sqrt{\lambda}}\left(4(6+\lambda) - 8(3-\lambda)\cos\sqrt{\lambda} - \sqrt{\lambda}(24-\lambda)\sin\sqrt{\lambda}\right),$$

and arrive at the FD given in (2.179).

Chapter 3

3.2.1 We can deduce that

$$\begin{aligned}\int_0^1 q(t)(m(t) - q(t))\,dt &= \int_0^1 \sum_{n=1}^{\infty} c_n f_n(t)\left(m(t) - \sum_{m=1}^{\infty} c_m f_m(t)\right)dt \\ &= \sum_{n=1}^{\infty} c_n^2 - \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c_m c_n \int_0^1 f_m(t) f_n(t)\,dt \\ &= \sum_{n=1}^{\infty} c_n^2 - \sum_{n=1}^{\infty} c_n^2 = 0.\end{aligned}$$

3.2.2 Putting $\alpha = a^2 + ab + b^2/3$ and $A^2 = 4a^2/3 + 5ab/3 + 8b^2/15$, consider

$$\begin{aligned}\psi(\theta) &= \mathrm{E}\left(e^{i\theta(S_1-\alpha)/A}\right) \\ &= \left(\cos\sqrt{2i\theta/A}\right)^{-1/2}\exp\left[-\frac{b(2a+b)}{2} + ab\sec\sqrt{2i\theta/A}\right. \\ &\quad \left. + \frac{2i\theta a^2/A + b^2}{2}\frac{\tan\sqrt{2i\theta/A}}{\sqrt{2i\theta/A}} - \frac{i\theta\alpha}{A}\right],\end{aligned}$$

where

$$\cos\sqrt{2i\theta/A} = 1 - \frac{2i\theta}{2A} + O(A^{-2}),$$

$$\sec\sqrt{2i\theta/A} = 1 + \frac{2i\theta}{2A} + \frac{5(2i\theta/A)^2}{24} + O(A^{-3}),$$

$$\frac{\tan\sqrt{2i\theta/A}}{\sqrt{2i\theta/A}} = 1 + \frac{2i\theta}{3A} + \frac{2}{15}\left(\frac{2i\theta}{A}\right)^2 + O(A^{-3}).$$

Then we obtain, as $|ab| \to \infty$ so that $A \to \infty$, $\psi(\theta) \to e^{-\theta^2/2}$, which means that $(S_1 - \alpha)/A \Rightarrow \mathrm{N}(0,1)$.

Alternatively, it can be shown more easily by noting that

$$S_1 = \int_0^1 W^2(t)\,dt + 2\int_0^1 (a+bt)W(t)\,dt + \alpha \;\Rightarrow\; \mathrm{N}(\alpha, A^2),$$

where

$$A^2 = 4\mathrm{Var}\left(\int_0^1 (a+bt)W(t)\,dt\right) = 8\int_0^1 \left(\int_0^t (a+bs)s\,ds\right)(a+bt)\,dt.$$

3.2.3 The integral equation (3.16) with $K(s,t) = \min(s,t) - st$ and $m(t) = a+bt$ is equivalent to $h''(t;\lambda) + \lambda h(t;\lambda) = -a - bt$ with $h(0;\lambda) = h(1;\lambda) = 0$. The general solution is given by

$$h(t;\lambda) = c_1 \cos\sqrt{\lambda}\,t + c_2 \sin\sqrt{\lambda}\,t - \frac{a+bt}{\lambda}.$$

It follows from the boundary conditions that

$$c_1 = \frac{a}{\lambda}, \qquad c_2 = \frac{a+b}{\lambda\sin\sqrt{\lambda}} - \frac{a}{\lambda}\cot\sqrt{\lambda}.$$

Then we obtain

$$\begin{aligned}
&\frac{\lambda}{2}\int_0^1 \left(m^2(t) + \lambda h(t;\lambda)m(t)\right)\,dt \\
&= \frac{1}{\cos\sqrt{\lambda}+1}\Big[a(a+b)\sqrt{\lambda}\sin\sqrt{\lambda} \\
&\qquad + \frac{b^2}{2}\left(\frac{\sqrt{\lambda}\cos\sqrt{\lambda}\sin\sqrt{\lambda}}{\cos\sqrt{\lambda}-1} + \cos\sqrt{\lambda} + 1\right)\Big] \\
&= \exp\left[\frac{b^2}{2} + \frac{a(a+b)\sqrt{\lambda}}{\sin\sqrt{\lambda}} - \sqrt{\lambda}\left(a(a+b) + \frac{b^2}{2}\right)\cot\sqrt{\lambda}\right],
\end{aligned}$$

which yields the c.f. of S_1 given in (3.22).

Putting $\alpha = a^2 + ab + b^2/3$ and $B^2 = a^2/3 + ab/3 + 4b^2/45$, consider

$$\begin{aligned}\psi(\theta) &= \mathrm{E}\left(e^{i\theta(S_1-\alpha)/B}\right)\\ &= \left(\frac{\sin\sqrt{2i\theta/B}}{\sqrt{2i\theta/B}}\right)^{-1/2}\exp\left[\frac{b^2}{2} + a(a+b)\sqrt{2i\theta/B}\,\mathrm{cosec}\sqrt{2i\theta/B}\right.\\ &\quad\left. - \left(a(a+b)+\frac{b^2}{2}\right)\sqrt{2i\theta/B}\cot\sqrt{2i\theta/B} - \frac{i\theta\alpha}{B}\right],\end{aligned}$$

where

$$\begin{aligned}\frac{\sin\sqrt{2i\theta/B}}{\sqrt{2i\theta/B}} &= 1 - \frac{2i\theta}{6B} + O(B^{-2}),\\ \sqrt{2i\theta/B}\,\mathrm{cosec}\sqrt{2i\theta/B} &= 1 + \frac{2i\theta}{6B} + \frac{7}{360}\left(\frac{2i\theta}{B}\right)^2 + O(B^{-3}),\\ \sqrt{2i\theta/B}\cot\sqrt{2i\theta/B} &= 1 - \frac{2i\theta}{3B} - \frac{1}{45}\left(\frac{2i\theta}{B}\right)^2 + O(B^{-3}).\end{aligned}$$

Then we obtain, as $|ab| \to \infty$ so that $B \to \infty$, $\psi(\theta) \to e^{-\theta^2/2}$, which means that $(S_1 - \alpha)/B \Rightarrow \mathrm{N}(0, 1)$.

Alternatively, it can be shown more easily that

$$S_1 = \int_0^1 Y^2(t)\,dt + 2\int_0^1 (a+bt)Y(t)\,dt + \alpha \;\Rightarrow\; \mathrm{N}(\alpha, B^2),$$

where

$$B^2 = 4\mathrm{Var}\left(\int_0^1 (a+bt)Y(t)\,dt\right) = 8\int_0^1\left(\int_0^t (a+bs)(s-st)\,ds\right)(a+bt)\,dt.$$

3.2.4 We have

$$S_1 = \int_0^1 Y^2(t)\,dt - 2\gamma\int_0^1 t(1-t)Y(t)\,dt + \gamma^2\int_0^1 t^2(1-t)^2\,dt,$$

where

$$\begin{aligned}\mathrm{Var}\left(\int_0^1 t(1-t)Y(t)\,dt\right) &= \int_0^1\int_0^1 st(1-s)(1-t)[\min(s,t) - st]\,ds\,dt\\ &= \frac{17}{5040},\\ \int_0^1 t^2(1-t)^2\,dt &= \frac{1}{30}.\end{aligned}$$

Then it holds that

$$\frac{S_1 - \gamma^2/30}{\sqrt{4\gamma^2 \times 17/5040}} = \frac{S_1 - \gamma^2/30}{\sqrt{\gamma^2 \times 17/1260}} \ \Rightarrow\ \mathrm{N}(0,1).$$

This result can also be obtained from the c.f. of S_1 given in (3.26).

3.2.5 Putting $\sigma = \sqrt{a^2/2\pi^2}$, we prove that $S = (S_1 - a^2/2)/\sigma \ \Rightarrow\ \mathrm{N}(0,1)$ as $|a| \to \infty$. In fact, it follows from (3.28) that

$$\begin{aligned}
\mathrm{E}\left(e^{i\theta S}\right) &= \left(\frac{\sin\sqrt{2i\theta/\sigma}}{\sqrt{2i\theta/\sigma}}\right)^{-1/2} \exp\left[\frac{a^2 2i\theta/\sigma}{4}\left(1 - \frac{2i\theta/\sigma}{4\pi^2}\right)^{-1} - \frac{a^2 i\theta}{2\sigma}\right] \\
&= \left(\frac{\sin\sqrt{2i\theta/\sigma}}{\sqrt{2i\theta/\sigma}}\right)^{-1/2} \exp\left[\frac{a^2}{2\sigma^2\pi^2}\frac{(i\theta)^2}{2} + O(\sigma^{-1})\right] \\
&\to \exp\left[-\frac{\theta^2}{2}\right].
\end{aligned}$$

3.3.1 It can be easily established that the integral equation leads to the differential equation with the two boundary conditions. Conversely, putting $g(t) = \Gamma(0,t;\lambda)$ and using

$$\lambda g(t) = -\left(\frac{g'(t)}{t^{2m}}\right)', \quad \lambda t^{2m} g(t) = -g''(t) + \frac{2m}{t}\, g'(t),$$

we obtain

$$\begin{aligned}
R &= \lambda \int_0^1 K(s,t) g(s)\, ds \\
&= \frac{\lambda}{2m+1}\left[\int_0^1 g(s)\, ds - t^{2m+1}\int_0^t g(s)\, ds - \int_t^1 s^{2m+1} g(s)\, ds\right] \\
&= \frac{-1}{2m+1}\left[\int_0^1 \left(\frac{g'(s)}{s^{2m}}\right)' ds - t^{2m+1}\int_0^t \left(\frac{g'(s)}{s^{2m}}\right)' ds \right. \\
&\qquad\qquad \left. - \int_t^1 \left(s g''(s) - 2m g'(s)\right)\, ds\right] \\
&= \frac{-1}{2m+1}(1 - t^{2m+1}) + g(t) = -K(0,t) + g(t),
\end{aligned}$$

which gives the integral equation to be proved.

3.3.2 Consider $\mathrm{E}\left(e^{i\theta(S_2-\mu)/\sigma}\right)$, where $\mu = a^2/(2m+1)$ and $\sigma^2 = 4a^2/(m+$

$1)(4m+3)$. Noting that

$$J_\nu(\kappa) = \left(\frac{\kappa}{2}\right)^{\nu} \frac{1}{\Gamma(\nu+1)} \left(1 - \frac{\Gamma(\nu+1)}{\Gamma(\nu+2)} \frac{\kappa^2}{4} + O(\kappa^4)\right),$$
$$J_{-\nu}(\kappa) = \left(\frac{\kappa}{2}\right)^{-\nu} \frac{1}{\Gamma(-\nu+1)} \left(1 - \frac{\Gamma(-\nu+1)}{\Gamma(-\nu+2)} \frac{\kappa^2}{4} + O(\kappa^4)\right),$$

where $\kappa = \eta/\sqrt{\sigma} = \sqrt{2i\theta/\sigma}/(m+1)$, we obtain

$$\begin{aligned}
A &= \Gamma(-\nu+1)J_{-\nu}(\kappa) \bigg/ \left(\frac{\kappa}{2}\right)^{-\nu} = 1 + O\left(|a|^{-1}\right),\\
B &= \frac{ia^2\theta/\sigma}{2m+1} \frac{\Gamma(\nu+1)J_\nu(\kappa)}{\Gamma(-\nu+1)J_{-\nu}(\kappa)} \left(\frac{\kappa}{2}\right)^{-2\nu}\\
&= \frac{ia^2\theta/\sigma}{2m+1} \left[1 + \left(\frac{1}{1-\nu} - \frac{1}{1+\nu}\right)\frac{\kappa^2}{4} + O(\kappa^4)\right]\\
&= \frac{i\mu\theta}{\sigma} - \frac{\theta^2}{2} + O\left(|a|^{-1}\right),\\
\mathrm{E}\left(e^{i\theta(S_2-\mu)/\sigma}\right) &= A^{-1/2}e^{B}e^{-i\mu\theta/\sigma} \to \exp\left[-\frac{\theta^2}{2}\right] \quad (|a| \to \infty),
\end{aligned}$$

which gives the required result. Alternatively, we can use (3.35) to compute $a^2K(0,0) = a^2/(2m+1)$ and

$$\mathrm{Var}\left(2a\int_0^1 K(0,t)\,dW(t)\right) = 4a^2\int_0^1 K^2(0,t)\,dt = \frac{4a^2}{(m+1)(4m+3)}.$$

3.3.3 We prove that the differential equation with the two boundary conditions leads to the integral equation (3.32). Putting $g(t) = \Gamma(0,t;\lambda)$ and using

$$\lambda g(t) = \frac{2}{m}\left(\frac{g'(t)}{t^{m-1}}\right)', \quad \lambda t^m g(t) = \frac{2}{m}\left(tg''(t) - (m-1)g'(t)\right),$$

we obtain

$$\begin{aligned}
\lambda\int_0^1 K(s,t)g(s)\,ds &= \frac{\lambda}{2}\left[t^m\int_0^t g(s)\,ds + \int_t^1 s^m g(s)\,ds\right]\\
&= \frac{1}{m}\left[t^m\int_0^t \left(\frac{g'(s)}{s^{m-1}}\right)' ds + \int_t^1 \left(sg''(s) - (m-1)g'(s)\right) ds\right]\\
&= g(t) - \frac{1}{2}t^m = g(t) - K(0,t).
\end{aligned}$$

3.4.1 Let us define

$$S_{N3} = \sum_{n=1}^{N} \left(\frac{1}{\lambda_n} Z_n^2 + \frac{2c_n}{\sqrt{\lambda_n}} Z_n Z \right) + Z^2 \int_0^1 m^2(t)\, dt = \boldsymbol{W}' A \boldsymbol{W},$$

where $\boldsymbol{W} = (Z_1, \ldots, Z_N, Z)'$ and

$$A = \begin{pmatrix} \Lambda & \boldsymbol{h} \\ \boldsymbol{h}' & q \end{pmatrix}, \qquad \Lambda = \mathrm{diag}\left(\frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_N} \right),$$

$$\boldsymbol{h} = \left(\frac{c_1}{\sqrt{\lambda_1}}, \ldots, \frac{c_N}{\sqrt{\lambda_N}} \right)', \quad c_n = \int_0^1 f_n(t) m(t)\, dt, \quad q = \int_0^1 m^2(t)\, dt.$$

Since

$$\begin{aligned} \mathrm{E}\left(e^{i\theta S_{N3}}\right) &= |I_{N+1} - 2i\theta A|^{-1/2} \\ &= \left[|I_N - 2i\theta\Lambda| \left\{ 1 - 2iq\theta + 4\theta^2 \boldsymbol{h}'(I_N - 2i\theta\Lambda)^{-1}\boldsymbol{h} \right\} \right]^{-1/2} \\ &= \prod_{n=1}^{N} \left(1 - \frac{2i\theta}{\lambda_n} \right)^{-1/2} \left(1 - 2iq\theta + 4\theta^2 \sum_{n=1}^{N} \frac{c_n^2}{\lambda_n - 2i\theta} \right)^{-1/2} \end{aligned}$$

and $S_{N3} \Rightarrow S_3$ as $N \to \infty$, we have, using (3.17),

$$\begin{aligned} \lim_{N\to\infty} \mathrm{E}\left(e^{i\theta S_{N3}}\right) &= \prod_{n=1}^{\infty} \left(1 - \frac{2i\theta}{\lambda_n} \right)^{-1/2} \left(1 - 2iq\theta + 4\theta^2 \sum_{n=1}^{\infty} \frac{c_n^2}{\lambda_n - 2i\theta} \right)^{-1/2} \\ &= \left(D(2i\theta) \right)^{-1/2} \left(1 - 2i\theta \int_0^1 m^2(t)\, dt \right. \\ &\qquad \left. + 4\theta^2 \int_0^1 h(t; 2i\theta) m(t)\, dt \right)^{-1/2}, \end{aligned}$$

which establishes the required result described in Theorem 3.12.

3.4.2 It follows from Theorem 3.12 that the c.f. of S_3 is given by

$$\mathrm{E}\left(e^{i\theta S_3}\right) = (D(2i\theta))^{-1/2} \left(1 - 2i\theta \int_0^1 m^2(t)\, dt + 4\theta^2 \int_0^1 h(t; 2i\theta) m(t)\, dt \right)^{-1/2},$$

where $D(\lambda) = \cos\sqrt{\lambda}$ and (3.20) gives

$$\begin{aligned} &\lambda \int_0^1 m^2(t)\, dt + \lambda^2 \int_0^1 h(t; \lambda) m(t)\, dt \\ &= \lambda^2 \left[-\frac{b(2a+b)}{\lambda^2} + 2ab \frac{\sec\sqrt{\lambda}}{\lambda^2} + (a^2\lambda + b^2) \frac{\tan\sqrt{\lambda}}{\lambda^2\sqrt{\lambda}} \right], \end{aligned}$$

where $\lambda = 2i\theta$. Then we obtain the c.f. of S_3 described in this problem.

3.4.3 We obtain, from (3.24),

$$\lambda \int_0^1 m^2(t)\,dt + \lambda^2 \int_0^1 h(t;\lambda)m(t)\,dt = \frac{a^2\lambda \sin\mu/\mu}{\cos\mu - \gamma\sin\mu/\mu},$$

where $\lambda = 2i\theta$ and $a^2 = -1/2\gamma$. Then, noting that the FD of $\mathrm{Cov}(X(s), X(t))$ is given by

$$D(\lambda) = e^{\gamma}\left(\cos\mu - \gamma\frac{\sin\mu}{\mu}\right), \quad \mu = \sqrt{\lambda - \gamma^2},$$

Theorem 3.12 yields the c.f. of S_3 described in this problem.

3.5.1 Let us define

$$S_{N4} = \sum_{n=1}^{N} \frac{1}{\lambda_n}\left(Z_n^2 + 2af_n(0)Z_nZ\right) + bZ^2 = \boldsymbol{W}'A\boldsymbol{W},$$

where $\boldsymbol{W} = (Z_1, \ldots, Z_N, Z)'$ and

$$A = \begin{pmatrix} \Lambda & \boldsymbol{h} \\ \boldsymbol{h}' & b \end{pmatrix}, \qquad \Lambda = \mathrm{diag}\left(\frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_N}\right),$$

$$\boldsymbol{h} = \left(\frac{af_1(0)}{\lambda_1}, \ldots, \frac{af_N(0)}{\lambda_N}\right)'.$$

Since

$$\begin{aligned}
\mathrm{E}\left(e^{i\theta S_N}\right) &= |I_{N+1} - 2i\theta A|^{-1/2} \\
&= \left[|I_N - 2i\theta\Lambda|\left\{1 - 2ib\theta + 4\theta^2\boldsymbol{h}'(I_N - 2i\theta\Lambda)^{-1}\boldsymbol{h}\right\}\right]^{-1/2} \\
&= \prod_{n=1}^{N}\left(1 - \frac{2i\theta}{\lambda_n}\right)^{-1/2}\left(1 - 2ib\theta + 4a^2\theta^2\sum_{n=1}^{N}\frac{f_n^2(0)}{\lambda_n(\lambda_n - 2i\theta)}\right)^{-1/2}
\end{aligned}$$

and $S_{N4} \Rightarrow S_4$ as $N \to \infty$, we have

$$\lim_{N\to\infty} \mathrm{E}\left(e^{i\theta S_{N4}}\right) = (D(2i\theta))^{-1/2}\left(1 - 2ib\theta + 4a^2\theta^2\sum_{n=1}^{\infty}\frac{f_n^2(0)}{\lambda_n(\lambda_n - 2i\theta)}\right)^{-1/2},$$

where

$$\sum_{n=1}^{\infty}\frac{f_n^2(0)}{\lambda_n(\lambda_n - 2i\theta)} = \frac{1}{2i\theta}\sum_{n=1}^{\infty}\left(\frac{f_n^2(0)}{\lambda_n - 2i\theta} - \frac{f_n^2(0)}{\lambda_n}\right) = \frac{1}{2i\theta}\,\left(\Gamma(0,0;2i\theta) - K(0,0)\right).$$

Then we obtain the c.f. of S_4 described in Theorem 3.16.

3.5.2 It is recognized that S_4 is obtained from S_2 in (3.33) by replacing a by aZ, where $a = 1/\sqrt{-2\gamma}$. Thus it follows from (3.34) that

$$\mathrm{E}\left(e^{i\theta S_4}\middle| Z\right) = e^{-\gamma/2}\left(\cos\nu - \gamma\frac{\sin\nu}{\nu}\right)^{-1/2}\exp\left[\frac{ia^2Z^2\theta\ \sin\nu/\nu}{\cos\nu - \gamma\ \sin\nu/\nu}\right].$$

Taking expectations on both sides with respect to Z, we obtain the c.f. of S_4 described in Example 3.17.

3.5.3 It follows from (3.50) with $b = a^2K(0,0)$ that

$$\mathrm{E}\left(e^{i\theta S_4}\right) = (D(2i\theta))^{-1/2}\left(1 - 2ia^2\Gamma(0,0;2i\theta)\right)^{-1/2},$$

where $D(\lambda)$ is given in (2.133), whereas $\Gamma(0,0;\lambda)$ is given in (3.38). Thus we obtain the c.f. of S_4 described in (3.53).

3.5.4 It follows from (3.50) with $K(0,0) = 0$ and $b = 0$ that

$$\mathrm{E}\left(e^{i\theta S_4}\right) = (D(2i\theta))^{-1/2}\exp\left\{-\frac{i\theta}{2(m+1)}\right\}\left(1 - 2ia^2\Gamma(0,0;2i\theta)\right)^{-1/2},$$

where $D(\lambda)$ is given in (2.143), whereas $\Gamma(0,0;\lambda)$ is given in (3.43). Thus we obtain the c.f. of S_4 described in (3.55).

Chapter 4

4.2.1 Noting that $(x-\alpha)/\beta = -\cot\pi t$, we have

$$\begin{aligned}
\left.\frac{\partial F(x;\boldsymbol{\xi})}{\partial\alpha}\right|_{x(t;\boldsymbol{\xi})} &= \left.\frac{-1}{\beta\pi}\frac{1}{1+(x-\alpha)^2/\beta^2}\right|_{x(t;\boldsymbol{\xi})} = -\frac{1}{\beta\pi}\frac{1}{1+\cot^2\pi t} \\
&= \frac{-1}{\beta\pi}\ \sin^2\pi t, \\
\left.\frac{\partial F(x;\boldsymbol{\xi})}{\partial\beta}\right|_{x(t;\boldsymbol{\xi})} &= \left.\frac{-(x-\alpha)}{\beta^2\pi}\frac{1}{1+(x-\alpha)^2/\beta^2}\right|_{x(t;\boldsymbol{\xi})} = \frac{\cot\pi t}{\beta\pi}\frac{1}{1+\cot^2\pi t} \\
&= \frac{1}{2\beta\pi}\ \sin 2\pi t.
\end{aligned}$$

4.2.2 Using the formula described in the problem, we obtain

$$
\begin{aligned}
\mathrm{E}\left[\left(\frac{\partial \log f}{\partial \alpha}\right)^2\right] &= \frac{4}{\beta\pi}\int_{-\infty}^{\infty} \frac{(x-\alpha)^2/\beta^4}{(1+(x-\alpha)^2/\beta^2)^2}\frac{1}{1+(x-\alpha)^2/\beta^2}\,dx \\
&= \frac{8}{\beta^2\pi}\int_0^{\infty}\frac{u^2}{(1+u^2)^3}\,du = \frac{8}{\beta^2\pi}\times\frac{1}{2}\times B\left(3-\frac{3}{2},\frac{3}{2}\right) \\
&= \frac{1}{2\beta^2}.
\end{aligned}
$$

Similarly we have

$$
\begin{aligned}
\mathrm{E}\left[\left(\frac{\partial \log f}{\partial \beta}\right)^2\right] &= \mathrm{E}\left[\left(-\frac{1}{\beta}-\frac{2(X-\alpha)^2/\beta^3}{1+(X-\alpha)^2/\beta^2}\right)^2\right] \\
&= \mathrm{E}\left[\frac{1}{\beta^2}-\frac{4(X-\alpha)^2/\beta^4}{1+(X-\alpha)^2/\beta^2}+\frac{4(X-\alpha)^4/\beta^6}{(1+(X-\alpha)^2/\beta^2)^2}\right] \\
&= \frac{1}{\beta^2}+\frac{8}{\beta^2\pi}\int_0^{\infty}\left(\frac{-u^2}{(1+u^2)^2}+\frac{u^4}{(1+u^2)^3}\right)du \\
&= \frac{1}{\beta^2}+\frac{8}{\beta^2\pi}\left(-\frac{1}{2}B\left(\frac{1}{2},\frac{3}{2}\right)+\frac{1}{2}B\left(\frac{1}{2},\frac{5}{2}\right)\right) \\
&= \frac{1}{2\beta^2}.
\end{aligned}
$$

On the other hand, it holds that

$$
\begin{aligned}
\mathrm{E}\left[\left(\frac{\partial \log f}{\partial \alpha}\right)\left(\frac{\partial \log f}{\partial \beta}\right)\right] &= \mathrm{E}\left[\frac{2(X-\alpha)/\beta^2}{1+(X-\alpha)^2/\beta^2}\left(-\frac{1}{\beta}+\frac{2(X-\alpha)^2/\beta^3}{1+(X-\alpha)^2/\beta^2}\right)\right] \\
&= 0
\end{aligned}
$$

because the integrand is an odd function.

4.2.3 It is clear that the integral equation leads to the differential equation with the two boundary conditions. Conversely, denoting by R the right side of the integral equation to be proved and using $\lambda h(s) = \lambda h(s;\lambda) = -h''(s) - m(s)$, we have

$$
\begin{aligned}
R &= \int_0^1 K(s,t)m(s)\,ds + \int_0^1 K(s,t)(-h''(s)-m(s))\,ds \\
&= -\int_0^1 K(s,t)h''(s)\,ds \\
&= -\int_0^t sh''(s)\,ds - t\int_t^1 h''(s)\,ds + t\int_0^1 sh''(s)\,ds \\
&= h(t),
\end{aligned}
$$

which is the left side of the integral equation to be proved.

4.2.4 Putting $T = (R^2(\boldsymbol{\xi}_0) - \mu)/\sigma$, where

$$\mu = \frac{3(\gamma_1/\beta_0)^2 + (\gamma_2/\beta_0)^2}{8\pi^2}, \quad \sigma^2 = \left(\frac{1}{12\pi^2} + \frac{5}{8\pi^4}\right)\left(\frac{\gamma_1}{\beta_0}\right)^2 + \frac{1}{8\pi^4}\left(\frac{\gamma_2}{\beta_0}\right)^2,$$

we have, from (4.14),

$$\begin{aligned} \mathrm{E}\left(e^{i\theta T}\right) = & \left(\frac{\sin\sqrt{\lambda/\sigma}}{\sqrt{\lambda/\sigma}}\right)^{-1/2} \exp\left[\left(\frac{\gamma_1}{\beta_0}\right)^2 \left\{\frac{\lambda/\sigma}{4(4\pi^2 - \lambda/\sigma)} + \frac{4\pi^2}{(4\pi^2 - \lambda/\sigma)^2}\right.\right. \\ & \left.\left. \times \frac{1 - \cos\sqrt{\lambda/\sigma}}{\sin\sqrt{\lambda/\sigma}/\sqrt{\lambda/\sigma}}\right\} + \left(\frac{\gamma_2}{\beta_0}\right)^2 \frac{\lambda/\sigma}{4(4\pi^2 - \lambda/\sigma)} - \frac{\lambda\mu}{2\sigma}\right], \end{aligned}$$

where $\lambda = 2i\theta$. We obtain, as $\sigma \to \infty$,

$$\begin{aligned} \frac{\sin\sqrt{\lambda/\sigma}}{\sqrt{\lambda/\sigma}} &= 1 + O(\sigma^{-1}), \\ \frac{\lambda/\sigma}{4(4\pi^2 - \lambda/\sigma)} &= \frac{\lambda}{16\pi^2\sigma} + \frac{\lambda^2}{64\pi^4\sigma^2} + O(\sigma^{-3}), \\ \frac{4\pi^2}{(4\pi^2 - \lambda/\sigma)^2} &= \frac{1}{4\pi^2}\left(1 + \frac{\lambda}{2\pi^2\sigma} + \frac{3\lambda^2}{16\pi^4\sigma^2} + O(\sigma^{-3})\right), \\ \frac{1 - \cos\sqrt{\lambda/\sigma}}{\sin\sqrt{\lambda/\sigma}/\sqrt{\lambda/\sigma}} &= \frac{\lambda}{2\sigma} + \frac{\lambda^2}{24\sigma^2} + O(\sigma^{-3}). \end{aligned}$$

Thus we have

$$\begin{aligned} \mathrm{E}\left(e^{i\theta T}\right) = & \left(1 + O(\sigma^{-1})\right)^{-1/2} \exp\left[\frac{\lambda^2}{8\sigma^2}\left\{\left(\frac{1}{12\pi^2} + \frac{5}{8\pi^4}\right)\left(\frac{\gamma_1}{\beta_0}\right)^2\right.\right. \\ & \left.\left. + \frac{1}{8\pi^4}\left(\frac{\gamma_2}{\beta_0}\right)^2\right\} + O(\sigma^{-3})\right] \to \exp\left(\frac{\lambda^2}{8}\right) = \exp\left(-\frac{\theta^2}{2}\right). \end{aligned}$$

4.3.1 Putting $\psi(t) = \sqrt{2}\sin^2 \pi t$, we have, from Darling's formula in (2.159),

$$D(\lambda) = \frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}} - 2\int_0^1 \left(\int_0^t \psi'(s)\, \cos\sqrt{\lambda} s\, ds\right) \psi'(t)\, \cos\sqrt{\lambda}(1-t)\, dt,$$

where $\psi'(t) = \sqrt{2}\pi \sin 2\pi t$. Using computerized algebra we can obtain the FD given in (4.23). We also have, from Mercer's theorem,

$$K(s,t) = \min(s,t) - st - \frac{1}{2\pi^2}\, \sin 2\pi s\, \sin 2\pi t = \sum_{n=1}^{\infty} \frac{f_n(s) f_n(t)}{\lambda_n} - \frac{f_2(s) f_2(t)}{\lambda_2},$$

where $\lambda_n = n^2\pi^2$ and $f_n(t) = \sqrt{2}\sin n\pi t$. Since $K(s,t)$ excludes the eigenvalue λ_2 from the FD of $\min(s,t) - st$, we can also obtain the FD in (4.23).

4.3.2 It can be shown that the integral equation leads to

$$h''(t) + \lambda h(t) = -m(t) + 2\lambda a \sin 2\pi t, \tag{A.6}$$

with conditions

$$h(0) = h(1) = 0, \quad a = \int_0^1 h(s)\ \sin 2\pi s\, ds. \tag{A.7}$$

Conversely, assume that (A.6) and (A.7) hold. Then

$$h(s) = c_1 \cos\sqrt{\lambda}s + c_2 \sin\sqrt{\lambda}s + \frac{\gamma}{2\beta\pi}\left(\frac{1}{\lambda} + \frac{\cos 2\pi s}{4\pi^2 - \lambda}\right) - 2\lambda a \frac{\sin 2\pi s}{4\pi^2 - \lambda}$$

so that it follows from the last condition in (A.7) that

$$a = \int_0^1 h(s)\sin 2\pi s\, ds = \frac{2\pi}{4\pi^2 - \lambda}\left((1 - \cos\sqrt{\lambda})c_1 - c_2 \sin\sqrt{\lambda}\right) - \frac{\lambda a}{4\pi^2 - \lambda}.$$

Since it follows from the two boundary conditions in (A.7) that

$$h(0) - h(1) = (1 - \cos\sqrt{\lambda})c_1 - c_2 \sin\sqrt{\lambda} = 0,$$

we conclude that $a = 0$. Denoting by R the right side of the integral equation, we have

$$\begin{aligned} R &= \int_0^1 K(s,t)m(s)\, ds + \int_0^1 K(s,t)(-h''(s) - m(s) + 2\lambda a \sin 2\pi s)\, ds \\ &= -\int_0^1 K(s,t)h''(s)\, ds + 2\lambda a \int_0^1 K(s,t)\sin 2\pi s\, ds \\ &= h(t) + 2\lambda a \frac{\sin 2\pi t}{4\pi^2} = h(t), \end{aligned}$$

which is the left side of the integral equation to be proved.

4.3.3 It is clear that the integral equation leads to the differential equation

$$h''(t) + \lambda h(t) = -m(t) - 4\lambda a \cos 2\pi t, \quad m(t) = \frac{\gamma}{2\beta\pi}\ \sin 2\pi t \tag{A.8}$$

with conditions

$$h(0) = h(1) = 0, \quad a = \int_0^1 h(s)\ \sin^2 \pi s\, ds. \tag{A.9}$$

Conversely, assume that (A.8) and (A.9) hold. Then we have

$$h(s) = c_1 \cos \sqrt{\lambda} s + c_2 \sin \sqrt{\lambda} s + \frac{\gamma}{2\beta\pi} \frac{\sin 2\pi s}{4\pi^2 - \lambda} + 4\lambda a \frac{\cos 2\pi s}{4\pi^2 - \lambda}. \quad \text{(A.10)}$$

It follows from the last condition in (A.9) that

$$\begin{aligned} a &= \int_0^1 h(s) \sin^2 \pi s \, ds \\ &= \frac{2\pi^2}{\sqrt{\lambda}(4\pi^2 - \lambda)} \left(c_1 \sin \sqrt{\lambda} + c_2 (1 - \cos \sqrt{\lambda}) \right) - \frac{\lambda a}{4\pi^2 - \lambda} \end{aligned}$$

so that

$$a = \frac{1}{2\sqrt{\lambda}} \left(c_1 \sin \sqrt{\lambda} + c_2 (1 - \cos \sqrt{\lambda}) \right),$$

from which it holds that

$$\begin{aligned} \int_0^1 h(s) \cos 2\pi s \, ds &= \frac{-\sqrt{\lambda}}{4\pi^2 - \lambda} \left(c_1 \sin \sqrt{\lambda} + c_2 (1 - \cos \sqrt{\lambda}) \right) + \frac{2\lambda a}{4\pi^2 - \lambda} \\ &= 0. \end{aligned}$$

Denoting by R the right side of the integral equation, we have

$$\begin{aligned} R &= \int_0^1 K(s,t) m(s) \, ds + \int_0^1 K(s,t)(-h''(s) - m(s) - 4\lambda a \cos 2\pi s) \, ds \\ &= -\int_0^1 K(s,t) h''(s) \, ds - 4\lambda a \int_0^1 K(s,t) \cos 2\pi s \, ds \\ &= h(t) + 4 \sin^2 \pi t \int_0^1 h(s) \cos 2\pi s = h(t), \end{aligned}$$

which is the left side of the integral equation to be proved.

4.3.4 For $m(s)$ and $h(s)$ given in (A.8) and (A.10), respectively, we obtain

$$\begin{aligned} \int_0^1 h(s) m(s) \, ds &= \int_0^1 h(s) \frac{\gamma}{2\beta\pi} \sin 2\pi s \, ds \\ &= \frac{\gamma}{\beta(4\pi^2 - \lambda)} \left(c_1 (1 - \cos \sqrt{\lambda}) - c_2 \sin \sqrt{\lambda} \right) \\ &\quad + \frac{\gamma^2}{8\beta^2\pi^2} \frac{1}{4\pi^2 - \lambda}. \end{aligned}$$

Since the two boundary conditions $h(0) = h(1) = 0$ give $c_1(1 - \cos \sqrt{\lambda}) - c_2 \sin \sqrt{\lambda} = 0$, we arrive at the desired result.

Chapter 5

5.2.1 Since $L(\boldsymbol{\delta})$ is given by

$$L(\boldsymbol{\delta}) = -\frac{n}{2}\log 2\pi\sigma_\varepsilon^2 - \frac{1}{2}\log|I_n + \rho D_x CC' D_x| \\ -\frac{1}{2\sigma_\varepsilon^2}(\boldsymbol{y} - \beta_0\boldsymbol{x} - Z\boldsymbol{\gamma})'(I_n + \rho D_x CC' D_x)^{-1}(\boldsymbol{y} - \beta_0\boldsymbol{x} - Z\boldsymbol{\gamma}),$$

the MLEs of $\sigma_\varepsilon^2, \beta_0$ and $\boldsymbol{\gamma}$ under H_0 are given by

$$\tilde{\sigma}_\varepsilon^2 = \frac{1}{n}\boldsymbol{y}'M\boldsymbol{y}, \quad (\tilde{\beta}_0, \tilde{\boldsymbol{\gamma}}')' = \begin{pmatrix} \boldsymbol{x}'\boldsymbol{x} & \boldsymbol{x}'Z \\ Z'\boldsymbol{x} & Z'Z \end{pmatrix}^{-1} \begin{pmatrix} \boldsymbol{x}'\boldsymbol{y} \\ Z'\boldsymbol{y} \end{pmatrix}.$$

Then it it holds that, putting $\tilde{\boldsymbol{v}} = \boldsymbol{y} - \tilde{\beta}_0\boldsymbol{x} - Z\tilde{\boldsymbol{\gamma}} = M\boldsymbol{y}$,

$$\left.\frac{\partial L(\boldsymbol{\delta})}{\partial\rho}\right|_{H_0} = -\frac{1}{2}\operatorname{tr}(D_x CC' D_x) + \frac{1}{2\tilde{\sigma}_\varepsilon^2}\tilde{\boldsymbol{v}}' D_x CC' D_x \tilde{\boldsymbol{v}} \\ = -\frac{1}{2}\operatorname{tr}(D_x CC' D_x) + \frac{n}{2}\frac{\boldsymbol{y}'MD_x CC' D_x M\boldsymbol{y}}{\boldsymbol{y}'M\boldsymbol{y}}.$$

5.2.2 For Case (a) consider

$$\frac{1}{n}\boldsymbol{y}'M\boldsymbol{y} = \frac{1}{n}(\boldsymbol{\varepsilon} + \rho D_x C\boldsymbol{\eta})'M(\boldsymbol{\varepsilon} + \rho D_x C\boldsymbol{\eta}) \\ \overset{\mathfrak{D}}{=} \frac{1}{n}\boldsymbol{\zeta}'(I_n + \rho D_x CC' D_x)^{1/2} M (I_n + \rho D_x CC' D_x)^{1/2}\boldsymbol{\zeta} \\ \overset{\mathfrak{D}}{=} \frac{1}{n}\boldsymbol{\zeta}'M\boldsymbol{\zeta} + \frac{c}{n^{2m+3}}\boldsymbol{\zeta}'MD_x CC' D_x M\boldsymbol{\zeta},$$

where $\boldsymbol{\zeta} \sim \mathrm{N}(\boldsymbol{0}, \sigma_\varepsilon^2 I_n)$. It holds that $\boldsymbol{\zeta}'M\boldsymbol{\zeta}/n \to \sigma_\varepsilon^2$ in probability because

$$\frac{1}{n}\boldsymbol{\zeta}'M\boldsymbol{\zeta} = \frac{1}{n}\boldsymbol{\zeta}'\boldsymbol{\zeta} - \frac{1}{n}\boldsymbol{\zeta}'(I_n - M)\boldsymbol{\zeta} = \frac{1}{n}\boldsymbol{\zeta}'\boldsymbol{\zeta} + O_p(n^{-1}) \;\to\; \sigma_\varepsilon^2.$$

On the other hand, we have

$$\frac{1}{n^{2m+2}}\boldsymbol{\zeta}'MD_x CC' D_x M\boldsymbol{\zeta} \overset{\mathfrak{D}}{=} \frac{1}{n^{2m+2}}\boldsymbol{y}'MD_x CC' D_x M\boldsymbol{y},$$

which is assumed to converge to a nondegenerate distribution. Thus it holds that $\boldsymbol{y}'M\boldsymbol{y}/n \to \sigma_\varepsilon^2$ in probability. Cases (b) and (c) can be dealt with similarly.

5.3.1 Since $\boldsymbol{x} = (1^m, 2^m, \ldots, n^m)'$ and

$$A_n = C' D_x (I_n - \boldsymbol{x}\boldsymbol{x}'/\boldsymbol{x}'\boldsymbol{x}) D_x C = C' D_x D_x C - C' D_x \boldsymbol{x}\boldsymbol{x}' D_x C/\boldsymbol{x}'\boldsymbol{x},$$

we can deduce that the (j,k)th element of A_n is given by

$$A_n(j,k) = \sum_{\ell=\max(j,k)}^{n} \ell^{2m} - \sum_{\ell=j}^{n} \ell^{2m} \sum_{\ell=k}^{n} \ell^{2m} \Bigg/ \sum_{\ell=1}^{n} \ell^{2m}.$$

5.3.2 Define $b(\theta) = -\theta^2 + 2ic\theta/(m+1)^2$. Noting that $b(\theta) \to -\theta^2$ and $\nu \to 1$ as $m \to \infty$, and $\Gamma(\nu+1)J_\nu(z)/(z/2)^\nu \to 1$ as $z \to 0$, we have

$$\mathrm{E}\left(e^{i\theta(m+1)^2 R(m;c)}\right) = [D_1(\theta)D_2(\theta)]^{-1/2},$$

where, for $\theta > 0$,

$$\begin{aligned} D_1(\theta) &= D\left((m+1)^2\left(i\theta + \sqrt{b(\theta)}\right)\right) \\ &= \Gamma(\nu+1)J_\nu\left(\sqrt{i\theta + \sqrt{b(\theta)}}\right) \Bigg/ \left(\frac{1}{2}\sqrt{i\theta + \sqrt{b(\theta)}}\right)^\nu \to 2J_1(\sqrt{2i\theta})/\sqrt{2i\theta}, \\ D_2(\theta) &= D\left((m+1)^2\left(i\theta - \sqrt{b(\theta)}\right)\right) \\ &= \Gamma(\nu+1)J_\nu\left(\sqrt{i\theta - \sqrt{b(\theta)}}\right) \Bigg/ \left(\frac{1}{2}\sqrt{i\theta - \sqrt{b(\theta)}}\right)^\nu \to 1. \end{aligned}$$

When $\theta < 0$, we have $D_1(\theta) \to 1$ and $D_2(\theta) \to 2J_1(\sqrt{2i\theta})/\sqrt{2i\theta}$. Thus it holds that

$$\lim_{m\to\infty} \mathrm{E}\left(e^{i\theta(m+1)^2 R(m;c)}\right) = \left(2J_1(\sqrt{2i\theta})/\sqrt{2i\theta}\right)^{-1/2}.$$

5.3.3 The (j,k)th element $A_n(j,k)$ of $A_n = C'MC$ is given by

$$\begin{aligned} n + 1 - \max(j,k) - \frac{1}{n\sum_{\ell=1}^n \ell^{2q} - \left(\sum_{\ell=1}^n \ell^q\right)^2} \Bigg[(n-k+1)\Bigg\{(n-j+1)\sum_{\ell=1}^{n} \ell^{2q} \\ - \sum_{\ell=j}^{n} \ell^q \sum_{\ell=1}^{n} \ell^q\Bigg\} + \sum_{\ell=k}^{n} \ell^q \Bigg\{n\sum_{\ell=j}^{n} \ell^q - (n-j+1)\sum_{\ell=1}^{n} \ell^q\Bigg\}\Bigg]. \end{aligned}$$

Then we can find $K(s,t;q)$ such that $\left|A_n(j,k)/n - K(j/n,k/n;q)\right|$ converges uniformly to 0, where $K(s,t;q)$ is given by

$$\begin{aligned}
& 1-\max(s,t)-\frac{(q+1)^2}{q^2}(1-s)(1-t) \\
& +\frac{2q+1}{q^2}\left[(1-s)(1-t^{q+1})+(1-t)(1-s^{q+1})-(1-s^{q+1})(1-t^{q+1})\right] \\
= \;& \min(s,t)-st+(1-s)(1-t)-\frac{(q+1)^2}{q^2}(1-s)(1-t) \\
& +\frac{2q+1}{q^2}\left[(1-s)(1-t^{q+1})+(1-t)(1-s^{q+1})-(1-s^{q+1})(1-t^{q+1})\right] \\
= \;& \min(s,t)-st-\frac{2q+1}{q^2}st(1-s^q)(1-t^q).
\end{aligned}$$

5.3.4 Since

$$(X'X)^{-1} \approx \begin{pmatrix} n & n^2/2 & n^3/3 \\ n^2 & n^3/3 & n^4/4 \\ n^3/3 & n^4/4 & n^5/5 \end{pmatrix}^{-1} = \frac{2160}{n^9}\begin{pmatrix} n^8/240 & -n^7/60 & n^6/72 \\ -n^7/60 & 4n^6/45 & -n^5/12 \\ n^6/72 & -n^5/12 & n^4/12 \end{pmatrix},$$

and

$$C'X = \begin{pmatrix} \vdots & \vdots & \vdots \\ n-j+1 & \sum_{\ell=j}^{n} \ell & \sum_{\ell=j}^{n} \ell^2 \\ \vdots & \vdots & \vdots \end{pmatrix},$$

the (j,k)th element $H_n(j,k)$ of $C'X(X'X)^{-1}X'C$ is given by

$$\begin{aligned}
H_n(j,k) \;\approx\; & \frac{2160}{n^9}\left[(n-k)\left(\frac{(n-j)n^8}{240}-\frac{(n^2-j^2)n^7}{120}+\frac{(n^3-j^3)n^6}{216}\right)\right. \\
& +\frac{n^2-k^2}{2}\left(-\frac{(n-j)n^7}{60}+\frac{2(n^2-j^2)n^6}{45}-\frac{(n^3-j^3)n^5}{36}\right) \\
& \left.+\frac{n^3-k^3}{3}\left(\frac{(n-j)n^6}{72}-\frac{(n^2-j^2)n^5}{24}+\frac{(n^3-j^3)n^4}{36}\right)\right].
\end{aligned}$$

Thus we can obtain the uniform limit $K(s,t)$ of $A_n(j,k)/n$ as

$$\begin{aligned}
K(s,t) \;=\; & 1-\max(s,t)-\Big[(1-t)\left(9(1-s)-18(1-s^2)+10(1-s^3)\right) \\
& +(1-t^2)\left(-18(1-s)+48(1-s^2)-30(1-s^3)\right) \\
& +(1-t^3)\left(10(1-s)-30(1-s^2)+20(1-s^3)\right)\Big] \\
=\; & \min(s,t)-st-2st(1-s)(1-t)(4-5s-5t+10st).
\end{aligned}$$

5.3.5 Since $My = M(\varepsilon + C\eta) \overset{\mathcal{D}}{=} M(I_n + \rho CC')^{1/2}\zeta$, it holds that

$$
\begin{aligned}
& P\left(\frac{1}{n^{1+\tau}}\frac{y'MCC'My}{y'My} \le x\right) \\
= \ & P\left((\varepsilon + C\eta)'\left(\frac{xM}{n} - \frac{MCC'M}{n^{2+\tau}}\right)(\varepsilon + C\eta) \ge 0\right) \\
= \ & P\left(\zeta' M\left(\frac{xM}{n} - \frac{MCC'M}{n^{2+\tau}}\right)M(I_n + \rho CC')\zeta \ge 0\right) \\
= \ & P\left(\zeta'\left(\frac{xM}{n} + \frac{\rho xMCC'M}{n} - \frac{MCC'M}{n^{2+\tau}} - \frac{\rho(MCC'M)^2}{n^{2+\tau}}\right)\zeta \ge 0\right) \\
= \ & P\big(\zeta' Q(\tau)\zeta \ge 0\big),
\end{aligned}
$$

where $\rho = c/n^{\kappa}$ has been inserted.

5.4.1 Define $b(\theta) = -\theta^2 + 2ic\theta/(m+1)$. Noting that $b(\theta) \to -\theta^2$ and $\nu \to 1$ as $m \to \infty$, and $\Gamma(-\nu+1)J_{-\nu}(z)/(z/2)^{-\nu} \to 1$ as $z \to 0$, we have

$$\mathrm{E}\left(e^{i\theta(m+1)S(m;c)}\right) = [D_1(\theta)D_2(\theta)]^{-1/2},$$

where, for $\theta > 0$,

$$
\begin{aligned}
D_1(\theta) &= D\left((m+1)\left(i\theta + \sqrt{b(\theta)}\right)\right) \\
&= \Gamma(-\nu+1)J_{-\nu}\left(\frac{\sqrt{(m+1)(i\theta+\sqrt{b(\theta)})}}{m+1}\right)\Bigg/\left(\frac{\sqrt{(m+1)(i\theta+\sqrt{b(\theta)})}}{2(m+1)}\right)^{-\nu} \\
&= \Gamma(-\nu+1)\sum_{k=0}^{\infty}\frac{\left(-\left(i\theta+\sqrt{b(\theta)}\right)/4(m+1)\right)^k}{k!\,\Gamma(-\nu+1+k)} \\
&\to 1 - i\theta, \\
D_2(\theta) &= D\left((m+1)\left(i\theta - \sqrt{b(\theta)}\right)\right) \\
&\to D(0) = 1.
\end{aligned}
$$

When $\theta < 0$, we have $D_1(\theta) \to 0$ and $D_2(\theta) \to 1 - i\theta$. Thus it holds that

$$\lim_{m\to\infty}\mathrm{E}\left(e^{i\theta(m+1)S(m)}\right) = (1 - i\theta)^{-1/2}.$$

5.4.2 We can easily obtain the mean and variance by using

$$\mathrm{E}(R(m;0)) = \int_0^1 K(t,t;m)\,dt, \quad \mathrm{Var}(R(m;0)) = 2\int_0^1\int_0^1 K^2(s,t;m)\,ds\,dt.$$

Alternatively, we have $\mathrm{E}\left(e^{i\theta R(m;0)}\right) = (D_R(2i\theta))^{-1/2}$, where

$$\begin{aligned} D_R(\lambda) &= \Gamma(\nu+1)J_\nu\left(\frac{\sqrt{\lambda}}{m+1}\right)\Bigg/\left(\frac{\sqrt{\lambda}}{2(m+1)}\right)^{\nu} \\ &= 1 - \frac{\Gamma(\nu+1)}{\Gamma(\nu+2)}\frac{\lambda}{4(m+1)^2} + \frac{\Gamma(\nu+1)}{2\Gamma(\nu+3)}\frac{\lambda^2}{16(m+1)^4} + \lambda^3 O\left(m^{-6}\right) \\ &= 1 - \frac{\lambda}{2(m+1)(4m+3)} + \frac{\lambda^2}{8(m+1)^2(4m+3)(6m+5)} + \lambda^3 O\left(m^{-6}\right). \end{aligned}$$

We also have $\mathrm{E}\left(e^{i\theta S(m;0)}\right) = (D_S(2i\theta))^{-1/2}$, where

$$\begin{aligned} D_S(\lambda) &= \Gamma(-\nu+1)J_{-\nu}\left(\frac{\sqrt{\lambda}}{m+1}\right)\Bigg/\left(\frac{\sqrt{\lambda}}{2(m+1)}\right)^{-\nu} \\ &= 1 - \frac{\Gamma(-\nu+1)}{\Gamma(-\nu+2)}\frac{\lambda}{4(m+1)^2} + \frac{\Gamma(-\nu+1)}{2\Gamma(-\nu+3)}\frac{\lambda^2}{16(m+1)^4} + \lambda^3 O\left(m^{-5}\right) \\ &= 1 - \frac{\lambda}{2(m+1)} + \frac{\lambda^2}{8(m+1)^2(2m+3)} + \lambda^3 O\left(m^{-5}\right). \end{aligned}$$

Then moments can be derived from Theorem 2.4.

5.4.3 Since $z = (1^q, 2^q, \ldots, n^q)'$ and

$$A_n = C'(I_n - zz'/z'z)C = C'C - C'zz'C/z'z,$$

we deduce that the (j,k)th element of A_n is given by

$$A_n(j,k) = n + 1 - \max(j,k) - \sum_{\ell=j}^{n} \ell^q \sum_{\ell=k}^{n} \ell^q \Bigg/ \sum_{\ell=1}^{n} \ell^{2q}.$$

Thus $A_n(j,k)/n$ converges uniformly to

$$\begin{aligned} K(s,t;q) &= 1 - \max(s,t) - \int_s^1 u^q\, du \int_t^1 v^q\, dv \Bigg/ \int_0^1 w^{2q}\, dw \\ &= 1 - \max(s,t) - \frac{2q+1}{(q+1)^2}(1 - s^{q+1})(1 - t^{q+1}). \end{aligned}$$

5.4.4 Since

$$(Z'Z)^{-1} \approx \begin{pmatrix} n^3/3 & n^4/4 \\ n^4/4 & n^5/5 \end{pmatrix}^{-1} = \frac{240}{n^8}\begin{pmatrix} n^5/5 & -n^4/4 \\ -n^4/4 & n^3/3 \end{pmatrix},$$

and the jth row of $C'Z$ is given by $(\sum_{\ell=j}^n \ell,\ \sum_{\ell=j}^n \ell^2)$, the (j,k)th element

$H_n(j,k)$ of $C'Z(Z'Z)^{-1}Z'C$ is given by

$$
\begin{aligned}
H_n(j,k) \;\approx\; & \frac{240}{n^8}\left[\frac{n^2-k^2}{2}\left(\frac{(n^2-j^2)n^5}{10}-\frac{(n^3-j^3)n^4}{12}\right)\right. \\
& \left.+\frac{n^3-k^3}{3}\left(-\frac{(n^2-j^2)n^4}{8}+\frac{(n^3-j^3)n^3}{9}\right)\right].
\end{aligned}
$$

Thus we can obtain the uniform limit $K(s,t)$ of $A_n(j,k)/n$ as

$$
\begin{aligned}
K(s,t) \;=\; & 1-\max(s,t)-240\left[\frac{1-t^2}{2}\left(\frac{1-s^2}{10}-\frac{1-s^3}{12}\right)\right. \\
& \left.+\frac{1-t^3}{3}\left(-\frac{1-s^2}{8}+\frac{1-s^3}{9}\right)\right] \\
=\; & 1-\max(s,t)+\frac{2}{9}(1-s)(1-t) \\
& \times\left(5(s^2+t^2-8s^2t^2+st(s+t))-4(1+s)(1+t)\right).
\end{aligned}
$$

Chapter 6

6.1.1 Putting $\boldsymbol{\eta} = \boldsymbol{y} - X\boldsymbol{\beta}$, we have

$$
\begin{aligned}
\frac{\partial L}{\partial \alpha} \;=\; & -\frac{1}{2}\mathrm{tr}\left(\Omega^{-1}(\alpha)\frac{\partial \Omega(\alpha)}{\partial \alpha}\right)+\frac{1}{2\sigma_\varepsilon^2}\boldsymbol{\eta}'\Omega^{-1}(\alpha)\frac{\partial \Omega(\alpha)}{\partial \alpha}\Omega^{-1}(\alpha)\boldsymbol{\eta}, \\
\frac{\partial^2 L}{\partial \alpha^2} \;=\; & -\frac{1}{2}\mathrm{tr}\left(-\left(\Omega^{-1}(\alpha)\frac{\partial \Omega(\alpha)}{\partial \alpha}\right)^2+\Omega^{-1}(\alpha)\frac{\partial^2 \Omega(\alpha)}{\partial \alpha^2}\right) \\
& +\frac{1}{2\sigma_\varepsilon^2}\boldsymbol{\eta}'\left(-2\left(\Omega^{-1}(\alpha)\frac{\partial \Omega(\alpha)}{\partial \alpha}\right)^2\Omega^{-1}(\alpha)+\Omega^{-1}(\alpha)\frac{\partial^2 \Omega(\alpha)}{\partial \alpha^2}\Omega^{-1}(\alpha)\right)\boldsymbol{\eta},
\end{aligned}
$$

which yields

$$
\left.\frac{\partial^2 L}{\partial \alpha^2}\right|_{H_0} = -\frac{1}{2}\mathrm{tr}(-I_n+2\Omega^{-1})+\frac{1}{2\tilde{\sigma}_\varepsilon^2}\,\tilde{\boldsymbol{\eta}}'(-2\Omega^{-1}+2\Omega^{-2})\tilde{\boldsymbol{\eta}}. \qquad \text{(A.11)}
$$

Since

$$
\mathrm{tr}(\Omega^{-1}) = \mathrm{tr}\left(CC'-\frac{1}{n+1}C\boldsymbol{e}\boldsymbol{e}'C'\right) = \frac{n(n+2)}{6}, \quad \tilde{\sigma}_\varepsilon^2 = \frac{1}{n}\,\tilde{\boldsymbol{\eta}}'\Omega^{-1}\tilde{\boldsymbol{\eta}},
$$

substituing these into (A.11) gives the required result.

6.1.2 It holds that

$$\begin{aligned}\tilde{\boldsymbol{\eta}}'\Omega^{-1}\tilde{\boldsymbol{\eta}} &\stackrel{\mathcal{D}}{=} \boldsymbol{\zeta}'\Omega^{1/2}(\alpha)\tilde{N}'\Omega^{-1}\tilde{N}\Omega^{1/2}(\alpha)\boldsymbol{\zeta} \\ &\stackrel{\mathcal{D}}{=} \boldsymbol{\zeta}'M\Omega^{-1/2}\Omega(\alpha)\Omega^{-1/2}M\boldsymbol{\zeta} \\ &\stackrel{\mathcal{D}}{=} \boldsymbol{\zeta}'M\Omega^{-1/2}\left(\alpha\Omega + (1-\alpha)^2 I_n\right)\Omega^{-1/2}M\boldsymbol{\zeta} \\ &= \alpha\boldsymbol{\zeta}'M\boldsymbol{\zeta} + \frac{c^2}{n^2}\boldsymbol{\zeta}'M\Omega^{-1}M\boldsymbol{\zeta},\end{aligned}$$

where $M = I_n - \Omega^{-1/2}X(X'\Omega^{-1}X)^{-1}X'\Omega^{-1/2} = M^2$. It holds that $\alpha\boldsymbol{\zeta}'M\boldsymbol{\zeta}/n \to \sigma_\varepsilon^2$ in probability, whereas $\boldsymbol{\zeta}'M\Omega^{-1}M\boldsymbol{\zeta}/n^3 = O_p(n^{-1})$, which establishes (6.12).

6.1.3 Consider

$$\frac{1}{n^\tau} R_n = \frac{\boldsymbol{\zeta}'\left[\dfrac{\alpha}{n^{\tau+2}} A_n^2\Omega + \dfrac{c^2}{n^{\tau+2\kappa+2}} A_n^2\right]\boldsymbol{\zeta}}{\boldsymbol{\zeta}'\left[\dfrac{\alpha}{n} A_n\Omega + \dfrac{c^2}{n^{2\kappa+1}} A_n\right]\boldsymbol{\zeta}}.$$

Since it holds that

$$\frac{1}{n^\tau} R_n = \begin{cases} \dfrac{\dfrac{c^2}{n^4}\boldsymbol{\zeta}'A_n^2\boldsymbol{\zeta} + o_p(1)}{\dfrac{\alpha}{n}\boldsymbol{\zeta}'A_n\Omega\boldsymbol{\zeta} + o_p(1)} & (1/2 < \kappa < 1,\ \tau = 2 - 2\kappa) \\[2ex] \dfrac{\dfrac{c^2}{n^4}\boldsymbol{\zeta}'A_n^2\boldsymbol{\zeta} + o_p(1)}{\sigma_\varepsilon^2 + \dfrac{c^2}{n^2}\boldsymbol{\zeta}'A_n\boldsymbol{\zeta} + o_p(1)} & (\kappa = 1/2,\ \tau = 1), \end{cases}$$

and, for $0 \le \kappa < 1/2$ and $\tau = 1$,

$$\frac{1}{n^\tau} R_n = \frac{\boldsymbol{\zeta}'\left[\dfrac{\alpha}{n^{4-2\kappa}} A_n^2\Omega + \dfrac{c^2}{n^4} A_n^2\right]\boldsymbol{\zeta}}{\boldsymbol{\zeta}'\left[\dfrac{\alpha}{n^{2-2\kappa}} A_n\Omega + \dfrac{c^2}{n^2} A_n\right]\boldsymbol{\zeta}} = \frac{\dfrac{c^2}{n^4}\boldsymbol{\zeta}'A_n^2\boldsymbol{\zeta} + o_p(1)}{\dfrac{c^2}{n^2}\boldsymbol{\zeta}'A_n\boldsymbol{\zeta} + o_p(1)},$$

we obtain the required convergence.

6.2.1 The jth row of $C'(\alpha)$ and the kth column of $C(\alpha)$ are given, respectively, by

$$(0, \ldots, 0, 1, \alpha, \ldots, \alpha^{n-j}), \quad (0, \ldots, 0, 1, \alpha, \ldots, \alpha^{n-k})',$$

so that the (j,k)th element H_{jk} of $C'(\alpha)C(\alpha)$ is, for $j \le k$,

$$H_{jk} = \sum_{i=0}^{n-k} \alpha^{k-j+2i}.$$

We have

$$\begin{aligned}\left.\frac{dH_{jk}}{d\alpha}\right|_{\alpha=1} &= \left.\sum_{i=0}^{n-k} (k-j+2i)\alpha^{k-j+2i-1}\right|_{\alpha=1} = (n-j)(n-k+1) \\ &= (n-j+1)(n-k+1) - (n-k+1),\end{aligned}$$

which gives the required result.

6.2.2 It is clear that $\gamma = 1$ for Case (A). For Case (B) we have

$$\boldsymbol{e}'M\boldsymbol{e} = \boldsymbol{e}'\boldsymbol{e} - (\boldsymbol{e}'C\boldsymbol{e})^2/\boldsymbol{e}'C'C\boldsymbol{e} = n - (n^2/2)^2/(n^3/3) + o(n) = n/4 + o(n)$$

so that $\gamma = 1/4$. For Case (C) we have

$$\boldsymbol{e}'M\boldsymbol{e} = \boldsymbol{e}'\boldsymbol{e} - (\boldsymbol{e}'C\boldsymbol{d})^2/(\boldsymbol{d}'C'C\boldsymbol{d}) = n - (n^3/6)^2/(n^5/20) + o(1) = 4n/9 + o(n)$$

so that $\gamma = 4/9$. For Case (D) we have

$$\begin{aligned}\boldsymbol{e}'M\boldsymbol{e} &= \boldsymbol{e}'\boldsymbol{e} - \boldsymbol{e}'C(\boldsymbol{e},\boldsymbol{d})\left[\begin{pmatrix}\boldsymbol{e}'\\ \boldsymbol{d}'\end{pmatrix} C'C(\boldsymbol{e},\boldsymbol{d})\right]^{-1}\begin{pmatrix}\boldsymbol{e}'\\ \boldsymbol{d}'\end{pmatrix} C'\boldsymbol{e} \\ &= n - (n^2/2, n^3/6)\begin{pmatrix} n^3/3 & n^4/8 \\ n^4/8 & n^5/20\end{pmatrix}^{-1}\begin{pmatrix} n^2/2 \\ n^3/6\end{pmatrix} + o(n) \\ &= n/9 + o(n)\end{aligned}$$

so that $\gamma = 1/9$.

6.2.3 Using (6.23), we have

$$\begin{aligned}\frac{1}{n}\boldsymbol{y}'C'MC\boldsymbol{y} &\overset{\mathcal{D}}{=} \frac{1}{n}\boldsymbol{\varepsilon}'C'MC\Psi(\alpha)\boldsymbol{\varepsilon} \\ &= \frac{1}{n}\boldsymbol{\varepsilon}'C'MC\left(\alpha(C'C)^{-1} + \alpha\frac{c}{n}\boldsymbol{e}_1\boldsymbol{e}_1' + \frac{c^2}{n^2}I_n\right)\boldsymbol{\varepsilon} \\ &= \frac{\alpha}{n}\boldsymbol{\varepsilon}'C'M(C^{-1})'\boldsymbol{\varepsilon} + \frac{\alpha c}{n^2}\boldsymbol{\varepsilon}'C'M\boldsymbol{e}\boldsymbol{e}_1'\boldsymbol{\varepsilon} + \frac{c^2}{n^3}\boldsymbol{\varepsilon}'C'MC\boldsymbol{\varepsilon} \\ &\overset{\mathcal{D}}{=} \frac{\alpha}{n}\boldsymbol{\varepsilon}'M\boldsymbol{\varepsilon} + o_p(1) \;\to\; \sigma_\varepsilon^2 \quad \text{in probability.}\end{aligned}$$

6.2.4 Using (6.23), we consider

$$\begin{aligned}\lim_{n\to\infty}\gamma_n &= \lim_{n\to\infty}\frac{1}{n}\,\boldsymbol{e}'MC\left[\alpha(C'C)^{-1}+\alpha\frac{c}{n}\,\boldsymbol{e}_1\boldsymbol{e}_1'+\frac{c^2}{n^2}\,I_n\right]C'M\boldsymbol{e}\\ &= \lim_{n\to\infty}\frac{\alpha}{n}\,\boldsymbol{e}'M\boldsymbol{e}+\lim_{n\to\infty}\frac{\alpha c}{n^2}(\boldsymbol{e}'M\boldsymbol{e})^2+\lim_{n\to\infty}\frac{c^2}{n^3}\,\boldsymbol{e}'MCC'M\boldsymbol{e}\\ &= \gamma+c\gamma^2+\lim_{n\to\infty}\frac{c^2}{n^3}\,\boldsymbol{e}'MCC'M\boldsymbol{e},\end{aligned}$$

where $\gamma=\lim_{n\to\infty}\boldsymbol{e}'M\boldsymbol{e}/n$ is available in the text. Concentrate on the last term. For Case (A) we have $\gamma=1$ and

$$\boldsymbol{e}'MCC'M\boldsymbol{e}=\boldsymbol{e}'CC'\boldsymbol{e}=\sum_{j=1}^{n}j^2=n^3/3+o(n^3)$$

so that $\gamma(A;c)=1+c+c^2/3$. For Case (B) we have $\gamma=1/4$ and

$$\boldsymbol{e}'MCC'M\boldsymbol{e}=\sum_{j=1}^{n}j^2-\frac{3}{n}\sum_{j=1}^{n}(n-j+1)\sum_{i=j}^{n}i+\left(\frac{3}{2n}\right)^2\sum_{j=1}^{n}\left(\sum_{i=j}^{n}i\right)^2+o(n^3)$$

so that $\lim_{n\to\infty}\boldsymbol{e}'MCC'M\boldsymbol{e}/n^3=1/120$. Thus $\gamma(B;c)=(1+c/4+c^2/30)/4$. For Case (C) we have $\gamma=4/9$ and

$$\begin{aligned}\boldsymbol{e}'MCC'M\boldsymbol{e} &= \sum_{j=1}^{n}j^2-\frac{20}{3n^2}\sum_{j=1}^{n}(n-j+1)\sum_{i=j}^{n}\frac{i(i+1)}{2}\\ &\quad+\left(\frac{10}{3n^2}\right)^2\sum_{j=1}^{n}\left(\sum_{i=j}^{n}\frac{i(i+1)}{2}\right)^2+o(n^3),\end{aligned}$$

which gives $\lim_{n\to\infty}\boldsymbol{e}'MCC'M\boldsymbol{e}/n^3=2/63$. Thus $\gamma(C;c)=4(1+4c/9+c^2/14)/9$. For Case (D) we have $\gamma=1/9$ and

$$\begin{aligned}\boldsymbol{e}'MCC'M\boldsymbol{e} &= \sum_{j=1}^{n}j^2-\sum_{j=1}^{n}(n-j+1)\left(\frac{2(n^2-j^2)}{n}-\frac{10(n^3-j^3)}{9n^2}\right)\\ &\quad+\sum_{j=1}^{n}\left(\frac{2(n^2-j^2)}{n}-\frac{10(n^3-j^3)}{9n^2}\right)^2+o(n^3),\end{aligned}$$

which gives $\lim_{n\to\infty}\boldsymbol{e}'MCC'M\boldsymbol{e}/n^3=1/945$. Thus $\gamma(D;c)=(1+c/9+c^2/105)/9$.

Chapter 7

7.1.1 We prove that the differential equation with the two boundary conditions implies the integral equation. Denote by R the right side of the integral equation to be proved. Then it follows that

$$
\begin{aligned}
R \;=\; & \lambda \int_0^1 \left(b_0 e^{-c|s-t|} + b_1(s) + b_2(s)e^{-ct}\right) f(s)\,ds \\
=\; & \frac{\lambda x}{2c}\left(e^{-ct}\int_0^t e^{cs} f(s)\,ds + e^{ct}\int_t^1 e^{-cs} f(s)\,ds\right) \\
& +\lambda\left[-\frac{x}{c^2}\int_0^1 f(s)\,ds + \left(\frac{x}{c^2}+\frac{1}{2c}\right)\int_0^1 e^{-cs} f(s)\,ds\right. \\
& \left. +\frac{x}{c^2}e^{-ct}\int_0^1 f(s)\,ds - \left(\frac{x}{c^2}+\frac{x}{2c}+\frac{1}{c}+\frac{1}{2}\right)e^{-ct}\int_0^1 e^{-cs} f(s)\,ds\right].
\end{aligned}
$$

Since

$$
\begin{aligned}
(\lambda x - c^2)\int_0^t e^{cs} f(s)\,ds \;&=\; \int_0^t e^{cs}(\lambda a - f''(s))\,ds \\
&=\; \frac{\lambda a}{c}(e^{ct}-1) - e^{ct}(f'(t) - cf(t)) \\
&\quad +f'(0) - cf(0) - c^2\int_0^t e^{cs} f(s)\,ds,
\end{aligned}
$$

we have

$$
\lambda x \int_0^t e^{cs} f(s)\,ds = \frac{\lambda a}{c}(e^{ct}-1) - e^{ct}(f'(t) - cf(t)) + f'(0) - cf(0).
$$

We also have

$$
\begin{aligned}
(\lambda x - c^2)\int_t^1 e^{-cs} f(s)\,ds \;&=\; \int_t^1 e^{-cs}(\lambda a - f''(s))\,ds \\
&=\; -\frac{\lambda a}{c}(e^{-c} - e^{-ct}) - e^{-ct}(f'(1) - f'(t) - cf(t)) \\
&\quad -ce^{-c}f(1) - c^2\int_t^1 e^{-cs} f(s)\,ds,
\end{aligned}
$$

so that

$$
\begin{aligned}
\lambda x \int_t^1 e^{-cs} f(s)\,ds \;&=\; -\frac{\lambda a}{c}(e^{-c} - e^{-ct}) - e^{-ct}(f'(1) - f'(t) - cf(t)) \\
&\quad -ce^{-c}f(1).
\end{aligned}
$$

Using these relations and the two boundary conditions together with the definition of a, we arrive at $R = f(t)$ after some manipulations.

7.1.2 Since it follows from Theorem 7.1 that

$$\phi(\theta; A, c, x) = \mathrm{E}(e^{i\theta(xV(A)-U(A))}) = e^{(c+i\theta)/2}\left[\cos\mu + (c+i\theta)\frac{\sin\mu}{\mu}\right]^{-1/2},$$

where $\mu = \sqrt{2i\theta x - c^2}$, the joint c.f. of $U(A)$ and $V(A)$ is given by

$$\psi(\theta_1, \theta_2) = \phi(-\theta_1; A, c, -\theta_2/\theta_1) = e^{(c-i\theta_1)/2}\left[\cos\nu + (c-i\theta_1)\frac{\sin\nu}{\nu}\right]^{-1/2},$$

where $\nu = \sqrt{2i\theta_2 - c^2}$. To deal with $\sqrt{c}(U(A)+cV(A))/cV(A)$, consider

$$\begin{aligned}\psi\left(\sqrt{c}\theta_1, c\sqrt{c}\theta_1 + c\theta_2\right) &= \mathrm{E}\left[\exp\left\{i\theta_1\sqrt{c}(U(A)+cV(A)) + i\theta_2 cV(A)\right\}\right]\\ &= \exp\left[\frac{1}{2}\left(c - i\sqrt{c}\theta_1\right)\right]H(\eta),\end{aligned}$$

where

$$\begin{aligned}\eta &= \left(c^2 - 2ic\sqrt{c}\theta_1 - 2ic\theta_2\right)^{1/2}\\ &= c - i\sqrt{c}\theta_1 - i\theta_2 + \frac{\theta_1^2}{2} + O\left(\frac{1}{\sqrt{c}}\right),\\ \frac{1}{\eta} &= \frac{1}{c}\left(1 + \frac{i\theta_1}{\sqrt{c}} + O\left(\frac{1}{c}\right)\right),\\ H(\eta) &= \left[\cosh\eta + \left(c - i\sqrt{c}\theta_1\right)\frac{\sinh\eta}{\eta}\right]^{-1/2}\\ &= e^{-\eta/2}\left(1 + O\left(\frac{1}{\sqrt{c}}\right)\right)\\ &= \exp\left[-\frac{c}{2} + \frac{i}{2}\sqrt{c}\theta_1 + \frac{1}{2}i\theta_2 - \frac{\theta_1^2}{4} + O\left(\frac{1}{\sqrt{c}}\right)\right]\left(1 + O\left(\frac{1}{\sqrt{c}}\right)\right).\end{aligned}$$

Then we obtain

$$\psi\left(\sqrt{c}\theta_1, c\sqrt{c}\theta_1 + c\theta_2\right) \;\to\; \exp\left[\frac{1}{2}i\theta_2 - \frac{\theta_1^2}{4}\right],$$

which shows that $cV(A)$ converges to $1/2$ in probability, whereas

$$\sqrt{c}\,(U(A) + cV(A)) \;\Rightarrow\; \mathrm{N}\left(0, \frac{1}{2}\right), \quad \sqrt{c}\left(\frac{U(A)}{cV(A)} + 1\right) \;\Rightarrow\; \mathrm{N}(0,2).$$

7.2.1 For $\boldsymbol{x}_j = 1$ it holds that

$$\begin{aligned}
\tilde{M} &= (CC')^{=1} - (CC')^{-1}\boldsymbol{e}\boldsymbol{e}'(CC')^{-1} \\
&= (C')^{-1}\left(I_n - \boldsymbol{e}_1\boldsymbol{e}_1'\right)C^{-1}, \quad \boldsymbol{e}_1 = (1,0,\ldots,0), \\
\tilde{M}\boldsymbol{y} &= \tilde{M}\boldsymbol{u} = (C')^{-1}(0, u_2 - u_1, u_3 - u_2, \ldots, u_n - u_{n-1})', \\
\boldsymbol{y}'\tilde{M}\boldsymbol{y} &= \boldsymbol{u}'\tilde{M}\boldsymbol{u} = \sum_{j=2}^{n}(u_j - u_{j-1})^2, \\
\boldsymbol{d}'\tilde{M}\boldsymbol{y} &= \boldsymbol{d}'\tilde{M}\boldsymbol{u} = u_n - u_1,
\end{aligned}$$

which gives

$$R_n = \frac{(u_n - u_1)^2}{\sum_{j=2}^{n}(u_j - u_{j-1})^2}.$$

Then it is seen that, as $n \to \infty$ under $\rho = 1 - c/n$, $R_n \Rightarrow Y^2(1)$ because of the same reasoning as in the case of $\boldsymbol{x}_j = 0$, where $dY(t) = -cY(t)\,dt + dW(t)$ with $Y(0) = 0$.

7.2.2 The kernel $e^{-c(2-s-t)}$ is degenerate and it follows from the definition of the FD given in (2.5) that

$$D(\lambda) = 1 - \lambda\int_0^1 e^{-2c(1-t)}\,dt = 1 - \frac{1 - e^{-2c}}{2c}\,\lambda.$$

Alternatively, noting that

$$Y^2(1) = \left(\int_0^1 e^{-c(1-t)}\,dW(t)\right)^2 \sim \frac{1 - e^{-2c}}{2c}\,\chi^2(1),$$

we have $\mathrm{E}\left(e^{i\theta Y^2(1)}\right) = \left(1 - 2i\theta(1 - e^{-2c})/2c\right)^{-1/2} = (D(2i\theta))^{-1/2}$, from which $D(\lambda)$ is obtained.

7.5.1 We obtain

$$\frac{\ell_n(\alpha)}{\ell_n(\beta)} = \exp\left[\frac{\beta^2 - \alpha^2}{2n^2}\sum_{j=1}^{n} y_{j-1}^2 + \frac{\beta - \alpha}{n}\sum_{j=1}^{n} y_{j-1}(y_j - y_{j-1})\right],$$

where it holds by the CMT that, under $\rho = 1 - \gamma/n$,

$$\left(\frac{1}{n^2}\sum_{j=1}^{n} y_{j-1}^2,\ \frac{1}{n}\sum_{j=1}^{n} y_{j-1}(y_j - y_{j-1})\right) \Rightarrow \left(\int_0^1 X^2(t)\,dt,\ \int_0^1 X(t)\,dX(t)\right).$$

Thus we establish the required result using again the CMT.

7.5.2 It holds that

$$\begin{aligned}
U &= \int_0^1 H(X(t))\,dX(t)\\
&= \int_0^1 \left(X(t) + (6t-4)\int_0^1 X(s)\,ds - (12t-6)\int_0^1 sX(s)\,ds\right) dX(t)\\
&= \frac{1}{2}(X^2(1)-1) + \left(2X(1) - 6\int_0^1 X(s)\,ds\right)\int_0^1 X(s)\,ds\\
&\qquad - \left(6X(1) - 12\int_0^1 X(s)\,ds\right)\int_0^1 sX(s)\,ds\\
&= -\frac{1}{2} + \frac{1}{2}X_1^2 + 2X_1X_2 - 6X_2^2 - 6X_1X_3 + 12X_2X_3,\\
V &= \int_0^1 H^2(X(t))\,dt\\
&= \int_0^1 X^2(t)\,dt - 4\left(\int_0^1 X(t)\,dt\right)^2 - 12\left(\int_0^1 tX(t)\,dt\right)^2\\
&\qquad +12\int_0^1 X(t)\,dt\int_0^1 tX(t)\,dt\\
&= \int_0^1 X^2(t)\,dt - 4X_2^2 - 12X_3^2 + 12X_2X_3.
\end{aligned}$$

Then, using Theorem 7.4, we have

$$\begin{aligned}
\mathrm{E}\left(e^{\theta_1 U + \theta_2 V}\right) &= e^{-\theta_1/2}\exp\left[\theta_1\left\{\frac{1}{2}Z_1^2 - 6Z_2^2 + 2Z_1Z_2 - 6Z_1Z_3 + 12Z_2Z_3\right\}\right.\\
&\qquad \left. + \frac{\beta - c}{2}(Z_1^2 - 1) + \theta_2(-4Z_2^2 - 12Z_3^2 + 12Z_2Z_3)\right],
\end{aligned}$$

which leads to the required expression.

7.5.3 It holds that

$$\begin{aligned}
Z_1 &= Z(1) = e^{-\beta}\int_0^1 e^{\beta s}\,dW(s) = \int_0^1 a_1(s)\,dW(s),\\
Z_2 &= \int_0^1 Z(t)\,dt = \int_0^1 \left(e^{-\beta t}\int_0^t e^{\beta s}\,dW(s)\right) dt\\
&= \int_0^1 \left(\int_s^1 e^{-\beta t}\,dt\right) e^{\beta s}\,dW(s) = \frac{1}{\beta}\int_0^1 \left(1 - e^{-\beta(1-s)}\right) dW(s)\\
&= \int_0^1 a_2(s)\,dW(s),
\end{aligned}$$

$$
\begin{aligned}
Z_3 &= \int_0^1 tZ(t)\,dt = \int_0^1 \left(te^{-\beta t} \int_0^t e^{\beta s}\,dW(s) \right) dt \\
&= \int_0^1 \left(\int_s^1 te^{-\beta t}\,dt \right) e^{\beta s}\,dW(s) \\
&= \int_0^1 \left(\frac{s - e^{-\beta(1-s)}}{\beta} + \frac{1 - e^{-\beta(1-s)}}{\beta^2} \right) dW(s) = \int_0^1 a_3(s)\,dW(s).
\end{aligned}
$$

Then we compute

$$
\mathrm{Cov}(Z_j, Z_k) = \int_0^1 a_j(s)a_k(s)\,ds \qquad (j,k = 1,2,3),
$$

which yields the covariance matrix Ω described in the text.

7.5.4 Let us put

$$
dY^0(t) = dW(t), \quad dY^\gamma(t) = \gamma Y^\gamma(t)\,dt + dW(t), \quad Y^0(0) = Y^\gamma(0) = 0.
$$

Then Girsanov's theorem yields

$$
\begin{aligned}
m_1(\theta; x) &= e^{-b\theta/2}\mathrm{E}\left[\exp\left\{ \left(\theta \left(ax + \frac{c^2\theta}{2} \right) + \frac{\gamma^2}{2} \right) \int_0^1 (Y^\gamma(t))^2\,dt \right.\right. \\
&\qquad\qquad \left.\left. - \frac{b\theta + \gamma}{2}\,(Y^\gamma(1))^2 + \frac{\gamma}{2} \right\} \right] \\
&= e^{(-b\theta+\gamma)/2}\mathrm{E}\left[\exp\left\{ -\frac{b\theta + \gamma}{2}\,(Y^\gamma(1))^2 \right\} \right],
\end{aligned}
$$

where $\gamma = \sqrt{-\theta(2ax + c^2\theta)}$. Since $Y^\gamma(1) \sim \mathrm{N}(0, (e^{2\gamma} - 1)/2)$, we have

$$
\begin{aligned}
m_1(\theta; x) &= e^{-b\theta/2} \left[e^{-\gamma} \left(1 + (b\theta + \gamma)\frac{e^{2\gamma} - 1}{2\gamma} \right) \right]^{-1/2} \\
&= e^{-b\theta/2} \left[e^{-\gamma} + \frac{(b\theta + \gamma)(e^{\gamma} - e^{-\gamma})}{2\gamma} \right]^{-1/2} \\
&= e^{-b\theta/2} \left[\cosh\gamma + b\theta\frac{\sinh\gamma}{\gamma} \right]^{-1/2} = e^{-b\theta/2} \left[\cos\mu + b\theta\frac{\sin\mu}{\mu} \right]^{-1/2},
\end{aligned}
$$

where $\mu = \sqrt{\theta(2ax + c^2\theta)}$. This last result yields (7.49).

7.5.5 Putting

$$
dY^0(t) = dW(t), \quad dY^\gamma(t) = \gamma Y^\gamma(t)\,dt + dW(t), \quad Y^0(0) = Y^\gamma(0) = 0,
$$

Girsanov's theorem yields

$$
\begin{aligned}
m_2(\theta; x) &= e^{-b\theta/2}\mathrm{E}\left[\exp\left\{\left(c_1 + \frac{\gamma^2}{2}\right)\int_0^1 (Y^\gamma(t))^2\, dt - c_1\left(\int_0^1 Y^\gamma(t)\, dt\right)^2\right.\right. \\
&\qquad \left.\left. + \left(c_2 - \frac{\gamma}{2}\right)(Y^\gamma(1))^2 + c_3 Y^\gamma(1)\int_0^1 Y^\gamma(t)\, dt + \frac{\gamma}{2}\right\}\right] \\
&= e^{(-b\theta+\gamma)/2}\mathrm{E}\left[\exp\left\{-c_1\left(\int_0^1 Y^\gamma(t)\, dt\right)^2 + \left(c_2 - \frac{\gamma}{2}\right)(Y^\gamma(1))^2\right.\right. \\
&\qquad \left.\left. + c_3 Y^\gamma(1)\int_0^1 Y^\gamma(t)\, dt\right\}\right],
\end{aligned}
$$

where $c_1 = \theta(ax + c^2\theta/2), c_2 = -b\theta/2, c_3 = b\theta$, and $\gamma = \sqrt{-2c_1}$. Since

$$
\boldsymbol{Z} = \begin{pmatrix} Y^\gamma(1) \\ \int_0^1 Y^\gamma(t)\, dt \end{pmatrix} \sim \mathrm{N}(\boldsymbol{0}, \Sigma), \quad \Sigma = \begin{pmatrix} \dfrac{e^{2\gamma} - 1}{2\gamma} & \dfrac{(e^\gamma - 1)^2}{2\gamma^2} \\ \dfrac{(e^\gamma - 1)^2}{2\gamma^2} & \dfrac{2\gamma + (e^\gamma - 1)(e^\gamma - 3)}{2\gamma^3} \end{pmatrix},
$$

we have

$$
m_2(\theta; x) = e^{(-b\theta+\gamma)/2}\mathrm{E}\Big[\exp\{\boldsymbol{Z}' A \boldsymbol{Z}\}\Big] = e^{(-b\theta+\gamma)/2}\Big|I_2 - 2A\Sigma\Big|^{-1/2},
$$

where

$$
A = \begin{pmatrix} c_2 - \dfrac{\gamma}{2} & \dfrac{1}{2}c_3 \\ \dfrac{1}{2}c_3 & -c_1 \end{pmatrix}.
$$

Then we arrive at (7.52) after some manipulations.

Chapter 8

8.1.1 Consider

$$
\begin{aligned}
f(t) &= \lambda\int_0^1 K_g(s,t;x) f(s)\, ds \\
&= \lambda x\left[\int_0^t A_1(s,t) f(s)\, ds + \int_t^1 A_2(s,t) f(s)\, ds\right] \\
&\qquad -\frac{\lambda(1-t)^g}{2(g!)^2}\int_0^1 (1-s)^g f(s)\, ds,
\end{aligned}
$$

where

$$A_1(s,t) = \frac{1}{(g!)^2} \int_t^1 ((s-u)(t-u))^g \; du,$$
$$A_2(s,t) = \frac{1}{(g!)^2} \int_s^1 ((s-u)(t-u))^g \; du.$$

Then we have, for $j = 0, 1, \ldots, g$,

$$\begin{aligned} f^{(j)}(t) \;=\; & \lambda x \left[\int_0^t \frac{\partial^j A_1(s,t)}{\partial t^j} f(s)\, ds + \int_t^1 \frac{\partial^j A_2(s,t)}{\partial t^j} f(s)\, ds \right] \\ & - \frac{\lambda g(g-1)\cdots(g-j+1)(-1)^j(1-t)^{g-j}}{2(g!)^2} \int_0^1 (1-s)^g f(s)\, ds, \end{aligned}$$

which yields

$$f^{(j)}(1) = 0 \quad (j = 0, 1, \ldots, g-1), \quad f^{(g)}(1) = \frac{\lambda(-1)^{g-1}}{2g!} \int_0^1 (1-s)^g f(s)\, ds.$$

We also have, for $j = 1, 2, \ldots, g+1$,

$$f^{(g+j)}(t) = \frac{\lambda x(-1)^j}{(g+1-j)!} \int_0^t (s-t)^{g+1-j} f(s)\, ds,$$

which yields

$$f^{(g+j)}(0) = 0 \quad (j = 1, \ldots, g+1),$$

and

$$f^{(2g+1)}(t) = \lambda x(-1)^{g+1} \int_0^t f(s)\, ds, \quad f^{(2g+2)}(t) = \lambda x(-1)^{g+1} f(t).$$

Conversely, define

$$C_j(s,t) = \frac{\partial^j A_1(s,t)}{\partial s^j} = \frac{g(g-1)\cdots(g-j+1)}{(g!)^2} \int_t^1 (s-u)^{g-j}(t-u)^g \, du,$$
$$D_j(s,t) = \frac{\partial^j A_2(s,t)}{\partial s^j} = \frac{g(g-1)\cdots(g-j+1)}{(g!)^2} \int_s^1 (s-u)^{g-j}(t-u)^g \, du.$$

It holds that $C_j(t,t) = D_j(t,t)$ and $C_j(s,1) = D_j(1,t) = C_{g+1+j}(s,t) = 0$ for $j = 0, 1, \ldots, g$, whereas

$$D_{g+1}(s,t) = \frac{-1}{g!}(t-s)^g, \quad D_{g+1+j}(t,t) = 0 \quad (j = 0, 1, \ldots, g-1),$$

and $D_{2g+1}(s,t) = (-1)^{g+1}$. Then we have, by integration by parts,

$$\begin{aligned}
R &= \lambda \int_0^1 K_g(s,t;x) f(s)\, ds \\
&= (-1)^{g+1} \left[\int_0^t A_1(s,t) f^{(2g+2)}(s)\, ds + \int_t^1 A_2(s,t) f^{(2g+2)}(s)\, ds \right] \\
&\qquad - \frac{\lambda(1-t)^g}{2(g!)^2} \int_0^1 (1-s)^g f(s)\, ds \\
&= (-1)^{g+2} \left[\int_0^t C_1(s,t) f^{(2g+1)}(s)\, ds + \int_t^1 D_1(s,t) f^{(2g+1)}(s)\, ds \right] \\
&\qquad - \frac{\lambda(1-t)^g}{2(g!)^2} \int_0^1 (1-s)^g f(s)\, ds,
\end{aligned}$$

where we have used $A_1(t,t) = A_2(t,t)$, $A_2(1,t) = 0$, and $f^{(2g+1)}(0) = 0$. Proceeding further by integration by parts, we obtain

$$\begin{aligned}
R &= - \left[\int_0^t C_g(s,t) f^{(g+2)}(s)\, ds + \int_t^1 D_g(s,t) f^{(g+2)}(s)\, ds \right] \\
&\qquad - \frac{\lambda(1-t)^g}{2(g!)^2} \int_0^1 (1-s)^g f(s)\, ds \\
&= D_{g+1}(1,t) f^{(g)}(1) - \int_t^1 D_{g+2}(s,t) f^{(g)}(s)\, ds \\
&\qquad - \frac{\lambda(1-t)^g}{2(g!)^2} \int_0^1 (1-s)^g f(s)\, ds \\
&= - \int_t^1 D_{g+2}(s,t) f^{(g)}(s)\, ds = (-1)^g \int_t^1 D_{2g+1}(s,t) f'(s)\, ds \\
&= (-1)^{2g+1} \int_t^1 f'(s)\, ds = -f(1) + f(t) = f(t),
\end{aligned}$$

which gives the required result.

8.1.2 The integral equation

$$f(t) = \lambda \int_0^1 L_0(s,t;x) f(s)\, ds = \lambda \int_0^1 \left[x \min(s,t) - \frac{1}{2} \right] f(s)\, ds$$

is equivalent to

$$f''(t) + \lambda x f(t) = 0, \quad f'(1) = 0, \quad f'(0) = -2x f(0).$$

The general solution to the differential equation is

$$f(t) = c_1 \cos \sqrt{\lambda x} t + c_2 \sin \sqrt{\lambda x} t,$$

and it follows from the two boundary conditions that $M(\lambda)\boldsymbol{c} = \mathbf{0}$, where $\boldsymbol{c} = (c_1, c_2)'$ and

$$M(\lambda) = \begin{pmatrix} 2x & \sqrt{\lambda x} \\ -\sin\sqrt{\lambda x} & \cos\sqrt{\lambda x} \end{pmatrix}.$$

Computing $|M(\lambda)|$ we arrive at the FD given in the problem.

8.1.3 Noting that

$$\mathrm{E}\left(e^{\theta(xV-U)}\right) = (D_0(2\theta; x))^{-1/2} = \left[\cos\sqrt{2\theta x} + \theta\,\frac{\sin\sqrt{2\theta x}}{\sqrt{2\theta x}}\right]^{-1/2},$$

we have

$$\psi(\theta_1, -\theta_2) = \mathrm{E}\left(e^{\theta_1 U - \theta_2 V}\right) = \left[\cosh\sqrt{2\theta_2} - \theta_1\frac{\sinh\sqrt{2\theta_2}}{\sqrt{2\theta_2}}\right]^{-1/2}.$$

Then it follows that

$$\begin{aligned}
\mathrm{E}\left(\frac{U}{V}\right) &= \int_0^\infty \left.\frac{\partial\psi(\theta_1, -\theta_2)}{\partial\theta_1}\right|_{\theta_1=0} d\theta_2 \\
&= \frac{1}{2}\int_0^\infty \frac{\sinh\sqrt{2\theta_2}}{\sqrt{2\theta_2}}\left(\cosh\sqrt{2\theta_2}\right)^{-3/2} d\theta_2 \\
&= \frac{1}{2}\int_1^\infty u^{-3/2}\,du = 1,
\end{aligned}$$

where we have put $\cosh\sqrt{2\theta_2} = u$ so that $1 < u < \infty$ and $du/d\theta_2 = \sinh\sqrt{2\theta_2}/\sqrt{2\theta_2}$. It is clear that $\mathrm{E}(U) = \mathrm{E}(V) = 1/2$.

To prove that $R = U/V$ and V are not independent, we show that $\mathrm{E}(R^2V^2) \neq \mathrm{E}(R^2)\mathrm{E}(V^2)$. It holds that $\mathrm{E}(R^2V^2) = \mathrm{E}(U^2) = \mathrm{E}(W^4(1))/4 = 3/4$ and

$$\begin{aligned}
\mathrm{E}(V^2) = \mathrm{E}\left[\left(\int_0^1 W^2(t)\,dt\right)^2\right] &= \int_0^1\int_0^1 \mathrm{E}(W^2(s)W^2(t))\,ds\,dt \\
&= \int_0^1\int_0^1 \left[\mathrm{E}(W^2(s))\mathrm{E}(W^2(t)) + 2\mathrm{E}^2(W(s)W(t))\right] ds\,dt \\
&= \int_0^1\int_0^1 [st + 2(\min(s,t))^2]\,ds\,dt = \frac{7}{12}.
\end{aligned}$$

On the other hand, since we obtain

$$\begin{aligned}
\mathrm{E}(R^2) &= \mathrm{E}\left[\left(\frac{1}{2}W^2(1)\Big/\int_0^1 W^2(t)\,dt\right)^2\right] \\
&= \int_0^\infty \theta_2 \frac{\partial^2}{\partial\theta_1^2}\left[\cosh\sqrt{2\theta_2} - \theta_1\frac{\sinh\sqrt{2\theta_2}}{\sqrt{2\theta_2}}\right]^{-1/2}\Bigg|_{\theta_1=0} d\theta_2 \\
&= \frac{3}{8}\int_0^\infty \frac{\tanh^2\sqrt{2\theta_2}}{\sqrt{\cosh\sqrt{2\theta_2}}}\,d\theta_2 = 1.891,
\end{aligned}$$

we have

$$\mathrm{E}(R^2V^2) = \frac{1}{4} \neq \mathrm{E}(R^2)\mathrm{E}(V^2) = 1.891 \times \frac{7}{12} = 1.103.$$

8.2.1 Using (8.21) we obtain, as $n \to \infty$,

$$\begin{aligned}
\frac{1}{n^{2d+1}}\,\mathrm{Var}\left(\sum_{j=1}^n v_j\right) &= \frac{1}{n^{2d+1}}\left[2\sum_{h=1}^{n-1}(n-h)\gamma_v(h) + n\gamma_v(0)\right] \\
&= \frac{\sigma_\varepsilon^2\Gamma(1-2d)}{\Gamma(d)\Gamma(1-d)}\,\frac{2}{n}\sum_{h=1}^n\left(1-\frac{h}{n}\right)\left(\frac{h}{n}\right)^{2d-1} + o(1) \\
&\to \frac{2\sigma_\varepsilon^2\Gamma(1-2d)}{\Gamma(d)\Gamma(1-d)}\int_0^1 (1-t)t^{2d-1}\,dt \\
&= \frac{\sigma_\varepsilon^2\Gamma(1-2d)}{(2d+1)\Gamma(1+d)\Gamma(1-d)}.
\end{aligned}$$

8.2.2 The first equality is evident. We have, as $|h| \to \infty$,

$$(h \pm 1)^{2H} = |h|^{2H} \pm 2H|h|^{2H-1} + H(2H-1)|h|^{2H-2} + O(|h|^{2H-3}),$$

which establishes the second equality.

8.2.3 We first note that the infinite series on the left side converges absolutely when $0 < H < 1/2$. Putting $\gamma_\Delta(h) = \mathrm{Cov}(\Delta B_H(j), \Delta B_H(j+h))$, it holds that, as $n \to \infty$,

$$\begin{aligned}
\sum_{h=-n}^n \gamma_\Delta(h) &= 2\sum_{h=1}^n \gamma_\Delta(h) + \gamma_\Delta(0) \\
&= \sum_{h=1}^n \left[|h+1|^{2H} + |h-1|^{2H} - 2|h|^{2H}\right] + 1
\end{aligned}$$

$$= (n+1)^{2H} - n^{2H} = 2Hn^{2H-1} + O(n^{2H-2}) \to 0.$$

8.2.4 Putting $\alpha = H - 1/2$ and $I(x < a) = 1$ for $x < a$ and 0 for $x \geq a$, it follows from (8.24) that, for $s < t$,

$$\begin{aligned}
\mathrm{Var}\,(B_H(t) - B_H(s)) &= \frac{1}{A_H^2} \int_{-\infty}^{\infty} \{(t-u)^\alpha I(u < t) - (s-u)^\alpha I(u < s)\}^2 \, du \\
&= \frac{1}{A_H^2} \int_{-\infty}^{\infty} \{(t-s-v)^\alpha I(v < t-s) - (-v)^\alpha I(v < 0)\}^2 \, dv \\
&= \frac{(t-s)^{2H}}{A_H^2} \int_{-\infty}^{\infty} \{(1-u)^\alpha I(u < 1) - (-u)^\alpha I(u < 0)\}^2 \, du \\
&= (t-s)^{2H}.
\end{aligned}$$

8.2.5 It holds that, when $0 < H < 1$, $X_j = n^H \left(B_H(t_j) - B_H(t_{j-1})\right)$ is a zero-mean stationary Gaussian process with $\mathrm{E}(X_j^2) = n^{2H}|t_j - t_{j-1}|^{2H} = 1$. Then it follows from the weak law of large numbers that

$$\begin{aligned}
\mathrm{plim}_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} X_j^2 &= \mathrm{plim}_{n\to\infty} \frac{1}{n^{1-2H}} \sum_{j=1}^{n} \left(B_H(t_j) - B_H(t_{j-1})\right)^2 \\
&= \mathrm{E}(X_j^2) = 1,
\end{aligned}$$

which gives the required result.

8.2.6 We have

$$\begin{aligned}
\int_0^s \int_0^t |u-v|^{2H-2} \, du \, dv &= \int_0^s \left[\int_0^v (v-u)^{2H-2} \, du + \int_v^t (u-v)^{2H-2} \, du \right] dv \\
&= \frac{1}{2H-1} \int_0^s \left[v^{2H-1} + (t-v)^{2H-1} \right] dv \\
&= \frac{1}{2H(2H-1)} \left[s^{2H} + t^{2H} - |t-s|^{2H} \right].
\end{aligned}$$

8.4.1 Consider, for $s \le t$,

$$
\begin{aligned}
&\mathrm{E}\,(B_H(s),\, M_H(t)) \\
&= \mathrm{E}\left[\kappa_H^{-1}\int_0^s\int_0^t (v(t-v))^{1/2-H}\, dB_H(u)\, dB_H(v)\right] \\
&= \kappa_H^{-1} H(2H-1)\int_0^s\int_0^t (v(t-v))^{1/2-H}|v-u|^{2H-2}\, du\, dv \\
&= \kappa_H^{-1} H(2H-1)\int_0^s\int_0^1 (b(1-b))^{1/2-H}\left|b-\frac{u}{t}\right|^{2H-2}\, db\, du \\
&= \kappa_H^{-1} H(2H-1)B(H-1/2, 3/2-H)\, s \;=\; s,
\end{aligned}
$$

which is the required result, where we have used the formula given in (8.46).

8.4.2 It follows from the definition of $B_H(t)$ and $N_H(t)$ that

$$
\begin{aligned}
\mathrm{Cov}(B_H(t), N_H(t)) &= \mathrm{E}\left[\frac{b_H}{A(H)}\int_0^t\int_0^t (t-u)^{H-1/2}v^{1/2-H}\, dW(u)\, dW(v)\right] \\
&= \frac{b_H}{A(H)}\int_0^t (t-u)^{H-1/2}u^{1/2-H}\, du \\
&= \frac{b_H}{A(H)}\int_0^1 (t(1-v))^{H-1/2}\,(tv)^{1/2-H}t\, dv \\
&= \frac{b_H}{A(H)}\, B(3/2-H, H+1/2)\, t = \frac{\Gamma^2(3/2-H)}{\Gamma(2-2H)}\, t.
\end{aligned}
$$

Noting that $\mathrm{Var}(B_H(t)) = t^{2H}$ and $\mathrm{Var}(N_H(t)) = a_H t^{2-2H}$, we have

$$
\begin{aligned}
\mathrm{Corr}(B_H(t), N_H(t)) &= \frac{1}{\sqrt{t^{2H}a_H t^{2-2H}}}\frac{\Gamma^2(3/2-H)}{\Gamma(2-2H)}\, t \\
&= \frac{\Gamma^2(3/2-H)}{\Gamma(2-2H)}\Bigg/\sqrt{a_H}.
\end{aligned}
$$

8.4.3 We have

$$
\begin{aligned}
&\int_0^1 C_H^2(t)\, dt \\
&= 2(1-H)\int_0^1 t^{4H-2}\left(\int_0^t\int_0^t (uv)^{1/2-H}\, dW(u)\, dW(v)\right) dt \\
&= 2(1-H)\int_0^1\int_0^1\left(\int_{\max(u,v)}^1 t^{4H-2}\, dt\right)(uv)^{1/2-H}\, dW(u)\, dW(v)
\end{aligned}
$$

$$= \frac{2(1-H)}{4H-1} \int_0^1 \int_0^1 \left[1 - (\max(u,v))^{4H-1}\right] (uv)^{1/2-H} \, dW(u) \, dW(v).$$

We also have, for $s < t$,

$$\text{Cov}\,(C_H(s), C_H(t)) = 2(1-H)(st)^{2H-1} \int_0^s u^{1-2H} \, du = (st)^{2H-1} \, s^{2-2H},$$

which establishes the required relations.

8.4.4 We prove that the differential equation with the two boundary conditions leads to the integral equation given by

$$t^{2\alpha} h(t) = \lambda \kappa^2 \left[\int_0^1 h(s) \, ds - t^{4\alpha+1} \int_0^t h(s) \, ds - \int_t^1 s^{4\alpha+1} h(s) \, ds \right], \tag{A.12}$$

where $\alpha = H - 1/2$ and $\kappa^2 = (1-2\alpha)/(4\alpha+1)$. We first note that

$$\left(2\alpha t^{-2\alpha-1} h(t) + t^{-2\alpha} h'(t)\right)' = -\lambda \kappa^2 (4\alpha+1) h(t).$$

Thus we have

$$\lambda \kappa^2 h(t) = -\frac{1}{4\alpha+1} \left(2\alpha t^{-2\alpha-1} h(t) + t^{-2\alpha} h'(t)\right)' = -\frac{1}{4\alpha+1} g'(t).$$

Then the right side of (A.12) is given by

$$-\frac{1}{4\alpha+1} \left[\int_0^1 g'(s) \, ds - t^{4\alpha+1} \int_0^t g'(s) \, ds - \int_t^1 s^{4\alpha+1} g'(s) \, ds \right],$$

which is shown to be equal to the left side of (A.12) by using the two boundary conditions.

8.4.5 The integral equation with the kernel $L_H(s,t)$ is shown to be equivalent to

$$f''(t) - \frac{2H-1}{t} f'(t) + \left(2(1-H)\lambda t^{2H-1} + \frac{2H-1}{t^2}\right) f(t) = 0$$

with the boundary conditions

$$\lim_{t \to 0} t^{2-2H} f'(t) = 0, \quad f'(1) = (2H-1) f(1).$$

The general solution to the above differential equation is given by

$$f(t) = t^H \left(c_1 J_{\nu-1}(\eta t^\gamma) + c_2 J_{1-\nu}(\eta t^\gamma)\right),$$

where $\eta = \sqrt{2(1-H)\lambda}/(H+1/2)$, $\gamma = H + 1/2$, and $\nu = (2H-1/2)/(H+$

1/2). We have, from the two boundary conditions, the homogeneous equations in $\boldsymbol{c} = (c_1, c_2)'$:

$$M(\lambda)\,\boldsymbol{c} = \begin{pmatrix} 1 & 0 \\ a & \dfrac{2(1-H)}{\eta(H+1/2)} J_{1-\nu}(\eta) - J_{2-\nu}(\eta) \end{pmatrix} \boldsymbol{c} = \boldsymbol{0},$$

where a is some constant. Since it holds [Watson (1958)] that

$$J_{-\nu}(\eta) = \frac{2(1-\nu)}{\eta} J_{1-\nu}(\eta) - J_{2-\nu}(\eta) = \frac{2(1-H)}{\eta(H+1/2)} J_{1-\nu}(\eta) - J_{2-\nu}(\eta),$$

we have $|M(\lambda)| = J_{-\nu}(\eta)$ and obtain the FD given in the problem.

8.4.6 We have

$$\begin{aligned} \mathrm{Var}(S_H) &= 2\int_0^1\int_0^1 \mathrm{Cov}^2\left(B_H(s), B_H(t)\right)\, ds\, dt \\ &= \frac{1}{2}\int_0^1\int_0^1 \left[s^{2H} + t^{2H} - |s-t|^{2H}\right]^2 ds\, dt, \end{aligned}$$

and arrive at the required result after some algebra. To compute $\mathrm{Var}(T_H)$, consider the FD of $K_H(s,t)$ given by

$$\begin{aligned} D_H(\lambda) &= \Gamma(1-\nu) J_{-\nu}(\eta) \Big/ \left(\frac{\eta}{2}\right)^{-\nu} = \Gamma(1-\nu) \sum_{k=0}^{\infty} \frac{(-1)^k (\eta/2)^{2k}}{k!\,\Gamma(-\nu+k+1)} \\ &= \Gamma(1-\nu)\left[\frac{1}{\Gamma(1-\nu)} - \frac{(\eta/2)^2}{\Gamma(2-\nu)} + \frac{(\eta/2)^4}{2\Gamma(3-\nu)} - \frac{(\eta/2)^6}{6\Gamma(4-\nu)} + \cdots\right] \\ &= 1 - \frac{(\eta/2)^2}{1-\nu} + \frac{(\eta/2)^4}{2(2-\nu)(1-\nu)} - \frac{(\eta/2)^6}{6(3-\nu)(2-\nu)(1-\nu)} + \cdots \\ &= 1 - d_1\lambda + \frac{d_2}{2}\lambda^2 - \frac{d_3}{6}\lambda^3 + \cdots, \end{aligned}$$

where $\eta = \sqrt{2(1-H)\lambda}/(H+1/2)$, $\nu = (2H-1/2)/(H+1/2)$, $d_1 = 1/(2H+1)$, and $d_2 = 2(1-H)/3(2H+1)^2$. Then it follows from Theorem 2.4 that

$$\mathrm{Var}(T_H) = 2(d_1^2 - d_2) = 2\left(\frac{1}{(2H+1)^2} - \frac{2(1-H)}{3(2H+1)^2}\right) = \frac{2}{3(2H+1)}.$$

8.5.1 We prove that the differential equation with the two boundary conditions leads to the integral equation given by

$$t^{2\alpha}h(t) = \lambda\left[\frac{x}{4\alpha+1}\left\{\int_0^1 h(s)\,ds - t^{4\alpha+1}\int_0^t h(s)\,ds - \int_t^1 s^{4\alpha+1}h(s)\,ds\right\}\right.$$
$$\left. -\frac{1}{2}\int_0^1 h(s)\,ds\right], \qquad \text{(A.13)}$$

where $\alpha = H - 1/2$. We first note that

$$G'(t) = \left(2\alpha t^{-2\alpha-1}h(t) + t^{-2\alpha}h'(t)\right)' = -2\alpha(2\alpha+1)t^{-2\alpha-2}h(t) + t^{-2\alpha}h''(t)$$
$$= t^{-2\alpha}\left(h''(t) - \frac{4\alpha^2+2\alpha}{t^2}h(t)\right) = -\lambda x h(t).$$

Then the right side of (A.13) is given by

$$\frac{1}{4\alpha+1}\left\{-\int_0^1 G'(s)\,ds + t^{4\alpha+1}\int_0^t G'(s)\,ds + \int_t^1 s^{4\alpha+1}G'(s)\,ds\right\}$$
$$+\frac{1}{2x}\int_0^1 G'(s)\,ds,$$

which is shown to be equal to the left side of (A.13) by using the two boundary conditions.

8.5.2 The general solution to the differential equation is given by

$$h(t) = t^{\beta}\left(c_1 J_{\nu}(\kappa t^{\gamma}) + c_2 J_{-\nu}(\kappa t^{\gamma})\right),$$

where $\beta = 1/2, \gamma = H+1/2, \kappa = \sqrt{\lambda x}/(H+1/2)$, and $\nu = (2H-1/2)/(H+1/2)$. Then it follows from the boundary condition $\lim_{t\to 0} G(t) = 0$ that, as $t \to 0$,

$$G(t) \to (2H-1)\left(c_1\frac{\kappa^{\nu}/2^{\nu}}{\Gamma(1+\nu)} + c_2\frac{\kappa^{-\nu}/2^{-\nu}}{\Gamma(1-\nu)}t^{-4H+1}\right)$$
$$+c_1\frac{(\beta+\gamma\nu)\kappa^{\nu}/2^{\nu}}{\Gamma(1+\nu)} + c_2\frac{(\beta-\gamma\nu)\kappa^{-\nu}/2^{-\nu}}{\Gamma(1-\nu)}t^{-4H+1}$$
$$= c_1(2H-1+\beta+\gamma\nu)\frac{\kappa^{\nu}/2^{\nu}}{\Gamma(1+\nu)}$$
$$+c_2(2H-1+\beta-\gamma\nu)\frac{\kappa^{-\nu}/2^{-\nu}}{\Gamma(1-\nu)}t^{-4H+1}$$
$$= c_1(4H-1)\frac{\kappa^{\nu}/2^{\nu}}{\Gamma(1+\nu)} = 0.$$

The other boundary condition yields, allowing for $c_1 = 0$,

$$
\begin{aligned}
&(2H - 1 - 2x)h(1) + h'(1) \\
= \ &(2H - 1 - 2x)c_2 J_{-\nu}(\kappa) \\
&+c_2 \left[\beta t^{\beta-1} J_{-\nu}(\kappa t^\gamma) + t^\beta \gamma\kappa t^{\gamma-1}\left\{-J_{1-\nu}(\kappa t^\gamma) - \frac{\nu}{\kappa t^\gamma} J_{-\nu}(\kappa t^\gamma)\right\}\right]_{t=1} \\
= \ &c_2\left[(2H - 1 - 2x + \beta - \gamma\nu)J_{-\nu}(\kappa) - \gamma\kappa J_{1-\nu}(\kappa)\right] \\
= \ &c_2\left[-2x J_{-\nu}(\kappa) - \sqrt{\lambda x} J_{1-\nu}(\kappa)\right] \ = \ 0.
\end{aligned}
$$

Then we have the equation in $\boldsymbol{c} = (c_1, c_2)'$ given in the problem, where we have used the relation

$$J'_{-\nu}(z) = -J_{-\nu+1}(z) - \frac{\nu}{z} J_{-\nu}(z).$$

8.5.3 Let us put

$$I = \int_0^\infty f(x)\, dx = \frac{1}{2}(H + 1/2) \int_0^\infty g(x)\, (h(x))^{-3/2}\, dx,$$

where

$$
\begin{aligned}
f(x) &= \frac{(\xi/2)^{-\nu/2-1} J_{1-\nu}(\xi)(J_{-\nu}(\xi))^{-3/2}}{4(H + 1/2)\sqrt{\Gamma(1-\nu)}}, \quad \xi = \frac{\sqrt{-2x}}{H + 1/2}, \\
g(x) &= \frac{\Gamma(1-\nu)}{2(H + 1/2)^2} J_{1-\nu}(\xi) \Bigg/ \left(\frac{\xi}{2}\right)^{1-\nu}, \\
h(x) &= \Gamma(1-\nu) J_{-\nu}(\xi) \Bigg/ \left(\frac{\xi}{2}\right)^{-\nu} = \prod_{n=1}^\infty \left(1 - \frac{\xi^2}{c_n^2}\right) \\
&= \prod_{n=1}^\infty \left(1 + \frac{2x}{c_n^2 (H + 1/2)^2}\right) \ \geq \ 1,
\end{aligned}
$$

with $c_1 < c_2 < \cdots$ being the positive zeros of $J_{-\nu}(\xi)$. Then, using the relation in (2.127), we have

$$\frac{d\{J_{-\nu}(\xi)/\xi^{-\nu}\}}{dx} = -\xi^\nu J_{1-\nu}(\xi)\frac{d\xi}{dx} = \frac{1}{(H + 1/2)^2} J_{1-\nu}(\xi)/\xi^{1-\nu}.$$

Thus we obtain $dh(x) = g(x)\, dx$, where $h(x) \geq 1$. Then it follows that

$$
\begin{aligned}
I &= \frac{1}{2}(H + 1/2) \int_0^\infty (h(x))^{-3/2}\, dh(x) \\
&= \frac{1}{2}(H + 1/2) \int_1^\infty u^{-3/2}\, du = H + 1/2.
\end{aligned}
$$

Appendix B

Graphs of Distributions of Brownian Functionals

In this appendix we present graphs of probability densities of some statistics dealt with in the main text. In each figure the associated FD or the c.f. is shown together with the equation number referred to in the text. Means, variances, and percent points are also reported for some distributions. Densities presented are computed by numerical integration, which is carried out by different methods depending on the property of statistics. More specifically, the statistics are classified into the following three cases:

$$S_j = \begin{cases} V & (j = 1) \\ U + V & (j = 2) \\ U/V & (j = 3), \end{cases}$$

where U and V are random variables with $-\infty < U < \infty$ and $V > 0$. Then we can compute the distribution function $F_j(x)$ of S_j as follows:

$$F_j(x) = P(S_j < x) = \begin{cases} \dfrac{1}{\pi} \displaystyle\int_0^\infty \mathrm{Re}\left[\frac{1 - e^{-i\theta x}}{i\theta} \,\mathrm{E}\left(e^{i\theta V}\right) \right] d\theta & (j = 1) \\ \dfrac{1}{2} - \dfrac{1}{\pi} \displaystyle\int_0^\infty \frac{1}{\theta} \,\mathrm{Im}\left[e^{-i\theta x} \mathrm{E}\left(e^{i\theta(U+V)}\right) \right] d\theta & (j = 2) \\ \dfrac{1}{2} + \dfrac{1}{\pi} \displaystyle\int_0^\infty \frac{1}{\theta} \,\mathrm{Im}\left[\mathrm{E}\left(e^{i\theta(xV-U)}\right) \right] d\theta & (j = 3), \end{cases}$$

where the case of $j = 1$ is Lévy's inversion formula, whereas the cases of $j = 2, 3$ are due to Imhof (1961).

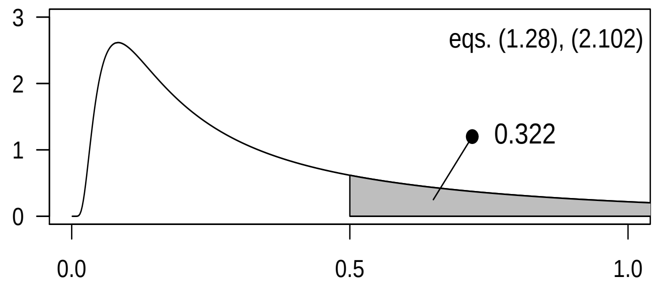
Numerical integration for the above integrals was done by Simpson's rule by taking care in the actual computation because the c.f. is usually the square root of a complex-valued function, which any computer cannot evaluate properly. We also need to examine the behavior of the integrand. It is sometimes the case that the function oscillates and converges to 0 quite slowly, for which Euler's transformation can be used to overcome the difficulty. It is also convenient to apply change of variables like $\theta = u^2$ so that the integrand vanishes at the origin. More computational details can be found in Tanaka (1996, 2017). In any case the probability density $f_j(x) = dF_j(x)/dx$ can be easily obtained from numerical derivatives of the distribution function $F_j(x)$ by computing $f_j(x) \approx (F_j(x+\delta) - F_j(x))/\delta$ for small δ.

Figure 1 $S_1 = \int_0^1 \int_0^1 [1 - \max(s,t)]\, dW(s)\, dW(t)$

$$= \int_0^1 W^2(t)\, dt,$$

$$D(\lambda) = \cos\sqrt{\lambda}, \quad \text{Mean} = \frac{1}{2}, \quad \text{Var} = \frac{1}{3},$$

$$x(0.9) = 1.1958, \quad x(0.95) = 1.6557, \quad x(0.99) = 2.7875$$



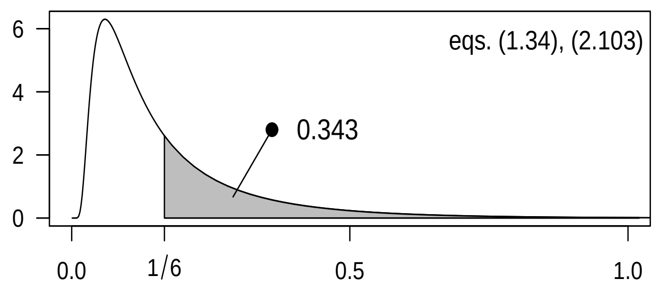


Figure 2 $S_2 = \int_0^1 \int_0^1 [\min(s,t) - st]\, dW(s)\, dW(t)$

$$= \int_0^1 \left(W(t) - \int_0^1 W(s)\, ds \right)^2 dt,$$

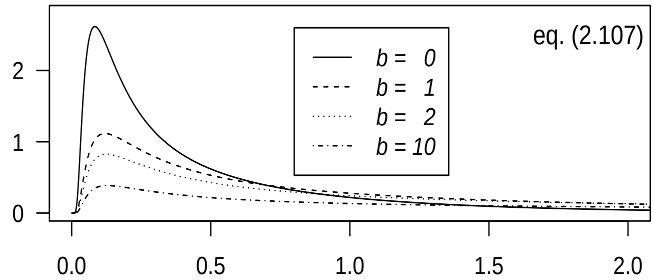
$$D(\lambda) = \frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}}, \quad \text{Mean} = \frac{1}{6}, \quad \text{Var} = \frac{1}{45},$$

$$x(0.9) = 0.3473, \quad x(0.95) = 0.4614, \quad x(0.99) = 0.7435$$

Figure 3 $S_3 = \int_0^1 \int_0^1 [1 - \max(s,t) + b]\, dW(s)\, dW(t)$

$D(\lambda) = \cos\sqrt{\lambda} - b\sqrt{\lambda}\sin\sqrt{\lambda}$,

Mean $= \frac{1}{2} + b$, Var $= \frac{1}{3} + 2b^2 + \frac{4}{3}b$



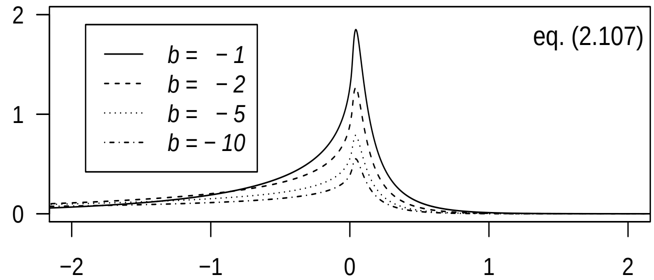


Figure 4 $S_4 = \int_0^1 \int_0^1 [1 - \max(s,t) + b]\, dW(s)\, dW(t)$

$D(\lambda) = \cos\sqrt{\lambda} - b\sqrt{\lambda}\sin\sqrt{\lambda}$,

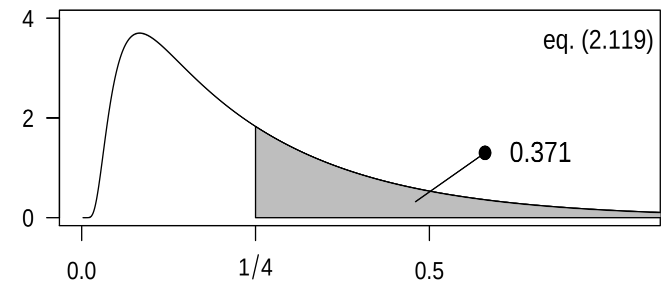
Mean $= \frac{1}{2} + b$, Var $= \frac{1}{3} + 2b^2 + \frac{4}{3}b$

Figure 5 $S_5 = \int_0^1 \int_0^1 \frac{1}{4}\,[1 - 2|s-t|]\, dW(s)\, dW(t)$

$$= \int_0^1 \left(W(t) - \frac{1}{2}W(1)\right)^2 dt,$$

$$D(\lambda) = \left(\cos\frac{\sqrt{\lambda}}{2}\right)^2, \quad \text{Mean} = \frac{1}{4}, \quad \text{Var} = \frac{1}{24},$$

$x(0.9) = 0.5156, \quad x(0.95) = 0.6560, \quad x(0.99) = 0.9822$



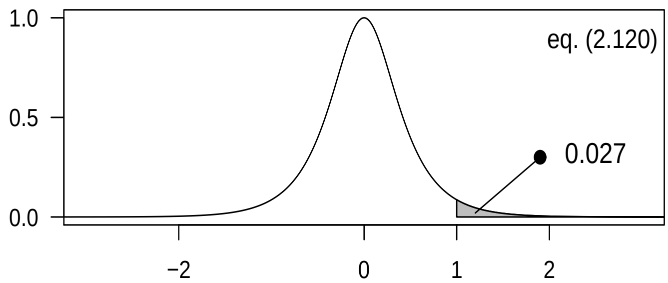


Figure 6 $S_6 = \frac{1}{2}\int_0^1 \left[W_1(t)\, dW_2(t) - W_2(t)\, dW_1(t)\right],$

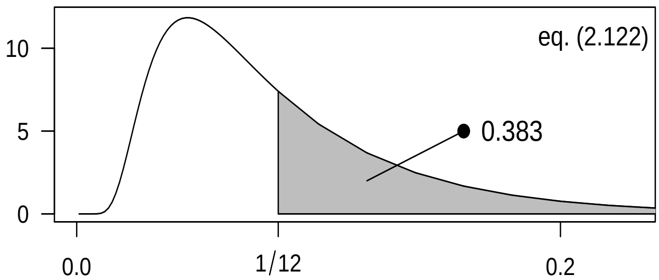
$$\mathrm{E}\left(e^{i\theta S_6}\right) = \left(\cosh\frac{\theta}{2}\right)^{-1}, \quad f(x) = \frac{1}{\cosh \pi x}, \quad \text{Var} = \frac{1}{4},$$

$x(0.9) = 0.5866, \quad x(0.95) = 0.8092, \quad x(0.99) = 1.3221$

Figure 7 $S_7 = \int_0^1 \int_0^1 \left[\min(s,t) - st + \frac{1}{2}(s^2 + t^2 - s - t) + \frac{1}{12} \right] dW(s)\, dW(t)$

$$= \int_0^1 \left(W(t) - tW(1) - \int_0^1 (W(s) - sW(1))\, ds \right)^2 dt,$$

$$D(\lambda) = \left(\sin \frac{\sqrt{\lambda}}{2} \Big/ \frac{\sqrt{\lambda}}{2} \right)^2, \quad \text{Mean} = \frac{1}{12}, \quad \text{Var} = \frac{1}{360}$$



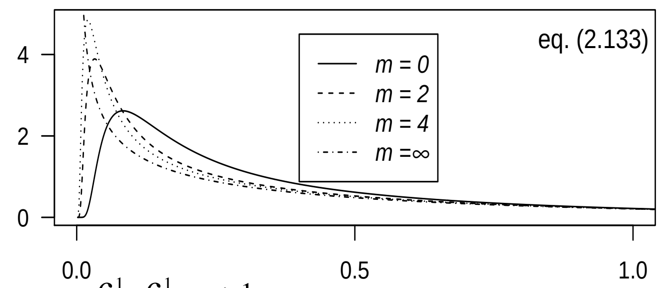


Figure 8 $S_8 = \int_0^1 \int_0^1 \frac{m+1}{2m+1} \left[1 - (\max(s,t))^{2m+1} \right] dW(s)\, dW(t)$

$$= \int_0^1 (m+1) \left(t^m W(t) \right)^2 dt,$$

$$D(\lambda) = \Gamma(-\nu + 1)\ J_{-\nu} \left(\frac{\sqrt{\lambda}}{\sqrt{m+1}} \right) \Big/ \left(\frac{\sqrt{\lambda}}{2\sqrt{m+1}} \right)^{-\nu}, \quad \nu = \frac{2m+1}{2(m+1)},$$

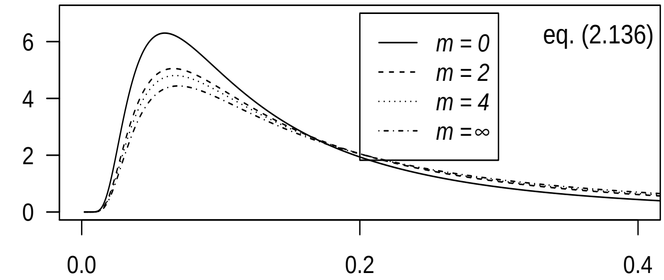
$$\text{Mean} = \frac{1}{2}, \quad \text{Var} = \frac{m+1}{2m+3}$$

Figure 9 $S_9 = \int_0^1 \int_0^1 (m+1)^2 s^m t^m [\min(s,t) - st]\, dW(s)\, dW(t)$

$$\stackrel{\mathcal{D}}{=} \int_0^1 (m+1)^2 \Big(t^m (W(t) - tW(1))\Big)^2 dt,$$

$$D(\lambda) = \Gamma\left(\frac{2m+3}{2(m+1)}\right) J_{1/2(m+1)}\left(\sqrt{\lambda}\right) \Big/ \left(\frac{\sqrt{\lambda}}{2}\right)^{1/2(m+1)},$$

$$\text{Mean} = \frac{m+1}{2(2m+3)}, \quad \text{Var} = \frac{(m+1)^3}{(2m+3)^2(4m+5)}$$



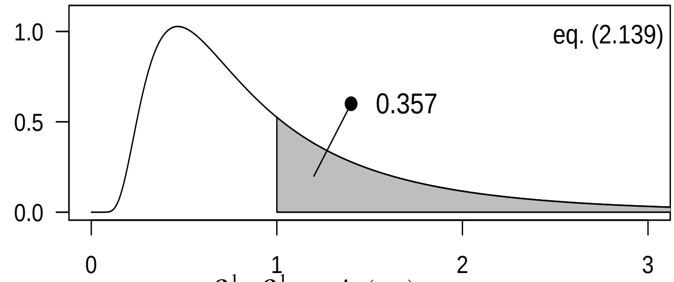


Figure 10 $S_{10} = \int_0^1 \int_0^1 \frac{\min(s,t) - st}{\sqrt{st(1-s)(1-t)}}\, dW(s)\, dW(t)$

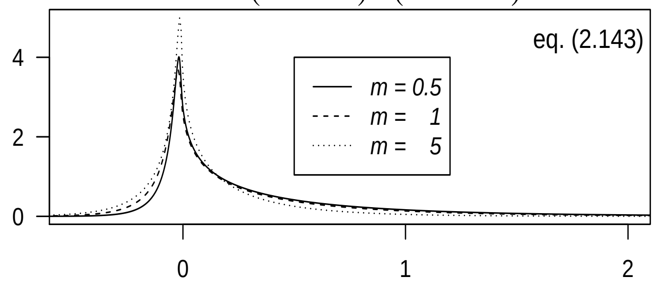
$$\stackrel{\mathcal{D}}{=} \int_0^1 \frac{(W(t) - tW(1))^2}{t(1-t)}\, dt \stackrel{\mathcal{D}}{=} \sum_{n=1}^{\infty} \frac{Z_n^2}{n(n+1)},$$

$$D(\lambda) = -\cos\frac{\pi\sqrt{1+4\lambda}}{2} \Big/ \pi\lambda, \quad \text{Mean} = 1, \quad \text{Var} = \frac{2\pi^2}{3} - 6$$

Figure 11 $S_{11} = \int_0^1 \int_0^1 \frac{1}{2} (\max(s,t))^m \, dW(s)\, dW(t)$

$$= \int_0^1 t^m W(t)\, dW(t) + \frac{1}{2(m+1)}, \quad \text{Mean} = \text{Var} = \frac{1}{2(m+1)},$$

$$D(\lambda) = \Gamma(-\nu)\ J_{-\nu-1}\left(\frac{\sqrt{-2\lambda m}}{m+1}\right) \Big/ \left(\frac{\sqrt{-2\lambda m}}{2(m+1)}\right)^{-\nu-1}, \quad \nu = \frac{m}{m+1}$$



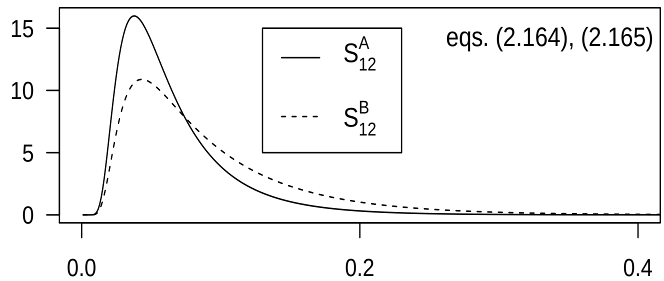


Figure 12 $S_{12}^A = \int_0^1 \int_0^1 \left[\min(s,t) - st - \frac{2}{\pi^2} \sin \pi s\ \sin \pi t\right] dW(s)\, dW(t),$

$$D^A(\lambda) = \frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}} \Big/ \left(1 - \frac{\lambda}{\pi^2}\right), \quad \text{Mean} = \frac{\pi^2 - 6}{6\pi^2}, \quad \text{Var} = \frac{1}{45} - \frac{2}{\pi^4},$$

$$S_{12}^B = \int_0^1 \int_0^1 \left[\min(s,t) - st - \frac{2}{\pi^2} \sin^2 \pi s\ \sin^2 \pi t\right] dW(s)\, dW(t),$$

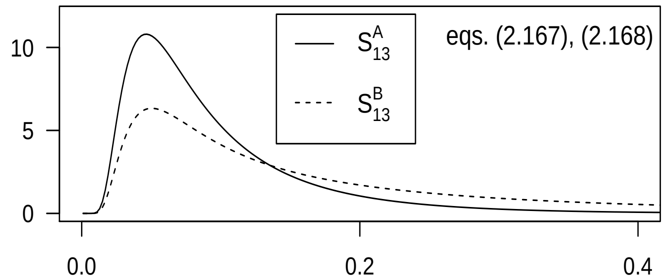
$$D^B(\lambda) = \left(\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}} + \frac{1 - \cos\sqrt{\lambda}}{\pi^2(1 - \lambda/4\pi^2)}\right) \Big/ \left(1 - \frac{\lambda}{4\pi^2}\right), \quad \text{Mean} = \frac{2\pi^2 - 9}{12\pi^2}$$

Figure 13 $S_{13}^A = \int_0^1 \int_0^1 \left[1 - \max(s,t) - \frac{8}{\pi^2}\cos\frac{\pi s}{2}\cos\frac{\pi t}{2}\right] dW(s)\, dW(t),$

$$D^A(\lambda) = \cos\sqrt{\lambda} \Big/ \left(1 - \frac{4\lambda}{\pi^2}\right), \quad \text{Mean} = \frac{\pi^2 - 8}{2\pi^2}, \quad \text{Var} = \frac{\pi^4 - 18\pi^2 + 48}{3\pi^4},$$

$$S_{13}^B = \int_0^1 \int_0^1 \left[1 - \max(s,t) - \frac{8}{\pi^2}\left(\cos\frac{\pi s}{2}\cos\frac{\pi t}{2}\right)^2\right] dW(s)\, dW(t),$$

$$D^B(\lambda) = \left(\cos\sqrt{\lambda} + \frac{2\sqrt{\lambda}\sin\sqrt{\lambda}}{\pi^2(1 - \lambda/\pi^2)}\right) \Big/ \left(1 - \frac{\lambda}{\pi^2}\right), \quad \text{Mean} = \frac{\pi^2 - 6}{2\pi^2}$$



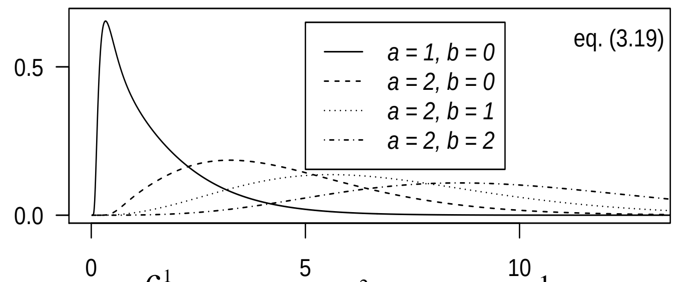


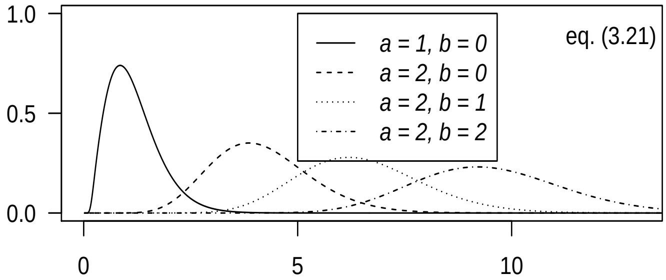
Figure 14 $S_{14} = \int_0^1 \left(W(t) + a + bt\right)^2 dt, \quad \text{Mean} = \frac{1}{2} + a^2 + ab + \frac{1}{3}b^2,$

$$\mathrm{E}\left(e^{i\theta S_{14}}\right) = \left(\cos\sqrt{\lambda}\right)^{-1/2} \exp\left[-\frac{b(2a+b)}{2} + ab\sec\sqrt{\lambda} + \frac{a^2\lambda + b^2}{2}\frac{\tan\sqrt{\lambda}}{\sqrt{\lambda}}\right] \qquad (\lambda = 2i\theta)$$

Figure 15 $S_{15} = \int_0^1 \left(W(t) - tW(1) + a + bt\right)^2 dt,$

$$\mathrm{E}\left(e^{i\theta S_{15}}\right) = \left(\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}}\right)^{-1/2} \exp\left[\frac{b^2}{2} + a(a+b)\sqrt{\lambda}\csc\sqrt{\lambda} - \sqrt{\lambda}\left(a(a+b) + \frac{b^2}{2}\right)\cot\sqrt{\lambda}\right] \quad (\lambda = 2i\theta),$$

$$\text{Mean} = \frac{1}{6} + a^2 + ab + \frac{1}{3}b^2$$



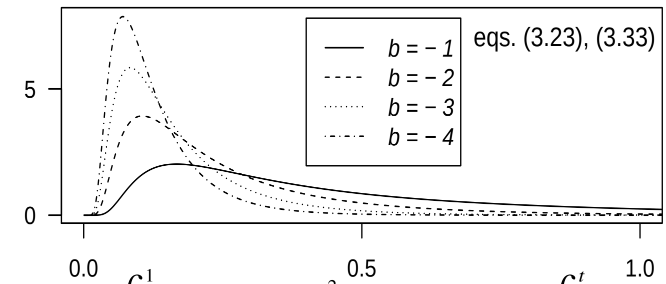


Figure 16 $S_{16} = \int_0^1 \left(X(t) + a\,e^{bt}\right)^2 dt, \quad X(t) = e^{bt}\int_0^t e^{-bs}\,dW(s),$

$$\mathrm{E}\left(e^{i\theta S_{16}}\right) = \left(\cos\mu - b\frac{\sin\mu}{\mu}\right)^{-1/2} \exp\left[-\frac{b}{2} + \frac{ia^2\theta\sin\mu/\mu}{\cos\mu - b\sin\mu/\mu}\right],$$

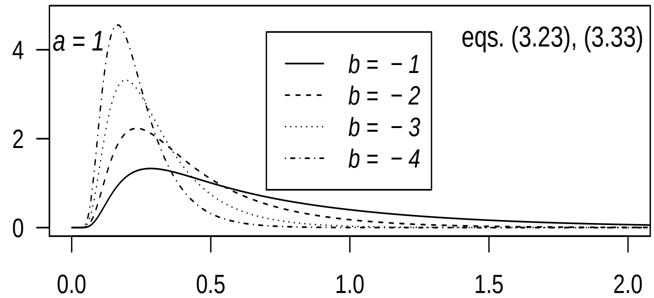
$$\mu = \sqrt{2i\theta - b^2}, \quad a = \sqrt{-1/(2b)} \quad (b < 0),$$

$$\text{Mean} = -\frac{1}{2b}, \quad \text{Var} = \frac{-e^{4b} + 4e^{2b} - 4b - 3}{8b^4}$$

Figure 17 $S_{17} = \int_0^1 \left(X(t) + a\,e^{bt}\right)^2 dt, \quad X(t) = e^{bt}\int_0^t e^{-bs}\,dW(s),$

$$\mathrm{E}\left(e^{i\theta S_{17}}\right) = \left(\cos\mu - b\frac{\sin\mu}{\mu}\right)^{-1/2} \exp\left[-\frac{b}{2} + \frac{ia^2\theta\sin\mu/\mu}{\cos\mu - b\sin\mu/\mu}\right],$$

$$\mu = \sqrt{2i\theta - b^2}, \quad \text{Mean} = \frac{e^{2b}-1}{2b}\left(a^2 + \frac{1}{2b}\right) - \frac{1}{2b}$$



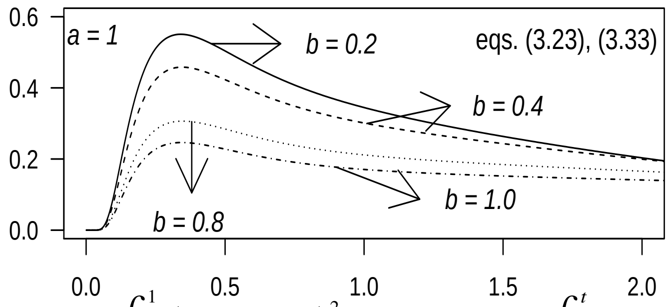


Figure 18 $S_{18} = \int_0^1 \left(X(t) + a\,e^{bt}\right)^2 dt, \quad X(t) = e^{bt}\int_0^t e^{-bs}\,dW(s),$

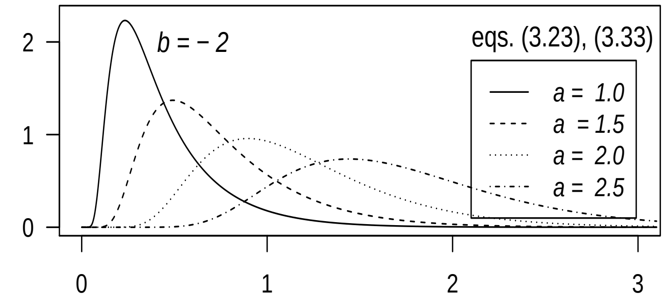
$$\mathrm{E}\left(e^{i\theta S_{18}}\right) = \left(\cos\mu - b\frac{\sin\mu}{\mu}\right)^{-1/2} \exp\left[-\frac{b}{2} + \frac{ia^2\theta\sin\mu/\mu}{\cos\mu - b\sin\mu/\mu}\right],$$

$$\mu = \sqrt{2i\theta - b^2}, \quad \text{Mean} = \frac{e^{2b}-1}{2b}\left(a^2 + \frac{1}{2b}\right) - \frac{1}{2b}$$

Figure 19 $S_{19} = \int_0^1 \left(X(t) + a\,e^{bt}\right)^2 dt, \quad X(t) = e^{bt}\int_0^t e^{-bs}\,dW(s),$

$$\mathrm{E}\left(e^{i\theta S_{19}}\right) = \left(\cos\mu - b\frac{\sin\mu}{\mu}\right)^{-1/2} \exp\left[-\frac{b}{2} + \frac{ia^2\theta\sin\mu/\mu}{\cos\mu - b\sin\mu/\mu}\right],$$

$$\mu = \sqrt{2i\theta - b^2}, \quad \text{Mean} = \frac{e^{2b}-1}{2b}\left(a^2 + \frac{1}{2b}\right) - \frac{1}{2b}$$



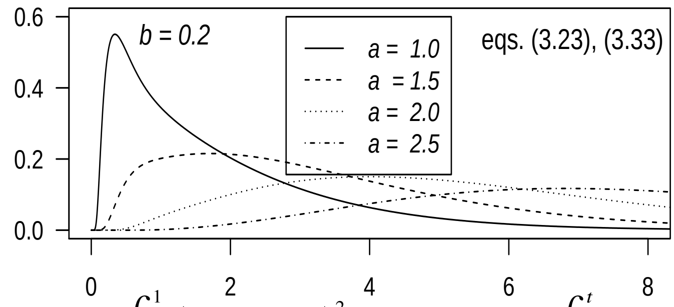


Figure 20 $S_{20} = \int_0^1 \left(X(t) + a\,e^{bt}\right)^2 dt, \quad X(t) = e^{bt}\int_0^t e^{-bs}\,dW(s),$

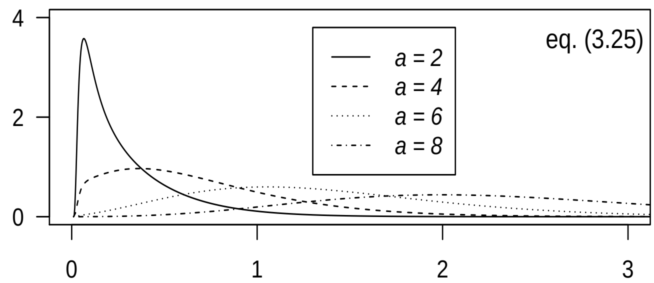
$$\mathrm{E}\left(e^{i\theta S_{20}}\right) = \left(\cos\mu - b\frac{\sin\mu}{\mu}\right)^{-1/2} \exp\left[-\frac{b}{2} + \frac{ia^2\theta\sin\mu/\mu}{\cos\mu - b\sin\mu/\mu}\right],$$

$$\mu = \sqrt{2i\theta - b^2}, \quad \text{Mean} = \frac{e^{2b}-1}{2b}\left(a^2 + \frac{1}{2b}\right) - \frac{1}{2b}$$

Figure 21 $S_{21} = \int_0^1 \left(W(t) - tW(1) - at(1-t)\right)^2 dt,$

$$\mathrm{E}\left(e^{i\theta S_{21}}\right) = \left(\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}}\right)^{-1/2} \exp\left[\frac{4a^2}{\lambda^2}\left(\frac{1-\cos\sqrt{\lambda}}{\sin\sqrt{\lambda}/\sqrt{\lambda}} - \frac{\lambda(\lambda+12)}{24}\right)\right],$$

$$\lambda = 2i\theta, \quad \text{Mean} = \frac{1}{6} + \frac{1}{30}a^2, \quad \text{Var} = \frac{1}{45} + \frac{17}{1260}a^2$$



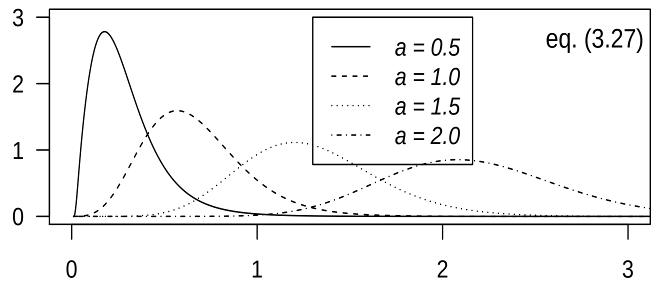


Figure 22 $S_{22} = \int_0^1 \left(W(t) - tW(1) - a\sin 2\pi t\right)^2 dt,$

$$\mathrm{E}\left(e^{i\theta S_{22}}\right) = \left(\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}}\right)^{-1/2} \exp\left[\frac{a^2\pi^2\lambda}{4\pi^2-\lambda}\right],$$

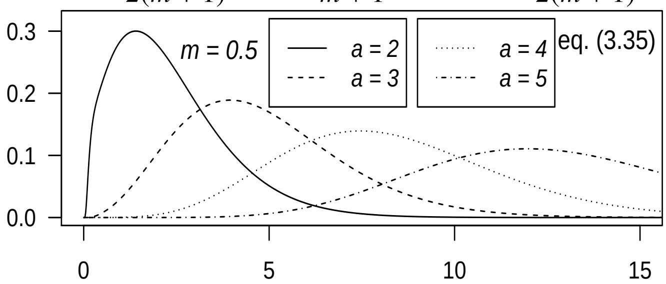
$$\lambda = 2i\theta, \quad \text{Mean} = \frac{1}{6} + \frac{1}{2}a^2, \quad \text{Var} = \frac{1}{45} + \frac{1}{2\pi^2}a^2$$

Figure 23 $S_{23} = \int_0^1 t^{2m}\Big(W(t)+a\Big)^2\,dt,$

$$\mathrm{E}\left(e^{i\theta S_{23}}\right) = \left[\Gamma(-\nu+1)\,J_{-\nu}(\eta)\Big/\left(\frac{\eta}{2}\right)^{-\nu}\right]^{-1/2}$$

$$\times\exp\left[\frac{ia^2\theta\,\Gamma(\nu+1)}{(2m+1)\,\Gamma(-\nu+1)}\,\frac{J_\nu(\eta)}{J_{-\nu}(\eta)}\left(\frac{\eta}{2}\right)^{-2\nu}\right],$$

$$\nu = \frac{2m+1}{2(m+1)},\quad \eta = \frac{\sqrt{2i\theta}}{m+1},\quad \text{Mean} = \frac{1}{2(m+1)} + \frac{a^2}{2m+1}$$



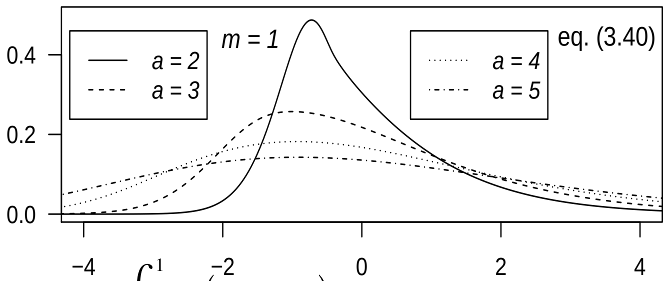


Figure 24 $S_{24} = \int_0^1 t^{m}\Big(W(t)+a\Big)\,dW(t),$

$$\mathrm{E}\left(e^{i\theta S_{24}}\right) = \left[\Gamma(-\nu)J_{-\nu-1}(\eta)\Big/\left(\frac{\eta}{2}\right)^{-\nu-1}\right]^{-1/2}$$

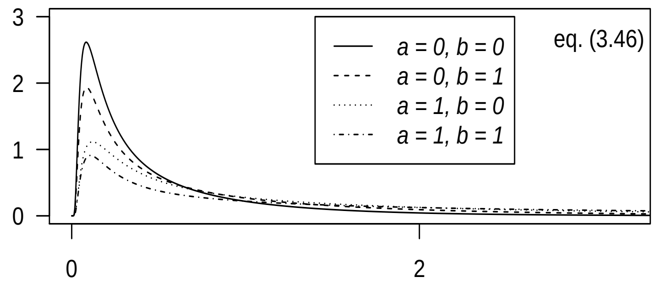
$$\times\exp\left[\frac{ia^2\theta}{2}\frac{\Gamma(\nu+1)}{\Gamma(-\nu+1)}\frac{J_{\nu+1}(\eta)}{J_{-\nu-1}(\eta)}\left(\frac{\eta}{2}\right)^{-2\nu} - \frac{i\theta}{2(m+1)}\right],$$

$$\nu = \frac{m}{m+1},\quad \eta = \frac{\sqrt{-4i\theta m}}{m+1},\quad \text{Mean} = 0$$

Figure 25 $S_{25} = \int_0^1 \left(W(t) + Z(a+bt)\right)^2 dt, \quad Z \sim \mathrm{N}(0,1),$

$$\mathrm{E}\left(e^{i\theta S_{25}}\right) = \left(\left(1 + b(2a+b)\right)\cos\sqrt{\lambda} - 2ab - (a^2\lambda + b^2)\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}}\right)^{-1/2},$$

$$\lambda = 2i\theta, \quad \text{Mean} = \frac{1}{2} + a^2 + ab + \frac{1}{3}b^2$$



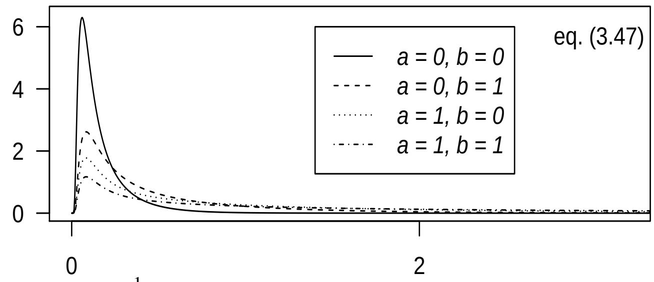


Figure 26 $S_{26} = \int_0^1 \left(W(t) - tW(1) + Z(a+bt)\right)^2 dt, \quad Z \sim \mathrm{N}(0,1),$

$$\mathrm{E}\left(e^{i\theta S_{26}}\right) = \left((1-b^2)\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}} - 2a(a+b) + \left(2a(a+b) + b^2\right)\cos\sqrt{\lambda}\right)^{-1/2},$$

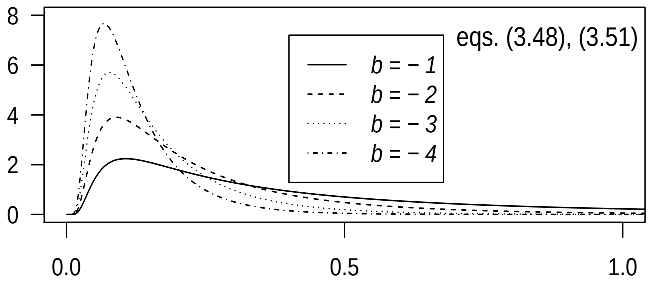
$$\lambda = 2i\theta, \quad \text{Mean} = \frac{1}{6} + a^2 + ab + \frac{1}{3}b^2$$

Figure 27 $S_{27} = \int_0^1 \left(X(t) + \frac{Z}{\sqrt{-2b}}\, e^{bt} \right)^2 dt, \quad b < 0, \quad Z \sim \mathrm{N}(0,1),$

$$X(t) = e^{bt} \int_0^t e^{-bs}\, dW(s),$$

$$\mathrm{E}\left(e^{i\theta S_{27}}\right) = e^{-b/2} \left(\cos\nu - \left(b - \frac{i\theta}{b} \right) \frac{\sin\nu}{\nu} \right)^{-1/2}, \quad \nu = \sqrt{2i\theta - b^2},$$

$$\text{Mean} = -\frac{1}{2b}, \quad \text{Var} = \frac{1}{4b^4} \left(e^{2b} - 2b - 1 \right)$$



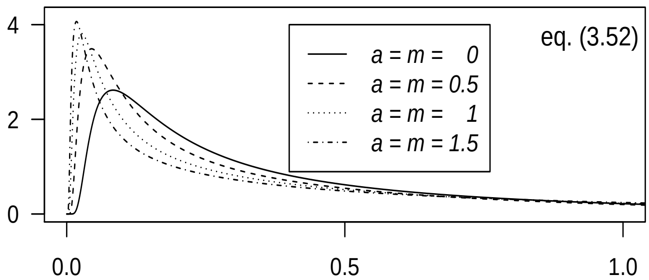


Figure 28 $S_{28} = \int_0^1 t^{2m} \left(W(t) + aZ \right)^2 dt, \quad Z \sim \mathrm{N}(0,1),$

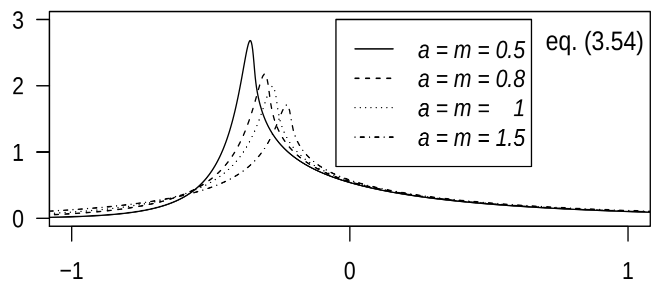
$$\mathrm{E}\left(e^{i\theta S_{28}}\right) = \left(\frac{\Gamma(-\nu+1)\, J_{-\nu}(\eta)}{(\eta/2)^{-\nu}} - \frac{2ia^2\theta\Gamma(\nu+1) J_\nu(\eta)}{(2m+1)(\eta/2)^\nu} \right)^{-1/2},$$

$$\nu = \frac{2m+1}{2(m+1)}, \quad \eta = \frac{\sqrt{2i\theta}}{m+1}, \quad \text{Mean} = \frac{1}{2(m+1)} + \frac{1}{2m+1}\, a^2$$

Figure 29 $S_{29} = \int_0^1 t^m \Big(W(t) + aZ\Big)\, dW(t), \quad Z \sim \mathrm{N}(0,1),$

$$\mathrm{E}\left(e^{i\theta S_{29}}\right) = \left[\left(\frac{\Gamma(-\nu)J_{-\nu-1}(\eta)}{(\eta/2)^{-\nu-1}} + \frac{ia^2\theta\Gamma(\nu+1)J_{\nu+1}(\eta)}{\nu(\eta/2)^{\nu-1}}\right) e^{i\theta/(m+1)}\right]^{-1/2},$$

$$\text{Mean} = 0, \quad \text{Var} = \frac{1}{2(m+1)} + \frac{1}{2m+1}\, a^2$$



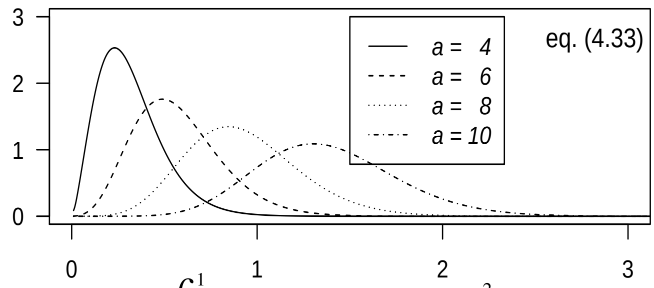


Figure 30 $S_{30} = \int_0^1 \Big(Z(t) + a\sin 2\pi t/(2\pi)\Big)^2 dt,$

$$\mathrm{Cov}(Z(s), Z(t)) = \min(s,t) - st - 2\sin^2 \pi s \sin^2 \pi t/\pi^2,$$

$$\mathrm{E}\left(e^{i\theta S_{30}}\right) = \left[\left(\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}} + \frac{1-\cos\sqrt{\lambda}}{\pi^2(1-\lambda/4\pi^2)}\right)\Big/\left(1 - \frac{\lambda}{4\pi^2}\right)\right]^{-1/2}$$

$$\times \exp\left[\frac{a^2\lambda}{4(4\pi^2-\lambda)}\right], \quad \lambda = 2i\theta,$$

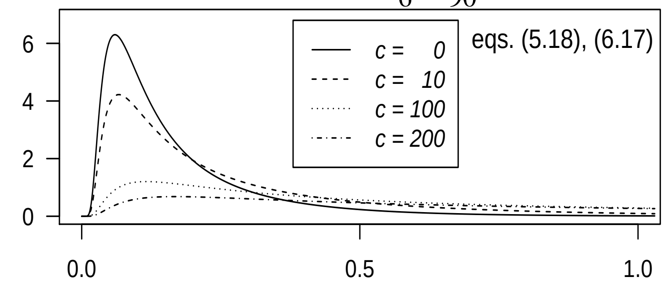
$$\text{Mean} = \frac{1}{6} - \frac{3}{4\pi^2} + \frac{a^2}{8\pi^2}, \quad \text{Var} = \frac{1}{45} - \frac{1}{6\pi^2} - \frac{1-a^2}{8\pi^4}$$

Figure 31 $S_{31} = \int_0^1 \int_0^1 \left(K(s,t) + cK^{(2)}(s,t)\right) dW(s)\, dW(t),$

$$K(s,t) = \min(s,t) - st, \quad K^{(2)}(s,t) = \int_0^1 K(s,u)K(u,t)\, du,$$

$$\mathrm{E}\left(e^{i\theta S_{31}}\right) = \left[D\left(i\theta + \sqrt{-\theta^2 + 2ic\theta}\right) D\left(i\theta - \sqrt{-\theta^2 + 2ic\theta}\right)\right]^{-1/2},$$

$$D(\lambda) = \sin\sqrt{\lambda}/\sqrt{\lambda}, \quad \text{Mean} = \frac{1}{6} + \frac{c}{90}$$



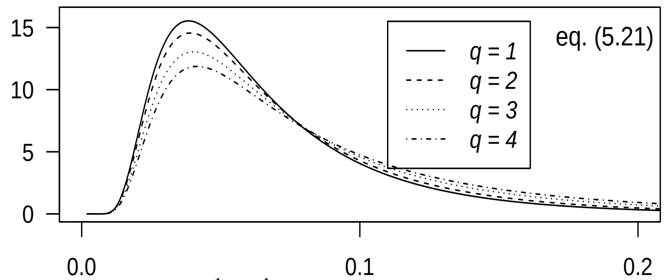


Figure 32 $S_{32} = \int_0^1 \int_0^1 K_q(s,t)\, dW(s)\, dW(t),$

$$K_q(s,t) = \min(s,t) - st - \frac{2q+1}{q^2}\, st(1-s^q)(1-t^q),$$

$$D(\lambda) = \frac{12}{\lambda^2}\left(2 - \sqrt{\lambda}\sin\sqrt{\lambda} - 2\cos\sqrt{\lambda}\right) \quad (q = 1),$$

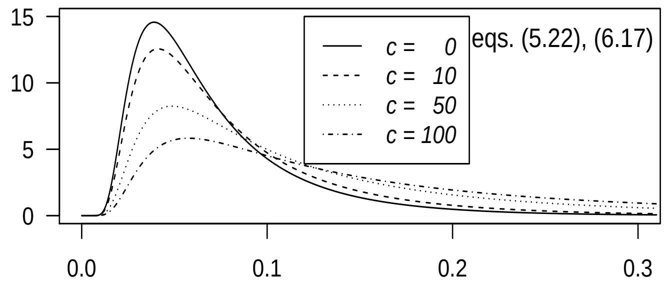
$$\text{Mean} = \frac{1}{15}, \quad \text{Var} = \frac{11}{6300} \quad (q = 1)$$

Figure 33 $S_{33} = \int_0^1 \int_0^1 \left(K(s,t) + cK^{(2)}(s,t)\right) dW(s)\, dW(t),$

$$K(s,t) = \min(s,t) - st - \frac{5}{4}\, st(1-s^2)(1-t^2),$$

$$D(\lambda) = \frac{15}{\lambda^3}\left(\sqrt{\lambda}(3-\lambda)\sin\sqrt{\lambda} - 3\lambda\cos\sqrt{\lambda}\right),$$

$$\text{Mean} = \frac{1}{14} + \frac{1}{882}\, c$$



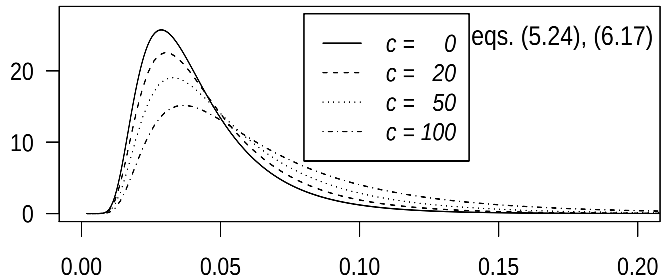


Figure 34 $S_{34} = \int_0^1 \int_0^1 \left(K(s,t) + cK^{(2)}(s,t)\right) dW(s)\, dW(t),$

$$K(s,t) = \min(s,t) - st - 2st(1-s)(1-t)(4-5s-5t+10st),$$

$$D(\lambda) = \frac{720}{\lambda^4}\left(4(6+\lambda) - \sqrt{\lambda}(24-\lambda)\sin\sqrt{\lambda} - 8(3-\lambda)\cos\sqrt{\lambda}\right),$$

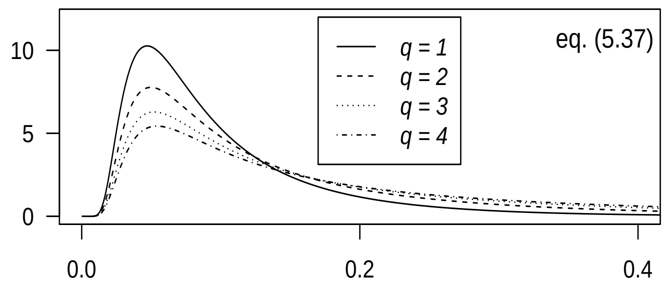
$$\text{Mean} = \frac{3}{70} + \frac{11}{44100}\, c$$

Figure 35 $S_{35} = \int_0^1 \int_0^1 K_q(s,t)\, dW(s)\, dW(t),$

$$K_q(s,t) = 1 - \max(s,t) - \frac{2q+1}{(q+1)^2}\,(1 - s^{q+1})(1 - t^{q+1}),$$

$$D(\lambda) = \frac{3}{\lambda^2}\left(\sqrt{\lambda}\sin\sqrt{\lambda} - \lambda\cos\sqrt{\lambda}\right) \quad (q = 1),$$

$$\text{Mean} = \frac{1}{10}, \quad \text{Var} = \frac{1}{175} \quad (q = 1)$$



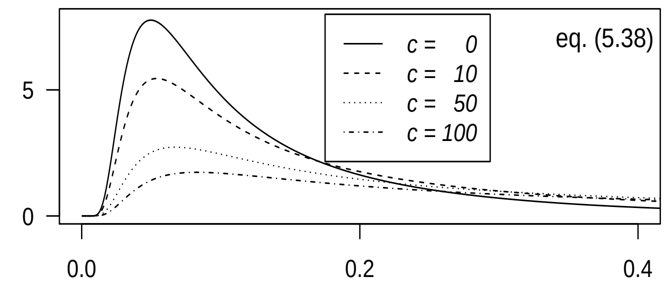


Figure 36 $S_{36} = \int_0^1 \int_0^1 \left(K(s,t) + cK^{(2)}(s,t)\right) dW(s)\, dW(t),$

$$K(s,t) = 1 - \max(s,t) - \frac{5}{9}\,(1 - s^3)(1 - t^3),$$

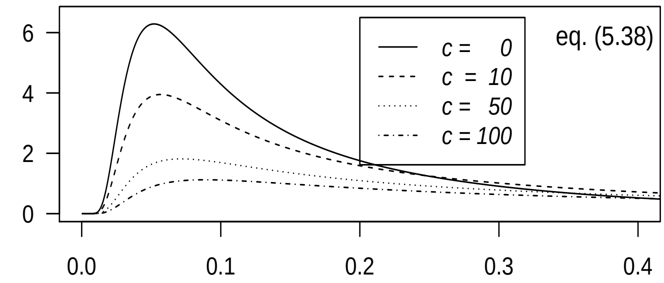
$$D(\lambda) = \frac{20}{\lambda^3}\left(-2\lambda + \sqrt{\lambda}(1+\lambda)\sin\sqrt{\lambda} + \lambda\left(1 - \frac{\lambda}{3}\right)\cos\sqrt{\lambda}\right),$$

$$\text{Mean} = \frac{1}{7} + \frac{277}{31752}\,c$$

Figure 37 $S_{37} = \int_0^1 \int_0^1 \left(K(s,t) + cK^{(2)}(s,t)\right) dW(s)\, dW(t),$

$$K(s,t) = 1 - \max(s,t) - \frac{7}{16}(1-s^4)(1-t^4),$$

$$D(\lambda) = \frac{126}{\lambda^4}\left(\sqrt{\lambda}(2 - 2\lambda + \frac{\lambda^2}{2}) \sin\sqrt{\lambda} - \lambda(2 - \frac{4\lambda}{3} + \frac{\lambda^2}{10}) \cos\sqrt{\lambda}\right)$$



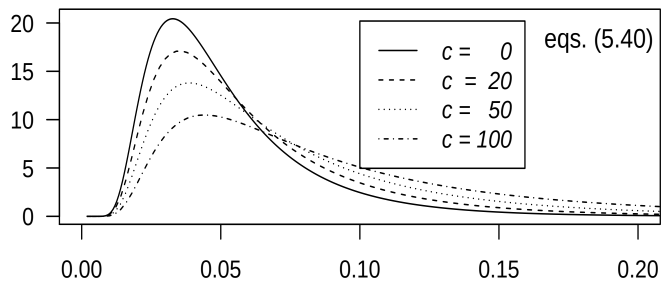


Figure 38 $S_{38} = \int_0^1 \int_0^1 \left(K(s,t) + cK^{(2)}(s,t)\right) dW(s)\, dW(t),$

$$K(s,t) = 1 - \max(s,t) + \frac{2}{9}(1-s)(1-t)$$

$$\times (5(s^2 + t^2 - 8s^2t^2 + st(s+t)) - 4(1+s)(1+t)),$$

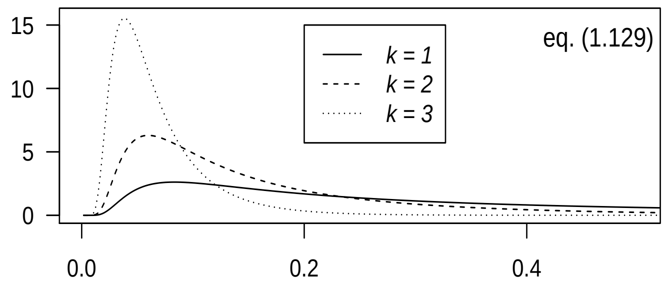
$$D(\lambda) = \frac{80}{\lambda^4}(12(2+\lambda) - 4\sqrt{\lambda}(6+\lambda)\sin\sqrt{\lambda} - (24-\lambda^2)\cos\sqrt{\lambda})$$

Figure 39 $S_{39} = \int_0^1 G_k^2(t)\,dt,$

$$G_1(t) = W(t), \quad G_2(t) = W(t) - \int_0^1 W(s)\,ds,$$

$$G_3(t) = W(t) + (6t-4)\int_0^1 W(s)\,ds - (12t-6)\int_0^1 sW(s)\,ds,$$

$$\text{Mean} = \frac{1}{2}, \frac{1}{6}, \frac{1}{15} \quad (k = 1, 2, 3)$$



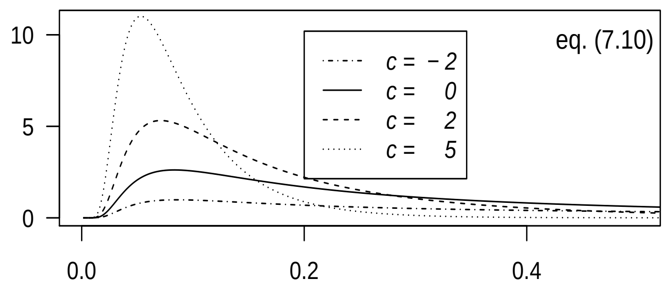


Figure 40 $S_{40} = \int_0^1 Y^2(t)\,dt, \quad Y(t) = e^{-ct}\int_0^t e^{cs}\,dW(s),$

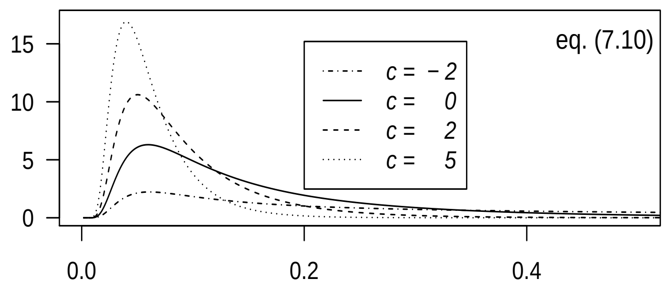
$$D(\lambda) = e^{-c}\left(\cos\mu + c\frac{\sin\mu}{\mu}\right), \quad \mu = \sqrt{\lambda - c^2},$$

$$\text{Mean} = \frac{1}{4c^2e^{2c}}\left((2c-1)e^{2c} + 1\right)$$

Figure 41 $S_{41} = \int_0^1 \left(Y(t) - \int_0^1 Y(s)\, ds \right)^2 dt, \quad Y(t) = e^{-ct} \int_0^t e^{cs}\, dW(s),$

$$D(\lambda) = e^{-c} \left(\frac{\lambda - c^3}{\mu^2} \frac{\sin\mu}{\mu} - \frac{c^2}{\mu^2} \cos\mu - \frac{2c\lambda}{\mu^4} (\cos\mu - 1) \right), \quad \mu = \sqrt{\lambda - c^2},$$

$$\text{Mean} = \frac{1}{4c^3 e^{2c}} \left((2c^2 - 5c + 6) e^{2c} - 8e^c + c + 2 \right)$$



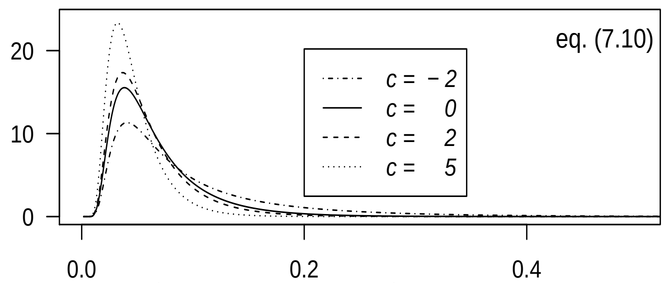


Figure 42 $S_{42} = \int_0^1 \left(Y(t) + (6t - 4) \int_0^1 Y(s)\, ds - (12t - 6) \int_0^1 sY(s)\, ds \right)^2 dt,$

$$D(\lambda) = e^{-c} \left(\frac{c^5 - 4\lambda(c^2 - 3c - 3)}{\mu^4} \frac{\sin\mu}{\mu} - \frac{24\lambda^2(c+1)}{\mu^6} \left(\frac{\sin\mu}{\mu} + \frac{\cos\mu}{\mu^2} - \frac{1}{\mu^2} \right) \right.$$
$$\left. + \left(\frac{c^4}{\mu^4} + \frac{8\lambda c^3}{\mu^6} \right) \cos\mu + \frac{4\lambda c^2 (c+2)}{\mu^6} \right),$$

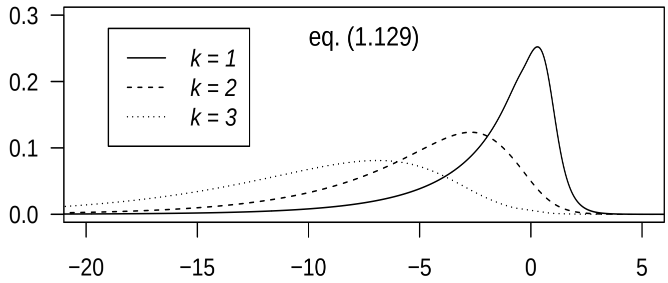
$$\text{Mean} = \frac{1}{4c^5 e^{2c}} \left((2c^4 - 9c^3 + 24c^2 - 24c - 24) e^{2c} + 16(c^2 + 3c) e^c \right.$$
$$\left. + c^3 + 8c^2 + 24c + 24 \right)$$

Figure 43 $S_{43} = \int_0^1 G_k(t)\,dW(t) \Big/ \int_0^1 G_k^2(t)\,dt,$

$$G_1(t) = W(t), \quad G_2(t) = W(t) - \int_0^1 W(s)\,ds,$$

$$G_3(t) = W(t) + (6t-4)\int_0^1 W(s)\,ds - (12t-6)\int_0^1 sW(s)\,ds,$$

Mean = −1.7814, −5.3791, −10.2455 $(k = 1, 2, 3)$



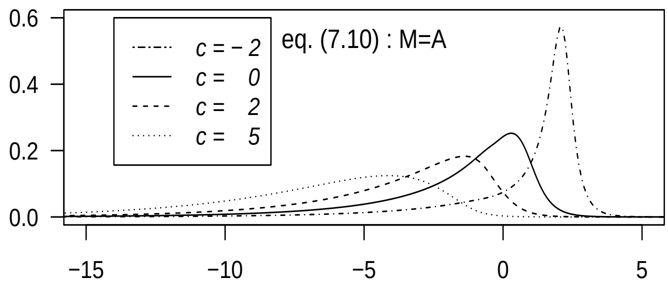


Figure 44 $S_{44} = \int_0^1 Y(t)\,dY(t) \Big/ \int_0^1 Y^2(t)\,dt,$

$$Y(t) = e^{-ct}\int_0^t e^{cs}\,dW(s),$$

Mean = 0.7488, −1.7814, −3.9299, −6.9758 $(c = -2, 0, 2, 5)$,

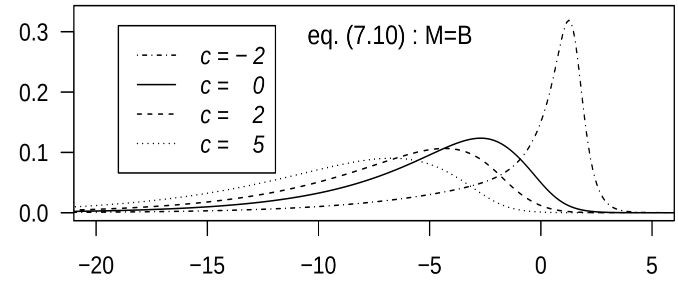
$x(0.01) = -13.6954, \quad x(0.05) = -8.0391, \quad x(0.1) = -5.7137 \quad (c = 0)$

Figure 45 $S_{45} = \int_0^1 G(t)\,dY(t) \Big/ \int_0^1 G^2(t)\,dt,$

$$G(t) = Y(t) - \int_0^1 Y(s)\,ds, \quad Y(t) = e^{-ct}\int_0^t e^{cs}\,dW(s),$$

Mean = −1.1866, −5.3791, −7.3742, −9.9345 $(c = -2, 0, 2, 5)$,

$x(0.01) = -20.6259, \quad x(0.05) = -14.0936, \quad x(0.1) = -11.2506 \quad (c = 0)$



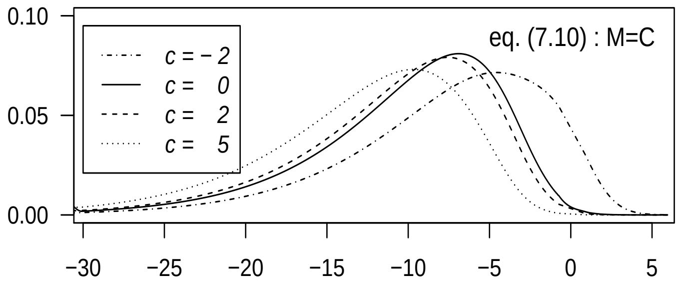


Figure 46 $S_{46} = \int_0^1 H(t)\,dY(t) \Big/ \int_0^1 H^2(t)\,dt, \quad Y(t) = e^{-ct}\int_0^t e^{cs}\,dW(s),$

$$H(t) = Y(t) + (6t-4)\int_0^1 Y(s)\,ds - (12t-6)\int_0^1 sY(s)\,ds,$$

Mean = −7.6596, −10.2455, −10.9622, −13.0996 $(c = -2, 0, 2, 5)$,

$x(0.01) = -29.3586, \quad x(0.05) = -21.7112, \quad x(0.1) = -18.2453 \quad (c = 0)$

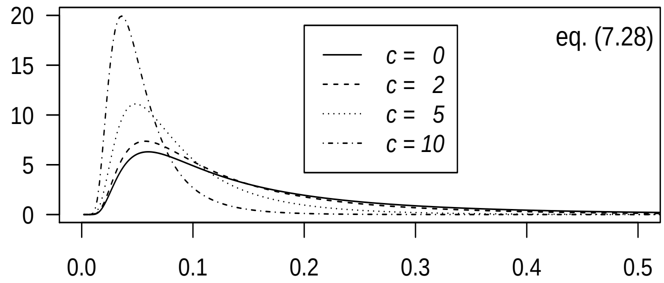
Figure 47 $S_{47} = \int_0^1 \left(Y(t) - tY(1)\right)^2 dt,$

$$Y(t) = e^{-ct} \int_0^t e^{cs}\, dW(s), \quad \mu = \sqrt{\lambda - c^2}, \quad \lambda = 2i\theta,$$

$$D(\lambda) = e^{-c}\left[\frac{c^4}{\mu^4}\cos\mu - \left(\frac{3c^3 - \lambda(c^2+3c+3)}{3\mu^2} + \frac{\lambda c^2}{\mu^4}\right)\frac{\sin\mu}{\mu}\right],$$

Mean = 0.1667, 0.1386, 0.0913, 0.0552 $(c = 0, 2, 5, 10)$,

$x(0.01) = 0.03446, \quad x(0.05) = 0.05646, \quad x(0.1) = 0.07654 \quad (c = 0)$



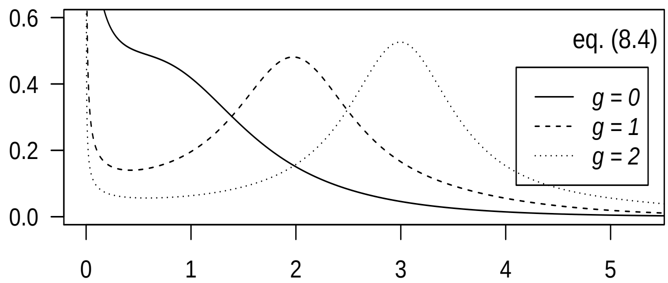


Figure 48 $S_{48} = \frac{1}{2} F_g^2(1) \Big/ \int_0^1 F_g^2(t)\, dt,$

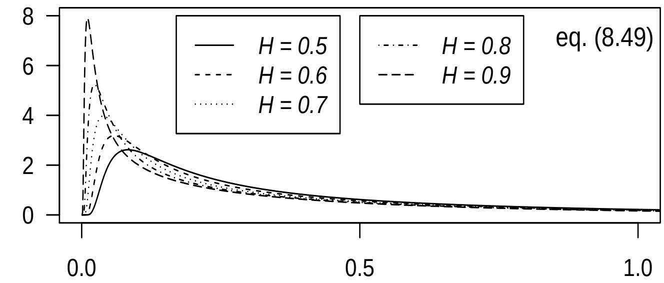
$$F_g(t) = \int_0^t F_{g-1}(s)\, ds = \int_0^t \frac{(t-s)^g}{g!}\, dW(s),$$

Mean = $g + 1$

Figure 49 $S_{49} = \int_0^1 \int_0^1 (st)^{2H-1} (\min(s,t))^{2-2H} \, dW(s) \, dW(t),$

$$D(\lambda) = \Gamma(1-\nu) J_{-\nu}(\eta(\lambda)) \Big/ \left(\frac{\eta(\lambda)}{2}\right)^{-\nu}, \quad \text{Mean} = \frac{1}{2H+1},$$

$$\lambda = 2i\theta, \quad \eta(\lambda) = \frac{\sqrt{2(1-H)\lambda}}{H+1/2}, \quad \nu = \frac{2H-1/2}{H+1/2}$$



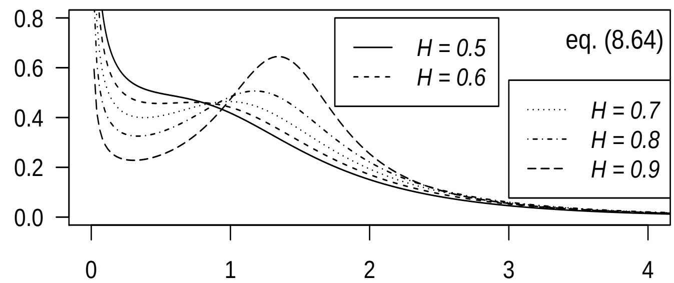


Figure 50 $S_{50} = \dfrac{\frac{1}{2}\left(\int_0^1 u^{1/2-H} \, dW(u)\right)^2}{\int_0^1 t^{4H-2} \left(\int_0^t u^{1/2-H} \, dW(u)\right)^2 dt}, \quad \text{Mean} = H + \frac{1}{2},$

$$D(\lambda; x) = \Gamma(1-\nu) \left(J_{-\nu}(\kappa) + \frac{\lambda}{2} \frac{J_{1-\nu}(\kappa)}{\sqrt{\lambda x}}\right) \Big/ \left(\frac{\kappa}{2}\right)^{-\nu},$$

$$\lambda = 2i\theta, \quad \nu = \frac{2H-1/2}{H+1/2}, \quad \kappa = \frac{\sqrt{\lambda x}}{H+1/2}$$